Exercise I. Conductivity of a solid containing tiny holes.

We consider a body with constant conductivity but not necessarily isotropic. Its reference configuration Ω_ε is defined as the intersection of an open set Ω and a $\ll$ grid $\gg$ $\varepsilon\omega$, where ω is a Y-periodic open set, Y being the unit cube.

$$\Omega_\varepsilon = \Omega \cap \varepsilon\omega,$$
$$\varepsilon\omega = \{x \in \mathbb{R}^N : \varepsilon^{-1}x \in \omega\},$$
$$\omega \ Y\text{-periodic.}$$

The body is heated by the application of a volumic heat flux $f \in L^2(\Omega_\varepsilon)$ and maintained to a constant zero temperature its external boundary $\Gamma_\varepsilon^D = \partial\Omega \cap \varepsilon\omega$.

The temperature u_ε is solution of the PDE

$$\left\{\begin{array}{rcll} -\operatorname{div}(A\nabla u_\varepsilon) & = & f & \text{in } \Omega_\varepsilon, \\ A\nabla u_\varepsilon.n & = & 0 & \text{on } \Gamma_\varepsilon^N \\ u_\varepsilon & = & 0 & \text{on } \Gamma_\varepsilon^D \end{array}\right. \tag{1}$$

where $\Gamma_\varepsilon^N = \Omega \cap \partial\varepsilon\omega$.

Our aim is to derive the behavior of the solution u_ε as ε goes to zero. To this end, we assume that u_ε is given by the following two scaled asymptotic development

$$u_\varepsilon(x) = \sum_{i=0}^{\infty} u_i(x, x/\varepsilon)\varepsilon^i, \tag{2}$$

with $(u_i)_{i\in\mathbb{N}} \in V^{\mathbb{N}}$ and

$$V := \{v : \mathbb{R}^n \times \omega \to \mathbb{R} \text{ such that } v(x, y+k) = v(x,y) \text{ for all } k \in \mathbb{N}^n\}.$$

1. Initial Problem

Prove that u_ε can be characterized as the minimizer of an energy J_ε over the space

$$X_\varepsilon = \{u \in H^1(\Omega_\varepsilon \cup \Omega_{ext}) \text{ such that } u = 0 \text{ on } \Omega_{ext}\},$$

where $\Omega_{ext} = \mathbb{R}^n \setminus \Omega$.

2. Development of the energy

We denote by $J(\varepsilon) : V^{\mathbb{N}} \to \mathbb{R}$ the energy expressed in terms of the asymptotic development of u_ε, that is

$$J(\varepsilon)((v_i)_{i\in\mathbb{N}}) = J_\varepsilon(v_\varepsilon),$$

where v_ε is defined by

$$v_\varepsilon = \sum_{i=0}^{\infty} v_i(x, x/\varepsilon)\varepsilon^i.$$

and V is the space of Y-periodic functions from $\mathbb{R}^n \times \omega$ with values in $\mathbb{R}$.
a. Prove that $J(\varepsilon)$ admits an asymptotic expansion with respect to ε of the following type

$$J(\varepsilon)((u_i)_{i\in\mathbb{N}}) = \sum_{k=-2}^{\infty} \varepsilon^k J_k^\varepsilon((u_i)_{i\in\mathbb{N}}),$$

where the functional J_k^ε depends on ε through Ω_ε and x/ε only. Determine J_k^ε (Hint : To simplify the computations, the convention $u_k = 0$ for all negative k could be used).
b. (Optional) Prove that if $v(x, y)$ is a continuous function from $\Omega \times \omega$ with values in $\mathbb{R}$ that is Y-periodic with respect to the second variable, then

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} v(x, x/\varepsilon)\, dx \to |Y|^{-1} \int_{Y\cap\omega} \int_{\Omega} v(x, y)\, dx\, dy.$$

c. We denote by J_k the functions from $V^{\mathbb{N}}$ with values in $\mathbb{R}$ defined by

$$J_k((u_i)_{i\in\mathbb{N}}) = \lim_{\varepsilon\to 0} J_k^\varepsilon((u_i)_{i\in\mathbb{N}}).$$

3. Solving the first minimization problems
From the hypothesis made on the solution, it can be established that the asymptotic expansion $(u_i)_{i\in\mathbb{N}}$ of the solution u_ε of the initial problem is such that

$$(u_i)_{i\in\mathbb{N}} \in \bigcap_k M_k,$$

where M_k is the set of solutions of the minimization problem

$$M_k = \{(u_i)_{i\in\mathbb{N}} \in M_{k-1} \text{ such that } J_k^\varepsilon(u_i) = \inf_{(v_i)\in M_{k-1}} J_k^\varepsilon(v_i) \text{ for all } \varepsilon\}.$$

We intend to successively solve this sequence of minimization problems.
a. We assume ω to be connected. Prove that

$$M_{-2} = \left\{(u_i) \in V^{\mathbb{N}} \text{ such that } u_0(x, y) = u_0(x)\right\},$$

that is, the minimizers of M_{-2} are the elements $(u_i)_{i\in\mathbb{N}}$ of $V^{\mathbb{N}}$ such that $u_0(x, y)$ is independent of y.
c. Prove that the second minimization problem is trivial, that is $M_{-1} = M_{-2}$.
e. Determine the third minimization problem M_0, prove that the function $y \to u_1(x, y)$ is solution of a minimization problem that depends only on $\nabla_x u_0(x)$ and that this problem admits a unique solution up to a constant.
f. Deduce from the minimization problem satisfied by u_0 and the conductivity homogenized law A^*. Prove that this problem admits a unique solution.

Exercise II. Polycrystals.

Our aim is to determine the behavior of a two-dimensional polycrystal locally periodic, whose conductivity reads as

$$A^\varepsilon(x) = A(x, x/\varepsilon)$$

where A is defined over $\Omega \times Y$, Ω being the reference configuration of the polycrystal, Y the periodicity cell $]0,1[^2$ and

$$A(x,y) = R(x,y)^T A^0 R(x,y),$$

with

$$A^0 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

$0 < \alpha \leq \beta$ and $R(x,y)$ are the matrices of rotation

$$R(x,y) = \begin{pmatrix} \cos(\theta(x,y)) & \sin(\theta(x,y)) \\ -\sin(\theta(x,y)) & \cos(\theta(x,y)) \end{pmatrix}$$

For ε small enough, the behavior of the polycrystal could be approximated by an homogeneous body of conductivity

$$A^*(x)\xi.\xi = \min_{w \in H^1_\#(Y)} \int_Y A(x,y)(\xi + \nabla w).(\xi + \nabla w)dy. \tag{3}$$

1. Prove that

$$\alpha \,\mathrm{Id} \leq A^* \leq \beta \,\mathrm{Id}\,.$$

2. We want to determine the conductivity law for a rank one laminate in the direction e_1, made of a mixture of two materials of conductivity A^0 and $\Pi^T A^0 \Pi$ where Π is the rotation of angle $\pi/2$, in respective proportions θ and $(1-\theta)$.

Applying the lamination formula of the course, prove that

$$A^* = \begin{pmatrix} \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1} & 0 \\ 0 & \theta\beta + (1-\theta)\alpha \end{pmatrix}.$$

Compute the Jacobian of A^*.

3. We want to prove that all polycrystal of the previous type are such that

$$\det(A^*(x)) = \alpha\beta.$$

Prove first that

$$A^{*-1}\sigma.\sigma = \min_{\substack{\tau(y) \in L^2_\#(Y)^2 \\ \operatorname{div}\tau = 0 \text{ et } \int_Y \tau dy = 0}} \int_Y A^{-1}(x,y)(\sigma+\tau).(\sigma+\tau)dy.$$

To this end, we can use the principle of complementary energy , which applied to the minimization problem (3) leads to

$$A^*\xi.\xi = \max_{\substack{\tau(y) \in L^2_{\#}(Y)^2 \\ \operatorname{div}\tau = 0}} -\int_Y A(x,y)^{-1}\tau.\tau dy + 2\int_Y \tau.\xi dy.$$

Prove that $\Pi(A(x,y))^{-1}\Pi^T = A(x,y)/\alpha\beta$. Deduce that $\Pi(A^*(x,y))^{-1}\Pi^T = A^*(x,y)/\alpha\beta$. We could use to this end the fact that for all fields τ such that $\operatorname{div}\tau = 0$ and $\int_Y \tau dx = 0$, there exists $w \in H^1_{\#}(Y)$ such that $\tau = \Pi(\nabla w)$. Conclude.