Exercise I. Optimization of the smallest eigenvalue of a membrane. A real $\lambda$ is said to be an eigenvalue of a membrane $\Omega$ of thickness $h$ if there exists $u \neq 0$ such that
\[
\begin{cases}
-\text{div}(h\nabla u) = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
We seek to maximize the first eigenvalue of the membrane on the set of elements $h$ belonging to
\[
U_{ad} = \left\{ h \in L^\infty(\Omega) : h_{\text{min}} \leq h \leq h_{\text{max}}, \int_{\Omega} h dx \leq V \right\},
\]
where $h_{\text{min}} > 0$ and $h_{\text{max}}$ denote respectively the minimal and maximal thickness of the membrane whereas $V$ stands for the maximal admissible volume (or weight).

We recall that the smallest eigenvalue can be defined by
\[
\lambda(h) = \min_{u \in H^1_0(\Omega) \atop \|u\|_{L^2}=1} \int_{\Omega} h|\nabla u|^2.
\]

1. We recall that the smallest eigenvalue of the Laplacian is simple.

Thus, for all $h$, there exists a unique $u(h) \in H^1_0(\Omega)$, solution of (1) such that $u(h) \geq 0$ almost everywhere and $\|u(h)\|_{L^2(\Omega)} = 1$. We assume in this question that the functions $\lambda(h)$ and $u(h)$ are both differentiable with respect to $h$.

a. Deriving the variational formulation satisfied by the couple $(u, \lambda)$, prove that
\[
\lambda'(h) = |\nabla u|^2.
\]
b. Suggest a algorithm of maximization of $\lambda(h)$.

2. (More tricky) The aim of the following questions consists in proving that the functions $\lambda(h)$ and $u(h)$ are indeed differentiable with respect to $h$.

We “recall” the following compactness result: If $u_n$ is a bounded sequence in $H^1_0(\Omega)$, there exists a subsequence $u_{n_k}$ of $u_n$, and an element $u \in H^1_0(\Omega)$ such that $u_{n_k}$ converges toward $u$ in $L^2(\Omega)$. Moreover, for all $v \in L^2(\Omega)^N$,
\[
(\nabla u_{n_k}, v)_{L^2} \to (\nabla u, v)_{L^2}.
\]

Continuity. We consider a sequence $h_n$ of elements of $U_{ad}$, converging toward an element $h$ in $L^\infty(\Omega)$. We set $\lambda_n = \lambda(h_n)$ and $u_n = u(h_n)$.

a. Prove that $\lambda_n$ is a bounded sequence.
b. Prove that a subsequence $u_{n_k}$ can be extract from $u_n$ such that if converges in $L^2(\Omega)$ toward an element $u \in H^1_0(\Omega)$.
c. Prove that
\[
\int_{\Omega} h|\nabla u|^2 dx \leq \liminf_{n} \int_{\Omega} h_{n_k}|\nabla u_{n_k}|^2 dx.
\]
d. Prove that \( \lambda(h) = \lim \inf \lambda_n \) and that \( u(h) = u \).

e. Conclude.

**Derivability.**

a. Using the variational formulation satisfied by \( u(h) \), prove (formally) that \( u_k = u'(h), k > \) is such that for all \( \phi \in H^1_0(\Omega) \),

\[
a(u_k, \phi) = L(\phi)
\]

where

\[
a(w, \phi) = \int_{\Omega} h\nabla w \cdot \nabla \phi - \lambda w \phi dx
\]

and

\[
L(\phi) = \left( \int_{\Omega} k |\nabla u(h)|^2 dx \right) \int_{\Omega} u(h) \phi dx - \int_{\Omega} k \nabla u(h) \cdot \nabla \phi dx.
\]

Prove that there exists a unique solution \( w \in H^1_0(\Omega) \) to this variational problem such that

\[
\int_{\Omega} w u dx = 0.
\]

To this end, the quotient space \( V \) of \( H^1_0(\Omega) \) by \( u(h) \) endowed with the norm

\[
\|\phi\|_V^2 = \inf_{\alpha \in \mathbb{R}} \int_{\Omega} k |\nabla (\phi - \alpha u(h))|^2 dx
\]

could be introduced.

b. Using the variational formulations satisfied by \( u(h + k) \) and \( w \), prove that there exists a constant \( C_0 \) such that for all \( \phi \in H^1_0(\Omega) \),

\[
a(u(h + k) - w, \phi) \leq C_0 \left( \lambda(h + k) - \lambda(h) - \int_{\Omega} k |\nabla u(h)|^2 dx \right) \|\phi\|_{L^2(\Omega)} + o(k) \|\phi\|_{H^1(\Omega)}
\]

where

\[
\lim_{\|k\|_{L^\infty} \to 0} o(k)/\|k\|_{L^\infty(\Omega)} = 0.
\]

Deduce that there exists a constant \( C_1 \) such that

\[
\|u(h + k) - u\|_V \leq C_1 \left( \lambda(h + k) - \lambda(h) - \int_{\Omega} k |\nabla u(h)|^2 dx + o(k) \right).
\]

c. Prove that

\[
\lambda(h + k) - \lambda(h) - \int_{\Omega} k |\nabla u(h)|^2 dx = a(u(h + k) - w, u(h + k) - w) + o(k).
\]

d. Deduce from the questions b. and c. that

\[
\|u(h + k) - u\|_V = o(k).
\]
Exercise II. Tomography.

The tomography consists in determining the internal structure of an object by the application of a tension to its surface and the measure of the induced current. Let $\Omega$ be a body made of two material $A$ and $B$ of respective conductivity $a_{\text{min}}$ and $a_{\text{max}}$. We apply a potential $u_0$ at the boundary of $\Omega$ and measure the induced surface current $g(u_0)$. We want to determine the respective repartition of material $A$ and $B$ within $\Omega$, that is the domain $\omega_A$ occupied by $A$.

We denote $u_1$ the potential, solution of the PDE

$$\begin{cases} -\nabla \cdot (a \nabla u_1) = 0 & \text{in } \Omega \\ u_1(x) = u_0 & \text{on } \partial \Omega \end{cases} \quad (2)$$

where $a = \chi_{\omega_A} a_{\text{min}} + (1 - \chi_{\omega_A}) a_{\text{max}}$ and $\chi_{\omega_A}$ is the characteristic function of $\omega_A$ and $u_0 \in H^1(\Omega)$. Moreover, we denote $u_2$ the solution of the PDE with Neumann conditions given by $g(u_0)$

$$\begin{cases} -\nabla \cdot (a \nabla u_2) = 0 & \text{in } \Omega \\ \frac{\partial u_2}{\partial n}(x) = g(u_0) & \text{on } \partial \Omega \end{cases} \quad (3)$$

Finally, we introduce the functional

$$J(a) = \int_\Omega a \nabla (u_1(a) - u_2(a)) \cdot \nabla (u_1(a) - u_2(a)) \, dx.$$ 

In order to determine a conductivity compatible with the measure, we look for a minimizer of $J$ with respect to $a$.

1. Determine the variational formulations associated with PDE (2) and (3).
2. Determine (formally) the differentials of $u_1(a)$ and $u_2(a)$ with respect to the conductivity $a$.
3. Justify the previous computations by proving first that $u_1$ and $u_2$ are bounded and do continuously depend on $a$, than by proving that they are differentiable with respect to $a$ for all $a \in \mathcal{U}_{\text{ad}}$, where

$$\mathcal{U}_{\text{ad}} = \{a \in L^\infty(\Omega) : a_{\text{min}} \leq a \leq a_{\text{max}}\}.$$

4. Prove that $J$ is differentiable with respect to $a$ for all $a \in \mathcal{U}_{\text{ad}}$, express the differential of the function in terms of the differentials of $u_0$ and $u_1$. Is the obtained expression of any use from the practical viewpoint ?
5. Define the Lagrangian associated to the minimization problem of

$$j(a, u_1, u_2) = \int_\Omega a \nabla (u_1 - u_2) \cdot \nabla (u_1 - u_2) \, dx,$$

with respect to $(a, u_1, u_2)$ under the constraints $u_1$ solution of (2) and $u_2$ solution solution of (3). Deduce a new expression of the differential of $J$ in terms of two adjoint states $p_1$ and $p_2$ to precise.

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