Exercise I. Conductivity of a solid containing tiny holes.

We consider a body with constant conductivity but not necessarily isotropic. Its reference configuration $\Omega_\varepsilon$ is defined as the intersection of an open set $\Omega$ and a grid $\varepsilon \omega$ where $\omega$ is a $Y$-periodic open set, $Y$ being the unit cube.

$$\Omega_\varepsilon = \Omega \cap \varepsilon \omega, \quad \varepsilon \omega = \{ x \in \mathbb{R}^N : \varepsilon^{-1} x \in \omega \},$$

$\omega Y$-periodic.

The body is heated by the application of a volumic heat flux $f \in L^2(\Omega_\varepsilon)$ and maintained to a constant zero temperature its external boundary $\Gamma^D_\varepsilon = \partial \Omega \cap \varepsilon \omega$.

The temperature $u_\varepsilon$ is solution of the PDE

$$\begin{cases}
- \text{div}(A \nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon, \\
A \nabla u_\varepsilon \cdot n = 0 & \text{on } \Gamma^N_\varepsilon \\
u_\varepsilon = 0 & \text{on } \Gamma^D_\varepsilon
\end{cases} \quad (1)$$

where $\Gamma^N_\varepsilon = \Omega \cap \partial \varepsilon \omega$.

Our aim is to derive the behavior of the solution $u_\varepsilon$ as $\varepsilon$ goes to zero. To this end, we assume that $u_\varepsilon$ is given by the following two scaled asymptotic development

$$u_\varepsilon(x) = \sum_{i=0}^\infty u_i(x, x/\varepsilon) \varepsilon^i, \quad (2)$$

with $(u_i)_{i \in \mathbb{N}} \in V^N$ and

$$V := \{ v : \mathbb{R}^n \times \omega \to \mathbb{R} \text{ such that } v(x, y + k) = v(x, y) \text{ for all } k \in \mathbb{N}^n \}.$$  

1. Initial Problem

Prove that $u_\varepsilon$ can be characterized as the minimizer of an energy $J_\varepsilon$ over the space

$$X_\varepsilon = \{ u \in H^1(\Omega_\varepsilon \cup \Omega_{\text{ext}}) : u = 0 \text{ on } \Omega_{\text{ext}} \},$$

where $\Omega_{\text{ext}} = \mathbb{R}^n \setminus \Omega$.

2. Development of the energy

We denote by $J(\varepsilon) : V^N \to \mathbb{R}$ the energy expressed in terms of the asymptotic development of $u_\varepsilon$, that is

$$J(\varepsilon)((v_i)_{i \in \mathbb{N}}) = J_\varepsilon(v_\varepsilon),$$

where $v_\varepsilon$ is defined by

$$v_\varepsilon = \sum_{i=0}^\infty v_i(x, x/\varepsilon) \varepsilon^i.$$
and $V$ is the space of $Y$-periodic functions from $\mathbb{R}^n \times \omega$ with values in $\mathbb{R}$.

a. Prove that $J(\varepsilon)$ admits an asymptotic expansion with respect to $\varepsilon$ of the following type

$$J(\varepsilon)((u_i)_{i \in \mathbb{N}}) = \sum_{k=-2}^{\infty} \varepsilon^k J_k((u_i)_{i \in \mathbb{N}}),$$

where the functional $J_k$ depends on $\varepsilon$ through $\Omega_\varepsilon$ and $x/\varepsilon$ only. Determine $J_k$ (Hint: To simplify the computations, the convention $u_k = 0$ for all negative $k$ could be used).

b. (Optional) Prove that if $v(x,y)$ is a continuous function from $\Omega \times \omega$ with values in $\mathbb{R}$ that is $Y$-periodic with respect to the second variable, then

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} v(x,x/\varepsilon) \, dx = |Y|^{-1} \int_{\Omega \cap \omega} \int_{Y} v(x,y) \, dx \, dy.$$

c. We denote by $J_k$ the functions from $V^\mathbb{N}$ with values in $\mathbb{R}$ defined by

$$J_k((u_i)_{i \in \mathbb{N}}) = \lim_{\varepsilon \to 0} J_k((u_i)_{i \in \mathbb{N}}),$$

3. Solving the first minimization problems

From the hypothesis made on the solution, it can be established that the asymptotic expansion $(u_i)_{i \in \mathbb{N}}$ of the solution $u_\varepsilon$ of the initial problem is such that

$$(u_i)_{i \in \mathbb{N}} \in \bigcap_k M_k,$$

where $M_k$ is the set of solutions of the minimization problem

$$M_k = \{(u_i)_{i \in \mathbb{N}} \in M_{k-1} \text{ such that } J_k(u_i) = \inf_{(v_i)_{i \in \mathbb{N}}} J_k(v_i) \text{ for all } \varepsilon\}.$$

We intend to successively solve this sequence of minimization problems.

a. We assume $\omega$ to be connected. Prove that

$$M_{-2} = \{(u_i) \in V^\mathbb{N} \text{ such that } u_0(x,y) = u_0(x)\},$$

that is, the minimizers of $M_{-2}$ are the elements $(u_i)_{i \in \mathbb{N}}$ of $V^\mathbb{N}$ such that $u_0(x,y)$ is independent of $y$.

c. Prove that the second minimization problem is trivial, that is $M_{-1} = M_{-2}$.

e. Determine the third minimization problem $M_0$, prove that the function $y \to u_1(x,y)$ is solution of a minimization problem that depends only on $\nabla \cdot u_0(x)$ and that this problem admits a unique solution up to a constant.

f. Deduce from the minimization problem satisfied by $u_0$ and the conductivity homogenized law $A^*$. Prove that this problem admits a unique solution.
Exercise II. Polycrystals.

Our aim is to determine the behavior of a two-dimensional polycrystal locally periodic, whose conductivity reads as

\[ A^\varepsilon(x) = A(x, x/\varepsilon) \]

where \( A \) is defined over \( \Omega \times Y \), \( \Omega \) being the reference configuration of the polycrystal, \( Y \) the periodicity cell \([0, 1]^2\) and

\[ A(x, y) = R(x, y)^T A^0 R(x, y), \]

with

\[ A^0 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \]

\( 0 < \alpha \leq \beta \) and \( R(x, y) \) are the matrices of rotation

\[ R(x, y) = \begin{pmatrix} \cos(\Theta(x, y)) & \sin(\Theta(x, y)) \\ -\sin(\Theta(x, y)) & \cos(\Theta(x, y)) \end{pmatrix} \]

For \( \varepsilon \) small enough, the behavior of the polycrystal could be approximated by an homogeneous body of conductivity

\[ A^*(x) = \min_{w \in H^1_{\#}(Y)} \int_Y A(x, y)(\xi + \nabla w)(\xi + \nabla w)dy. \quad (3) \]

1. Prove that

\[ \alpha \Id \leq A^* \leq \beta \Id. \]

2. We want to determine the conductivity law for a rank one laminate in the direction \( e_1 \), made of a mixture of two materials of conductivity \( A^0 \) and \( \Pi^T A^0 \Pi \) where \( \Pi \) is the rotation of angle \( \pi/2 \), in respective proportions \( \theta \) and \( (1 - \theta) \).

Applying the lamination formula of the course, prove that

\[ A^* = \begin{pmatrix} \frac{\alpha}{\sigma} + \frac{1-\theta}{\sigma} & 0 \\ 0 & \theta \beta + (1-\theta)\alpha \end{pmatrix}. \]

Compute the Jacobian of \( A^* \).

3. We want to prove that all polycrystal of the previous type are such that

\[ \det(A^*(x)) = \alpha \beta. \]

Prove first that

\[ A^{-1}(x) \sigma = \min_{\tau(y) \in L^2_{\#}(Y)^2} \int_Y A^{-1}(x, y)(\sigma + \tau)(\sigma + \tau)dy. \]

\[ \text{div } \tau = 0 \text{ et } \int_Y \tau dy = 0 \]
To this end, we can use the principle of complementary energy, which applied to the minimization problem (3) leads to

\[
A^* \xi \cdot \xi = \max_{\tau(y) \in L^2_#(Y)^2} \tau(y) - \int_Y A(x,y)^{-1} \tau \cdot \tau dy + 2 \int_Y \tau \xi \cdot dy.
\]

Prove that \(\Pi(A(x,y)^{-1}) = A(x,y) / \alpha \beta\). Deduce that \(\Pi(A^*(x,y)^{-1}) = A^*(x,y) / \alpha \beta\). We could use to this end the fact that for all fields \(\tau\) such that \(\text{div} \tau = 0\) and \(\int_Y \tau dx = 0\), there exists \(w \in H^{1/2}_#(Y)\) such that \(\tau = \Pi(\nabla w)\). Conclude.