TC2 Optimal design of structures O.Pantz
Convex Optimization

Exercise I. Optimization of a heater.

We want to optimize the temperature T of a room Ω by the mean of a heater ω whose heat flux $v \in L^2(\omega)$ is controlled. An air stream of velocity u fills the room. We assume that the air is incompressible, that is u is assumed to be divergence free. Moreover, u is also assumed to be regular. The temperature on the boundary of the room is equal to the external temperature, assumed to be zero. The temperature in the room satisfies the following convection-diffusion equation

$$-\Delta T + u \cdot \nabla T = 1_\omega v.$$

a. Determine the variational problem satisfied by the solution T of the convection-diffusion equation. Prove that it admits an unique solution depending continuously on the data.

b. We want to optimize the value of v in order to maintain the temperature T to a desired value T_0. To this end, we introduce the cost function

$$J(v) = \int_\Omega |T(v) - T_0|^2 dx$$

which we want to minimize. Compute the derivatives of $T(v)$ and $J(v)$ with respect to v. Can the expressions obtained be used to implement a gradient type algorithm applied to the minimization of J ?

c. The gradient of J can be explicitly expressed by introducing an adjoint state. To this end, we first introduce the Lagrangian

$$\mathcal{L}(v, T, p) = \int_\Omega |T - T_0|^2 dx - \int_\Omega \nabla p \cdot \nabla T + (u \cdot \nabla T) p dx + \int_\omega p v dx,$$

where $T, p \in H_0^1(\Omega)$ and $v \in L^2(\omega)$. Prove that finding the minimizer of J is equivalent of solving the following min-max problem

$$\min_{v, T \in H_0^1(\Omega)} \sup_{p \in H_0^1(\Omega)} \mathcal{L}(v, T, p).$$

Determine the derivative of $\mathcal{L}$ with respect to T and v.

d. By noticing that $J(v) = \mathcal{L}(v, T(v), p)$ for all $p \in H_0^1(\Omega)$, find a new expression of the differential of J depending on the derivatives of $\mathcal{L}$. Prove that a particular choice for p enables us to get rid of the term that depends on $\partial T / \partial v$. Deduce a new (and workable) version of the gradient of J.

Exercise II. Optimization of a non-linear system

If linear PDEs do correctly describe the behavior of physical systems close to their equilibrium state, non linear phenomena can appear in more general situations. The aim of this exercise is to extend the analysis performed on a linear system in the previous exercise to some non-linear cases.

Let Ω be a bounded open set of $\mathbb{R}^N$, $p \in \mathbb{N}$ such that $p > 2$ and $f \in L^2(\Omega)$ the control. We consider the PDE (whose unknown is u)

$$\begin{aligned} -\operatorname{div}((1+|\nabla u|^{p-2})\nabla u) &= f \text{ in } \Omega \\ u &= 0 \text{ sur } \partial\Omega. \end{aligned} \tag{1}$$

For $|\nabla u| << 1$ (small perturbations), we recover the standard Poisson equation.

1. Solving system (1)

a. Clarkson Inequality (Optional).

Prove that for all a and $b \in \mathbb{R}$,

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}(|a|^p + |b|^p).$$

Hint: Prove that for all $\alpha, \beta \geq 0$, $\alpha^p + \beta^p \leq (\alpha^2+\beta^2)^{p/2}$ for $\beta = 1$ then for all $\beta \geq 0$.

b. Prove that (1) admits an unique solution. To this end, we will show that (1) is equivalent to a minimization problem of a functional G to determine over the Banach space $W_0^{1,p}(\Omega)$. We *recall* that

$$W_0^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega), \text{ et } u = 0 \text{ on } \partial\Omega\},$$

is the space of L^p functions with L^p weak derivatives. It is a *uniformly convex* Banach space endowed with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$$

Moreover, we recall that the Poincaré inequality is still valid for all p such that $1 \leq p < \infty$ (Ω is assumed to be bounded and regular), that is, there exists $C > 0$ (depending on Ω and p) such that for all $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^p}.$$

Moreover, let us *recall* that the existence Theorem of a minimizer for strongly convex functions, known in the context of Hilbert spaces, is still valid for *uniformly convex* Banach spaces (see MAP431 Tutorial Classes for a *basic* proof of this).

2. Compute (formally) the derivative of u with respect to f.

3. We seek for the best control $f \in L^2(\Omega)$ so that the state of the system u is as close as possible to a target function u_0. To this end, we introduce the cost function

$$J(f) = \int_\Omega |u(f) - u_0|^2 dx.$$

Compute (formally) the derivative of J with respect to f. Can the expression obtained be used to implement a gradient type algorithm applied to the minimization of J ?

4. In order to obtain a more convenient expression of the gradient of J, we are going to introduce as in the previous exercise an adjoint state.
a. Reformulate the minimization problem of J as a min-max problem. To this end, the Lagrangian $\mathcal{L}$ associated to the minimization of $\|u - u_0\|^2_{L^2(\Omega)}$ under the constraint "u solution of (1)" will be introduced.
b. Compute the partial derivatives of the Lagrangian introduced.
c. Give a new expression of the differential of J depending on an adjoint state to define. Prove that if u is regular the adjoint system admits an unique solution in $H^1_0(\Omega)$.

Remark: Even if the state equation is non linear, the adjoint problem is always linear.

Exercise III. Optimization of a Truss.
Notation: If i is a positive integer, we denote by $\mathbf{i}$ the set $\{1, \cdots, i\}$.

We want to optimize a truss in $\mathbb{R}^d$ ($d = 2$ or 3) submitted to loads located at its nodes. We assume the truss to be made of m beams, linked with one another by N nodes (with N big) $(a_i)_{i\in\mathbf{N}}$. For all $k \in \mathbf{m}$, we denote by $b_k = \{i, j\}$ the couple of labels i and j in $\mathbf{N}$ of its ends. We assume that the truss is clamped in s nodes. We denote by $N_{dl} = N - s$ the number of free nodes. Finally, we denote by $f_i \in \mathbb{R}^d$ the load applied to the node i.

1. Modeling

We consider the case of small displacements and assume that the beams only resist in traction and compression. In other words, a beam k such that $b_k = \{i, j\}$, exerts on the nodes i and j a force colinear to its direction $e_{ij} = (a_i - a_j)/|a_i - a_j|$. Thus, the forces applied to the beam k on i and j are respectively of the form $q_k e_{ji}$ and $q_k e_{ij}$ where q_k is a real.

The state of the deformed truss is then completely determined by the displacement $u \in \mathbb{R}^n$ of its nodes, where $n = dN_{dl}$ is the number of degree of freedom. For each beam k, we denote by q_k the stress it supports.
a. Prove that the equilibrium law of the forces on each node i can be written as

$$Bq = f.$$

b. We assume that the stress q_k to which the beam k is submitted is proportional to the extension Δl_k of the beam. More precisely, we have

$$q_k = Ex_k \Delta l_k / l_k^2,$$

where E is an elasticity moduli which only depends on the type of material used, l_k is the length of the beam at rest and x_k its volume.

Prove that

$$B^T u = \Delta l.$$

Deduce that the displacement of the beam is such that

$$K(x)u = f, \tag{2}$$

where $K(x)$ has the following form

$$K(x) = \sum_{k=1}^{m} x_k C_k^T C_k. \tag{3}$$

where C_k is a matrix $m \times n$ to determine.

2. Existence

We want to prove that a solution of the system (2) do actually exist, where $K(x)$ is defined by (3). In the following, we will assume that

$$\sum_{k=1}^{m} K_k,$$

with $K_k = C_k^T C_k$ is definite positive and that for all $k \in \mathbf{m}$ the volume x_k of the beam is k not equal to zero.

Prove that the equation (2) admits an unique solution.

3. Optimization, self-adjoint case: Minimization of the compliance

We want to minimize the compliance of the truss

$$J(x) = f^T u(x)$$

using the smallest among of material as possible. Note that the more the compliance is small, the more the structure is rigid. We introduce the admissible space

$$X_V = \{x \in \mathbb{R}^m : \sum_{k=1}^{m} x_k \leq V, \quad x_k \geq \varepsilon \text{ pour tout } k \in \mathbf{m}\}.$$

We assume that $\varepsilon m < V$, so that X_V is neither empty nor reduce to an unique element.

a. Prove that J admits at least one minimizer on X_V.

b. Determine the gradient of u with respect to x. Knowing that we never explicitly compute the inverse of $K(x)$ (which can be a very big matrix), how many linear systems do we have to solve to compute the gradient of u ?

c. Determine the gradient of J. Prove that the gradient of J can be compute by solving only the linear system (2).

d. Prove that

$$J(x) = -\min_{v \in \mathbb{R}^N} \left(v^T K(x) v - 2 f^T v \right).$$

Deduce that J is actually a convex function.

e. Prove that $x \in X_V$ minimize J on X_V if and only if there exists $\mu \in \mathbb{R}_+$, $(\lambda)_{k \in \mathbf{m}} \in \mathbb{R}_+^m$ such that

$$u^T K_k u = \mu - \lambda_k \text{ for all } k \in \mathbf{m},$$

$$\mu(V - \sum_{k=1}^{m} x_k) = 0,$$
$$\forall k \in \mathbf{m},\ \lambda_k(\varepsilon - x_k) = 0.$$

Hint: We recall that in the convex case (i.e. convex cost function and convex constraints), first order optimality conditions are equivalent to optimality (Kuhn and Tucker Theorem).

f. In the case $f \neq 0$, prove that if x is a minimizer of J on X_V, the constraint $V \geq \sum_k x_k$ is activated.

g. Propose an algorithm in order to compute the minimizer of J. One step of this algorithm is not completely obvious. Which one ? (we do not ask for a detailed description of this part of the algorithm, see the course for more details).

h. We recall that the projected gradient algorithm with fix step converges if the step is chosen small enough when applied to the minimization of strongly convex Lipschitzian functions. Is it the case of the function J considered ?

4. Optimization, non self-adjoint case: Target displacement

This time, we want to optimize the truss so that the displacement of some of its nodes are close to a target displacement

$$u_i \simeq u_i^0 \text{ pour tout } i \in I,$$

where I is a set of labels. To this end, we introduce the cost function J_c defined by

$$J_c(x) = |D(u - u_0)|^2,$$

where

$$D_{ij} = \begin{cases} 1 & \text{if } i = j \in I \\ 0 & \text{if } i = j \notin I. \end{cases}$$

Determine the derivatives of the cost function J_c using the solution $p \in \mathbb{R}^n$ of the dual state solution of

$$K(x)p = D(u - u_0).$$

5. Complementary energy

For all $v \in \mathbb{R}^n$, we set

$$E(v, x) = \frac{1}{2} v^T K(x) v - f^T v.$$

a. Prove that for all $x > 0$, $E(v, x)$ admits an unique minimizer with respect to v and that this minimum is the solution of the system(2).

b. Let

$$E(v, e, x) = \frac{1}{2} \sum_{k=1}^{m} x_k e_k^T e_k - f^T v$$

Prove that minimizing the functional $E(v,x)$ with respect to v is equivalent to the minimization of $E(v,e,x)$ under the constraint $e_k = C_k v$. Define the Lagrangian $\mathcal{M}(v,e,\tau,x)$ associated to this former problem.
c. We introduce the Lagrangian

$$\mathcal{L}(v,\tau,x) = \min_e \mathcal{M}(v,e,\tau,x).$$

Determine an explicit expression of $\mathcal{L}$. Check that

$$E(v,x) = \max_\tau \mathcal{L}(v,\tau,x).$$

d. Let

$$G(\tau,x) = \min_v \mathcal{L}(v,\tau,x).$$

Compute explicitly

G as a function of τ and x. Prove that the dual problem (i.e. maximization of G with respect to τ) admits an unique solution. Prove that

$$E(u,x) = G(\sigma,x),$$

where u is the solution of the problem (2) and $\sigma_k = x_k C_k u$. Deduce that $\mathcal{L}(\cdot,\cdot,x)$ admits an unique saddle point (x is considered as a fixed parameter).

6. Minimization of the compliance

We go back to the minimization problem of the compliance $J(x)$ of a truss.
a. Prove that the minimization problem of J over X_V is equivalent to the minimization problem of

$$H(\tau,x) = \sum_{k=1}^m x_k^{-1}|\tau_k|^2$$

on the set of $(x,\tau) \in X_V \times (\mathbb{R}^m)^m$ such that $\sum_{k=1}^m C_k^T \tau_k = f$.
b. Prove that for all $\tau \neq 0$, there exists an unique non negative real ℓ such that if

$$x_k = \max(\varepsilon, |\tau_k|\ell^{-1/2}), \tag{4}$$

$\sum_{k=1}^m x_k = V$.

Check that the minimum of H with respect to $x \in X_V$ (for a given τ) is reached on x defined by (4).
c. How can we compute the minimizer of H with respect to τ such that $\sum_k C_k^T \tau_k = f$ (for a given x)?
d. Propose a minimization algorithm of H for which two minimization problems are solved successively at each iteration.
e. Deduce for the optimality conditions satisfied by the optimal solution minimizing J (obtained in 3.e) that all fixed point of the proposed algorithm do minimize J on X_V.