# TOPOLOGY OPTIMIZATION WITH THE HOMOGENIZATION AND THE LEVEL-SET METHODS

G. ALLAIRE
*CMAP, Ecole Polytechnique*
*91128 Palaiseau, France*

**Abstract.** After a brief review of the homogenization and level-set methods for structural optimization we make some comparisons of their numerical results. The typical problem is to find the optimal shape of an elastic body which is both of minimum weight and maximal stiffness under specified loadings. This problem is known to be "ill-posed", namely there is generically no optimal shape and the solutions computed by classical numerical algorithms are highly sensitive to the initial guess and mesh-dependent. The homogenization method makes this problem well-posed by allowing microperforated composites as admissible designs. It induces new numerical algorithms which capture an optimal shape on a fixed mesh. The homogenization method is able to perform topology optimization since it places no explicit or implicit restriction on the topology of the optimal shape. The level-set method instead does not change the ill-posed nature of the problem. It is a combination of the level-set algorithm of Osher and Sethian with the classical shape gradient (or boundary sensitivity). Although this last method is not specifically designed for topology optimization, it can easily handle topology changes. Its cost is also moderate since the shape is captured on a fixed Eulerian mesh. We discuss their respective advantages and drawbacks.

## 1. Introduction

Shape optimization of elastic structures is a very important and popular field. The classical method of shape sensitivity (or boundary variation) has been much studied (see e.g. [10], [13], [16]). It is a very general method which can handle any type of objective functions and structural models, but it has two main drawbacks: its computational cost (because of remeshing) and its tendency to fall into local minima far away from global ones. The homogenization method (see e.g. [1], [2], [6], [7], [8]) is an adequate remedy to these drawbacks but it is mainly restricted to linear elasticity and particular objective functions (compliance, eigenfrequency, or compliant mechanism). Recently yet another method appeared based on the first approach of shape sensitivity but using the versatile level-set method for computational efficiency (see [4], [11], [15], [17]). The level-set method has been devised by Osher and Sethian [12] for numerically tracking fronts and free boundaries. In this paper we review and compare the homogenization method and this new level-set method for structural optimization.

## 2. Setting of the problem.

We consider the following structural optimization problem : find the optimal shape that minimizes a weighted sum of its elastic compliance and weight. As usual the compliance (i.e. the work done by the load) is a global measure of the design's rigidity. We introduce a working domain $Q \subset \mathbb{R}^d$ in which all admissible shapes $\Omega$ are included, i.e. $\Omega \subset Q$. The boundary of a shape $\Omega$ is made of three disjoint parts

$$\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D,$$

with Dirichlet boundary conditions on $\Gamma_D$, and Neumann boundary conditions on $\Gamma \cup \Gamma_N$. We assume that $\Gamma_D$ and $\Gamma_N$ are parts of $\partial Q$ and are supposed to be fixed. Only $\Gamma$ is allowed to vary in the optimization process.

The displacement field $u$ in $\Omega$ is the unique solution of the linearized elasticity system

$$\begin{cases} -\mathrm{div}\,(A\,e(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ (A\,e(u))n = f & \text{on } \Gamma_N \\ (A\,e(u))n = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where $A$ is the elasticity tensor of a linearly isotropic elastic material (with bulk and shear moduli $\kappa$ and $\mu$). Recall that the deformation tensor is $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$ and the stress tensor is $\sigma = Ae(u)$. The compliance of the structure $\Omega$ is

$$c(\Omega) = \int_{\Gamma_N} f \cdot u = \int_\Omega Ae(u) \cdot e(u) = \int_\Omega A^{-1}\sigma \cdot \sigma. \tag{2}$$

Introducing a positive Lagrange multiplier $\ell$, our structural optimization problem is to minimize, over all subsets $\Omega \subset Q$, the objective function $J(\Omega)$ equal to the weighted sum of the compliance and weight of $\Omega$. In other words we want to compute minimizers of

$$\inf_{\Omega\subset Q} \Big\{ J(\Omega) = c(\Omega) + \ell|\Omega| \Big\}. \tag{3}$$

The Lagrange multiplier $\ell$ has the effect of balancing the two contradictory objectives of rigidity and lightness of the shape (increasing its value decreases the weight).

As is well known, in absence of any supplementary (topological) constraints on the admissible designs $\Omega$, the objective function $J(\Omega)$ may have no minimizer, i.e. there is no optimal shape (see e.g. [1], [9]). The physical reason for this non-existence is that it is often advantageous to cut infinitely many small holes (rather than just a few big ones) in a given design in order to decrease the objective function. Thus, achieving the minimum may require a limiting procedure leading to a "generalized" design consisting of composite materials made by microperforation of the original material.

## 3. Homogenized formulation.

To cope with this physical behavior of nearly optimal shapes, we enlarge the space of admissible designs by permitting perforated composites from the start : this process is called relaxation. Such composite structures are determined by two functions $\theta(x)$ and $A^*(x)$ : $\theta$ is the local volume fraction of the original material, taking values between 0 and 1, and $A^*$ is the effective Hooke's law determined by the microstructure of perforations. In this section we briefly recall the main results on this so-called homogenization approach (see [1], [7], [9] and references therein).

A minimizing sequence of the objective function (3) can be regarded as a composite material obtained by microperforation of the original material $A$. The effective behavior of such a composite material is characterized by a material density $\theta(x) \in [0,1]$ and a Hooke's law $A^*(x)$ such that the average or macroscopic behavior of solutions of (1) are determined by the homogenized problem

$$\left\{\begin{array}{ll} \sigma = A^*(x)e(u) & e(u) = \frac{1}{2}(\nabla u + \nabla^t u) \\ \mathrm{div}\sigma = 0 & \text{in } Q \\ u = 0 & \text{on } \Gamma_D \\ \sigma n = f & \text{on } \Gamma_N \\ \sigma n = 0 & \text{on } \partial Q \setminus (\Gamma_D \cup \Gamma_N) . \end{array}\right. \tag{4}$$

The homogenized compliances is

$$\tilde{c}(\theta, A^*) = \int_{\Gamma_N} f \cdot u = \int_Q A^*(x)^{-1}\sigma \cdot \sigma,$$

where the stress $\sigma$ is solution of the homogenized equation (4). Remark that, for a given value $\theta$ of the density, there are many different possible effective Hooke's law $A^*$ in a set $G_\theta$, the so-called $G$-closure set at volume fraction $\theta$, which is the set of all possible homogenized Hooke's law with density $\theta$. We thus obtain the relaxed or homogenized functional

$$\min_{0\leq\theta\leq 1,\ A^*\in G_\theta} \left\{ \tilde{J}(\theta, A^*) = \tilde{c}(\theta, A^*) + \ell \int_Q \theta(x)\, dx \right\}.$$

The relaxed functional $\tilde{J}(\theta, A^*)$ has to be minimized over all admissible composite designs, i.e. over all density $\theta$ and effective Hooke's law $A^* \in G_\theta$. Although $G_\theta$ is not known explicitly, the minimization of the compliance $\tilde{c}(\theta, A^*)$ can be done analytically since optimal composites are shown to be the so-called sequential laminates (which have explicit elasticity tensors). Indeed, we rewrite the compliance as

$$\tilde{c}(\theta, A^*) = \min_{\substack{\mathrm{div}\sigma=0 \text{ in } Q \\ \sigma n = f \text{ on } \Gamma_N \\ \sigma n = 0 \text{ on } \Gamma}} \int_Q A^*(x)^{-1}\sigma \cdot \sigma. \tag{5}$$

Then, the two minimizations, in $(\theta, A^*)$ and in $\sigma$, can be switched. Since the microstructure can be optimized pointwise in the domain, the relaxed formulation becomes

$$\min_{\substack{\mathrm{div}\sigma=0 \text{ in } Q \\ \sigma n = f \text{ on } \Gamma_N \\ \sigma n = 0 \text{ on } \Gamma}} \int_Q \min_{0\leq\theta\leq 1,\ A^*\in G_\theta} \left( {A^*}^{-1}\sigma \cdot \sigma + \ell\theta \right) dx. \tag{6}$$

For a fixed stress $\sigma$, the minimization of ${A^*}^{-1}\sigma \cdot \sigma$ on $G_\theta$ is a classical problem in the theory of homogenization and composite materials [1], [9]. It amounts to find the most rigid composite of given density $\theta$ under the stress $\sigma$. In two dimensions, the result is

$$\min_{A^*\in G_\theta} {A^*}^{-1}\sigma \cdot \sigma = A^{-1}\sigma \cdot \sigma + \frac{(\kappa + \mu)(1-\theta)}{4\kappa\mu\theta} \left( |\sigma_1| + |\sigma_2| \right)^2 \tag{7}$$

where $\sigma_1$ and $\sigma_2$ are the eigenvalues of the 2 by 2 symmetric matrix $\sigma$. Furthermore, optimality in (7) is achieved for a so-called rank-2 sequential laminate aligned with the eigendirections of $\sigma$. In three dimensions, the result is more complicated, and we give it in the special case of Poisson's ratio equal to zero, i.e. $3\kappa = 2\mu$ (the general case is not much different in essence)

$$\min_{A^*\in G_\theta} {A^*}^{-1}\sigma \cdot \sigma \quad = \quad A^{-1}\sigma \cdot \sigma + \frac{(1-\theta)}{4\mu\theta} g^*(\sigma)$$

with

$$g^*(\sigma) = \begin{cases} \left( |\sigma_1| + |\sigma_2| + |\sigma_3| \right)^2 & \text{if } |\sigma_3| \leq |\sigma_1| + |\sigma_2| \\ 2\left( (|\sigma_1| + |\sigma_2|)^2 + |\sigma_3|^2 \right) & \text{if } |\sigma_3| \geq |\sigma_1| + |\sigma_2| \end{cases} \tag{8}$$

where the eigenvalues of $\sigma$ are labeled in such a way that $|\sigma_1| \leq |\sigma_2| \leq |\sigma_3|$. Furthermore, optimality in the first regime of (8) is achieved by a rank-3 sequential laminate aligned with the eigendirections of $\sigma$, while in the second regime it is achieved by a rank-2 sequential laminate aligned with the two first eigendirections of $\sigma$.

After this crucial step, the minimization in $\theta$ can easily be done by hand, which completes the explicit calculation of the relaxed formulation. From a mathematical point of view, one can prove that the relaxed formulation (6) admits a minimizer, that any minimizing sequence of the original problem (3) converges (in the sense of homogenization) to a minimizer of (6), and that the two infimum values of (6) and (3) are equal. However, in general there is no uniqueness of the minimizer.

## 4. Numerical algorithm for the homogenization method.

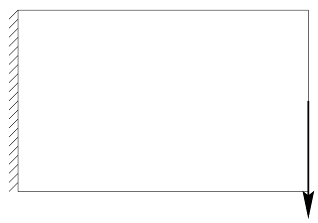

*Figure 1.* Boundary conditions of a 2-d cantilever.

The first main advantage of the homogenization method is to change a difficult "free-boundary" problem into a much easier "sizing" optimization problem in a fixed domain. The computational cost is thus very low compared to traditional algorithms since the mesh is fixed (shapes are captured rather than tracked). The second main advantage is that the homogenized formulation is well-posed. In practice, the resulting optimal shape is independent of the initial guess and the homogenization method thus performs topology optimization (the final optimal shape may have a topology completely different from that of the initial guess).

Let us describe briefly our favorite algorithm. It is an alternate direction algorithm: we start with an initial design (usually full material everywhere), then, at each iteration, we compute the stress $\sigma$ solution of a linear elasticity problem with a Hooke's law corresponding to the previous design, and we update the design variables $\theta$ and $A^*$ in terms of $\sigma$ by using the explicit formula for the optimal laminated composite material in (7) or (8). We iterate this process until convergence which is detected when the density variation becomes smaller than some threshold. This algorithm converges smoothly in a relatively small number of iterations (between 10 and 100,

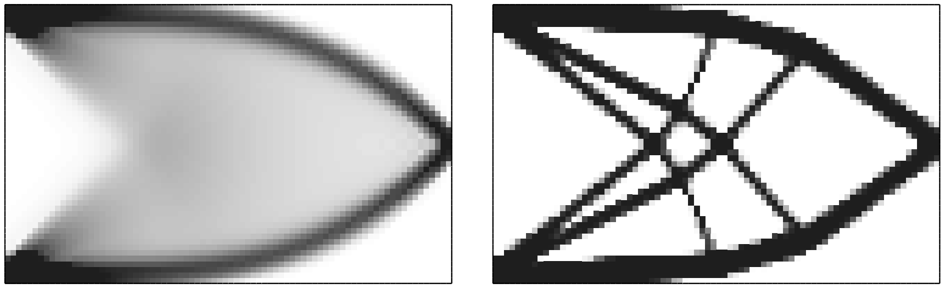

*Figure 2.* Homogenization method: optimal shape of the cantilever: composite (left) and penalized (right).

depending on the desired accuracy). Furthermore, in practice it is insensitive to the choice of initial guess and convergent under mesh refinement, suggesting that the numerical algorithm always picks up the same global minimum. However, as expected, it usually produces homogenized optimal designs that include large region of composite materials with intermediate density. From a practical point of view, this is an undesirable feature since the primal goal is to find an optimal shape, i.e. a density taking only the values 0 or 1 ! The remedy is to introduce a penalization technique that will get rid of composite materials. The strategy is the following : after convergence has been reached on a homogenized optimal design, we run a few more iterations (around 10) of our algorithm during which we force the density to take values close to 0 or 1. More specifically, denoting by $\theta_{opt}$ the true optimal density, the penalization procedure amounts to update the density at the value $\theta_{pen} = \frac{1-\cos(\pi\theta_{opt})}{2}$. There is no specific reason to choose a cosine-shape function for the penalized density, except that it works fine and yields surprisingly nice shapes featuring fine patterns instead of composite regions.

The success of this method is due to the fact that the relaxed design is characterized not only by a density $\theta$ but also by a microstructure $A^*$ which is hidden at the sub-mesh level. The penalization has the effect of reproducing this microstructure at the mesh level. Of course it is strongly mesh-dependent in the sense that the finer the mesh the more complicated the resulting "almost optimal" structure.

The homogenization method can be generalized to several other types of objective functions, including sum of compliances for multiple loads, eigenfrequencies, least square criteria for a target displacement or stress (see e.g. [1], [3]). However, in its rigorous setting it is restricted to a linear elasticity model.

## 5. Shape derivative

In this section we come back to the classical setting of shape sensitivity for structural optimization. We briefly recall how to compute a shape derivative for the objective function $J(\Omega)$ defined by (3). In order to define a shape derivative we follow the approach of Murat-Simon [10] (see also [13], [16]). Starting from a reference domain $\Omega_0$, we consider domains of the type

$$\Omega = (\,\mathrm{Id} + \tau)(\Omega_0), \tag{9}$$

where $\tau \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ (for sufficiently small $\tau$, $(\,\mathrm{Id} + \tau)$ is a diffeomorphism). We further restrict the class of domains by asking that they all share the same parts of the boundary $\Gamma_N$ and $\Gamma_D$: specifically, the map $\tau$ must vanish on $\Gamma_N \cup \Gamma_D$. The shape derivative of $J(\Omega)$ at $\Omega_0$ is then defined as the Fréchet derivative in $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ at 0 of the map $\tau \to J((\,\mathrm{Id} + \tau)(\Omega_0))$. This notion is well defined and a standard computation shows that the shape derivative of (3) is

$$\langle \frac{\partial J}{\partial \Omega}(\Omega_0), \tau \rangle = \int_\Gamma \Big( \ell - Ae(u) \cdot e(u) \Big) \tau \cdot n \, ds, \tag{10}$$

where $u$ is the solution of (1) in $\Omega_0$, $n$ is the unit exterior normal and $H$ the curvature of $\Gamma$. Remark that there is no adjoint state involved in (10) (indeed the minimization of (3) is a self-adjoint problem). Of course, the shape derivative can be computed for other objective functions and other model problems including non-linear elasticity [4], [5].

## 6. Numerical algorithm for the level-set method

We review the numerical implementation of a gradient method for the minimization of problem (3) as proposed in [4], [5]. The idea is to combine the shape derivative of Section 5 and the level-set method of Osher and Sethian [12].

In order to describe the boundary of $\Omega$ we introduce a level-set function $\psi$ defined on the working domain $Q$ such that

$$\begin{cases} \psi(x) = 0 & \Leftrightarrow x \in \partial\Omega \cap Q \\ \psi(x) < 0 & \Leftrightarrow x \in \Omega \\ \psi(x) > 0 & \Leftrightarrow x \in (Q \setminus \Omega) \end{cases}$$

The normal $n$ to the shape $\Omega$ is recovered as $\nabla\psi/|\nabla\psi|$ and the curvature $H$ is given by the divergence of $n$ (these quantities are evaluated by finite differences since our mesh is uniformly rectangular). Remark that, although $n$ and $H$ are defined on $\Gamma$, the level set method allows to define easily their extension in the whole domain $Q$. We fill the void part $Q \setminus \Omega$ with a very

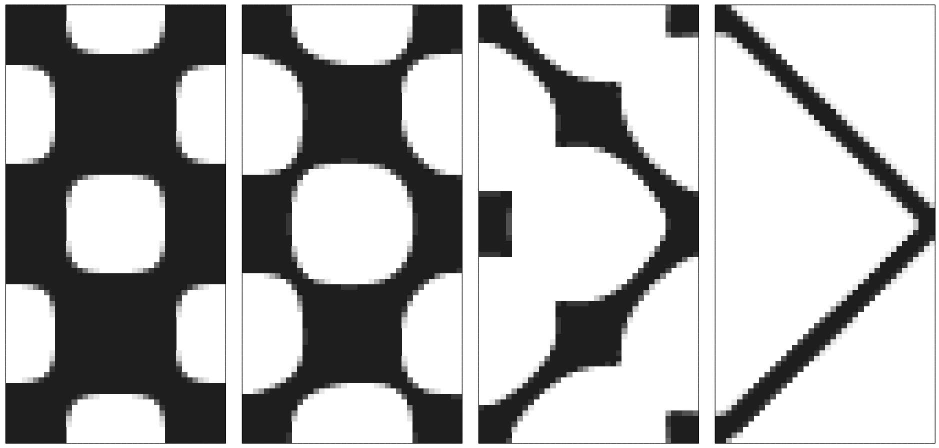

*Figure 3.* Short cantilever with the level-set method: initialization and successive iterations.

weak material with Hooke's law $B = 10^{-3}A$ and we perform the elasticity analysis on a fixed rectangular mesh in $Q$ (using $Q1$ finite elements). Since $n$ and $H$, as well as the displacement $u$, are computed everywhere in $Q$, formula (10) delivers a vector field $V$ throughout the domain and not only on the free boundary $\Gamma$, namely

$$V = v\, n \quad \text{with} \quad v = \ell - Ae(u) \cdot e(u)\ .$$

After evaluating the gradient of $J(\Omega)$, or equivalently this vector field $V$, we transport the level set function $\psi$ along this gradient flow $-V = -v\, n$. Since $n = \nabla\psi/|\nabla\psi|$, we end up with the following Hamilton-Jacobi equation

$$\frac{\partial \psi}{\partial t} - v|\nabla\psi| = 0, \tag{11}$$

where the time variable $t$ plays the role of the descent step in the gradient algorithm. Transporting $\psi$ by (11) is equivalent to move the boundary of $\Omega$ (the zero level set of $\psi$) along the descent gradient direction $-\frac{\partial J}{\partial \Omega}$. We solve (11) using a standard explicit upwind finite difference scheme (see e.g. [14]). Finally, our algorithm is an iterative method, structured as follows:

1. Initialization of the level-set function $\psi_0$ as the signed distance function to the boundary of an initial guess $\Omega_0$.
2. Iteration until convergence, for $k \geq 0$:
   (a) Computation of $u_k$ by solving a linear elasticity problem in $Q$ with Hooke's law

$$A_k(x) = A \text{ where } \psi_k(x) < 0 \quad A_k(x) = B \text{ where } \psi_k(x) > 0.$$

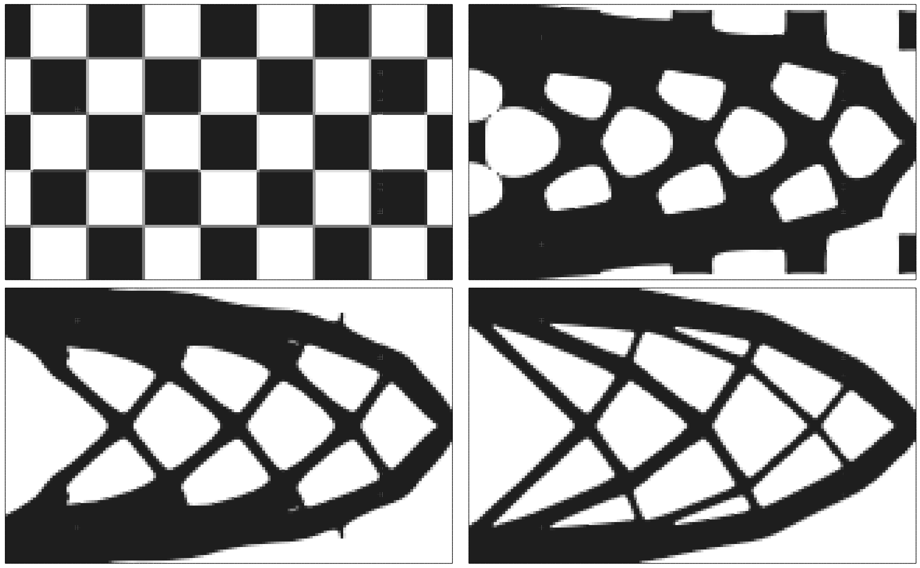

*Figure 4.* Medium cantilever with the level-set method: initialization with many holes and successive iterations.

(b) Deformation of the shape through the transport of the level set function: $\psi_{k+1}(x) = \psi(\Delta t_k, x)$ where $\psi(t, x)$ is the solution of (11) with velocity $v_k = \ell - Ae(u_k)\cdot e(u_k)$ and initial condition $\psi(0, x) = \psi_k(x)$. The time step $\Delta t_k$ is chosen such that $J(\Omega_{k+1}) \le J(\Omega_k)$.

This algorithm never creates new holes or boundaries if the time step $\Delta t_k$ satisfies a CFL condition for (11) (there is no nucleation mechanism for new holes). However the level set method is well known to handle easily topology changes, i.e. merging or cancellation of holes. In numerical practice, the number of holes always decrease in dimension $d = 2$, so the initialization must contain enough holes in order to obtain a good optimal shape (compare Figures 4 and 5). However, in dimension $d = 3$ new holes can appear by pinching thin plate-like objects, so the initial design is less critical (also still important). In any case, our algorithm is able to perform topology optimization. The algorithm converges smoothly to a (local) minimum which depends, of course, on the initial topology. The numerical results are very similar to those obtained by the homogenization method. In order to speed up the convergence we perform several (of the order of 20 in numerical practice) time steps of the transport equation for each elasticity analysis. The exact number of time steps is controlled by the decrease of the objective function.

From time to time, for stability reasons, we also reinitialize the level set function $\psi$ in order that it be the signed distance function to the boundary

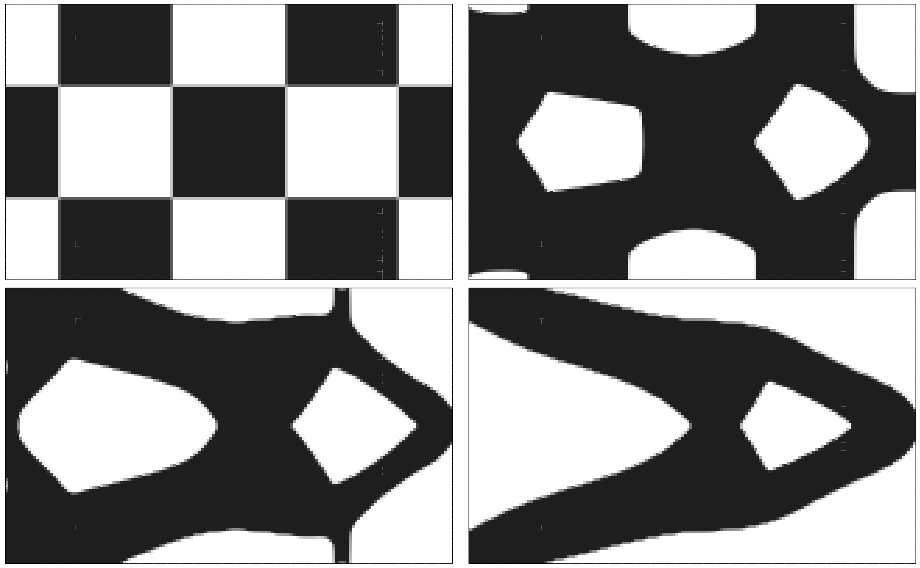

*Figure 5.* Medium cantilever with the level-set method: initialization with few holes and successive iterations.

of the current shape $\Omega$ (see [14]).

We give some numerical results for the compliance objective function (3) (more examples, including load dependent designs can be found in [5]). The boundary conditions for cantilever problems are displayed on Figure 1. The results are shown on Figures 4 and 5 for an increasing number of iterations. Figure 6 shows the history of the objective function for a medium cantilever optimized with the two methods discussed here: homogenization and level-set. A three-dimensional example is given on Figure 7.

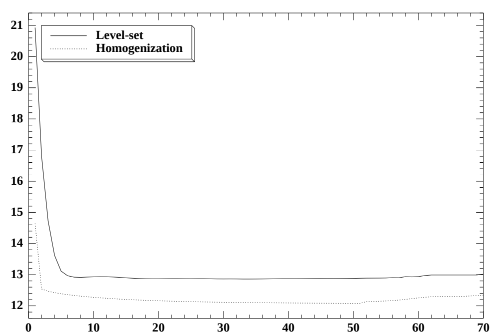


*Figure 6.* Convergence of the objective function for the two iterative methods.

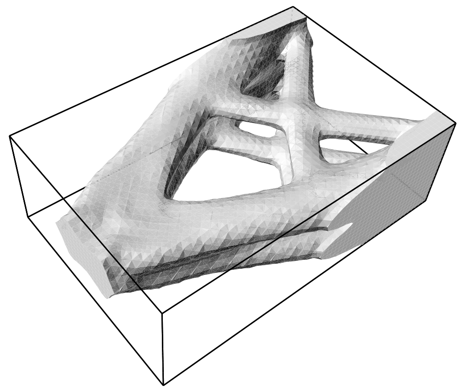

*Figure 7.* Three-dimensional cantilever by the level-set method.

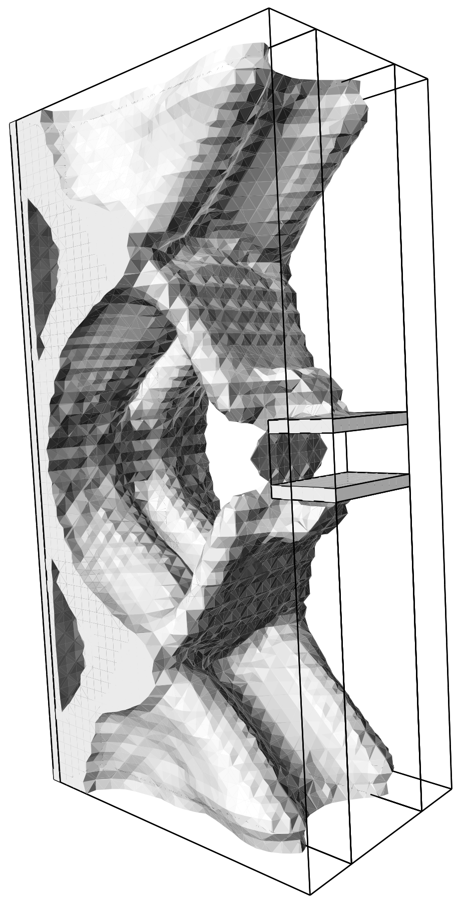

*Figure 8.* Three-dimensional gripping mechanism by the level-set method.

## 7. Comparisons and conclusions

We implemented the homogenization method and the level-set method in two and three space dimensions for shape and topology optimization. They both share the following advantages:

1. they allow for drastic topology changes during the optimization process,
2. their cost is moderate in terms of CPU time since they are Eulerian shape capturing methods.

However, they are very different with respect to the following points.

1. Influence of the initial design: since the homogenization method is a relaxation method, it is independent of the initial guess and it numerically converges to a global maximum ; on the contrary, the level-set method is very sensitive to the initial guess and easily get caught in local minima.
2. Generality of application: the homogenization method is mostly restricted to some specific objective functions and to the linear elasticity setting ; on the other hand, the level-set method can handle very general objective functions and mechanical models, including nonlinear elasticity (see Figure 9).

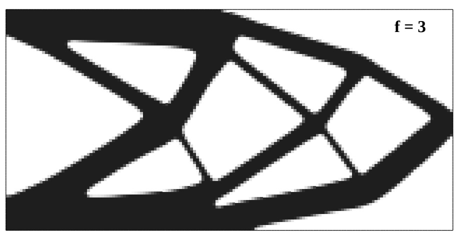


*Figure 9.* Optimal cantilever in non-linear elasticity (the bars under compression are thicker than those under traction).

The two methods discussed above are not in competition ; rather they are complementary. One should find a correct topology for a simple objective function in linear elasticity by applying the homogenization method, and then use it as an initial guess for the level-set method in the context of a more involved objective function and non-linear elasticity model.

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