STRUCTURAL OPTIMIZATION BY THE LEVEL SET METHOD

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-I- INTRODUCTION

Two main approaches in structural optimization:

1) Geometric optimization by boundary variations

☞ Hadamard method of shape sensitivity: Murat-Simon, Pironneau, Zolésio...

☞ Ill-posed problem: many local minima, no convergence under mesh refinement.

☞ * Very costly because of remeshing.

☞ * Very general: any model or objective function.

2) Topology optimization (the homogenization method)

☞ Developed by Murat-Tartar, Lurie-Cherkaev, Kohn-Strang, Bendsoe-Kikuchi...

☞ Well-posed problem ; topology changes.

☞ * Limited to linear models and simple objective functions.

☞ * Very cheap because it captures shapes on a fixed mesh.

GOAL OF THIS WORK

Combine the advantages of the two approaches:

- ☞ Fixed mesh: low computational cost.
- ☞ General method: based on shape differentiation.

Main tool: the level set method of Osher and Sethian.

- ☞ Some references: Sethian and Wiegmann (JCP 2000), Osher and Santosa (JCP 2001), Allaire, Jouve and Toader (CRAS 2002), Wang, Wang and Guo (CMAME 2003).
- ☞ Similar (but different) from the phase field approach of Bourdin and Chambolle (COCV 2003).
- ☞ Some drawbacks remain: reduction of topology rather than variation (mainly in 2-d), many local minima.

-II- SETTING OF THE PROBLEM

Structural optimization in linearized elasticity (to begin with).

Shape Ω with boundary

$$\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D,$$

with Dirichlet condition on Γ_D, Neumann condition on $\Gamma \cup \Gamma_N$. **Only Γ is optimized**.

$$\begin{cases} -\mathrm{div}\ (A\, e(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(A\, e(u)\big)n = g & \text{on } \Gamma_N \\ \big(A\, e(u)\big)n = 0 & \text{on } \Gamma \end{cases}$$

with $e(u) = \frac{1}{2}\left(\nabla u + \nabla^t u\right)$, and A an homogeneous isotropic elasticity tensor.

OBJECTIVE FUNCTIONS

Two examples:

Compliance or work done by the load

$$J(\Omega) = \int_{\Gamma_N} g \cdot u \, ds = \int_\Omega A\, e(u) \cdot e(u) \, dx,$$

A least square criteria (useful for designing mechanisms)

$$J(\Omega) = \left(\int_\Omega k(x) |u - u_0|^\alpha dx \right)^{1/\alpha},$$

with a target displacement u_0, $\alpha \geq 2$ and k a given weighting factor.

EXISTENCE THEORY

The "minimal" set of admissible shapes

$$\mathcal{U}_{ad} = \left\{ \Omega \subset D, \quad \mathrm{vol}(\Omega) = V_0, \ \Gamma_D \cup \Gamma_N \subset \partial\Omega \right\}$$

with D a bounded open set $\mathbb{R}^N$. Usually, the minimization problem has no solution in $\mathcal{U}_{ad}$.

There exists an optimal shape if further conditions are required:

1. a uniform cone condition (D. Chenais).
2. a perimeter constraint (L. Ambrosio, G. Buttazzo).
3. a bound on the number of connected components of $D \setminus \Omega$ in two space dimensions (A. Chambolle).

PROPOSED NUMERICAL METHOD

First step: we compute shape derivatives of the objective functions in a continuous framework.

Second step: we model a shape by a level-set function ; the shape is varied by advecting the level-set function following the flow of the shape gradient (the transport equation is of Hamilton-Jacobi type).

-III- SHAPE DIFFERENTIATION

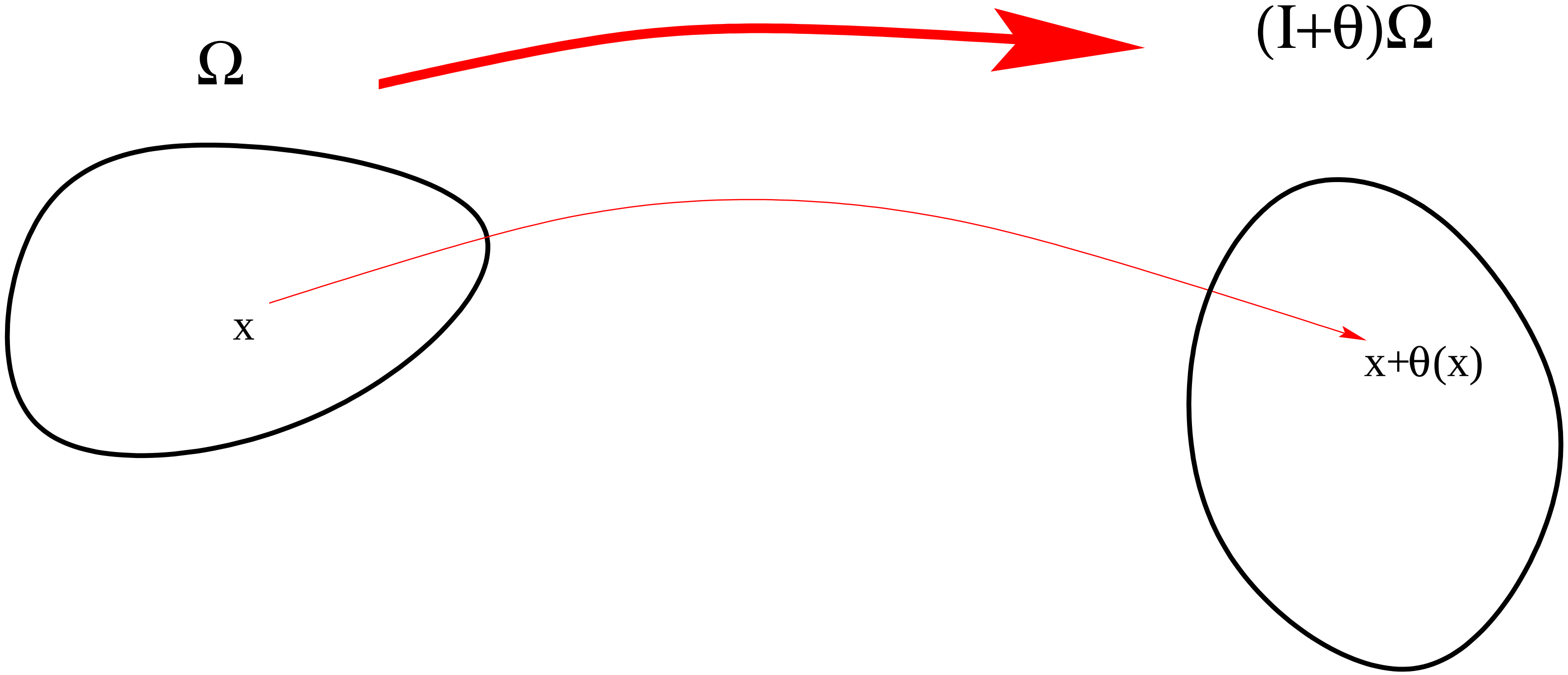
Framework of Murat-Simon:

Let Ω_0 be a reference domain. Consider its variations

$$\Omega = \big(Id + \theta\big)\Omega_0 \quad \text{with} \quad \theta \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N).$$

Lemma. For any $\theta \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ such that $\|\theta\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)} < 1$, $(Id+\theta)$ is a diffeomorphism in $\mathbb{R}^N$.

Definition: the shape derivative of $J(\Omega)$ at Ω_0 is the Fréchet differential of $\theta \to J\big((Id+\theta)\Omega_0\big)$ at 0.

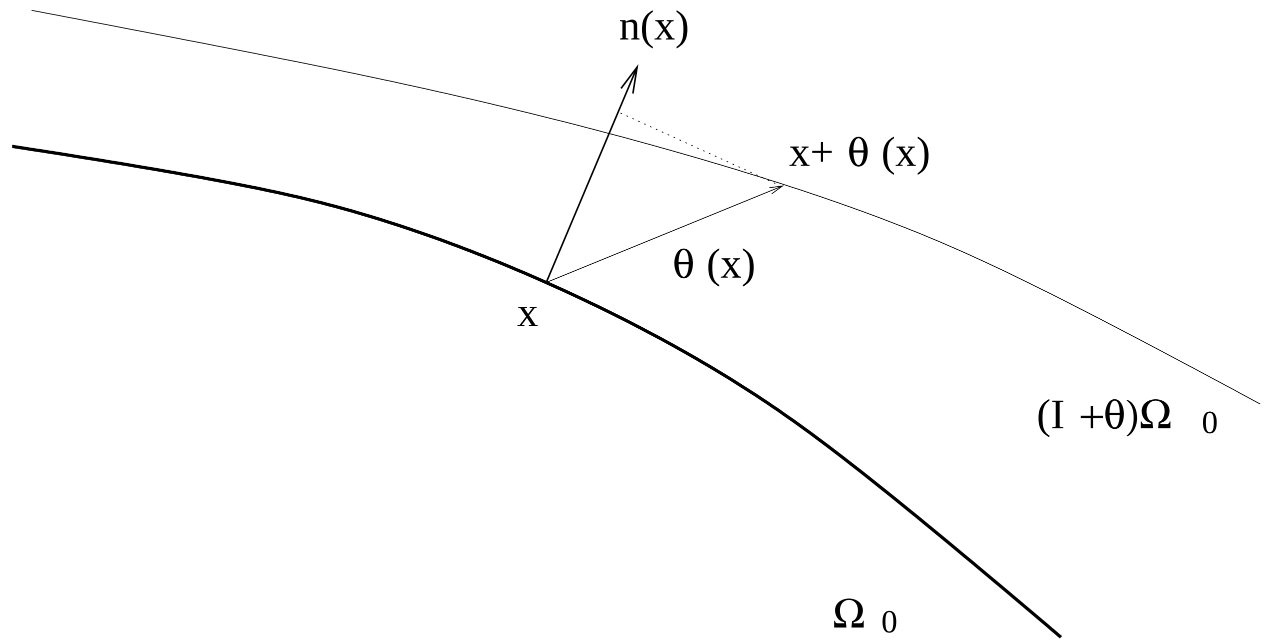


The set $\Omega = (Id + \theta)(\Omega_0)$ is defined by

$$\Omega = \{x + \theta(x) \mid x \in \Omega_0\}\,.$$

The vector field $\theta(x)$ is the displacement of Ω_0.

The derivative $J'(\Omega_0)(\theta)$ depends only on $\theta \cdot n$ on the boundary $\partial\Omega_0$.



Lemma. Let Ω_0 be a smooth bounded open set and $J(\Omega)$ a differentiable function at Ω_0. Its derivative satisfies

$$J'(\Omega_0)(\theta_1) = J'(\Omega_0)(\theta_2)$$

if $\theta_1, \theta_2 \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ are such that $\theta_2 - \theta_1 \in \mathcal{C}^1(\mathbb{R}^N;\mathbb{R}^N)$ and

$$\theta_1 \cdot n = \theta_2 \cdot n \quad \text{on } \partial\Omega_0.$$

Example 1 of shape derivative

Let Ω_0 be a smooth bounded open set and $f(x) \in W^{1,1}(\mathbb{R}^N)$. Define

$$J(\Omega) = \int_\Omega f(x)\, dx.$$

Then J is differentiable at Ω_0 and

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \operatorname{div}\, \big(\theta(x)\, f(x)\big)\, dx = \int_{\partial\Omega_0} \theta(x) \cdot n(x)\, f(x)\, ds$$

for any $\theta \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$.

Example 2 of shape derivative

Let Ω_0 be a smooth bounded open set and $f(x) \in W^{2,1}(\mathbb{R}^N)$. Define

$$J(\Omega) = \int_{\partial\Omega} f(x)\, ds.$$

Then J is differentiable at Ω_0 and

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \Big(\nabla f \cdot \theta + f\big(\mathrm{div}\,\theta - \nabla\theta n \cdot n\big)\Big)\, ds$$

for any $\theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$.

An integration by parts on the manyfold $\partial\Omega_0$ yields

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left(\frac{\partial f}{\partial n} + Hf \right) ds,$$

where H is the mean curvature of $\partial\Omega_0$ defined by $H = \mathrm{div}\, n$.

SHAPE DERIVATIVE OF THE COMPLIANCE

$$J(\Omega) = \int_{\Gamma_N} g \cdot u_\Omega \, ds = \int_\Omega A \, e(u_\Omega) \cdot e(u_\Omega) \, dx,$$

$$J'(\Omega_0)(\theta) = - \int_\Gamma Ae(u) \cdot e(u) \, \theta \cdot n \, ds,$$

where u is the state variable in Ω_0.

Remark: self-adjoint problem (no adjoint state is required).

SHAPE DERIVATIVE OF THE LEAST-SQUARE CRITERIA

$$J(\Omega) = \left(\int_\Omega k(x) |u_\Omega - u_0|^\alpha dx \right)^{1/\alpha},$$

$$J'(\Omega_0)(\theta) = \int_\Gamma \left(-Ae(p) \cdot e(u) + \frac{C_0}{\alpha} k |u - u_0|^\alpha \right) \theta \cdot n \, ds,$$

with the state u and the adjoint state p defined by

$$\begin{cases} -\mathrm{div}\, \left(A\, e(p) \right) = C_0 k(x) |u - u_0|^{\alpha-2} (u - u_0) & \text{in } \Omega_0 \\ p = 0 & \text{on } \Gamma_D \\ \big(Ae(p) \big) n = 0 & \text{on } \Gamma_N \cup \Gamma, \end{cases}$$

and $C_0 = \left(\int_{\Omega_0} k(x) |u(x) - u_0(x)|^\alpha dx \right)^{1/\alpha - 1}$.

SHAPE DERIVATIVES OF CONSTRAINTS

Volume constraint:

$$V(\Omega) = \int_\Omega dx,$$

$$V'(\Omega_0)(\theta) = \int_\Gamma \theta \cdot n \, ds$$

Perimeter constraint:

$$P(\Omega) = \int_{\partial\Omega} ds,$$

$$P'(\Omega_0)(\theta) = \int_\Gamma H \, \theta \cdot n \, ds$$

Idea of the proof.

The proof is classical.

Rigorous but lengthy proof:

- Change of variables: $x \in \Omega_0 \Rightarrow y = x + \theta(x) \in \Omega$. Rewrite all integrals in the fixed reference domain Ω_0.
- Write a variational formulation of the p.d.e. in Ω_0.
- Differentiate with respect to θ.

Formal but simpler proof (due to Céa) for $J(\Omega) = \int_\Omega j(x, u_\Omega)\, dx$:

- Write a Lagrangian for $(v, q) \in \left(H^1(\mathbb{R}^d; \mathbb{R}^d)\right)^2$

$$\mathcal{L}(\Omega, v, q) = \int_\Omega j(x, v)\, dx + \int_\Omega Ae(v) \cdot e(q)\, dx - \int_{\Gamma_N} q \cdot g\, ds - \int_{\Gamma_D} \Big(q \cdot Ae(v)n + v \cdot Ae(q)n \Big) ds.$$

- Stationarity of $\mathcal{L}$ gives the state and adjoint equations.
- Remark that $J(\Omega) = \mathcal{L}(\Omega, u_\Omega, p_\Omega)$, and thus

$$J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, u_\Omega, p_\Omega)$$

-IV- FRONT PROPAGATION BY LEVEL SET

Shape capturing method on a fixed mesh of a “large” box D.

A shape Ω is parametrized by a level set function

$$\begin{cases} \psi(x) = 0 & \Leftrightarrow x \in \partial\Omega \cap D \\ \psi(x) < 0 & \Leftrightarrow x \in \Omega \\ \psi(x) > 0 & \Leftrightarrow x \in (D \setminus \Omega) \end{cases}$$

The normal n to Ω is given by $\nabla\psi/|\nabla\psi|$ and the curvature H is the divergence of n. These formulas make sense everywhere in D on not only on the boundary $\partial\Omega$.

Hamilton Jacobi equation

Assume that the shape $\Omega(t)$ evolves in time t with a normal velocity $V(t,x)$. Then

$$\psi\Big(t, x(t)\Big) = 0 \quad \text{for any } x(t) \in \partial\Omega(t).$$

Deriving in t yields

$$\frac{\partial \psi}{\partial t} + \dot{x}(t) \cdot \nabla_x \psi = \frac{\partial \psi}{\partial t} + V n \cdot \nabla_x \psi = 0.$$

Since $n = \nabla_x \psi / |\nabla_x \psi|$ we obtain

$$\frac{\partial \psi}{\partial t} + V |\nabla_x \psi| = 0.$$

This Hamilton Jacobi equation is posed in the whole box D, and not only on the boundary $\partial\Omega$, if the velocity V is known everywhere.

Idea of the method

Shape derivative

$$J'(\Omega_0)(\theta) = \int_{\Gamma_0} j(u,p,n)\,\theta\cdot n\,ds.$$

Gradient algorithm for the shape:

$$\Omega_{k+1} = \Big(Id - j(u_k,p_k,n_k)n_k \Big)\Omega_k$$

since the normal n_k is "automatically" defined everywhere in D. In other words, the normal advection velocity of the shape is $-j$. Introducing a "pseudo-time" (a descent parameter), we solve the Hamilton-Jacobi equation

$$\frac{\partial \psi}{\partial t} - j|\nabla_x \psi| = 0 \quad \text{ in } D$$

Choice of the advection velocity

Simplest choice:

$$J'(\Omega_0)(\theta) = \int_\Gamma j\, \theta \cdot n\, ds \quad \Rightarrow \quad \theta = -j\, n.$$

However, j may be not smooth enough (typically $j \in L^1(\Omega_0)$ if there are "corners").

Classical trick: one can smooth the velocity. For example:

$$\begin{cases} -\Delta\theta = 0 & \text{in } \Omega_0 \\ \theta = 0 & \text{on } \Gamma_D \cup \Gamma_N \\ \frac{\partial\theta}{\partial n} = -j\, n & \text{on } \Gamma \end{cases}$$

It increases of one order the regularity of θ and

$$\int_{\Omega_0} |\nabla\theta|^2 dx = -\int_\Gamma j\, \theta \cdot n\, ds$$

which guarantees the decrease of J.

-V- NUMERICAL ALGORITHM

1. Initialization of the level set function ψ_0 (including holes).
2. Iteration until convergence for $k \geq 1$:
   (a) Computation of u_k and p_k by solving linearized elasticity problem with the shape ψ_k. Evaluation of the shape gradient = normal velocity V_k
   (b) Transport of the shape by V_k (Hamilton Jacobi equation) to obtain a new shape ψ_{k+1}.
   (c) (Occasionally, re-initialization of the level set function ψ_{k+1} as the signed distance to the interface).

Algorithmic issues

- ✗ Quadrangular mesh.
- ✗ Finite difference scheme, upwind of order 1, for the Hamilton Jacobi equation (ψ is discretized at the mesh nodes).
- ✗ Q1 finite elements for the elasticity problems in the box D

$$
\begin{cases}
-\text{div } (A^* e(u)) = 0 & \text{in } D \\
u = 0 & \text{on } \Gamma_D \\
\big(A^* e(u)\big) n = g & \text{on } \Gamma_N \\
\big(A^* e(u)\big) n = 0 & \text{on } \partial D \setminus (\Gamma_N \cup \Gamma_D).
\end{cases}
$$

- ✗ Elasticity tensor A^* defined as a "mixture" of A and a weak material mimicking holes

$$
A^* = \theta A \quad \text{with} \quad 10^{-3} \leq \theta \leq 1
$$

and $\theta =$ volume of the shape $\psi < 0$ in each cell (piecewise constant proportion).

Upwind scheme

$$\frac{\partial \psi}{\partial t} - j|\nabla_x \psi| = 0 \quad \text{in } D$$

solved by an explicit 1st order upwind scheme

$$\frac{\psi_i^{n+1} - \psi_i^n}{\Delta t} - \max(j_i^n, 0)\, g^+(D_x^+\psi_i^n, D_x^-\psi_i^n) - \min(j_i^n, 0)\, g^-(D_x^+\psi_i^n, D_x^-\psi_i^n) = 0$$

with $D_x^+\psi_i^n = \frac{\psi_{i+1}^n - \psi_i^n}{\Delta x}$, $D_x^-\psi_i^n = \frac{\psi_i^n - \psi_{i-1}^n}{\Delta x}$, and

$$g^-(d^+, d^-) = \sqrt{\min(d^+, 0)^2 + \max(d^-, 0)^2},$$

$$g^+(d^+, d^-) = \sqrt{\max(d^+, 0)^2 + \min(d^-, 0)^2}.$$

☞ 2nd order extension.

☞ Easy computation of the curvature (for perimeter penalization).

Re-initialization

In order to regularize the level set function (which may become too flat or too steep), we reinitialize it periodically by solving

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + \mathrm{sign}(\psi_0)\Big(|\nabla_x \psi| - 1\Big) = 0 & \text{for } x \in D,\ t > 0 \\ \psi(t=0,x) = \psi_0(x) \end{cases}$$

which admits as a stationary solution the signed distance to the initial interface $\{\psi_0(x) = 0\}$.

- ☞ Classical idea in fluid mechanics.
- ☞ A few iterations are enough.
- ☞ Improve the convergence of the optimization process (for fine meshes).

Choice of the descent step

Two different strategies:

At each elasticity analysis, we perform a single time step of transport:

- ☞ The descent step is controlled by the CFL of the transport equation.
- ☞ Smooth descent.
- ☞ Lengthy computations (sub-optimal descent step).

At each elasticity analysis, we perform many time steps of transport:

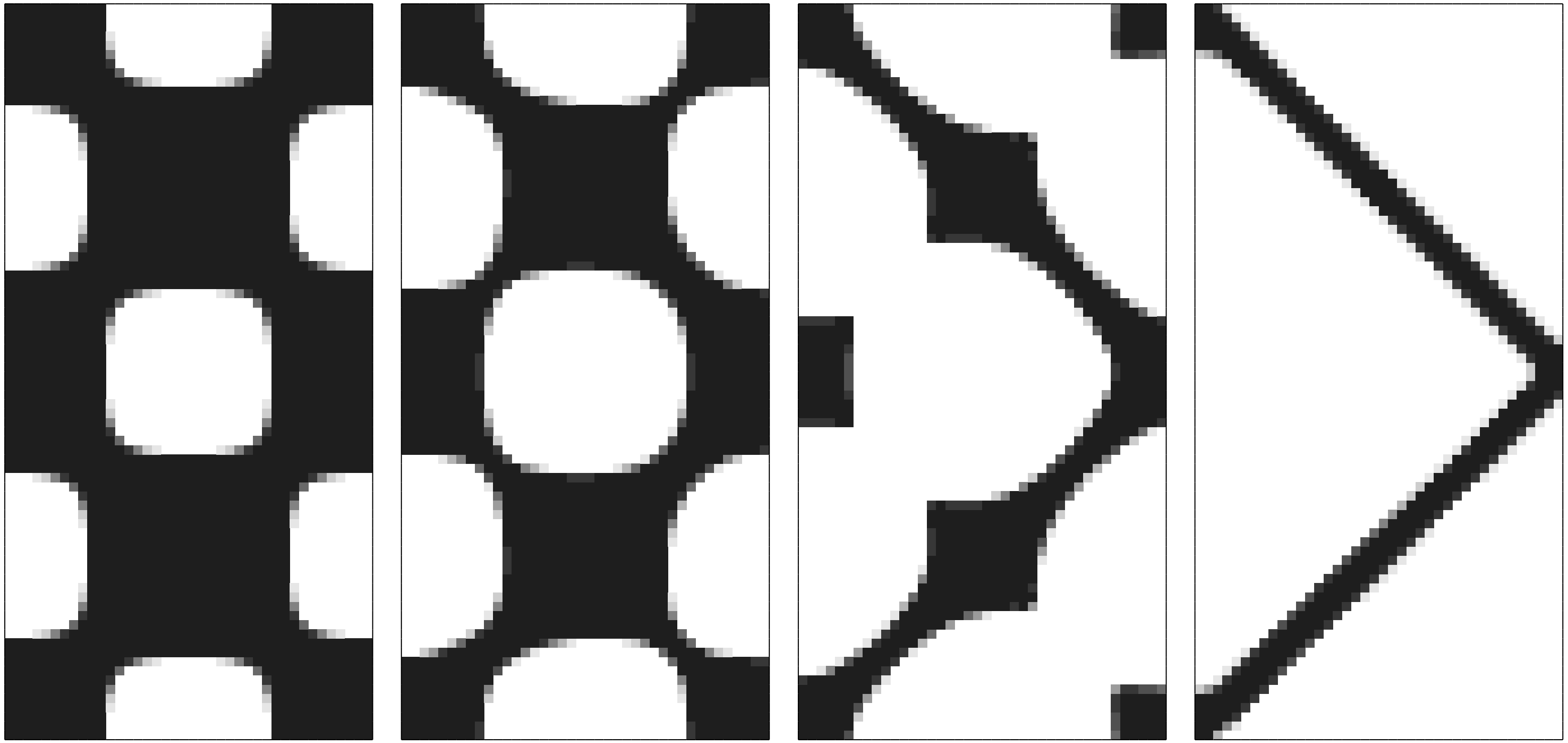
- ☞ The descent step is controlled by the decrease of the objective function.
- ☞ Fast descent but requires a good heuristic for monitoring the number of time steps.

NUMERICAL EXAMPLES

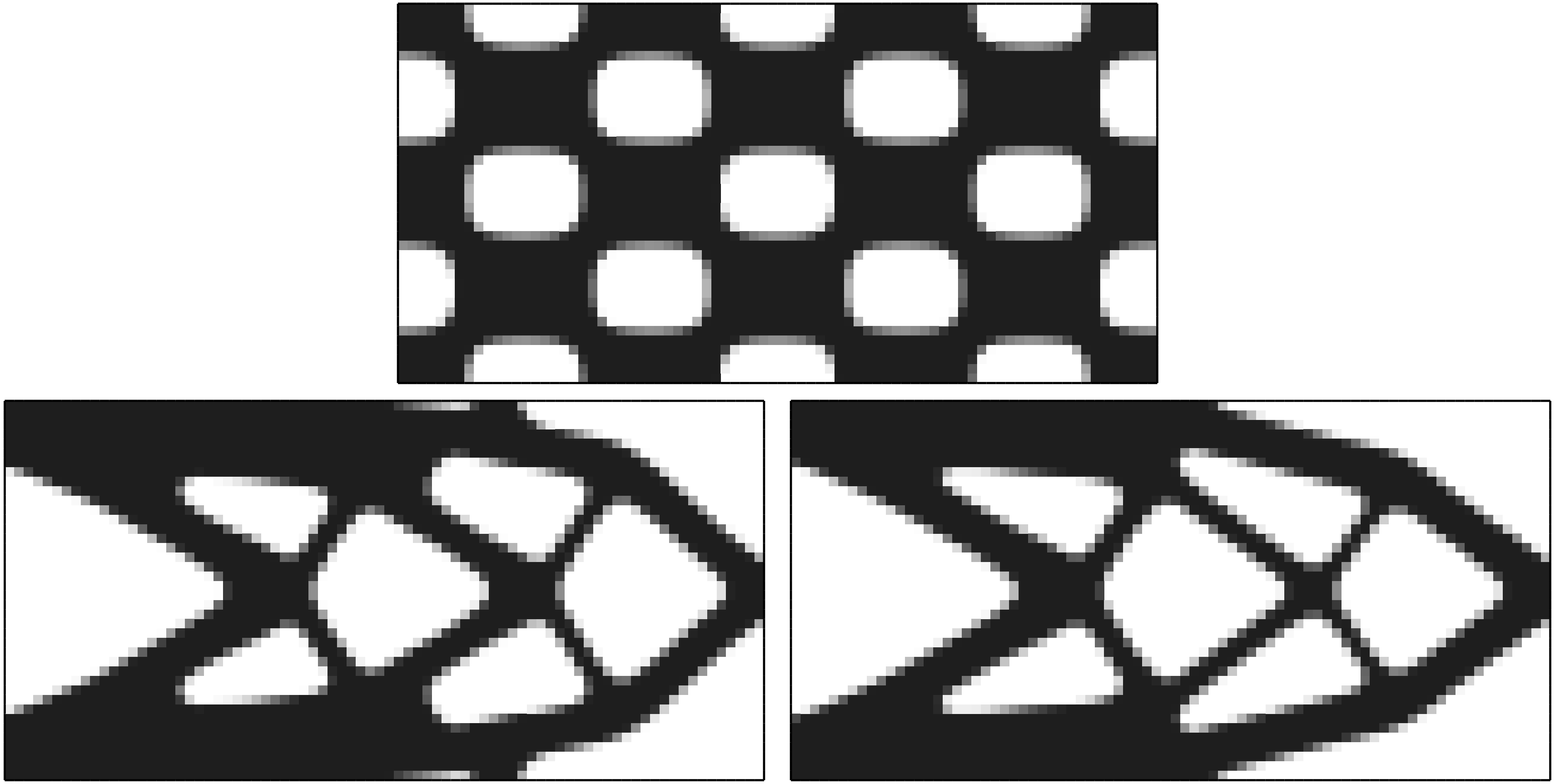
See the web page

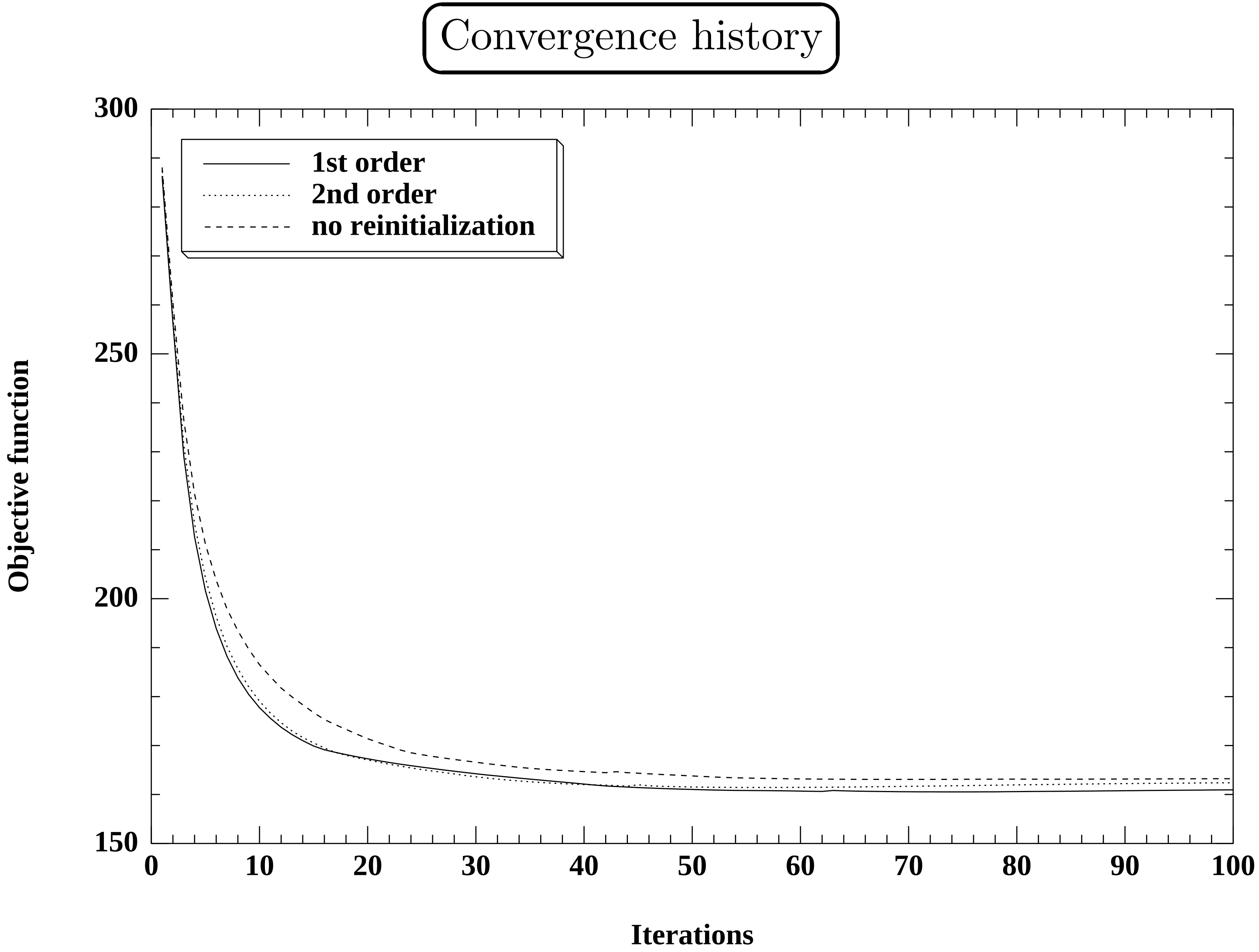
http://www.cmap.polytechnique.fr/~optopo/level_en.html

Short cantilever

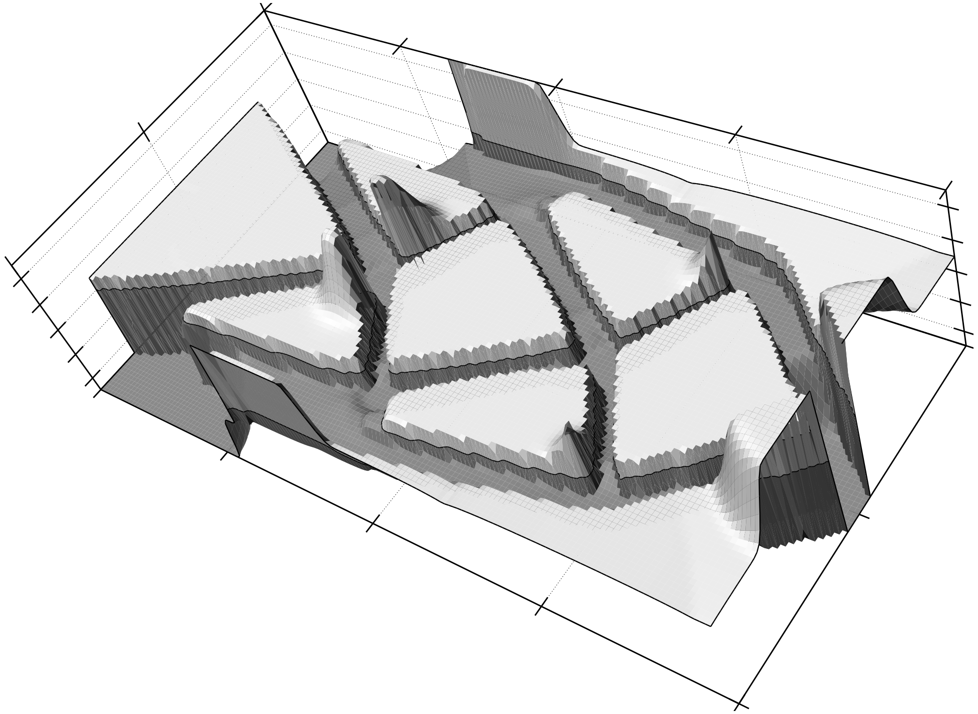


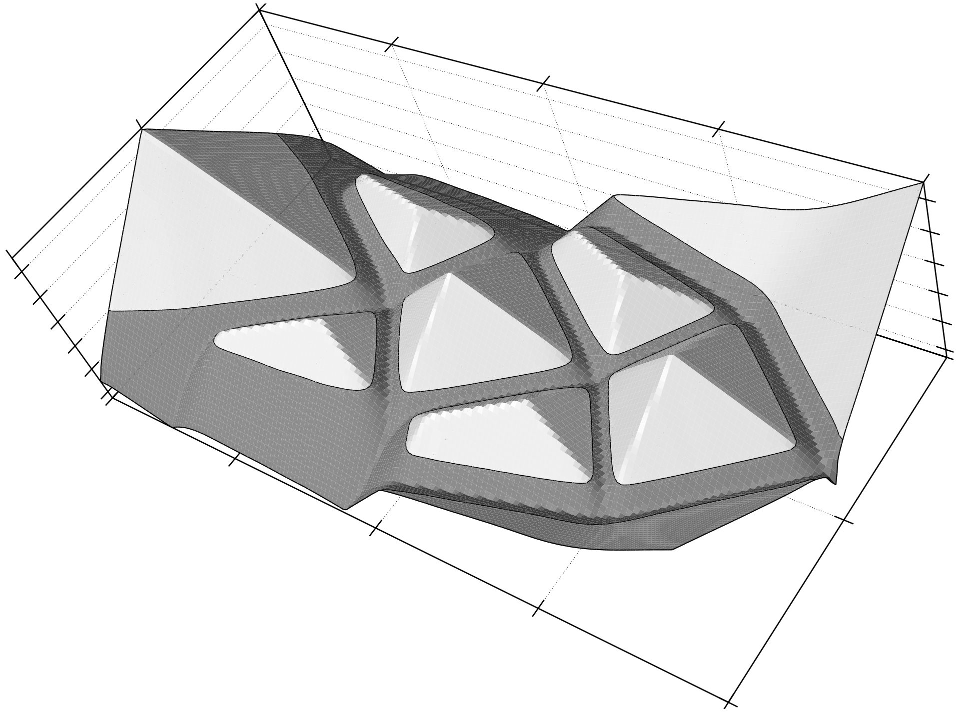
Medium cantilever: iterations 0, 10 and 50



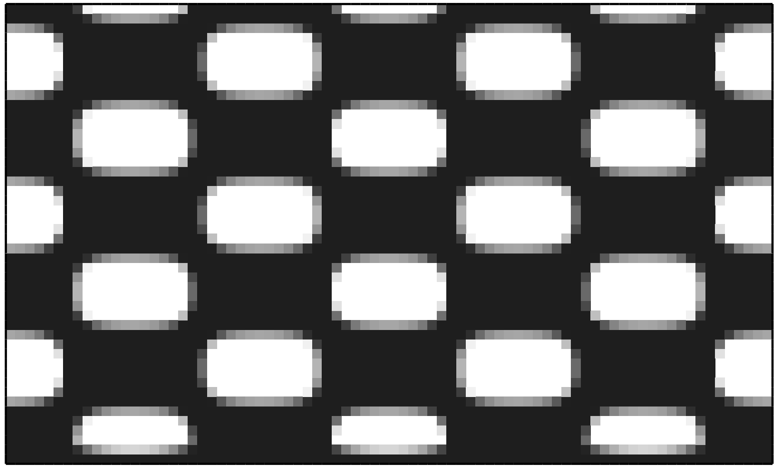

Convergence history
1st order
2nd order
no reinitialization
Objective function
Iterations
300
250
200
150
0
10
20
30
40
50
60
70
80
90
100

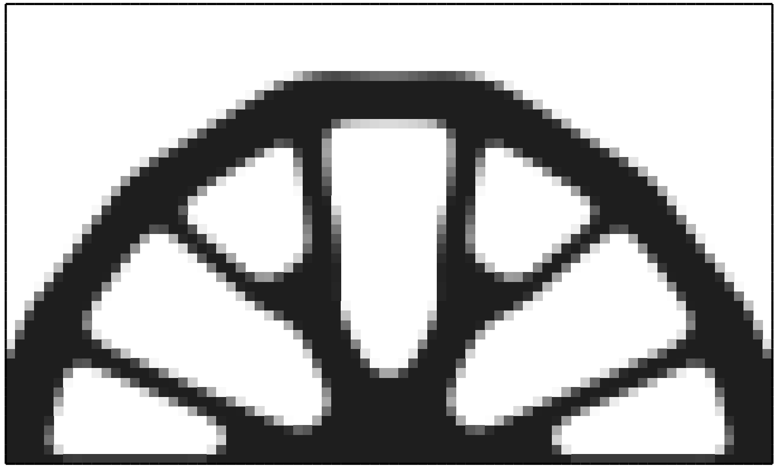
Influence of re-initialization

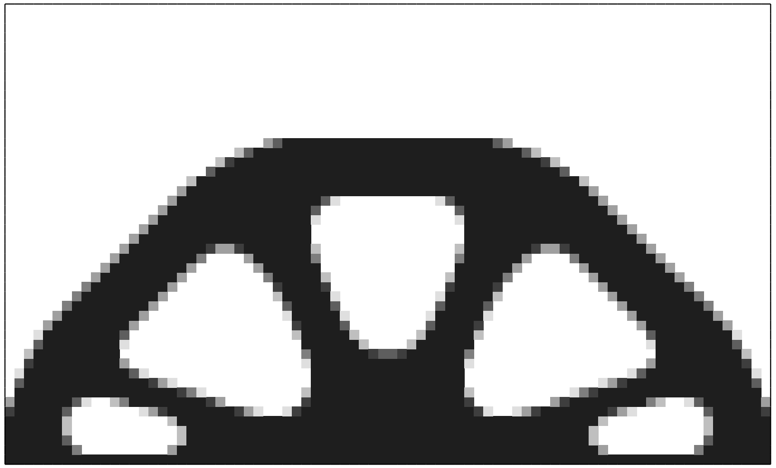




Influence of perimeter constraint







Design dependent loads - 1

Force g applied to the free boundary

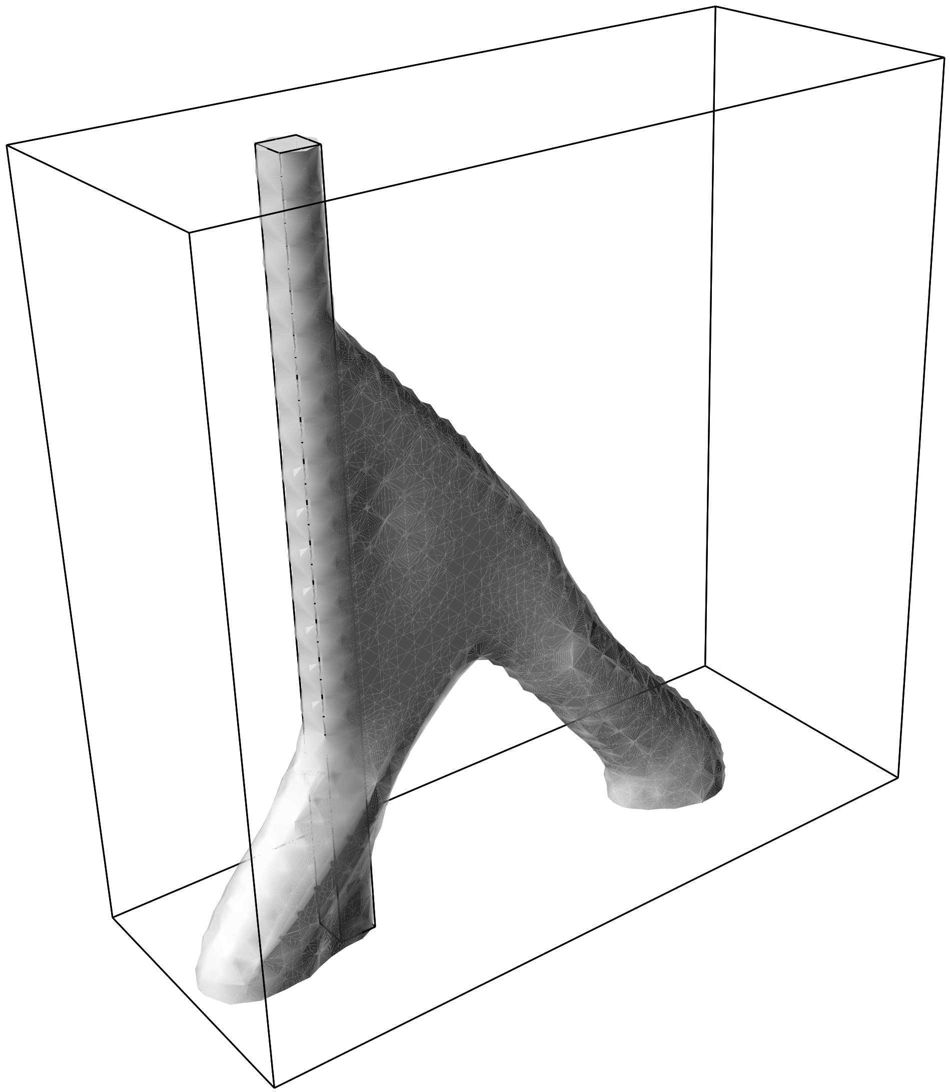
$$\begin{cases} -\mathrm{div}\ (A\,e(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(A\,e(u)\big)n = g & \text{on } \Gamma_N \cup \Gamma \end{cases}$$

Compliance minimization

$$J(\Omega) = \int_{\Gamma \cup \Gamma_N} g \cdot u\, ds = \int_\Omega A\,e(u) \cdot e(u)\, dx,$$

$$J'(\Omega_0)(\theta) = \int_{\Gamma_0} \left(2 \left[\frac{\partial (g \cdot u)}{\partial n} + H g \cdot u \right] - Ae(u) \cdot e(u) \right) \theta \cdot n\, ds,$$

Optimal mast under a uniform wind



Design dependent loads - 2

Pressure p_0 applied to the free boundary

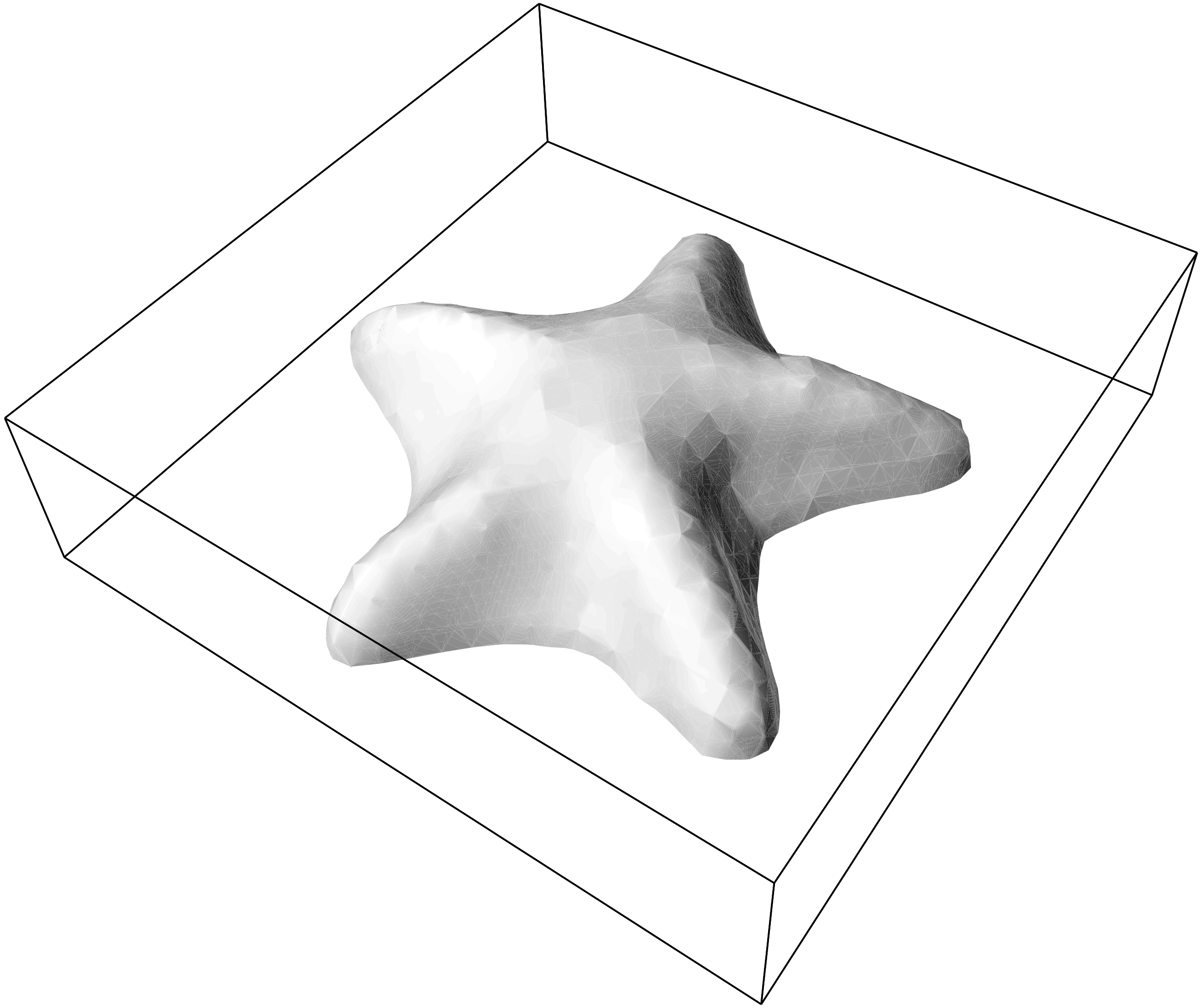
$$\begin{cases} -\mathrm{div}\ (A\,e(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(A\,e(u)\big)n = p_0\,n & \text{on } \Gamma_N \cup \Gamma \end{cases}$$

Compliance minimization

$$J(\Omega) = \int_{\Gamma\cup\Gamma_N} p_0\,n \cdot u\,ds = \int_\Omega A\,e(u)\cdot e(u)\,dx,$$

$$J'(\Omega_0)(\theta) = \int_{\Gamma_0} \theta\cdot n\Big(2\mathrm{div}\ (p_0 u) - Ae(u)\cdot e(u)\Big)ds$$

Sea star under a uniform pressure load

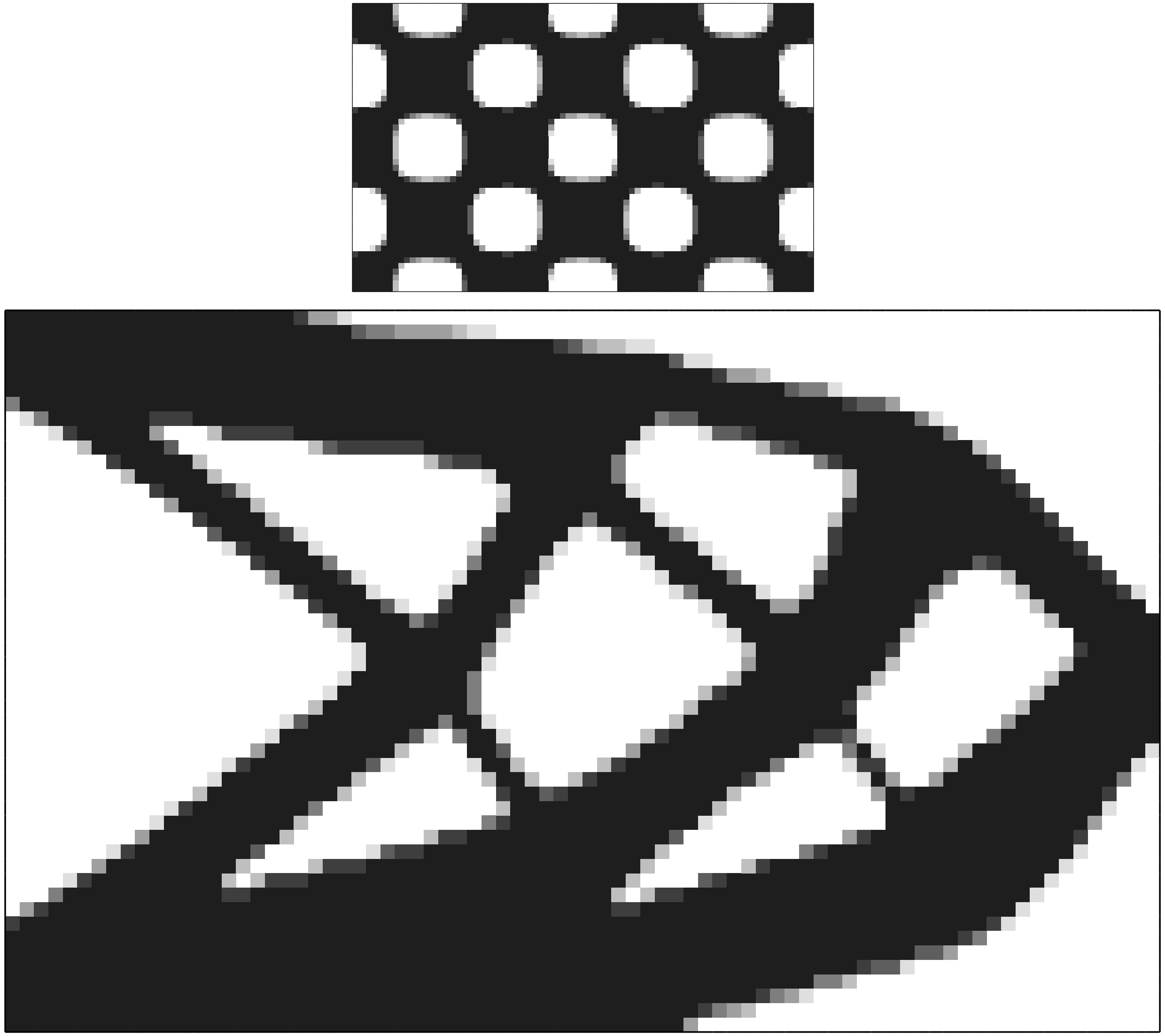


Non-linear elasticity

$$\left\{ \begin{array}{rcll} -\text{div}\ (T(F)) & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \Gamma_D \\ T(F)n & = & g & \text{on } \Gamma_N, \end{array} \right.$$

with the deformation gradient $F = (I + \nabla u)$ and the stress tensor

$$T(F) = F\Big(\lambda \text{Tr}(E) I + 2\mu E\Big) \quad \text{with} \quad E = \frac{1}{2}\big(F^T F - I\big)$$



Conclusion

- ☞ Efficient method.
- ☞ With a good initialization, comparable to the homogenization method.
- ☞ No nucleation mechanism.
- ☞ Can be pre-processed by the homogenization method.
- ☞ Can handle non-linear models, design dependent loads and any smooth objective function.