EQUILIBRIUM FLUCTUATIONS FOR INTERACTING
ORNSTEIN-UHLENBECK PARTICLES.

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ABSTRACT. We study the hydrodynamic density fluctuations of an infinite system of
interacting particles on $\mathbb{R}^d$. The particles interact between them through a two body
superstable potential, and with a surrounding fluid in equilibrium with a random vis-
cous force of the Ornstein-Uhlenbeck type. The stationary initial distribution is the
Gibbs measure associated with the potential and with a given temperature and fu-
gacity. We prove that the time-dependent density fluctuating field converges in law,
under diffusive scaling of space and time, to the solution of a linear stochastic partial
differential equation driven by white noise.

1. INTRODUCTION

Consider an aqueous suspension of particles in equilibrium at temperature $T = 1/\beta$.
Let the interaction between the particle be given by a two body potential $V$. We assume
that $V$ is superstable, positive and smooth with compact support. The interaction with
the fluid is modeled by an Ornstein-Uhlenbeck type force (linear viscosity plus white
noise) such that the particles velocities are maintained in equilibrium, i.e. distributed by
a Maxwellian distribution of temperature $T$ (cf. [13], [19]). The dynamics of this model
is given by the solution of the following infinite system of stochastic differential equations:

$$
\begin{align*}
\frac{dx_j(t)}{dt} &= v_j(t) \\
\frac{dv_j(t)}{dt} &= -\sum_{i\neq j} \nabla V(x_j(t) - x_i(t))dt - \gamma v_j(t)dt + \sqrt{\frac{2\gamma}{\beta}} dw_j(t)
\end{align*}
$$

(1.1)

Here $\{w_j(t)\}$ are independent standard Wiener processes, $\gamma$ is the friction coefficient. We
set here the mass of each particle as equal to 1. We refer at this system as the interacting
Ornstein-Uhlenbeck particles (IOU).

Consider the grand canonical Gibbs measures $\mu_{\beta}$ corresponding to the formal Hamiltonian

$$
\mathcal{H} = \sum_j \frac{v_j^2}{2} + \frac{1}{2} \sum_{i\neq j} V(x_j - x_i)
$$

(1.2)

with fugacity $\beta$ and inverse temperature $\beta$.

As in the case of the corresponding deterministic Hamiltonian system ($\gamma = 0$), the
proof of the existence of the dynamics given by (1.1) for a wide set of initial configurations
is a challenging problem. In [7], J. Fritz proves the existence and uniqueness of the solution of (1.1)
for a class of initial configurations that has probability 1 for any grand canonical Gibbs measure. The results contained in [7] are limited to dimension $d \leq 2$. 

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We thanks J. Fritz for fruitful discussions, in particular about the existence of the infinite dynamics.
A special thanks to L. Bertini for the help in the proof of the spectral gap estimate (cf. appendix B).
For our purposes it is enough to consider the existence of the \textit{equilibrium dynamics}: for a given \(\mu_{z,\beta}\) there exists a set of initial configurations and a set of realizations of the Wiener processes \(\{w_j\}\) such that they have full measure and for which (1.1) has a non exploding solution (see section 2 for a precise definition). We prove this existence theorem in the appendix A (section 7), by approximation with finite local dynamics.

The proof, that has the advantage that works in any dimension, follows the classical approach of Lanford ([10, 12]) adapted to our stochastic case. A certain care should be done in this proof for existence of the dynamics when dealing with random evolutions. In fact an approximation by local dynamics defined using reflection on hard walls would bring difficulties concerning the existence of these local dynamics (cf. the paper of R. Lang with the addition of T. Shiga [11], where a similar problem arise in the context of the interacting Brownian motions). In order to avoid such problems, we apply an idea of J. Fritz ([8, 9]; define special \textit{smooth} local dynamics such that the stationary Gibbs measures stay unmodified (cf. equations (7.38)).

The purpose of this paper is to study the macroscopic behavior of this system in equilibrium. Let us consider the positions of the particles under diffusive rescaling:

\[ x_j^\varepsilon(t) = \varepsilon x_j(\varepsilon^{-2}t) \]  

where \(\varepsilon\) is a scaling parameter. We are interested in the macroscopic limit as \(\varepsilon \to 0\).

The empirical distribution of the particles is a random positive measure on \(\mathbb{R}^d\) defined by

\[ n_\varepsilon(G) = \varepsilon^d \sum_j G(x_j^\varepsilon(t)) \]

where \(G\) is a smooth function with compact support. If the particles are distributed (in the original microscopic coordinates \(x_j\)) by the equilibrium Gibbs measure \(\mu_{z,\beta}\), by the law of large numbers, we have

\[ n_\varepsilon(G) \to \rho \int G(q) dq \quad \mu_{z,\beta} - a.s. \]

where \(\rho = \rho(z, \beta)\) is the average density of particles per unit volume for \(\mu_{z,\beta}\).

Then we consider the density fluctuation field

\[ \xi_\varepsilon(G) = \varepsilon^{-d/2} \left( n_\varepsilon(G) - \rho \int G(q) dq \right) \]

In equilibrium, the positions of the particles are distributed by \(\mu_{z,\beta}\), and because the mixing properties of this Gibbs measure one can show (cf. [1]) that \(\xi_\varepsilon(J)\), for a fixed \(t\), converges in law to a Gaussian field \(\xi\) on \(\mathbb{R}^d\) with mean zero and covariance

\[ <\xi(f)\xi(g)> = \chi \int f(q) g(q) \ dq \]  

where \(\chi = \chi(z, \beta)\) is the compressibility of \(\mu_{z,\beta}\).

We prove in this paper that \(\xi_\varepsilon\), as a distribution valued process, converges in law to the solution \(\xi_t\) of the stochastic linear partial differential equation:

\[ \partial_t \xi = D(\rho) \Delta \xi + \sqrt{\frac{2\rho}{\gamma}} \nabla 
\]

where \(\{\delta\}_{i=1,\ldots,d}\) are \(\delta\)-correlated space–time white noise, i.e. \(d\)-vector Gaussian field on \(\mathbb{R}^{d+1}\) with covariance

\[ <j_i(q,t) j_h(q',s)> = \delta(q-q') \delta(t-s) \delta_{i,h}. \]
The diffusion coefficient \( D(\rho) \) is identified as the derivative of the thermodynamic pressure as a function of the density, as in [16].

The equation (1.5) should be intended in the weak sense, i.e. for any smooth test function \( G \)

\[
\xi_t(G) - \xi_0(G) = \int_0^t \xi_s(D\Delta G) \, ds + M_t(\nabla G)
\]

where \( M_t(\nabla G) \) is a continuous martingale with constant quadratic variation given by \( \frac{2\mu}{\gamma \rho} \int |\nabla G(q)|^2 dq \). Since (1.5) is obtained in equilibrium, the initial condition \( \xi_0 \) is distributed by the Gaussian field with covariance given by (1.4), which is in fact the invariant law for the evolution given by (1.5). This implies the following identity for the bulk diffusion coefficient:

\[
D = \frac{\rho}{\gamma \beta \chi}
\]  

(1.6)

If we consider the IOU system (1.1) in a long time scale, the velocities will relax to equilibrium. If we suppress them we obtain a closed evolution on the positions given by

\[
dx_j(t) = -\frac{1}{\gamma} \sum_{i \neq j} \nabla V(x_j(t) - x_i(t)) \, dt + \sqrt{\frac{2}{\gamma \beta}} \, dw_j(t)
\]  

(1.7)

We refer to this system as the interacting Brownian particles (IB). In [19] Spohn proved the convergence of the density fluctuation fields \( \xi_i^j \) for (1.7). The limit equation is still given by (1.5). The equivalence of the bulk diffusion in the two systems could also be seen from the hydrodynamic limit out of equilibrium (cf. [22, 16]).

As noticed first by Rost [18], the main point in the proof of a theorem of equilibrium fluctuations for a system with conserved quantities is the so-called Boltzmann-Gibbs principle. This principle is an estimate of the space-time variance of the difference between a non-conserved quantity (typically the flux of the density fluctuation field) and its linearization along the conserved quantities (here the density fluctuation field itself). This principle has been proven in various reversible models (in [2] for zero-range models, in [5] for speed-change exclusion models and in [19] for the interacting Brownian particle system (1.7)). In all these papers the approach to the proof of the Boltzmann-Gibbs principle is the following. One defines an Hilbert space of functions of the configurations, with a scalar product related to the static variance of these functions with respect to the stationary measure of the process. Then one needs to identify the subspace that is invariant under the action of the semigroup of the generator of the process with the one dimensional subspace generated by the density fluctuation field. This identification follows by a strong version of the equivalence of the canonical and grand canonical Gibbs measures. One of the difficulties in this approach is the selfadjoint extension of the generator of the process in this new Hilbert space (observe that all these results concern reversible models).

In [3], C. C. Chang introduced a different, and simpler, approach to the proof of the Boltzmann-Gibbs principle. In the Chang’s approach, by using the same strong form of the equivalence of ensembles, the problem is reduced to the estimate of the space-time variance in the canonical measure in a fixed finite volume. Then this estimate can be done by standard finite dimensional analysis (cf. [6] chapter 11, for a clear exposition of the method). Beside its simplicity, this approach has two other advantages: is avoids problems in defining the dynamics in a different Hilbert space, and it can be extended more easily to non-reversible models (cf. [4]).

Our proof follows more the direction of this second approach. The difficulty here is due to the fact that the generator of our Markov process is degenerate in the directions
of the positions of the particles (noise acts only on the velocities). This poses a problem even in the finite dimensional analysis needed to estimate the variance in the canonical measure on a finite box. We solve this problem by relating the (degenerate) generator of our process to the strictly elliptic and symmetric generator of the interacting Brownian particles (cf. proof of proposition 4.4). Then we need an estimate of the spectral gap of this symmetric operator, that we prove in appendix B.

For the strong form of the equivalence of ensembles, which is in fact a local central limit theorem for the Gibbs measures, we use the results by Spohn contained in [19].

The paper is organized as follows: in section 2 we give a precise definition of the model and of the dynamics in equilibrium. In section 3 we study the evolution equations of the fluctuation field. In section 4 we prove the Boltzmann-Gibbs principle for our system. In section 5 we prove the tightness of the distribution of the fluctuation field as a stochastic process with values in a certain negative Sobolev space. In section 6, by using the Boltzmann-Gibbs principle and the compactness result, we prove the convergence to the solution of (1.5). In appendix A we prove the existence of the dynamics in equilibrium, In appendix B we prove a lower bound on the spectral gap of the generator of the interacting Brownian motions (cf. (1.7)) in a finite box. We need this bound in the proof of the Boltzmann-Gibbs principle.

2. THE DYNAMICS IN EQUILIBRIUM

We assume that $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is a pair potential satisfying the following assumptions:

(i) $V$ is symmetric: $V(q) = V(-q)$,
(ii) $V$ is positive: $V \geq 0$,
(iii) $V$ has finite range: $V(q) = 0$ if $|q| \geq R$.
(iv) $V$ is three times continuously differentiable,
(v) $V$ is superstable, i.e. satisfies Ruelle's condition (SS) in [15].

Notice that the positivity implies immediately the lower regularity condition (LR) of [15]. Furthermore, because of (ii), if $V(0) > 0$, then $V$ is superstable.

The configuration space $\Omega$ is defined as the set of locally finite labeled configurations of particles $\omega = \{(x_i, v_i), i \in \mathbb{N}\}$, where $x_i = x_i(\omega)$ and $v_i = v_i(\omega)$ have values in $\mathbb{R}^d$, and the sequence $x_i = x_i(\omega)$ has no accumulation points. We will use the notation $\omega^x = \{x_i(\omega)\}$. Let $\Omega$ be equipped with the weak topology: $\lim_n \omega_n = \omega$ means that $\lim_n x_i(\omega_n) = x_i(\omega)$ and $\lim_n v_i(\omega_n) = v_i(\omega)$. We denote by $\mathcal{B}$ the corresponding Borel $\sigma$-algebra of subsets of $\Omega$.

A grand canonical Gibbs state for $V$ at temperature $\beta^{-1}$ and activity (or fugacity) $z$ is a probability measure $\mu$ on $(\Omega, \mathcal{B})$ that distribute $\omega^x$ according to a grand canonical Gibbsian point field with pair interacting potential $\beta V$ and activity $z$, while velocities are independent of positions and are identically distributed as independent Gaussian variables of zero means and variance $\beta^{-1}$. This means that the $\mu$-conditional probability to find $n$ particles in a finite volume region $\Lambda \subset \mathbb{R}^d$ with configuration $\{(x_1, v_1), \ldots, (x_n, v_n)\}$, conditioned to the configuration outside $\Lambda$, i.e. $\omega^{\Lambda^c} = \{(x_i(\omega), v_i(\omega)), x_i(\omega) \in \Lambda^c, i \in \mathbb{N}\}$, has density respect to the Lebesgue measure on $(\Lambda \times \mathbb{R}^d)^n$ given by

$$
\mu_\Lambda((x_1, v_1), \ldots, (x_n, v_n)|\omega^{\Lambda^c}) = \frac{1}{Z_{\Lambda}(\omega^{\Lambda^c}) n!} \exp \left[ -\beta h_n((x_1, v_1), \ldots, (x_n, v_n); \omega^{\Lambda^c}) \right]
$$  \hfill (2.8)


where

$$\mathcal{H}_n((x_1,v_1),\ldots,(x_n,v_n);\omega_N^x) = \sum_{j=1}^n \left\{ \frac{v_j^2}{2} + \frac{1}{2} \sum_{i=1}^n V(x_j - x_i) + \sum_{y_i \in \omega_N^x} V(x_j - y_i) \right\}$$

and $Z_A(\omega_N^x)$ is the corresponding normalization.

In order to simplify notations we fix the values $\gamma = \beta = 1$, and, since the activity $z \in (0,z_0)$ is fixed, we will omit to write the explicit dependence on $z$ and we write $\mu = \mu_{z,\beta}$, and $\rho = \rho(z,\beta)$ for the corresponding density of particles.

In the following we will assume that

$$0 < z < z_0 = 0.28 \left( e \int (1 - e^{-V(q)}) dq \right) \quad .$$

We need this condition on the activity in order to apply Spohn’s results on the equivalence of ensembles (cf. [19]). As a consequence we are in the low fugacity regime where the grand canonical Gibbs measure is unique.

We give now a precise definition of the dynamics in equilibrium associated to (1.1) and to a grand canonical Gibbs measure $\mu$.

Let $\mathcal{W}$ the probability measure induced on $\mathfrak{M} = \mathfrak{C}([0,\infty),\mathbb{R}^d)^\mathbb{N}$ by the infinite independent Wiener processes $\{w_i(\cdot), i \in \mathbb{N}\}$. On $\Omega \times \mathfrak{M}$ we define the product measure $\mathbb{P}_\mu = \mu \otimes \mathcal{W}$, and we denote $\mathbb{E}_\mu$ the corresponding expectation. On $\Omega \times \mathfrak{M}$ we also define the increasing filtration $\{\mathcal{F}_t\}_{t \geq 0}$, where $\mathcal{F}_t$ is the $\sigma$-algebra generated by $\{w_i(s), s \leq t, i \in \mathbb{N}\}$ and $\mathcal{B}$.

Given an initial configuration $\omega(0)$, a solution of (1.1) is a $\mathcal{F}_t$-adapted continuous stochastic process $\{\omega(t)\}$ with values in $\Omega$, which satisfies

$$x_i(\omega(t)) = x_i(\omega(0)) + \int_0^t v_i(\omega(s)) \, ds$$

$$v_i(\omega(t)) = v_i(\omega(0)) - \int_0^t \sum_{j \neq i} \nabla V(x_i(\omega(s)) - x_j(\omega(s))) \, ds$$

$$- \int_0^t v_i(\omega(s)) ds + \sqrt{2} \, w_i(t) \quad (2.9)$$

In the appendix A it is proven that there exists a set $M \subset \Omega \times \mathfrak{M}$ such that $\mathbb{P}_\mu(M) = 1$, and such that if $\{\omega(0),\{w_i(\cdot)\}_{i \in \mathbb{N}}\} \in M$, then (2.9) has a solution. This way is possible to define a strongly continuous semigroup of contraction operators $P^t$ on $L^2(\Omega,\mathcal{B},\mu)$. A straightforward calculation shows that the generator of this process (1.1) can be written as a sum of a symmetric operator and an antisymmetric one (with respect to $\mu$):

$$L = A + S$$

$$A = \{\mathfrak{H}, \cdot \} = \sum_j \left( v_j \cdot \nabla x_j - \sum_{i \neq j} \nabla V(x_j - x_i) \cdot \nabla v_j \right)$$

$$S = \sum_j (\Delta v_j - v_j \cdot \nabla v_j)$$

(2.10)

Observe that the antisymmetric operator $A$ is given by the Poisson brackets with the Hamiltonian $\mathfrak{H}$, i.e. is the generator of the corresponding deterministic Hamiltonian dynamics.
3. Time Evolution of the Fluctuation Field

For every configuration $\omega \in \Omega$ and $\epsilon > 0$, the density fluctuation field is the measure on $\mathbb{R}^d$ defined by

$$\xi^\epsilon(G)(\omega) = \epsilon^{-d/2} \left( \epsilon^d \sum_j G(\epsilon x_j(\omega)) - \rho \int G(q) \, dq \right)$$

where $G$ is a continuous function with compact support on $\mathbb{R}^d$.

Observe that $\xi^\epsilon(G)(\omega) = \xi^1(G_\epsilon)(\omega)$ where $G_\epsilon(q) = \epsilon^{d/2} G(\epsilon q)$. It is easy to see that $\xi^\epsilon(G) \in L^2(\Omega, \mu)$. In fact we have the following bound:

**Lemma 3.1.** There exists a constant $B = B(\rho) < \infty$ such that for any $G$:

$$\sup_{\epsilon > 0} \langle \xi^\epsilon(G)^2 \rangle \leq B \|G\|_{L^2}^2$$

(3.11)

**Proof.** Using the 2-point correlation function $\rho_2$ of the grand canonical Gibbs measure $\mu$, and Schwarz inequality and the translation invariance of $\mu$:

$$\langle \xi^\epsilon(G)^2 \rangle = \int \int G_\epsilon(q_1)G_\epsilon(q_2)(\rho_2(q_1, q_2) - \rho^2) \, dq_1 \, dq_2$$

$$\leq \left( \int G_\epsilon(q_1)^2 \, dq_1 \right) \int |\rho_2(0, q_2) - \rho^2| \, dq_2 = \|G\|_{L^2}^2 \int |\rho_2(0, q) - \rho^2| \, dq .$$

This last quantity is finite by lemma 4.4.8 of [14].

In order to simplify notations, we will denote $\xi^\epsilon_t = \xi^\epsilon(\omega(\epsilon^{-2}t))$. For a fixed arbitrary $T > 0$, we denote by $P^\epsilon$ the distribution, under $P_\mu$, of $\{\xi^\epsilon_t, 0 \leq t \leq T\}$ in $\mathcal{C}([0, T], M(\mathbb{R}^d))$.

Consider now a smooth test function $G$ with compact support. By a simple calculation we have

$$\xi^\epsilon_t(G) - \xi^\epsilon_0(G) = \sqrt{2} \epsilon^{d/2+1} \int_0^{\epsilon^{-2}t} \sum_j \nabla G(\epsilon x_j(s)) \cdot d\mathbf{w}_j(s) + \epsilon^2 \int_0^{\epsilon^{-2}t} \gamma_\epsilon(\omega(s)) \, ds$$

$$- \epsilon^{d/2+1} \sum_j \left( \nabla G(\epsilon x_j(\epsilon^{-2}t)) \cdot v_j(\epsilon^{-2}t) - \nabla G(x_j^0(0)) \cdot v_j(0) \right)$$

(3.12)

where

$$\gamma_\epsilon(\omega) = \epsilon^{d/2} \sum_j \left( \sum_{a, \sigma = 1}^d \partial_a \partial_{\sigma} G(\epsilon x_j) v_j^a \sigma - \epsilon^{-1} \sum_{i \neq j} \nabla G(\epsilon x_j) \cdot \nabla V(x_j - x_i) \right)$$

(3.13)

It is easy to see that variance of the last term in the rhs of (3.12) converges to 0 as $\epsilon \to 0$. In fact, by stationarity we have

$$\mathbb{E}_\mu \left( \epsilon^{d/2+1} \sum_j \left( \nabla G(\epsilon x_j(\epsilon^{-2}t)) \cdot v_j(\epsilon^{-2}t) - \nabla G(x_j^0(0)) \cdot v_j(0) \right) \right)^2$$

$$\leq 2\epsilon^2 \left( \epsilon^{d/2} \sum_j \nabla G(\epsilon x_j) \cdot v_j \right)^2 \mu$$

and by the independence of the velocities:

$$\epsilon^{d/2} \sum_j \nabla G(\epsilon x_j) \cdot v_j \mu = \epsilon^d \sum_j |\nabla G(\epsilon x_j)|^2 \mu = \rho \int |\nabla G(q)|^2 \, dq .$$

(3.14)
The main step we need to do, in order to obtain (1.5) as a macroscopic equation, is to prove that:

$$\lim_{\varepsilon \to 0} \mathbb{E}_\mu \left( \left[ \int_0^t (\gamma_\varepsilon(\omega(\varepsilon^{-2}s)) - D\xi_s^\varepsilon(\Delta G)) \, ds \right]^2 \right) = 0 \quad (3.15)$$

For any continuous function $f$ on $\mathbb{R}^d$ with compact support, let us define the local function

$$\Upsilon_{\alpha, \sigma}(f) = \sum_{i,j} (x_i^\sigma - x_j^\sigma) V_\alpha(x_i - x_j) \int_0^1 f(\lambda x_i + (1 - \lambda)x_j) d\lambda. \quad (3.16)$$

**Lemma 3.2.** Let $f$ be a continuous function with compact support such that $\int f(q) \, dq = 0$. Then there exists a constant $C = C(\rho)$ such that

$$< (\Upsilon_{\alpha, \sigma}(f))^2 >_\mu \leq C \|f\|_{L^2}^2.$$  

**Proof.** This is basically (4.13) in [19]. Since $\int f(q) \, dq = 0$, we have

$$< \Upsilon_{\alpha, \sigma}(f) > = \int_0^1 d\lambda \int\int (q_1^\sigma - q_2^\sigma) V_\alpha(q_1 - q_2)f(\lambda q_1 + (1 - \lambda)q_2) \rho_2(q_1, q_2) \, dq_1 \, dq_2$$

$$= \int_0^1 d\lambda \int dz \, z^\sigma V_\alpha(z) \rho_2(0, z) \int dq_2 f(\lambda z + q_2) = 0$$

Let us denote by $\tilde{\rho}_4$ the truncated 4-point correlation function of $\mu$, i.e.

$$\tilde{\rho}_4(q_1, q_2, q_3, q_4) = \rho_4(q_1, q_2, q_3, q_4) - \rho_2(q_1, q_2) \rho_2(q_3, q_4)$$

In order to simplify notations, let us define

$$g(q_1, q_2, \lambda) = (q_1^\sigma - q_2^\sigma) V_\alpha(q_1 - q_2)f(\lambda q_1 + (1 - \lambda)q_2).$$

Then, by using Jensen inequality, we have

$$< (\Upsilon_{\alpha, \sigma}(f))^2 >_\mu = < (\Upsilon_{\alpha, \sigma}(f) - < \Upsilon_{\alpha, \sigma}(f) >)^2 >_\mu$$

$$\leq \int_0^1 d\lambda \left( \sum_{(i,j)} g(x_i, x_j, \lambda) - \int g(q_1, q_2, \lambda) \rho_2(q_1, q_2) \, dq_1 \, dq_2 \right)^2 \mu \quad (3.17)$$

$$= \int_0^1 d\lambda \int\int\int\int dq_1 dq_2 dq_3 dq_4 g(q_1, q_2, \lambda) g(q_3, q_4, \lambda) \tilde{\rho}_4(q_1, q_2, q_3, q_4)$$

By a linear change of variables in the quadruple integration and translation invariance, the rhs of (3.17) is equal to

$$\int_0^1 d\lambda \int\int\int\int z_1^\sigma V_\alpha(z_1) z_2^\sigma V_\alpha(z_3) f(\lambda z_1 + z_2)f(\lambda z_3 + z_4) \times \rho_4(z_1, 0, z_3 + z_4, z_4) \, dz_1 \, dz_2 \, dz_3 \, dz_4$$

which is bounded by

$$\left( \int f(z)^2 \, dz \right) \int\int\int\int dz_1 dz_2 |z_1^\sigma V_\alpha(z_1)| |z_2^\sigma V_\alpha(z_3)| \int dz_4 |\tilde{\rho}_4(z_1, 0, z_3 + z_4, z_4)|$$

The integral $\int dz_4 |\tilde{\rho}_4(z_1, 0, z_3 + z_4, z_4)|$ is finite by thm. 4.4.8 of [14]. \qed
Defining $G_{\alpha,\sigma}^{\varepsilon}(q) = \varepsilon^{d/2} G_{\alpha,\sigma}(\varepsilon q)$, then using the symmetry of $V$ we can rewrite

$$
\varepsilon^{d/2-1} \sum_{j} \sum_{i \neq j} \nabla G(\varepsilon x_j) \cdot \nabla V(x_j - x_i) = \frac{1}{2} \sum_{\alpha,\sigma} Y_{\alpha,\sigma}(G_{\alpha,\sigma}^{\varepsilon})
$$

Let $h(q)$ be a positive continuous function with support in $B(0,1/2)$, the ball centered at the origin and of radius 1/2, and with total integral equal to 1. Since $\|G_{\alpha,\sigma}^{\varepsilon} - G_{\alpha,\sigma}^{\varepsilon} * h\|_{L^2}$ converges to 0 as $\varepsilon \to 0$, by Lemma 3.2 we have

$$
\lim_{\varepsilon \to 0} < (Y_{\alpha,\sigma}(G_{\alpha,\sigma}^{\varepsilon} - G_{\alpha,\sigma}^{\varepsilon} * h))^2 > \mu \longrightarrow 0
$$

This implies, using Schwarz inequality and stationarity, that in (3.15) we can substitute $\gamma_\varepsilon$ with

$$
\tilde{\gamma}_\varepsilon(\omega) = \sum_{\alpha,\sigma} \varepsilon^{d/2} \int G_{\alpha,\sigma}(\varepsilon q) \phi_{\alpha,\sigma}(\tau_q \omega) \, dq
$$

where $\phi_{\alpha,\sigma}$ is the local function given by

$$
\phi_{\alpha,\sigma}(\omega) = \sum_{j} h(x_j) v_j^\alpha v_j^\beta - \frac{1}{2} Y_{\alpha,\sigma}(h)
$$

(3.18)

Observe that, by the virial theorem (cf. [17, 21]),

$$
< \phi_{\alpha,\sigma} > \mu = \delta_{\alpha,\sigma} P(\rho)
$$

where $P$ is the thermodynamic pressure as a function of the density $\rho$. Consequently $D = P(\rho)$.

The limit (3.15) is a direct consequence of the Boltzmann-Gibbs principle that we enunciate and prove in the next section.

4. BOLTZMANN-GIBBS PRINCIPLE

In this section we will consider the following set of smooth local functions on $\Omega$. We say that a measurable function $\phi : \Omega \to \mathbb{R}$ is in $\mathcal{E}$ if it can be written as

$$
\phi(\omega) = H \left( \sum_{i} f_1(x_i(\omega), v_i(\omega)), \ldots, \sum_{i} f_m(x_i(\omega), v_i(\omega)) \right)
$$

where \{f_k(q,v), k = 1, \ldots, m\} are smooth functions on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support in $q$ and growth at most linear in $v$ ($|f(q,v)| \leq C(1 + |v|)$), and $H(y_1, \ldots, y_m) \in C^\infty(\mathbb{R}^m)$ such that there exists a constant $c < \infty$ such that $|H(y_1, \ldots, y_m)| \leq c \sum_{k=1}^m |y_k|$.

If $\Lambda$ is a finite region containing all the $q$-supports of $f_k$, then there exists a finite constant $C$ such that for any configuration $\omega$

$$
|\phi(\omega)| \leq \exp \left( C \sum_{x_j(\omega) \in \Lambda} (1 + |v_j(\omega)|) \right)
$$

(4.19)

By (4.19) and by superstability estimate (cf. [15]), it follows that $\phi \in L^p(\mu)$ for any $p < \infty$. We denote its expectation by

$$
\hat{\phi}(\rho) = < \phi > \mu
$$

as a function of the density $\rho$. Always by the superstability estimates, $\hat{\phi}(\rho)$ is a smooth function of $\rho$. 
Let us define
\[ \Phi(\omega) = \phi(\omega) - \hat{\phi}(\rho) - \hat{\phi}'(\rho) \left( \sum_j h(x_j) - \rho \right) \]
where \( h : \mathbb{R}^d \to \mathbb{R}_+ \) is a positive smooth function with support in \( B(0, 1/2) \), the ball centered at the origin and of radius 1/2, and with total integral \( \int h(q) dq = 1 \).

The main result of this section is contained in the following proposition:

**Proposition 4.1 (Boltzmann-Gibbs principle).** Let \( \phi \in \mathcal{E} \) and \( \Phi \) defined as above. Then for any smooth function \( G \) with compact support on \( \mathbb{R}^d \)
\[ \lim_{\varepsilon \to 0} \mathbb{E}_\mu \left[ \left( \varepsilon^{d/2} \int_0^t ds \int dq \, G(\varepsilon q) \Phi(\tau_q \omega(\varepsilon^{-2}s)) \right)^2 \right] = 0 \]

We will make a wide use of the following estimate:

**Lemma 4.2.** There exists a finite constant \( C \), depending only on \( V \) and \( \rho \), such that for any \( G \) in \( L^2(\mathbb{R}^d) \) and for any local function \( \psi(\omega) \) in \( L^2(\mu) \) whose support is contained in a finite set \( \Lambda \subset \mathbb{R}^d \), we have:
\[ \left\langle \left( \int G(q) \psi(\tau_q \omega) \, dq \right)^2 \right\rangle \leq C |\Lambda| < \psi^2 > \int G(q)^2 \, dq \]

**Proof.** This is a consequence of the exponential decay of the correlations, cf. lemma 4 in [19], and the proof is very close to lemma 9 in [19].
\[
\left\langle \left( \int G(q) \psi(\tau_q \omega) \, dq \right)^2 \right\rangle = \iint G(q)G(q') \, \psi(\tau_q \omega) \psi(\tau_{q'} \omega) > dq \, dq' \\
\leq \frac{1}{2} \iint (G(q)^2 + G(q')^2) \, |\psi(\tau_q \omega) \psi(\tau_{q'} \omega)| > dq \, dq' \\
= \int dq \, G(q)^2 \int dq' \, |\psi(q_{q-q'} \omega) \psi(\omega)| > | \]

By lemma 4 in [19] there exist \( c \) and \( \alpha \) positive constants, depending only on \( V \) and \( \rho \), such that
\[ |\psi(\tau_q \omega) \psi(\omega)| \leq \psi^2 > \mu \min \{ 1, c |\Lambda| e^{-ad(q_1, \Lambda)} e^{|\Lambda|} e^{-a d(q_2, \Lambda)} \} \]
where \( d \) is the distance on \( \mathbb{R}^d \), and \( \Lambda = \{ q \in \mathbb{R}^d : d(q, \Lambda) \leq R \} \). \( (R \) being the radius of the support of the interaction \( V \)). Then performing the integration in \( q' \) one obtains (4.20).

The Proposition 4.1 will be proven in several steps. In the first one, we will condition \( \Phi \) on the positions configuration, that we denote by \( \omega^x \). We will use then the following Lemma:

**Lemma 4.3.** Let \( \Psi \) a local function on \( L^2(\mu) \) such that its \( \mu \)-expectation conditioned on the positions configuration \( \omega^x \) is
\[ < \Psi |\omega^x > = 0 \quad \mu - a.s. \]

Then for any smooth function \( G \) with compact support on \( \mathbb{R}^d \)
\[ \lim_{\varepsilon \to 0} \mathbb{E}_\mu \left[ \left( \varepsilon^{d/2} \int_0^t ds \int dq \, G(\varepsilon q) \Psi(\tau_q \omega(\varepsilon^{-2}s)) \right)^2 \right] = 0 \]
Proof. Observe that the operator $S$, the symmetric part of the generator of the process defined by (2.10), has a spectral gap. Since $<\Psi|\omega^x >= 0$, $S^{-1}\Psi(\omega)$ is a function in $L^2(\mu)$. Recall we have defined $\varepsilon^{d/2}G(\varepsilon\omega) = G_\varepsilon(\omega)$. It follows that $S^{-1}\int dq\ G_\varepsilon(q)|\Psi(\tau_q\omega)\in L^2(\mu)$. Then (cf. lemma 4.3 of [4]):

$$\mathbb{E}_\mu\left[\left(\int_0^1 ds\int dq\ G_\varepsilon(q)\Psi(\tau_q\omega(\varepsilon^{-2}s))\right)^2\right]$$

$$\leq 14\varepsilon^2\int dq\ G_\varepsilon(q)|\Psi(\tau_q\omega)|^2$$

By the spectral gap of $S$ this last quantity is bounded by:

$$\leq 14\varepsilon^2\left\|\int dq\ G_\varepsilon(q)|\Psi(\tau_q\omega)\right\|_{L^2(\mu)}^2$$

By lemma 4.2 this is bounded by

$$14\varepsilon^2\left\|G\right\|_{L^2}^2 C(\Psi)$$

\[\square\]

As a consequence of lemma 4.3, we only need to prove Proposition 4.1 for the function

$$\bar{\Phi}(\omega^x) = <\Phi|\omega^x>$$

Observe that $\bar{\Phi}$ is still a local smooth function satisfying (4.19).

For $l$ large, let $\Lambda$ be a centered box of size $l$. The $\mu$ canonical expectation of $\bar{\Phi}$ conditioned on the configuration of the positions of the particles outside $\Lambda$ (denoted by $\omega_{\Lambda^c}^x$), and on the number of particles in $\Lambda$ is defined by:

$$\Gamma_\Lambda\bar{\Phi}(\omega^x) = <\bar{\Phi}|N_\Lambda(\omega), \omega_{\Lambda^c}^x>$$

By DLR equations, it can be defined for every $\omega_{\Lambda^c}^x$ and it does not depend on $\rho$. Always by DLR equations, one can see that $\Gamma_\Lambda\bar{\Phi}$ is smooth on the set of configurations where there are no particle on the border of $\Lambda$. Since this set has $\mu$ full measure, we have that $\Gamma_\Lambda$ is smooth $\mu$-a.s.

It is proven in [19], eq. (8.5) at pag.17, that

$$\lim_{\Lambda\uparrow\mathbb{R}^d} |\Lambda| < (\Gamma_\Lambda\bar{\Phi})^2 = 0$$

(4.21)

Then, using Lemma 4.2, (4.21) implies

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \lim_{\varepsilon\to 0} <\varepsilon^{d/2}\int dq\ G(\varepsilon q)|\Gamma_\Lambda\bar{\Phi}(\tau_q\omega^x)^2 \geq 0$$

(this is basically lemma 9 in [19]). So we can recenter $\bar{\Phi}$ around its canonical expectation $\Gamma_\Lambda\bar{\Phi}$, and Proposition 4.1 will follow from the following one:

**Proposition 4.4.** For any smooth function $G$ with compact support on $\mathbb{R}^d$ and for any finite box $\Lambda$

$$\lim_{\varepsilon\to 0}\mathbb{E}_\mu\left[\varepsilon^{d/2}\int_0^t ds\int dq\ G(\varepsilon q)(\bar{\Phi} - \Gamma_\Lambda\bar{\Phi})(\tau_q\omega^x(\varepsilon^{-2}s))\right]^2 = 0$$
Proof. We denote by
\[ f(x_1, \ldots, x_N; \omega^x_{\Lambda^c}) = (\tilde{\Phi} - \Gamma_{\Lambda} \tilde{\Phi})(\omega^x) \]
and we will consider $f$ as a function of the positions of the particles inside $\Lambda^n$, for any fixed exterior configuration $\omega^x_{\Lambda^c}$ and any fixed number of particles $N_{\Lambda} = n$ inside $\Lambda$. Since the set of configurations with particles on the boundary of $\Lambda$ have null measure with respect to $\mu$, we can consider only configurations $\omega^x_{\Lambda^c}$ without particle on the boundary of $\Lambda$, and the dependence of $f$ on $\omega^x_{\Lambda^c}$ is smooth on this set of configurations. Furthermore, for any such $\omega^x_{\Lambda^c}$ fixed, $f$ is a smooth function of the positions $(x_1, \ldots, x_n)$ on the interior of $\Lambda^n$.

Then, for any $\omega^x_{\Lambda^c}$ and $N_{\Lambda} = n$ fixed, we consider the elliptic operator on $\Lambda^n$
\[ L^W_{n, \omega^x_{\Lambda^c}} = \sum_{j=1}^{n} \left( \Delta x_j - \sum_{i \neq j} (\nabla V)(x_j - x_i) \cdot \nabla x_j \right) \]  
with Neumann boundary conditions. Then, by the properties of $f$, it exists a smooth function $u(x_1, \ldots, x_n; \omega^x_{\Lambda^c})$ solution of
\[ -L^W_{n, \omega^x_{\Lambda^c}} u = f(x_1, \ldots, x_n; \omega^x_{\Lambda^c}) \]  
We consider now $u$ as a local function of $\omega^x$ (i.e. as a function of $\{x_1, \ldots, x_N, \omega^x_{\Lambda^c}\}$).

By lemma 9.1 in appendix 9, there exist $c_1, c_2$ finite constant independent of $n, \omega^x_{\Lambda^c}$ and $u$ such that
\[ \Gamma_{\Lambda} \left( \sum_{k=1}^{n} |\nabla x_k|^2 \right) (n, \omega^x_{\Lambda^c}) \leq c_1 n \exp \left\{ c_2 \left( n + N_{\partial^+ R^+ \Lambda} (\omega^x_{\Lambda^c}) \right) \right\} \Gamma_{\Lambda} \left( f^2 \right) (n, \omega^x_{\Lambda^c}) \]
where $\partial^+ R^+ \Lambda = \Lambda(l + R) \setminus \Lambda$. Integrating with respect to $\mu$ we obtain
\[ \left\langle \sum_{x_k \in \Lambda} |\nabla x_k|^2 \right\rangle \leq c_1 \left\langle N_{\Lambda} e^{c_2 N_{\Lambda}(l + R)} f^2 \right\rangle \]  
and the left hand side is bounded by (4.19) and superstability estimates.

The trick now is to relate $L^W_{n, \omega^x_{\Lambda^c}}$ to the generator of our process $L$ (cf. (2.10)). Define the local function
\[ F(\omega) = \sum_{x_j \in \Lambda} v_j \cdot \nabla x_j u \]
By (4.24), $F \in L^2(\mu)$, and by the results contained in appendix 9 it is in the domain of the generator $L$. Then we apply $L$ to $F$ and we obtain
\[ LF(\omega) = \sum_{x_j \in \Lambda} \sum_k (v_k \cdot \nabla x_k)(v_j \cdot \nabla x_j) u - \sum_{x_j \in \Lambda} \left( \sum_{i \neq j} \nabla V(x_j - x_i) \right) \cdot \nabla x_j u - F \]  
The first term on the rhs of (4.25) can be rewritten as
\[ \sum_{x_j \in \Lambda} \sum_k \sum_{a, \sigma} \sum_{d \alpha} v_k^a v_j^\alpha \theta_{x_k} \theta_{x_j} u = \sum_{x_j \in \Lambda} \Delta x_j u + \Psi_1(\omega) \]
where, by lemma 9.2, $\Psi_1(\omega)$ is a local function in $L^2(\mu)$ such that
\[ \langle \Psi_1 | \omega^x \rangle = 0 \quad \mu - a.s. \]
Observe that also $F$ has the same property. In conclusion we can write
\[ (\tilde{\Phi} - \Gamma_{\Lambda} \tilde{\Phi})(\omega^x) = L^W_{N_{\Lambda}, \omega^x_{\Lambda^c}} u = LF(\omega) + \Psi(\omega) \]
where $\Psi(\omega)$ is a local function in $L^2(\mu)$ such that

$$<\Psi|\omega^x> = 0 \quad \mu-a.s.$$ 

So we can apply Lemma 4.3, and we are left to prove

$$\lim_{\varepsilon \to 0} \mathbb{E}_\mu \left[ (\varepsilon^{d/2} \int_0^t ds \int dq \, G(\varepsilon q)(L^F)(\tau_{\varepsilon}\omega(\varepsilon^{-2} s)))^2 \right] = 0$$

Integrating in time we can rewrite

$$\varepsilon^{d/2} \int_0^t ds \int dq \, G(\varepsilon q)(L^F)(\tau_{\varepsilon}\omega(\varepsilon^{-2} s))$$

$$= \varepsilon^{d/2+2} \int dq \, G(\varepsilon q)((\tau_{\varepsilon}\omega(\varepsilon^{-2} t)) - F(\tau_{\varepsilon}\omega(0))) + \mathcal{M}_t^\varepsilon$$

where $\mathcal{M}_t^\varepsilon$ is a martingale such that

$$\mathbb{E}[(\mathcal{M}_t^\varepsilon)^2] = t \varepsilon^{d+2} \sum_k \left( \int dq \, G(\varepsilon q) \partial_{v_k} F(\tau_{\varepsilon}\omega) \right)^2$$

$$= t \varepsilon^{d+2} \int dq \, dq' \, G(\varepsilon q) G(\varepsilon q') \sum_k \partial_{v_k} F(\tau_{\varepsilon}\omega) \partial_{v_k} F(\tau_{\varepsilon}\omega)$$

and by Schwarz inequality this is bounded by

$$t \varepsilon^{d+2} \int dq \, dq' \left| G(\varepsilon q) \right| \left| G(\varepsilon q') \right| < \left( \sum_k \left( \partial_{v_k} F(\tau_{\varepsilon}\omega) \right)^2 \right)^{1/2} \left( \sum_k \left( \partial_{v_k} F(\tau_{\varepsilon}\omega) \right)^2 \right)^{1/2}$$

$$= t \varepsilon^{d+2} \left( \varepsilon^{d/2} \int dq \left| G(\varepsilon q) \right| \left( \sum_k \left( \partial_{v_k} F(\tau_{\varepsilon}\omega) \right)^2 \right)^{1/2} \right)^2$$

Since

$$\sum_k \left( \partial_{v_k} F(\omega) \right)^2 = \sum_{\omega \in \Lambda} (\nabla_{\omega} u)^2$$

then, by (4.24), this is a local function in $L^1(\mu)$, and by Lemma 4.2, $\mathbb{E}[(\mathcal{M}_t^\varepsilon)^2]$ converges to 0 as $\varepsilon \to 0$. It follows by lemma 4.3 that the variance of the first term on the rhs of (4.27) goes to 0 as $\varepsilon \to 0$.

\[ \square \]

5. Tightness

For any $k \geq 0$ and $f, g \in C^\infty(\mathbb{R}^d)$ consider the scalar product

$$(g, f)_k = \int_{\mathbb{R}^d} g(q) \left( |q|^2 - \Delta \right)^k f(q) \, dq$$

and denote by $\mathcal{H}_k$ the corresponding closure. For any positive $k$ we denote by $\mathcal{H}_{-k}$ its dual space with respect to the $L^2(\mathbb{R}^d) \equiv \mathcal{H}_0$ scalar product.

It is convenient to represent the scalar product $(\cdot, \cdot)_k$ in the ON base of the Hermite polynomials, which are the eigenfunctions of $|q|^2 - \Delta$. Let $\vec{n}$ be a multi-index of $\mathbb{Z}^d$ and $|\vec{n}| = \sum_{i=1}^d n(i)$. We denote by $\lambda_{n(i)} = 2n(i) + 1$ for $n(i) \in \mathbb{Z}^+$ and $\lambda_{\vec{n}} = \sum_{i=1}^d \lambda_{n(i)}$.  

Define $h_n(q) = \prod_{i=1}^{d} h_n(i)(q_i)$ where $h_n$ is the $m^{th}$ normalized Hermite polynomial of order $m$ in $\mathbb{R}$.

$$\|f\|_k^2 = \int_{\mathbb{R}^d} f(q) (|q|^2 - \Delta)^k f(q) \, dq = \sum_{\bar{n} \in (\mathbb{Z}^+)^d} \lambda_{\bar{n}}^k \left( \int_{\mathbb{R}^d} f(q) h_n(q) \, dq \right)^2$$

This is valid also for negative $k$. So the $H_{-k}$-norm of a distribution $\xi$ on $\mathbb{R}^d$ can be written as

$$\|\xi\|_{-k}^2 = \sum_{\bar{n} \in (\mathbb{Z}^+)^d} \lambda_{\bar{n}}^{-k} \xi(h_{\bar{n}})^2$$

(5.29)

Recall that for a fixed arbitrary $T > 0$, we have denoted by $P^\varepsilon$ the distribution, under $\mathbb{P}_\mu$, of $\{\xi_{\varepsilon} = \xi^\varepsilon(\omega(\varepsilon^{-2}\varepsilon)), 0 \leq t \leq T\}$ in $\mathcal{C}([0, T], \mathcal{M}(\mathbb{R}^d))$.

**Proposition 5.1.** For any $k > d + 1$ and every $T > 0$, the sequence of probability measures $\{P^\varepsilon, \varepsilon \in (0, 1]\}$ has support in $\mathcal{C}([0, T], H_{-k})$ and is relatively compact in this space.

Proposition 5.1 is, by standard arguments, an immediate consequence of the following

**Proposition 5.2.** For any $k > d + 1$ and every $T > 0$, we have that

(i) $$\sup_{\varepsilon \in (0, 1)} \mathbb{E}_\mu \left( \sup_{t \in [0, T]} \|\xi_t^\varepsilon\|^2_{-k} \right) < +\infty$$

(ii) For any $R > 0$,

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \mathbb{P}_\mu \left( \sup_{t \in [0, T], \|\xi_t^\varepsilon - \xi_t^\delta\|_{-k} > R} \right) = 0$$

To prove proposition 5.2, we need the following key estimate

**Lemma 5.3.** Let $G \in L^2(\mathbb{R}^d, dq)$ with $\nabla G \in L^2(\mathbb{R}^d, dq)^d$. Then there exist a constant $B = B(\rho, T) < \infty$ such that

$$\sup_{\varepsilon \in (0, 1)} \mathbb{E}_\mu \left( \sup_{t \in [0, T]} \xi_t^\varepsilon(G)^2 \right) \leq B \int (G(q)^2 + |\nabla G(q)|^2) \, dq$$

**Proof.** Let us define $\mathcal{F}_\varepsilon(\omega) = \sum_j \nabla G(\varepsilon x_j) \cdot v_j$. By (1.1), we have

$$\xi_t^\varepsilon(G) = \xi_0^\varepsilon(G) + \varepsilon^{d/2+1} \int_0^{\varepsilon^{-2}t} \mathcal{F}_\varepsilon(\omega(s)) \, ds$$

(5.30)

Then

$$\mathbb{E}_\mu \left( \sup_{t \in [0, T]} \xi_t^\varepsilon(G)^2 \right) \leq 2 \left\langle \xi_t^\varepsilon(G)^2 \right\rangle + 2 \mathbb{E}_\mu \left( \varepsilon^{d/2+1} \int_0^{\varepsilon^{-2}t} \mathcal{F}_\varepsilon(\omega(s)) \, ds \right)^2$$

By lemma 3.1, there exists a constant $B'$ such that $\left\langle \xi_t^\varepsilon(G)^2 \right\rangle \leq B' \|G\|_{L^2}^2$. 


Since $\mathcal{F}_{\varepsilon} = -S\mathcal{F}_{\varepsilon}$, by lemma 4.3 of [4]

$$
\mathbb{E}_\mu \left( \sup_{t \in [0,T]} \left[ \varepsilon^{d/2+1} \int_0^{t_{\varepsilon^{-2t}}} \mathcal{F}_{\varepsilon}(\omega(s)) \, ds \right]^2 \right) \leq 14T\varepsilon^d \mathcal{F}_{\varepsilon}^{-1} \mathcal{F}_{\varepsilon} > 16T\varepsilon^d < \sum_j |\nabla G(\varepsilon x_j)|^2 = 16T \rho \| \nabla G \|^2_{L^2}.
$$

\[\square\]

Proof of proposition 5.2:

We start by proving (i). By (5.29) and lemma (5.3) we have

$$
\mathbb{E}_\mu \left[ \sup_{t \in [0,T]} \| \xi_t - \xi_s \|^2 - \right] \leq \sum_{\bar{n}} \lambda_{\bar{n}}^{-k} \mathbb{E}_\mu \left[ \sup_{t \in [0,T]} \left( \xi_t^\varepsilon(\bar{n}) - \xi_s^\varepsilon(\bar{n}) \right)^2 \right]
$$

$$
\leq B \sum_{\bar{n}} \lambda_{\bar{n}}^{-k} \left( 1 + \| h_{\bar{n}} \|_{L^2} \right)
$$

(5.31)

$$
\leq B \sum_{\bar{n}} \lambda_{\bar{n}}^{-k} \left( 1 + \lambda_{\bar{n}} \right)
$$

This series converges provided $k > d + 1$ and therefore (i) is proven.

We are left with the proof of (ii). Always by (5.29) we have

$$
\mathbb{E}_\mu \left[ \sup_{t \in [0,T]} \| \xi_t^\varepsilon - \xi_s^\varepsilon \|^2 - \right] \leq \sum_{\bar{n}} \lambda_{\bar{n}}^{-k} \mathbb{E}_\mu \left[ \sup_{t \in [0,T]} \left( \xi_t^\varepsilon(\bar{n}) - \xi_s^\varepsilon(\bar{n}) \right)^2 \right]
$$

(5.32)

For every $R \geq 1$,

$$
\sum_{\bar{n} \in \mathbb{N}^d : |\bar{n}| \geq R} \lambda_{\bar{n}}^{-k} \mathbb{E}_\mu \left[ \sup_{t \in [0,T]} \left( \xi_t^\varepsilon(\bar{n}) - \xi_s^\varepsilon(\bar{n}) \right)^2 \right]
$$

$$\leq 4 \sum_{\bar{n} \in \mathbb{N}^d : |\bar{n}| \geq R} \lambda_{\bar{n}}^{-k} \mathbb{E}_\mu \left[ \sup_{t \in [0,T]} \left( \xi_t^\varepsilon(\bar{n}) \right)^2 \right]
$$

By (5.31), since $k > d + 1$

$$
\lim_{R \to \infty} \limsup_{\varepsilon \to 0} \sum_{\bar{n} \in \mathbb{N}^d : |\bar{n}| \geq R} \lambda_{\bar{n}}^{-k} \mathbb{E}_\mu \left[ \sup_{t \in [0,T]} \left( \xi_t^\varepsilon(\bar{n}) \right)^2 \right] = 0
$$

(5.33)

Consequently, in order to prove (5.32), we only need to show that for every $\bar{n}$

$$
\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \mathbb{E}_\mu \left[ \sup_{t \in [0,T]} \left( \xi_t^\varepsilon(\bar{n}) - \xi_s^\varepsilon(\bar{n}) \right)^2 \right] = 0
$$

(5.34)

By (2.9)

$$
\xi_t^\varepsilon(\bar{n}) - \xi_s^\varepsilon(\bar{n}) = I_1^\varepsilon(t,s) + I_2^\varepsilon(t,s) + I_3^\varepsilon(t,s)
$$

where

$$
I_1^\varepsilon(t,s) = \sqrt{2} \varepsilon^{d/2+1} \int_{\varepsilon^{-2t}}^{\varepsilon^{-2s}} \sum_j \nabla h_{\bar{n}}(\varepsilon x_j(\tau)) \cdot dw_j(\tau)
$$
\[ I_2^e(t, s) = \int_{\varepsilon^{-2s}}^{e^{-2t}} \gamma_\varepsilon(\omega(u)) \, du \]

where \( \gamma_\varepsilon \) is given by

\[
\gamma_\varepsilon(\omega) = \varepsilon^{d/2} \sum_j \left( \sum_{\alpha, \sigma} \partial_\alpha \partial_\sigma h_{\varepsilon x_j} v_j^\alpha v_j^\sigma - \varepsilon^{-1} \sum_{i \neq j} \nabla h_{\varepsilon x_j} \cdot \nabla V(x_i - x_j) \right)
\]

and

\[ I_3^e(t, s) = - \left( \eta_\varepsilon(\omega(e^{-2t})) - \eta_\varepsilon(\omega(e^{-2s})) \right) \]

where we have defined

\[ \eta_\varepsilon(\omega) = \varepsilon^{d+1} \sum_j \nabla h_{\varepsilon x_j}(\omega) \cdot v_j(\omega) \]

Observe that, by Lemma 3.2, \( \gamma_\varepsilon \) is uniformly bounded in \( L^2(\mu) \), then by Cauchy-Schwarz inequality and stationarity

\[
\mathbb{E}_\mu \left[ \sup_{t, s \in [0, T], |t-s| \leq \delta} I_2^e(t, s)^2 \right] \leq \delta \int_0^T \mathbb{E}_\mu \left( \gamma_\varepsilon(\omega(e^{-2\tau}))^2 \right) \, d\tau \leq \delta T < \gamma_\varepsilon^2 \leq \delta T c(\pi) \quad (5.35)
\]

with \( c(\pi) \) is a constant independent of \( \varepsilon \) and \( \delta \). It follows that this term vanishes to 0 as \( \delta \) goes to 0.

For what concern \( I_2^e(t, s) \), we will prove that for any \( R > 0 \)

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \mathbb{P}_\mu \left( \sup_{t, s \in [0, T], |t-s| \leq \delta} |I_2^e(t, s)| > R \right) = 0
\]

This follows if we prove

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{1}{\delta} \mathbb{P}_\mu \left( \sup_{0 \leq t \leq s + \delta} |I_2^e(t, s)| > R \right) = 0
\]

By stationarity this is equivalent to prove

\[
\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}_\mu \left( \sup_{0 \leq t \leq \delta} |I_2^e(t, 0)| > R \right) = 0.
\]

Let us write \( I_2^e(t, s) \) as the difference \( M_1^e - M_2^e \), where \( M_1^e \) is a martingale with quadratic variation given by

\[
A_\varepsilon(t) = 2\varepsilon^d \int_0^t \sum_j (\nabla h_{\varepsilon x_j})^2 (\varepsilon x_j(\varepsilon^{-2}u)) \, du
\]

Observe that \( \sup_\varepsilon < A_\varepsilon(t) \leq Ct \mu n \). For any \( \ell > 0 \) define the stopping time

\[
\tau_{\varepsilon, \ell} = \inf \{ t : A_\varepsilon(t) > \ell \}
\]

and let \( \tilde{M}_{t, \ell}^\varepsilon = M_{t, \tau_{\varepsilon, \ell}}^\varepsilon \). Then for any fixed \( \ell \), \( \tilde{M}_{t, \ell}^\varepsilon \) is a martingale with quadratic variation bounded by \( \ell \), and by a standard estimate

\[
\mathbb{P}_\mu \left( \sup_{0 \leq t \leq \delta} |\tilde{M}_{t, \ell}^\varepsilon| \geq R \right) \leq 2e^{-\frac{R^2}{4\ell^2}} \quad (5.36)
\]

Then for any fixed \( \ell \) and any \( R > 0 \)

\[
\mathbb{P}_\mu \left( \sup_{0 \leq t \leq \delta} |M_t^\varepsilon| > R \right) \leq 2e^{-\frac{R^2}{4\ell^2}} + \mathbb{P}_\mu (\tau_{\varepsilon, \ell} < \delta)
\]
and
\[ \mathbb{P}_{\mu}(\tau_{\epsilon, \ell} < \delta) \leq \mathbb{P}_{\mu}(A_{\epsilon}(\delta) > \ell) \leq \frac{C \rho n \delta}{\ell} \]

It follows that for any \( R > 0 \)
\[ \lim_{\delta \to 0} \limsup_{\epsilon \to 0} \frac{1}{\delta} \mathbb{P}_{\mu} \left( \sup_{0 \leq t \leq \delta} |M_{t}^{\epsilon}| \geq R \right) = 0 \]

In order to study \( I_{2}^{\epsilon}(t, s) \), observe that
\[ \mathbb{E}_{\mu} \left[ \sup_{t, s \in [0, T]} I_{2}^{\epsilon}(t, s)^{2} \right] \leq 2 \mathbb{E}_{\mu} \left( \sup_{t \in [0, T]} \left[ \eta_{\epsilon}(\omega(\epsilon^{-2}t)) \right]^{2} \right) \]

Then the evolution equations for \( \eta_{\epsilon} \) says
\[ \eta_{\epsilon}(\omega(\epsilon^{-2}t)) = e^{-\epsilon^{-2}t} \eta_{\epsilon}(\omega(0)) + \int_{0}^{t} e^{-\epsilon^{-2}(t-r)} \gamma_{\epsilon}(\omega(\epsilon^{-2}r)) \, dr 
+ \int_{0}^{t} e^{-\epsilon^{-2}(t-r)} \, dM_{t}^{\epsilon} \]

For the first term of the rhs of the above expression, observe that \( \langle \eta_{\epsilon}^{2} \rangle \to 0 \) as \( \epsilon \to 0 \) (cf. (3.14)). About the second term, by Schwarz inequality we have
\[ \mathbb{E}_{\mu} \left( \sup_{t \in [0, T]} \left[ \int_{0}^{t} e^{-\epsilon^{-2}(t-r)} \gamma_{\epsilon}(\omega(\epsilon^{-2}r)) \, dr \right]^{2} \right) \leq T < \gamma_{\epsilon}^{2} \sup_{t \in [0, T]} \int_{0}^{t} e^{-2\epsilon^{-2}(t-r)} \, dr \to 0 \quad \epsilon \to 0 \]

About the martingale term, by Doob’s inequality:
\[ \mathbb{E}_{\mu} \left( \sup_{t \in [0, T]} \left[ \int_{0}^{t} e^{-\epsilon^{-2}(t-r)} \, dM_{r}^{\epsilon} \right]^{2} \right) \leq 8 \left( \int_{0}^{t} e^{-2\epsilon^{-2}(T-r)} \, dr \right) \rho \int (\nabla h_{\eta})^{2}(q) \, dq \]

that converge to 0 as \( \epsilon \to 0 \). \( \square \)

6. THE MACROSCOPIC EQUATION

As consequence of the results contained in sections 3 and 4 we have that for any test function \( G \in C^{2}(\mathbb{R}^{d}) \) with compact support
\[ \lim_{\epsilon \to 0} \mathbb{E}_{\mu} \left( \sup_{0 \leq t \leq T} \left| \xi_{t}^{\epsilon}(G) - \xi_{0}^{\epsilon}(G) - \int_{0}^{t} \xi_{s}^{\epsilon}(D\Delta G) \, ds - M_{t}^{\epsilon}(G) \right|^{2} \right) = 0 \]

where the martingale \( M_{t}^{\epsilon} \) is given by
\[ M_{t}^{\epsilon}(G) = \int_{0}^{t} \epsilon^{-2} \sqrt{2\epsilon^{d/2+1}} \sum_{j} (\nabla G)(\epsilon x_{j}(s)) \cdot dw_{j}(s) \]

The quadratic variation of \( M_{t}^{\epsilon} \) is given by
\[ A_{t}^{\epsilon} = 2\epsilon^{2} \int_{0}^{t} e^{-2\epsilon^{2} s} \sum_{j} (\nabla G)(\epsilon x_{j}(s))^{2} \, ds \]

It is easy to see that \( A_{t}^{\epsilon} \) converges to \( 2\rho t \int |(\nabla G)(q)|^{2} \, dq \) in \( L^{1}(\mathbb{P}_{\mu}) \) and a.s. Consequently \( M_{t}^{\epsilon}(G) \) converges to a martingale \( M_{t}(G) \) with constant quadratic variation given by \( 2\rho t \int |(\nabla G)(q)|^{2} \, dq \).
By the results in section 5, the laws $P^\varepsilon$ of $\xi^\varepsilon_t$ is tight in $C([0,T],H_{-d})$ for any $k > d+1$. It follows that any limit point $P$ of $P^\varepsilon$ is concentrated on the solutions of the equation

$$
\xi_t(G) - \xi_0(G) = \int_0^t \xi_s(D\Delta G) \, ds + M_t(G) \quad \forall G \in C^2_c(\mathbb{R}^d) \tag{6.37}
$$

Since these limits are obtained in the stationary state $\mu$, for each fixed time $t \in [0,T]$ the marginal distribution of $\xi_t$ is the law of the centered Gaussian field on $\mathbb{R}^d$ with covariance (cf. [1])

$$
E_P(\xi_t(G)\xi_t(F)) = \chi \int G(q)F(q) \, dq
$$

This permits to identify $P$ as the distribution of the stationary solution of (6.37), and $D$ with $\rho/\chi$. We summarize the final result in the following theorem.

**Theorem 6.1.** Let $k > d + 1$ and $T > 0$. The law $\xi^\varepsilon_t$ on $C([0,T],H_{-k})$ converges to the law of the Gaussian process with covariance given by

$$
E_P(\xi_t(G)\xi_t(F)) = \chi \int G(q)(e^{[t-s]D\Delta F})(q) \, dq
$$

with $D = \xi_t$. 

7. Appendix A: Existence of the dynamics

We prove here the existence of the *equilibrium dynamics* for the system of stochastic differential equations defined by (2.9). The idea is to approximate the infinite dynamics by some local dynamics that have $\mu$ as invariant measure.

Let $B(q,r)$ be the ball of radius $r$ centered at $q \in \mathbb{R}^d$. For every integer $n$ denote $B_n = B(0,2^n)$, let $a_n(q)$ a smooth positive function with support contained in $B_n$ and such that $a_n(q) = 1$ if $q \in B(0,2^n - 1)$. Given an initial configuration $\omega = \{x_i,v_i\}$, Let us consider the following local dynamics:

$$
\begin{align*}
\frac{d}{dt} x^n_i(t) &= a_n(x_i^n(t))v^n_i(t) \frac{d}{dt} \\
\frac{d}{dt} v^n_i(t) &= -a_n(x_i^n(t)) \sum_{j \neq i} \nabla V(x^n_j(t) - x^n_i(t)) + \nabla a_n(x_i^n(t)) + a_n(x_i^n(t)) dt
\end{align*} \tag{7.38}
$$

with initial conditions $x^n_i(0) = x_i(\omega)$ and $v^n_i(0) = v_i(\omega)$. Observe that particles outside $B_n = B(0,2^n)$ are frozen, while particles inside $B(0,2^n - 1)$ moves according to our original equations of motion.

The generator of the Markov process associated to (7.38) can be written as

$$
\begin{align*}
L_n = \sum_{i} e^{S_Bn_i(\omega)} & \left[ \partial_{v_i} a_n(x_i) e^{-S_Bn_i(\omega)} \partial_{v_i} + \partial_{v_i} a_n(x_i) e^{-S_Bn_i(\omega)} \partial_{x_i} - \partial_{x_i} a_n(x_i) e^{-S_Bn_i(\omega)} \partial_{v_i} \right]
\end{align*}
$$

where we have defined

$$
S_{B_n}(\omega) = \sum_{x_j(\omega) \in B_n} \left[ \frac{v_j(\omega)^2}{2} + \frac{1}{2} \sum_{x_j(\omega) \in B_n} V(x_j(\omega) - x_i(\omega)) + \sum_{x_k(\omega) \notin B_n} V(x_j(\omega) - x_k(\omega)) \right]
$$

Consequently, for any $m \in \mathbb{N}$ and any fixed configuration $\omega^\varepsilon_{B_n}$, the canonical Gibbs measure $\mu(\cdot|N_{B_n} = m, \omega^\varepsilon_{B(0,2^n)^c})$ is invariant for (7.38). Consequently $\mu$ is also invariant for any $n$. 
Step 1: We prove first some bounds on the velocities and the densities that are valid with $\mathbb{P}_\mu$-probability 1.

**Proposition 7.1.** Let $\tau \in \mathbb{R}^+$. For any constant $C > 0$ we have

$$\mathbb{P}_\mu \left( \exists n_0: \forall n \geq n_0 \sup_{i \neq j} \sup_{0 \leq t \leq \tau} |v_1^n(t)| \leq C n \right) = 1 \quad (7.39)$$

**Proof.** Let us fix an integer $n > 8\tau/C$ and consider

$$\mathbb{P}_\mu \left( \sup_{i \neq j} \sup_{0 \leq t \leq \tau} |v_1^n(t)| > C n \right) \leq \mathbb{E}_\mu \left( N_{B_n}(\omega(0)) \mathbb{P}_\mu \left( \sup_{0 \leq t \leq \tau} |v_1^n(t)| > C n \bigg| N_{B_n}(\omega(0)) \right) \right) \quad (7.40)$$

Since

$$\sup_{0 \leq t \leq \tau} |v_1^n(t)| \leq |v_1^n(0)| + \sup_{0 \leq t \leq \tau} |w_1(t)| + \int_0^\tau \left| \sum_{j \neq 1} \nabla V(x_1^n(s) - x_j^n(s)) \right| ds + 2\tau + \int_0^\tau |v_1^n(s)| ds$$

(7.40) is bounded above by

$$< N_{B_n} \mathbb{1}_{[v_1 > Cn/4]} >_\mu + C_1 \rho 2^{nd} \mathbb{E} \left( \sup_{0 \leq t \leq \tau} |w_1(t)| > Cn/4 \right)$$

$$+ \mathbb{E}_\mu \left( N_{B_n}(\omega(0)) \mathbb{P}_\mu \left( \int_0^\tau \left| \sum_{j \neq 1} \nabla V(x_1^n(s) - x_j^n(s)) \right| ds > Cn/4 \bigg| N_{B_n}(\omega(0)) \right) \right)$$

$$+ \mathbb{E}_\mu \left( N_{B_n}(\omega(0)) \mathbb{P}_\mu \left( \int_0^\tau |v_1^n(s)| ds \geq Cn/4 \bigg| N_{B_n}(\omega(0)) \right) \right) \quad (7.41)$$

where $C_1$ is a constant depending only on the dimension $d$. Since, under $\mu$, $v_1$ is Gaussian distributed and is independent from the positions, the last term on the right hand side of the above expression, is bounded by $C_1 \rho 2^{nd} e^{-C_2 n^2}$. The second term involving $w_1(t)$, by a standard estimate, satisfies a similar bound.

About the third term of the rhs of the above expression, by Chebichef inequality is bounded, for any $\alpha > 0$,

$$\mathbb{E}_\mu \left( N_{B_n} \exp \left\{ \alpha \left[ \int_0^\tau \left| \sum_{j \neq 1} \nabla V(x_1^n(s) - x_j^n(s)) \right| ds \right]^2 \right\} \right) e^{-\alpha C^2 n^2/16}$$

By Schwartz and Jensen inequality this is bounded by

$$\mathbb{E}_\mu \left( N_{B_n} \frac{1}{\tau} \int_0^\tau \exp \left\{ \alpha \tau^2 \left[ \sum_{j \neq 1} \nabla V(x_1^n(s) - x_j^n(s)) \right]^2 \right\} ds \right) e^{-\alpha C^2 n^2}$$

where $C_3$ depends only on $C$ and on $\tau$. Then by stationarity this is equal to

$$\left\langle N_{B_n} \exp \left\{ \alpha \tau^2 \left[ \sum_{j \neq 1} \nabla V(x_1 - x_j) \right]^2 \right\} \right\rangle e^{-\alpha C^2 n^2}$$
Since the range of the interaction is \( R \), posing \( \eta = \alpha \| \nabla V \|_\infty \tau^2 \), this last quantity is bounded by

\[
\left\langle N_{B_n} \exp \left[ \eta N_{B(x_1,R)}^2 \right] \right\rangle e^{-\alpha C_3 n^2} \\
\leq \left\langle N_{B_n}^2 \right\rangle^{1/2} \left\langle \exp \left[ 2\eta N_{B(x_1,R)}^2 \right] \right\rangle^{1/2} e^{-\alpha C_3 n^2} \\
\leq C_5 \rho 2^{2nd} \left\langle \exp \left[ 2\eta N_{B(0,1/2R)}^2 \right] \right\rangle^{1/2} e^{-\alpha C_3 n^2} \tag{7.42}
\]

By superstability estimates \( \left\langle \exp \left[ 2\eta N_{B(0,1/2R)}^2 \right] \right\rangle \) is finite for \( \eta \) enough small. Then we can correspondingly choose \( \alpha \) small and we obtain that (7.42) is bounded by \( C_6 \rho 2^{2nd} e^{-\alpha C_3 n^2} \).

We treat in a similar way the 4th term in (7.41) and, for \( \alpha \) small enough, we can bound it by

\[
\left\langle N_{B_n} e^{\alpha \tau^2 |v_1|^2} \right\rangle e^{-\alpha C_3 n^2} \leq C_7 \rho 2^{2nd} e^{-\alpha C_3 n^2}
\]

Then (7.39) follows by Borel-Cantelli lemma. \( \square \)

**Proposition 7.2.**

\[
\mu \left( \exists k \in \mathbb{N}, \forall n \in \mathbb{N} : \sup_{x \in B(0,2^n)} N_{B(x,n)} \leq kn^d \right) = 1 \tag{7.43}
\]

**Proof.** This is a consequence of Ruelle’s superstability estimates (cf. [15, 12]). \( \square \)

**Step 2:** We prove now the limits

\[
\lim_{n \to \infty} x_j^n(t) = x_j(t) \quad \lim_{n \to \infty} v_j^n(t) = v_j(t)
\]

exist \( P_\mu - a.s. \) and are uniform in \( t \).

After proposition 7.1 and 7.2 the proof is standard, with ideas similar to [10, 12]. Let us define

\[
\delta_1(n,r,t) = \sup_{x_i \in B(0,r)} \sup_{0 \leq s \leq t} |x_i^n(s) - x_i^{n-1}(s)|
\]

and

\[
\delta_2(n,r,t) = \sup_{x_i \in B(0,r)} \sup_{0 \leq s \leq t} |v_i^n(s) - v_i^{n-1}(s)|
\]

We will prove now that for any \( r_0 > 0 \) and \( t > 0 \):

\[
\sum_n \left( \delta_1(n,r_0,t) + \delta_2(n,r_0,t) \right) < \infty \quad P_\mu - a.s. \tag{7.44}
\]

All the inequalities in what follows should be considered valid for a set of full \( P_\mu \)-measure. We have immediately that

\[
\delta_1(n,r_0,t) \leq \int_0^t \delta_2(n,r_0,s) \, ds.
\]
By proposition 7.1 and proposition 7.2, for a given $C > 0$, there exist $k$ and $n_0$ integers such that for any $n \geq n_0$

$$
\delta_2(n, r_0, t) \leq \|\Delta V\|_\infty \int_0^t ds \sup_{x_i \in B(0, r_0)} \sum_{x_j \in B(x, R + 2Cn)} |x_j^n(s) - x_j^{n-1}(s)|
$$

$$
+ \int_0^t \delta_2(n, r_0, s) \, ds
$$

$$
\leq \int_0^t (\|\Delta V\|_\infty k n^d \delta_1(n, r_1, s) + \delta_2(n, r_0, s)) \, ds
$$

where $r_1 = r_0 + R + 2Ctn$. We have obtained

$$
\delta_1(n, r_0, t) + \delta_2(n, r_0, t) \leq \|\Delta V\|_\infty k n^d \int_0^t (\delta_1(n, r_1, s) + \delta_2(n, r_1, s)) \, ds
$$

We can assume that $2^n \geq r_1$ taking $n$ sufficiently large enough. And then we can iterate this relation for a number of times given by

$$
h(n) = \left[ \frac{2^n - r_0}{R + 2Ctn} \right]
$$

obtaining

$$
\delta_1(n, r_0, t) + \delta_2(n, r_0, t) 
\leq \left( \|\Delta V\|_\infty k n^d \right)^{h(n)} \int_0^t dt_1 \cdots \int_0^{t_{h(n)}} dt_{h(n)} (\delta_1(n, 2^n, t_{h(n)}) + \delta_2(n, 2^n, t_{h(n)}))
$$

Since $\delta_1(n, 2^n, t_{h(n)}) \leq 2^n$ and $\delta_2(n, 2^n, t_{h(n)}) \leq 2Cn$ we have

$$
\delta_1(n, r_0, t) + \delta_2(n, r_0, t) \leq 52^n (\|\Delta V\|_\infty k n^d)^{h(n)}
$$

By the choice done of $h(n)$, this implies (7.44).

8. Appendix B: Spectral gap for interacting Brownian particles

We prove here the spectral gap bound we used in section 4. In this appendix $\Lambda$ is a fixed centered cube of sidelength $2l$ ($|\Lambda| = (2l)^d$). We fix the number of particles in the box $\Lambda$ be equal to $n$, and an arbitrary configuration $\omega_{\Lambda}^x$ outside $\Lambda$. Let $m = m(\omega_{\Lambda}^x)$ the number of particles of the outside configuration $\omega_{\Lambda}^x$ that are at distance less or equal to $R$, the radius of the range of the interaction $V$.

Recall that $\Gamma_\Lambda(\cdot)(n, \omega_{\Lambda}^x)$ is the corresponding canonical Gibbs expectation.

**Theorem 8.1.** Let $f(x_1, \ldots, x_n)$ be a $C^4(\Lambda^n)$ function such that $\Gamma_\Lambda(f)(n, \omega_{\Lambda}^x) = 0$. Then

$$
\Gamma_\Lambda(f^2)(n, \omega_{\Lambda}^x) \leq 8d l^2 n e^{4(2n + m)} \|V\|_\infty \sum_{i=1}^n \Gamma_\Lambda \left( |\nabla \omega_i|^2 \right)(n, \omega_{\Lambda}^x)
$$

(8.45)

**Proof.** We denote by $\mu_\Lambda(dx) = \mu_\Lambda(dx_1, \ldots, dx_n|n, \omega_{\Lambda}^x)$ the canonical measure corresponding to $\Gamma_\Lambda(\cdot)(n, \omega_{\Lambda}^x)$. Observe that we can rewrite the canonical variance of $f$ as

$$
\Gamma_\Lambda(f^2)(n, \omega_{\Lambda}^x) = \frac{1}{2} \iint_{\Lambda^n \times \Lambda^n} (f(x) - f(x'))^2 \mu_\Lambda(dx) \mu_\Lambda(dx')
$$

$$
= \frac{1}{2} \iint_{\Lambda^n} (f(x) - f(x'))^2 e^{-\mathcal{H}_n(x, \omega_{\Lambda}^x) - \mathcal{H}_n(x', \omega_{\Lambda}^x)} \, dx \, dx'
$$

(8.46)
where we denoted

\[ \mathcal{H}_n(x, \omega_{\mathcal{X}}) = \sum_{(i,j)} V(x_i - x_j) + \sum_{i=1}^{n} \sum_{y_j \in \omega_{\mathcal{X}}^i} V(x_i - y_j) \]

and \( Z = Z_A(n, \omega_{\mathcal{X}}) \) is the corresponding canonical partition function.

For each couple of configurations \( x, x' \) we will choose a particular piecewise differentiable path that will connect these two configurations. The choice of this transformation follows a simple rule: we move each particle one by one along the coordinates axis. So, in order to simplify notation, it is convenient to consider \( x, x' \) as points in \([-l, l]^{nd}\) and write \( x = \{y_1, \ldots, y_{nd}\} \) and \( x' = \{y'_1, \ldots, y'_{nd}\} \). Then defining

\[ \xi_\alpha(t) = y_\alpha + t(y'_\alpha - y_\alpha) \quad t \in [0, 1] \quad \alpha = 1, \ldots, nd \]

we can rewrite the difference

\[ f(x) - f(x') = \sum_{\alpha=1}^{nd} \int_0^1 \frac{d}{dt} f(y_1, \ldots, y_{\alpha-1}, \xi_\alpha(t), y_{\alpha+1}, \ldots, y_{nd}) \, dt \]

and by Schwarz inequality

\[ (f(x) - f(x'))^2 \leq 2nd \sum_{\alpha=1}^{nd} \int_0^1 \left| f_\alpha(y_1, \ldots, y_{\alpha-1}, \xi_\alpha(t), y_{\alpha+1}, \ldots, y_{nd}) \right|^2 |\xi_\alpha'(t)|^2 \sqrt{1-t} \, dt \]

where we have denoted by \( f_\alpha \) the partial derivative of \( f \) with respect to \( y_\alpha \) and \( \xi_\alpha'(t) = y'_\alpha - y_\alpha \). Since \(|\xi_\alpha'(t)| \leq 2t\) we have that the right hand side of (8.47) is bounded by

\[ 8t^2 \sum_{\alpha=1}^{nd} \int_0^1 \left| f_\alpha(y_1, \ldots, y_{\alpha-1}, \xi_\alpha(t), y_{\alpha+1}, \ldots, y_{nd}) \right|^2 \sqrt{1-t} \, dt \]  

(8.48)

Since the interaction \( V \) is bounded, for any \( \xi_\alpha \in [-l, l] \) we have the uniform bound

\[ \mathcal{H}_n(x, \omega_{\mathcal{X}}) + \mathcal{H}_n(x', \omega_{\mathcal{X}}) \geq -4(2n + m) \|V\|_\infty 
+ \mathcal{H}_n(y_1, \ldots, y_{\alpha-1}, \xi_\alpha, y_{\alpha+1}, \ldots, y_{nd}, \omega_{\mathcal{X}}) + \mathcal{H}_n(y_1, \ldots, y_{\alpha-1}, y'_\alpha, y_{\alpha+1}, \ldots, y_{nd}, \omega_{\mathcal{X}}) \]

(8.49)

To prove this, observe that for any \( \theta, \eta \in \Lambda \) we can rewrite

\[ \mathcal{H}_n(x, \omega_{\mathcal{X}}) + \mathcal{H}_n(x', \omega_{\mathcal{X}}) 
= \mathcal{H}_n(x_1, \ldots, x_{i-1}, \theta, x_{i+1}, \ldots, x_n, \omega_{\mathcal{X}}) + \mathcal{H}_n(x'_1, \ldots, x'_{i-1}, \eta, x_{i+1}, \ldots, x_n, \omega_{\mathcal{X}}) 
+ \sum_{k=1}^{i-1} \left\{ (V(x_k - x_i) - V(x_k - \theta)) + (V(x'_k - x'_i) - V(x'_k - \eta)) \right\} 
+ \sum_{k=i+1}^{n} \left\{ (V(x_k - x_i) - V(x_k - \eta)) + (V(x'_k - x'_i) - V(x'_k - \theta)) \right\} 
+ \sum_{y_j \in \omega_{\mathcal{X}}^i} \left\{ (V(x_i - y_j) - V(\theta - y_j)) + (V(x'_i - y_j) - V(\eta - y_j)) \right\} \]

Then, by appropriate choice of \( \theta, \eta \), (8.49) follows.
Now, by the simple change of variable $y_\alpha \to \xi_\alpha(t)$, we have
\[
\int_{\Lambda^n} dy_1 \cdots dy_n d\xi_1 \cdots d\xi_n |f_\alpha(y_1', \ldots, y_{\alpha-1}', \xi_\alpha(t), y_{\alpha+1}, \ldots, y_n)|^2 
\times e^{-\mathcal{H}_n(y_1, \ldots, y_{\alpha-1}, \xi_\alpha(t), y_{\alpha+1}, \ldots, y_n, \omega_{\Lambda}^n)} - \mathcal{H}_n(y_1, \ldots, y_{\alpha-1}, \xi_\alpha, y_{\alpha+1}, \ldots, y_n, \omega_{\Lambda}^n) 
\]
\[
= \int_{-l}^l dy_1 \int_{-l}^l dy_\alpha \int_{-l}^l dy_{\alpha+1} \int_{-l}^l dy_n e^{-\mathcal{H}_n(y_1, \ldots, y_{\alpha-1}, \xi_\alpha, y_{\alpha+1}, \ldots, y_n, \omega_{\Lambda}^n)} 
\times \int_{-l}^l dy_1 \int_{-l}^l dy_\alpha \int_{-l}^l dy_{\alpha+1} \int_{-l}^l dy_n \int_{-l}^l dy_n d\xi_\alpha \frac{1-t}{1-t} 
\times |f_\alpha(y_1', \ldots, y_{\alpha-1}', \xi_\alpha, y_{\alpha+1}, \ldots, y_n)|^2 e^{-\mathcal{H}_n(y_1, \ldots, y_{\alpha-1}, \xi_\alpha, y_{\alpha+1}, \ldots, y_n, \omega_{\Lambda}^n)} 
\]
\[
\leq \frac{Z}{1-t} \int_{\Lambda^n} d\mathbf{x} |f_\alpha(\mathbf{x})|^2 e^{-\mathcal{H}_n(\mathbf{x}, \omega_{\Lambda}^n)} 
\]
(8.50)

Putting together (8.46), (8.48), (8.49) and (8.50), we obtain (8.45). \hfill \Box

9. Appendix C: Regularity of the solution of the Poisson equation

We prove here the regularity of the solution of the equation (4.23), as a consequence of the spectral gap bound of the previous section. Let $\Lambda$ be again the centered box of sidelength $2l$, and let $\Lambda_R$ be the centered box of sidelength $2(l+R)$. In $\Lambda_R$ we consider configurations with $n+m$ particles such that there are $n$ particles in $\Lambda$ (whose positions will be denoted by $\{x_1, \ldots, x_n\}$) and $m$ particles in $\Lambda_R \setminus \Lambda$ (whose positions will be denoted by $\omega_{\Lambda}^n = \{y_1, \ldots, y_m\}$). We do not consider configurations with particles on the boundary $\partial \Lambda$. Let $g(x_1, \ldots, x_n; y_1, \ldots, y_m)$ a smooth function of these configurations such that the canonical expectation $\Gamma_\Lambda(\mathbf{g})(n, \omega_{\Lambda}^n) = 0$. Recall we have defined the elliptic operator
\[
L^W_{n, \omega_{\Lambda}^n} = \sum_{j=1}^n \left( \Delta_{x_j} - (\nabla_{x_j} \bar{\mathbf{y}}_\Lambda) \cdot \nabla_{x_j} \right)
\]
where
\[
\nabla_{x_j} \bar{\mathbf{y}}_\Lambda = \sum_{i \neq j} \nabla V(x_j - x_i) + \sum_i \nabla V(x_j - y_i)
\]
Let $u(x_1, \ldots, x_n; y_1, \ldots, y_m)$ the solution of the equation
\[
L^W_{n, \omega_{\Lambda}^n} u(x_1, \ldots, x_n; y_1, \ldots, y_m) = g(x_1, \ldots, x_n; y_1, \ldots, y_m), \quad (x_1, \ldots, x_n) \in \Lambda^n \tag{9.51}
\]
with Neumann boundary conditions on $\partial \Lambda^n$. The position of the exterior particles $\omega_{\Lambda}^n = (y_1, \ldots, y_m) \in (\Lambda_R \setminus \Lambda)^m$ should be considered as exterior parameters in the equation (9.51).

Lemma 9.1.
\[
\Gamma_\Lambda \left( \sum_{j=1}^n |\nabla_{x_j} u|^2 \right)(n, \omega_{\Lambda}^n) \leq c_1 n e^{c_2(n+m)} \Gamma_\Lambda(g^2)(n, \omega_{\Lambda}^n) \tag{9.52}
\]
where $c_1 = 8dl^2$ and $c_2 = 8\|V\|_\infty.$
Proof. Multiplying (9.51) by \( u \) and integrating respect to the canonical Gibbs measure we obtain
\[
\Gamma_\Lambda \left( \sum_{j=1}^{n} |\nabla x_j u|^2 \right) (n, \omega_{X}^\Lambda) = \Gamma_\Lambda \left( u(-L^W_{n,\omega_{X}^\Lambda} u) \right) (n, \omega_{X}^\Lambda)
\]
\[
= \Gamma_\Lambda (ug) (n, \omega_{X}^\Lambda) \leq \left( \Gamma_\Lambda (u^2) (n, \omega_{X}^\Lambda) \right)^{1/2} \left( \Gamma_\Lambda \left( g^2 \right) (n, \omega_{X}^\Lambda) \right)^{1/2}
\]
By the spectral gap bound (8.45)
\[
\Gamma_\Lambda (u^2) (n, \omega_{X}^\Lambda) \leq c_1 ne^{c_2(n+m)} \Gamma_\Lambda \left( \sum_{j=1}^{n} |\nabla x_j u|^2 \right) (n, \omega_{X}^\Lambda)
\]
Inserting this in the previous inequality, we obtain the bound (9.52)

\[ \Box \]

**Lemma 9.2.** There exists a constant \( c_3 \) such that
\[
\sum_{j,k=1}^{n} \Gamma_\Lambda \left( |\nabla x_j \nabla x_k u|^2 \right) (n, \omega_{X}^\Lambda) \leq c_3 n^2 e^{2c_2(n+m)} \Gamma_\Lambda \left( g^2 + \sum_{k=1}^{n} |\nabla x_k g|^2 \right) (n, \omega_{X}^\Lambda) \quad (9.53)
\]

Proof. This is just a standard elliptic regularity argument. Fix a \( k = 1, \ldots, n \). Let \( \{e_1, \ldots, e_d\} \) the canonical base in \( \mathbb{R}^d \). The function \( u_{k,\alpha} = e_\alpha \cdot \nabla x_k u \) satisfies the equation
\[
L^W_{n,\omega_{X}^\Lambda} u_{k,\alpha} = e_\alpha \cdot \nabla x_k g - \sum_{j=1}^{n} V''(x_j - x_k) u_{j,\alpha}
\]
Defining \( g_{k,\alpha} \) the right hand side of this equation we observe that
\[
\sum_{\alpha=1}^{d} \Gamma_\Lambda (g_{k,\alpha}^2) \leq 2 \Gamma_\Lambda \left( |\nabla x_k g|^2 \right) + 2 \|V''\|_\infty^2 \Gamma_\Lambda \left( \sum_{j=1}^{n} |\nabla x_j u|^2 \right)
\]
\[
\leq 2 \Gamma_\Lambda \left( |\nabla x_k g|^2 \right) + 2c_1 \|V''\|_\infty^2 ne^{c_2(n+m)} \Gamma_\Lambda (g^2)
\]
By reapplying the same argument as in the proof of lemma 9.1, we have
\[
\Gamma_\Lambda \left( \sum_{j=1}^{n} |\nabla x_j u_{k,\alpha}|^2 \right) \leq c_1 ne^{c_2(n+m)} \Gamma_\Lambda (g_{k,\alpha}^2)
\]
and (9.53) follows from these last two inequality.

\[ \Box \]

**References**


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