TC1 Optimal design of structures O.Pantz
Basic Numerical Analysis Tools

Exercise I. Variational Formulation

Let Ω be a regular bounded and connected open set of $\mathbb{R}^n$. Let Γ_N and Γ_D such that $\partial\Omega = \Gamma_N \cup \Gamma_D$ of negligible intersection , each of them being of non zero surface measure. Let $g \in L^2(\Gamma_N)$ and $f \in L^2(\Omega)$. We consider the Partial Differential Equation

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \quad u = 0 \text{ on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = g \text{ on } \Gamma_N. \end{cases} \tag{1}$$

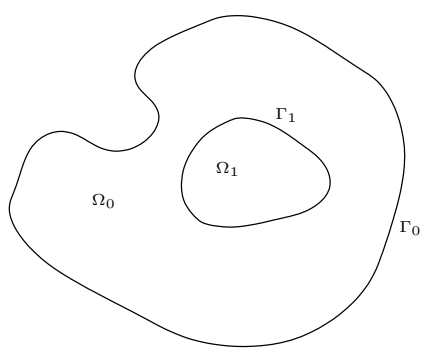
1. Give the variational formulation associated to (1).
2. Prove that there exists a unique solution u to the variational problem obtained. To this end, we could admit the following Lemma.

Lemma 1. *Let Ω be a connected open set of $\mathbb{R}^N$. Let $u \in H^1(\Omega)$ such that $\nabla u = 0$ then u is constant over Ω.*

3. If g is the trace of a $H^1(\Omega)$ map (that is if g belongs to $H^{1/2}(\Omega)$), the solution u of the variational formulation belongs to $H^2(\Omega)$. In this case, prove that it is also a solution of the initial PDE (1).

Exercise II. Non local limit conditions

Let Ω_0, Ω_1, be open bounded sets of $\mathbb{R}^2$ such that $\overline{\Omega}_1$ is included in Ω_0 assumed to be connected. We denote by Ω the open set $\Omega_0 - \overline{\Omega}_1$.



Let f be a map in $L^2(\Omega)$ and $q \in L^\infty(\Omega)$. We would like to establish the existence of a solution u of the following problem

$$\begin{cases} -\Delta u + q(x)u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = -\alpha \displaystyle\int_{\Gamma_1} u ds & \text{on } \Gamma_1 \\ u(x) = 0 & \text{on } \Gamma_0 \end{cases}$$

where α is a non negative real, $\Gamma_0 = \partial\Omega_0$ and $\Gamma_1 = \partial\Omega_1$.

1. *Variational Formulation.* Prove that all regular solutions of the PDE are solutions of a variational problem to determine. Conversely, prove that all regular solutions of the variational problem are solutions of the PDE.
2. *Existence.* We assume that there exists a positive real q_0 such that for almost all $x \in \Omega$,

$$q(x) \geq q_0.$$

Prove that there exists a unique solution u to the variational problem and that it depends continuously on the data.
3. *Poincaré like inequality.* Prove by contradiction (reductio ad absurdum), that there exists a constant C such that for all u such that $u(x) = 0$ on Γ_0,

$$\|u\|_{H^1(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

Can the conditions imposed to α and q be relaxed and still preserve the existence result ?

Exercise III. Eigenvalues of the Laplacian.

Let Ω be an regular bounded and connected open set of $\mathbb{R}^n$. A real λ is said to be an eigenvalue of the Laplacian with Dirichlet boundary conditions if there exists a non zero map $u \in H_0^1(\Omega)$ such that

$$-\Delta u = \lambda u.$$

The map u is called the eigenfunction associated to λ.
1.*Positivity of the eigenvalues.* Prove that all the eigenvalues of the Laplacian with Dirichlet BC is non negative (and even positive).
2.*Regularity of the eigenfunctions.* We recall the remarkable regularization property:

Theorem 1. *Let Ω be an regular open set of $\mathbb{R}^n$ and f a map of $H^m(\Omega)$, then the solution $u \in H_0^1(\Omega)$ of the PDE*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

belongs to $H^{m+2}(\Omega)$.

Moreover, we "recall" that all function that belongs to $H^m(\Omega)$ is continuous provided that $m > N/2$. Prove that the eigenfunctions of the Laplacian with Dirichlet boundary conditions belong to $\mathcal{C}^\infty(\overline{\Omega})$.
3. *First eigenvalue.* Let β defined by

$$\beta = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

Prove that β is positive and that there exists $u \in H_0^1(\Omega)$ such that

$$\beta = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

What is the relationship between β and the Poincaré inequality ?
4. *First eigenfunction.* Prove that the functional

$$J(u) = \int_\Omega |u|^2 dx$$

is Gâteaux differentiable, that is there exists a continuous linear form L on $H_0^1(\Omega)$ such that for all $w \in H_0^1(\Omega)$,

$$\lim_{\delta \to 0} \frac{J(u + \delta w) - J(u)}{\delta} = L(w)$$

Prove that the functional

$$F(u) = \int_\Omega |\nabla u|^2 dx$$

is also Gâteaux differentiable. Deduce that the function u maximizing J on the set

$$K = \{v \in H_0^1(\Omega) : F(u) \leq 1\},$$

is an eigenfunction of the Laplacian with Dirichlet boundary conditions.
5. *Weak maximum principle.* Let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ such that $f \geq 0$ and

$$-\Delta u = f.$$

Prove that $u(x) \geq 0$ almost everywhere. To this end, the following Lemma could be used.

Lemma 2. *Let Ω be an open set of $\mathbb{R}^n$ and $u \in H^1(\Omega)$. We set $u^- = \min(0, u)$. Then $u^- \in H^1(\Omega)$ and*

$$\nabla u^- = 1_{u<0} \nabla u.$$

There exists another maximum principle (called strong)

Theorem 2. *Let Ω be a connected regular bounded open set of $\mathbb{R}^N$. Let w be a map of class $\mathcal{C}^2$ on $\overline{\Omega}$ such that*

$$\begin{cases} -\Delta w \geq 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

then either w is equal to zero, or $w(x) > 0$ for all $x \in \Omega$.

6. *Uniqueness and positivity.* By using the strong maximum principle, prove that u is of constant sign on Ω. Conclude that β is a simple eigenvalue.

Exercise IV. Periodical problem in dimension 1
Let $\Omega =]0, 1[$. We seek for the solution $u \in H^1(\Omega)$ of the following PDE

$$\begin{cases} -u'' + u = f & \text{in } \Omega \\ u(1) = u(0) \\ u'(1) = u'(0) + r \end{cases}$$

with $f \in L^2(\Omega)$ and $r \in \mathbb{R}$.
1. Prove that there exists a constant C such that for all $u \in H^1(\Omega)$

$$\sup_{x\in\Omega} |u(x)| \leq C\|u\|_{H^1(\Omega)}.$$

2. Find the variational formulation associated to the PDE considered. Let us denote by a the bilinear form ℓ the linear form and V the Hilbert space thus introduced. Prove the equivalence between both formulations.
3. Prove that the variational problem admits a unique solution.
4. We would like to compute numerically an approximation of the solution of this problem. Let n be a positive integer. We set $h = 1/n$, and for all $i \in \{0, \cdots, n\}$, $x_i = ih$. Let V_h be the space defined by

$$V_h = \{u \in V : \ u_{|[x_i,x_{i+1}]} \in \Pi_1\},$$

where Π_1 is the set of polynomials of degree lower or equal to one. Prove that there exists a unique $u_h \in V_h$ such that

$$a(u_h, v_h) = \ell(v_h) \text{ for all } v_h \in V_h.$$

Prove that this problem is equivalent to find $U_h \in \mathbb{R}^n$ such that

$$A_h U_h = F_h,$$

where A_h is a $n \times n$ matrix and $F_h \in \mathbb{R}^n$. Compute explicitly the matrix A_h and express explicitly the vector F_h with respect to f and r.
5. Prove that there exists a positive constant C such that

$$\|u - u_h\|_{H^1} \leq C \inf_{v_h\in V_h} \|u - v_h\|_{H^1},$$

where u is the solution of the initial variational problem and u_h is the solution of the discrete system.
6. We define the interpolation operator $r_h : V \to V_h$ that maps each $v \in V$ to the element $r_h v \in V$ defined by

$$r_h v(x_i) = v(x_i) \text{ for all } i \in \{0, \cdots, n-1\}.$$

Verify that r_h is uniquely defined. Prove that for all $v \in V \cap H^2(\Omega)$,

$$\|v - r_h v\|_{H^1(\Omega)} \leq Ch\|v\|_{H^2(\Omega)}.$$

Deduce that u_h converges toward u.