Exercise I. Variational Formulation

Let $\Omega$ be a regular bounded and connected open set of $\mathbb{R}^n$. Let $\Gamma_N$ and $\Gamma_D$ such that $\partial\Omega = \Gamma_N \cup \Gamma_D$ of negligible intersection, each of them being of non zero surface measure. Let $g \in L^2(\Gamma_N)$ and $f \in L^2(\Omega)$. We consider the Partial Differential Equation

$$
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma_D, \\
\frac{\partial u}{\partial n} = g & \text{on } \Gamma_N.
\end{cases}
$$

1. Give the variational formulation associated to (1).
2. Prove that there exists a unique solution $u$ to the variational problem obtained. To this end, we could admit the following Lemma.

Lemma 1. Let $\Omega$ be a connected open set of $\mathbb{R}^N$. Let $u \in H^1(\Omega)$ such that $\nabla u = 0$ then $u$ is constant over $\Omega$.

3. If $g$ is the trace of a $H^1(\Omega)$ map (that is if $g$ belongs to $H^{1/2}(\Omega)$), the solution $u$ of the variational formulation belongs to $H^2(\Omega)$. In this case, prove that it is also a solution of the initial PDE (1).

Exercise II. Non local limit conditions

Let $\Omega_0$, $\Omega_1$, be open bounded sets of $\mathbb{R}^2$ such that $\overline{\Omega_1}$ is included in $\Omega_0$ assumed to be connected. We denote by $\Omega$ the open set $\Omega_0 - \overline{\Omega_1}$.

Let $f$ be a map in $L^2(\Omega)$ and $q \in L^\infty(\Omega)$. We would like to establish the existence of a solution $u$ of the following problem

$$
\begin{cases}
-\Delta u + q(x)u = f & \text{in } \Omega \\
\frac{\partial u}{\partial n} = -\alpha \int_{\Gamma_1} u ds & \text{on } \Gamma_1 \\
u(x) = 0 & \text{on } \Gamma_0
\end{cases}
$$

where $\alpha$ is a non negative real, $\Gamma_0 = \partial\Omega_0$ and $\Gamma_1 = \partial\Omega_1$. 

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1. **Variational Formulation.** Prove that all regular solutions of the PDE are solutions of a variational problem to determine. Conversely, prove that all regular solutions of the variational problem are solutions of the PDE.

2. **Existence.** We assume that there exists a positive real \( q_0 \) such that for almost all \( x \in \Omega \),

\[
q(x) \geq q_0.
\]

Prove that there exists a unique solution \( u \) to the variational problem and that it depends continuously on the data.

3. **Poincaré like inequality.** Prove by contradiction (reductio ad absurdum), that there exists a constant \( C \) such that for all \( u \) such that \( u(x) = 0 \) on \( \Gamma_0 \),

\[
\|u\|_{H^1(\Omega)} \leq C\|
abla u\|_{L^2(\Omega)}.
\]

Can the conditions imposed to \( \alpha \) and \( q \) be relaxed and still preserve the existence result?

**Exercise III.** Eigenvalues of the Laplacian.

Let \( \Omega \) be a regular bounded and connected open set of \( \mathbb{R}^n \). A real \( \lambda \) is said to be an eigenvalue of the Laplacian with Dirichlet boundary conditions if there exists a non zero \( u \in H^1_0(\Omega) \) such that

\[
-\Delta u = \lambda u.
\]

The map \( u \) is called the eigenfunction associated to \( \lambda \).

1. **Positivity of the eigenvalues.** Prove that all the eigenvalues of the Laplacian with Dirichlet BC is non negative (and even positive).

2. **Regularity of the eigenfunctions.** We recall the remarkable regularization property:

**Theorem 1.** Let \( \Omega \) be a regular open set of \( \mathbb{R}^n \) and \( f \) a map of \( H^m(\Omega) \), then the solution \( u \in H^1_0(\Omega) \) of the PDE

\[
\begin{cases}
-\Delta u = f & \text{ in } \Omega \\
 u = 0 & \text{ on } \partial \Omega
\end{cases}
\]

belongs to \( H^{m+2}(\Omega) \).

Moreover, we ”recall” that all function that belongs to \( H^m(\Omega) \) is continuous provided that \( m > N/2 \). Prove that the eigenfunctions of the Laplacian with Dirichlet boundary conditions belong to \( C^\infty(\Omega) \).

3. **First eigenvalue.** Let \( \beta \) defined by

\[
\beta = \inf_{u \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}
\]

Prove that \( \beta \) is positive and that there exists \( u \in H^1_0(\Omega) \) such that

\[
\beta = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.
\]

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What is the relationship between $\beta$ and the Poincaré inequality?

4. First eigenfunction. Prove that the functional

$$J(u) = \int_{\Omega} |u|^2 \, dx$$

is Gâteaux differentiable, that is there exists a continuous linear form $L$ on $H^1_0(\Omega)$ such that for all $w \in H^1_0(\Omega)$,

$$\lim_{\delta \to 0} \frac{J(u + \delta w) - J(u)}{\delta} = L(w)$$

Prove that the functional

$$F(u) = \int_{\Omega} |\nabla u|^2 \, dx$$

is also Gâteaux differentiable. Deduce that the function $u$ maximizing $J$ on the set

$$K = \{ v \in H^1_0(\Omega) : F(v) \leq 1 \}$$

is an eigenfunction of the Laplacian with Dirichlet boundary conditions.

5. Weak maximum principle. Let $f \in L^2(\Omega)$ and $u \in H^1_0(\Omega)$ such that $f \geq 0$ and

$$-\Delta u = f.$$

Prove that $u(x) \geq 0$ almost everywhere. To this end, the following Lemma could be used.

**Lemma 2.** Let $\Omega$ be an open set of $\mathbb{R}^n$ and $u \in H^1(\Omega)$. We set $u^- = \min(0, u)$. Then $u^- \in H^1(\Omega)$ and

$$\nabla u^- = \mathbb{1}_{u<0} \nabla u.$$

There exists another maximum principle (called strong)

**Theorem 2.** Let $\Omega$ be a connected regular bounded open set of $\mathbb{R}^N$. Let $w$ be a map of class $C^2$ on $\overline{\Omega}$ such that

$$\begin{cases}
-\Delta w \geq 0 & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega
\end{cases}$$

then either $w$ is equal to zero, or $w(x) > 0$ for all $x \in \Omega$.

6. Uniqueness and positivity. By using the strong maximum principle, prove that $u$ is of constant sign on $\Omega$. Conclude that $\beta$ is a simple eigenvalue.

**Exercise IV.** Periodical problem in dimension 1

Let $\Omega = ]0, 1[$. We seek for the solution $u \in H^1(\Omega)$ of the following PDE

$$\begin{cases}
-u'' + u = f & \text{in } \Omega \\
u(1) = u(0) \\
u'(1) = u'(0) + r
\end{cases}$$
with \( f \in L^2(\Omega) \) and \( r \in \mathbb{R} \).

1. Prove that there exists a constant \( C \) such that for all \( u \in H^1(\Omega) \)
   \[
   \sup_{x \in \Omega} |u(x)| \leq C\|u\|_{H^1(\Omega)}.
   \]

2. Find the variational formulation associated to the PDE considered. Let us
denote by \( a \) the bilinear form \( \ell \) the linear form and \( V \) the Hilbert space thus
introduced. Prove the equivalence between both formulations.

3. Prove that the variational problem admits a unique solution.

4. We would like to compute numerically an approximation of the solution
of this problem. Let \( n \) be a positive integer. We set \( h = 1/n \), and for all
\( i \in \{0, \cdots, n\} \), \( x_i = ih \). Let \( V_h \) be the space defined by
   \[
   V_h = \{ u \in V : u_{[x_i, x_{i+1}]} \in \Pi_1 \},
   \]
where \( \Pi_1 \) is the set of polynomials of degree lower or equal to one. Prove that
there exists a unique \( u_h \in V_h \) such that
\[
a(u_h, v_h) = \ell(v_h) \text{ for all } v_h \in V_h.
\]
Prove that this problem is equivalent to find \( U_h \in \mathbb{R}^n \) such that
   \[
   A_h U_h = F_h,
   \]
where \( A_h \) is a \( n \times n \) matrix and \( F_h \in \mathbb{R}^n \). Compute explicitly the matrix \( A_h \)
and express explicitly the vector \( F_h \) with respect to \( f \) and \( r \).

5. Prove that there exists a positive constant \( C \) such that
   \[
   \|u - u_h\|_{H^1} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{H^1},
   \]
where \( u \) is the solution of the initial variational problem and \( u_h \) is the solution
of the discrete system.

6. We define the interpolation operator \( r_h : V \to V_h \) that maps each \( v \in V \) to
the element \( r_h v \in V_h \) defined by
   \[
r_h v(x_i) = v(x_i) \text{ for all } i \in \{0, \cdots, n-1\}.
   \]
Verify that \( r_h \) is uniquely defined. Prove that for all \( v \in V \cap H^2(\Omega) \),
   \[
   \|v - r_h v\|_{H^1(\Omega)} \leq C \|v\|_{H^2(\Omega)}.
   \]
Deduce that \( u_h \) converges toward \( u \).