Construction of Minimization Sequences for Shape Optimization

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***Abstract*— In most shape optimization problems, the optimal solution does not belong to the set of genuine shapes but is a composite structure. The homogenization method consists in relaxing the original problem thereby extending the set of admissible structures to composite shapes. From the numerical viewpoint, an important asset of the homogenization method with respect to traditional geometrical optimization is that the computed optimal shape is quite independent from the initial guess (at least for the compliance minimization problem). Nevertheless, the optimal shape being a composite, a post-treatement is needed in order to produce an almost optimal non-composite (i.e. workable) shape. The classical approach consists in penalizing the intermediate densities of material, but the obtained result deeply depends on the underlying mesh used and the level of details is not controllable. In a previous work, we proposed a new post-treatement method for the compliance minimization problem of an elastic structure. The main idea is to approximate the optimal composite shape with a locally periodic composite and to build a sequence of genuine shapes converging toward this composite structure. This method allows us to balance the level of details of the final shape and its optimality. Nevertheless, it was restricted to particular optimal shapes, depending on the topological structure of the lattice describing the arrangement of the holes of the composite. In this article, we lift this restriction in order to extend our method to any optimal composite structure for the compliance minimization problem.**

I. Introduction

A paradigm of structural optimization is the compliance minimization problem in elasticity. Namely, we look for the optimal shape of a structure that is of maximal rigidity for a given weight (or quantity of material).

Let D be the optimization domain and let $\Omega \subset D$ be the reference configuration of an isotropic elastic body in $\mathbb{R}^2$. Let Γ, Γ_N and Γ_D be a partition of the boundary of Ω. Function $u(\Omega)$ denotes the displacement of the structure Ω which is assumed to be clamped on Γ_D and submitted to surface loads g on Γ_N:

$$\begin{cases} \operatorname{div}\sigma = 0 & \text{in } \Omega, \text{ with } \sigma = Ae(u), \\ u = 0 & \text{on } \Gamma_D, \\ \sigma \cdot n = g & \text{on } \Gamma_N, \\ \sigma \cdot n = 0 & \text{on } \Gamma = \partial\Omega \setminus (\Gamma_N \cup \Gamma_D), \end{cases} \tag{1}$$

where $e(u) = (\nabla u + \nabla u^T)/2$ is the linearized metric tensor, A is Hooke's law or the elasticity tensor defined by

$$A\xi = 2\mu\xi + \lambda(\operatorname{Tr}\xi)\,\mathrm{Id}, \tag{2}$$

with Lamé moduli λ and μ, and $\sigma = Ae(u)$ is the constraint tensor. We consider the following compliance minimization problem

$$\min_{|\Omega|=V,\ \Gamma_N \subset \partial\Omega} J(\Omega) := \int_{\Gamma_N} g \cdot u(\Omega)\, ds, \tag{3}$$

where V is a given volume.

It turns out that the optimal solution to the problem above is a shape made of composite material (see the books [1], [2], [3] and [4]). It can be computed by the homogenization method which consists in relaxing the problem by enlarging the set of admissible shapes to include composite shapes; see for instance [1] and [5]. However, our wish is to produce a sequence of workable shapes that converge toward the optimal shape instead of the optimal shape itself.

A naive way of tackling this issue consists in rebuilding on each triangle of the mesh a composite (laminate or periodical). This procedure depends on the size of the mesh and on the mesh itself. Moreover, the resulting shapes would have a heigh degree of geometrical complexity in order to approximate correctly the behavior of the underlying composite. Finaly, it leads to the appearance of boundary-layers that may not yield the correct result, depending on the cost function (for instance, if it depends on the gradient of the deformation). Another issue with the homogenization method is the identification of the optimal composite materials and their respective Hooke laws which may be an impediment to the construction of the optimal sequences. In the case of compliance minimization, optimal composites have been identified and their Hooke laws computed.

A means to circumvent the obstacles underlined above is to explicitly devise a set of convergent shape sequences for which the associated Hooke laws are well-known. Then, it suffices to minimize the cost function in this partially relaxed problem where admissible shapes are taken in the former set.

II. Shape sequences

The first step of the scheme presented in the introduction consists in defining sequences of shapes for which the behavior of the limit composite shape is computable. The set of composite shapes thus obtained has to be rich enough for the minimization of the cost function over this set to be the closest to the optimal cost.

The most simple composites for which an explicit converging shape sequence is known are periodic composites (presented in section II-A). However, this set is too restrictive and has to be enriched. In a second step, we introduce locally periodic shapes on a regular lattice, obtained by the transformation of a periodic composite by a diffeomorphism. For particular shape optimization problems, this set is large enough to capture an almost optimal solution. However, it remains limited in the general case. In particular, it does not allow the presence of singularities in the lattice which often precludes to obtain a satisfying solution. This problem is lifted in the last part of this section II-D, where locally periodic shapes on a lattice containing a finite number of singularities is introduced. We send the reader to [6] for a detailed description of the non singular case.

A. Periodic composites

Let $Y =]0,1[^2$ and let

$$\mathcal{U}_\sharp = \{\omega \subset \mathbb{R}^2 : x \in \omega \Leftrightarrow x + f_1 \in \omega \Leftrightarrow x + f_2 \in \omega\},$$

where (f_1, f_2) is the canonical basis of $\mathbb{R}^2$, be the set of Y-periodic open subset of $\mathbb{R}^2$ (where Y stands for the unit cube of $\mathbb{R}^2$). Homogeneous periodic solids are obtained as limits of the open sets of the form

$$\Omega_\varepsilon(\omega) = D \cap \omega_\varepsilon,$$

where D is the optimization domain,

$$\omega_\varepsilon = \{x \in \mathbb{R}^2 \ : \ \varepsilon^{-1} x \in \omega\}, \quad \text{and} \quad \omega \in \mathcal{U}_\sharp.$$

The sequence

$$\Omega_\varepsilon(\omega) = \{x \in D \ : \ \varepsilon^{-1} x \in \omega\}$$

is a shape sequence that converges to a periodic composite shape. Now, if we call $H^1_\sharp(Y)$ the space of Y-periodic H^1 functions, the Hooke law of the limit composite material reads as:

$$A^*\xi \cdot \xi = \inf_{u \in H^1_\#(\omega)} \int_{Y \cap \omega} A(\xi + e(u)) \cdot (\xi + e(u))\, dx, \tag{4}$$

for all symetric matrices $\xi \in \mathbb{R}^2 \times \mathbb{R}^2$. In the following, the Hooke law of the limit composite of a sequence of shapes (Ω_ε) will be denoted by

$$A^* = \lim_\varepsilon \Omega_\varepsilon.$$

B. Locally Periodic composites on structured lattices

This time around ω depends on x so that $\omega : D \to \mathcal{U}_\sharp$ is a function. We define

$$\Omega_\varepsilon(\omega) = \{x \in D \ : \ x \in \varepsilon\omega(x)\},$$

in the same fashion as above. The Hooke law A^* of the limit composite, in this case, depends on x, and is also given by formula (4).

C. Locally periodic composites on regular lattices

In order to enrich the space of limit composites, a simple idea consists in pulling back a locally periodic composite defined on a structured lattice by a diffeomorphism. Let $\varphi : D \to \mathbb{R}^2$ be a regular local diffeomorphism. In this paragraph, we consider the case where D is simply connected. The sequence of open sets (see Figure 1)

$$\Omega_\varepsilon(\varphi, \omega) = \{x \in D \ : \ x \in \varphi^{-1}(\varepsilon\omega(x))\} \tag{5}$$

converges to a composite shape whose Hooke law A^* is, at every point x, that of a periodic homogeneous shape $(v_1^*, v_2^*)^{-T}(\omega(x))$ of period $(v_1^*, v_2^*)^{-T}(Y)$, where

$$v^* = (v_1^*, v_2^*) = \det(D_x\varphi)^{-1/2} D_x\varphi^T \tag{6}$$

and is given by the following formula

$$A^*\xi\cdot\xi = \inf_{u \in H^1_\#(\omega)} \int_{Y \cap \omega(x)} A(\xi + e_{v^*}(u))\cdot(\xi + e_{v^*}(u))dx, \tag{7}$$

where for all matrix $F \in \mathcal{M}^2$ and all $u \in H^1_\#(\omega)$, we set

$$e_F(u) = \frac{1}{2}(\nabla u F^T + F \nabla u).$$

a) Local description of the composite shape: A basic approach, in order to minimize a cost function over the composite shapes obtained as limits of sequences of the form (5), will be to apply some gradient method directly over the variables (φ, ω) describing the shape sequences (and thus the limit composites). If possible, it will unfortunately leads to local minima: The computed solution will strongly depends on the initial guess. An alternative approach consists in describing the admissible shape only by using local variables, that is the local periodicity cell determined by v^* and the local micro-structure ω, then to minimize the cost function over the couples $(v^*, \omega) \in \mathcal{V}_{ad}$, where

$$\mathcal{V}_{ad} = \{(v^*, \omega) : D \to \mathrm{SL}(2, \mathbb{R}) \times \mathcal{U}_\sharp \quad \text{such that } v^* \text{ is integrable}\}, \tag{8}$$

a matrix field v^* being called integrable if there exists a local diffeomorphism φ from D into $\mathbb{R}^2$ such that (6) is verified. The main advantage of this approach is that the integrability condition can be strongly relaxed (or even dropped) at the beginning of the optimization process and only enforced at the end (potentially gradually). This enable us to avoid being trapped in a local minimum due to the global integrability constraint. Indeed, without this constraint, the optimization process is nothing else than a classical homogenization method, that is numerically well known to present few local minima (at least for the compliance minimization problem).

b) Integrability condition: The integrability condition (6) being implicit, it can prove to be difficult to be used directly. The following proposition gives an practical version of this constraint.

Proposition 1: Let D be a simply connected open subset of $\mathbb{R}^2$. A regular matrix field $v^* = (v_1^*, v_2^*)$ from D with values in $\mathrm{SL}(2,\mathbb{R})$ is integrable if and only if, there exists a regular map r from D into $\mathbb{R}$ such that

$$\nabla r = (\nabla \wedge v_1^*) v_2^* - (\nabla \wedge v_2^*) v_1^*. \qquad (9)$$

c) Recovering of the sequence of shapes: The next proposition characterizes the set of maps φ corresponding to a given integrable field v^*.

Proposition 2: Let D be a simply connected open subset of $\mathbb{R}^2$. Let v^* be a regular matrix field from D with values in $\mathrm{SL}(2,\mathbb{R})$ which verifies (9). Then there exists a local diffeomorphism from D into $\mathbb{R}^2$ such that

$$e^r v^* = D_x \varphi^T,$$

where r is a map such (9) is satisfied. Moreover, for all local diffeomorphisms ψ from D into $\mathbb{R}^2$ such that

$$v^* = \det(D_x\psi)^{-1/2} D_x \psi^T,$$

there exist $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}^2$ such that

$$\psi = e^{c_1}(\varphi + c_2).$$

The result stated in Proposition 2 is not surprising: A given composite can be obtained as limit of different sequences of shapes. It follows from the previous Proposition that for any regular local diffeomorphism φ from D into $\mathbb{R}^2$ and regular map ω from D into the set of periodic open sets $\mathcal{U}_\sharp$, we have

$$\lim_\varepsilon \Omega_{e^{-c_1}\varepsilon}(\varphi + c_2, \omega) = \lim_\varepsilon \Omega_\varepsilon(\varphi, \omega),$$

for all constants $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}^2$. Note that the sequence of shapes is morally independent of the constant c_1 which only perform a renumbering of the sequence.

D. *Locally periodic composites on non regular lattices*

Locally periodic composites on a regular lattice have a major limitation for shape optimization. The arrangement of periodicity cells has to fulfill the integrability condition (6), which imposes global topological constraints to the lattice defining the composite shape. Along all loops in the domain, the total rotation of the periodicity cell is always equal to zero and the number of right angles (each angle been counted positively or negatively according to its sign) to four. Such lattices are usually not optimal – even for the compliance minimization problem – and we would like to be able to consider non regular lattices, for which the total rotation of the periodicity cell along a loop is no necessarily null or equivalently, for which the number of right angles preformed by a loop could be different from four. Figure 2 represents two lattices containing a defect. The periodicity cell along the loops represented in Figure 3 performs a U-turns, and the number of right angles made by the loops is either equal to two or six, whereas it can be checked that for the regular meshes, the number of right angles performed by a loop is always equal to four. Non regular lattices could only be obtained if the field v^* describing the local periodicity cell is non regular – so the name – or if the domain is not simply connect (so that ”singularities” can be hidden in the holes).

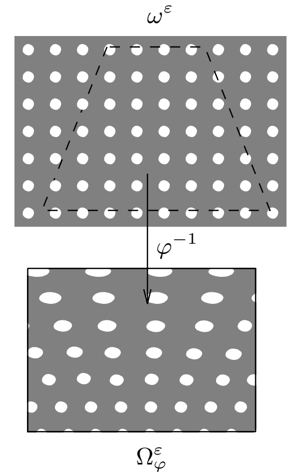

Fig. 1. Construction of $\Omega^\varepsilon_\varphi$

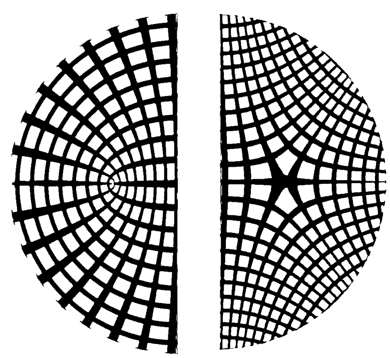

Fig. 2. Two non regular lattices

For now on, we consider not simply connected domain of optimization D, that is

$$D := D_0 \setminus (\bigcup_{i \in I} B_i), \qquad (10)$$

where the B_i are disjoint simply connected closed subsets of a simply connected domain D_0 of $\mathbb{R}^2$ and the set of indexes I is finite. As the cells of a non regular lattice could only be defined up to a rotation

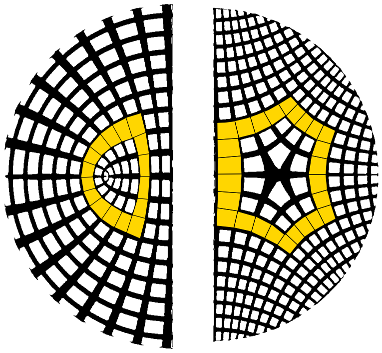

Fig. 3. As seen by following the paths draw on the lattices, periodicity cells perform U-turns around the singularities.

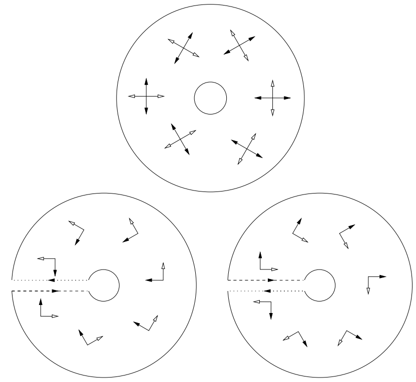

Fig. 4. Representation of the variety D_{v^*} by two charts (bottom), where v^* (top figure) is the field with value in the set of matrix of $\mathrm{SL}(2,\mathbb{R})$ defined up to a sign. The oriented dashed and dotted edges of the two charts are identified.

of angle π, we introduce the equivalence relation $\mathcal{R}$ defined by

$$(v^*,\omega)\mathcal{R}(w^*,\widetilde{\omega})$$

iif

$$(v^*,\omega)=(w^*,\widetilde{\omega}) \text{ or } (v^*,\omega)=(-w^*,s(\widetilde{\omega})),$$

where s is the rotation of angle π. Following the same steps as in the previous section, we introduce a set $\mathcal{V}^s_{ad}$ of admissible parameters describing our composite shape given by

$$\mathcal{V}^s_{ad}=\big\{(v^*,\omega):D\to \mathrm{SL}(2,\mathbb{R})\times\mathcal{U}_\sharp/\mathcal{R} \\ \text{such that } v^* \text{ is integrable}\big\}. \quad (11)$$

d) Recovering of the sequence of shapes: It remains to show how to construct a sequence of shapes converging toward the composite corresponding to a given element $(v^*,\omega)\in\mathcal{V}^s_{ad}$. To this end, several notations have to be introduced. First, we denote by $\mathcal{R}_1$ the equivalence relation on $\mathrm{SL}(2,\mathbb{R})$ defined by

$$v^*\mathcal{R}_1 w^* \text{ iff } v^*=w^* \text{ or } v^*=-w^*.$$

Next, for every $(v^*,\omega)\in\mathcal{V}^s_{ad}$, we set

$$D_{v^*}:=\big\{(x,w^*)\in D\times \mathrm{SL}(2,\mathbb{R}) \\ \text{such that } w^*(x)\mathcal{R}_1 v^*(x)\big\}.$$

Note that if $P_\mathcal{R}$ is the projection from $D\times\mathrm{SL}(2,\mathbb{R})$ onto D, the set $(P_\mathcal{R}|_{D_{v^*}})^{-1}(x)$ contains two elements for every $x\in D$. If v^* is regular enough, the set D_{v^*} is a variety, whose charts are given by the restriction of $P_\mathcal{R}$ to small enough open subsets. Note that the topology of D_{v^*} is not in general homeomorph to $D\times\{-1,1\}$ unless v^* has a representative that is continuous with values in $\mathrm{SL}(2,\mathbb{R})$ (see Figure 4).

For every subspace X, let

$$U(v^*,X):=\{\psi:D_{v^*}\to X \text{ such that} \\ \psi(x,-w)=-\psi(x,w^*) \text{ for all } (x,w^*)\in D_{v^*}\}.$$

We have the following proposition

Proposition 3: Let D be a domain of $\mathbb{R}^2$ defined by (10) and $(v^*,\omega)\in\mathcal{V}^s_{ad}$. Then there exists $\varphi\in U(v^*,\mathbb{R})$, a family $(c^i)\in(\mathbb{R}^2)^I$ and a family $(\psi^i)\in U(v^*,\mathbb{R}/\mathbb{Z})^I$ such that the shape sequence

$$\Omega_\varepsilon((v^*,\omega),\psi,\varphi,s)=\big\{x\in D: \\ \varepsilon^{-1}\varphi(x,v^*(x))+\sum_{i\in I}\left[\varepsilon^{-1}c^i\right]\psi^i(x,v^*(x))\in\omega(x)\big\}. \quad (12)$$

converges toward the composite shape whose Hooke law if given by (7).
Here $[\cdot]$ stands for the floor function. Finally, let us underline that the maps φ, the families $(\psi^i)_{i\in I}$ and $(c^i)_{i\in I}$ can be computed explicitly in function of v^*, but their construction will not be detailed here.

e) Representation of the local variables: Representing the local variables (v^*,ω) describing the limit composite can proven to be difficult, as they are defined up to the relation of equivalence $\mathcal{R}$. In order to circumvent this problem, we restrict our analysis to the case for which ω belongs to the set, denoted $\mathcal{U}^s_\sharp$, of Y-periodic open set of $\mathbb{R}^2$ that are symetric under rotations of angle π. Let us consider the factorization of v^* as

$$v^*=Q_\alpha e^M,$$

Q_α being the rotation

$$Q_\alpha=\alpha_1\,\mathrm{Id}+\alpha_2 R,$$

where

$$R=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

with $|\alpha|=1$ and M being a symetric matrix with trace equal to zero. Then an element $(v^*,\omega)\in\mathcal{V}^s_{ad}$ is uniquely represented by $(\beta,a,\omega)\in S^1\times\mathbb{R}^2\times\mathcal{U}^s_\sharp$, where $Q_\beta=Q_\alpha^2$ and

$$M=M_a:=\begin{pmatrix}a_1 & a_2\\ a_2 & -a_1\end{pmatrix}$$

It remains to rewrite the constraints (9) with respect to the variables β and a. We have

Proposition 4: Let v^* be a regular field from D into $\mathrm{SL}(2,\mathbb{R})$, defined by the variables β and a as above. Then v^* is integrable iff there exists a map r from D into $\mathbb{R}$ such that

$$\begin{aligned}\nabla r = Q_\beta\big(&f_1(|a|)M_a\nabla(|a|^2)\\ &+ f_2(2|a|)(\operatorname{div}(a), \nabla\wedge(a))^T - 1/2\operatorname{div}(Q_\beta^T)\big)\\ &\quad + f_2(|a|)^2(a\wedge\partial_2 a, -a\wedge\partial_1 a)^T\big),\end{aligned}\quad (13)$$

where

$$f_1(s) = (2s - \sinh(2s))/4s^3$$

and

$$f_2(s) = \sinh(s)/s.$$

III. Optimization Procedure

A. Recasting the problem

We are now in position to describe our optimization procedure. The new unknowns of our problem are the maps β, a and ω from D into (respectively) S^1, $\mathbb{R}^2$ and $\mathcal{U}^s_\sharp$. Let us denote by $J(\beta,a,\omega)$ the cost of the limit composite of the sequence of shapes described by those variables (see II-D.0.e) and by by $\mathcal{V}$ the set of parameters $(\beta,a,\omega)\in S^1\times\mathbb{R}^2\times\mathcal{U}^s_\sharp$ such that condition (13) is satisfied for a given map r. An optimal sequence of shapes can be obtained by computing

$$\inf_{(\beta,a,\omega)\in\mathcal{V}} J(\beta,a,\omega). \quad (14)$$

This problem is not necessarily well posed. Indeed, it could be of some interest to include many singularities in the shape. In order to avoid such a fate, we introduce a penalization of those singularities by the means of the Ginzburg-Landau functional.

$$G_\varepsilon(\beta) = \frac{1}{\ln\varepsilon}\int_D |\nabla\beta|^2\,dx + \frac{1}{\varepsilon^2\ln\varepsilon}(|\beta|^2-1)^2\,dx.$$

The first term imposes some regularity to β while the second one enforce β to be almost a unitary vector. As ε goes to zero, and for a sequence β_ε of maps from D into $\mathbb{R}^2$, the sequence $G_\varepsilon(\beta_\varepsilon)$ is bounded only if the number of singularities – that is where $\beta = 0$ – remains bounded. Moreover, we also have to add a (small) regularization term on a. Finally, we set

$$T(\beta,a,\omega) = J(\beta,s,\omega) + \alpha_1\|a\|^2_{H^1(D)^2} + \alpha_2 G_\varepsilon(\beta),$$

where α_1, α_2 and ε are positive reals and ε is chosen small. We could thus obtain an almost optimal minimization sequence by computing

$$\inf_{(\beta,a,\omega)\in\mathcal{V}_{opt}} T(\beta,a,\omega), \quad (15)$$

where $\mathcal{V}_{opt}$ is the set of maps from D into $\mathbb{R}^2\times\mathbb{R}^2\times\mathcal{U}^s_\sharp$ such that condition (13) is satisfied for a given map r.

B. Approximation of the Hooke laws

In this section, we present a simplified version of the optimization procedure introduced. It relies on the introduction of an approximated Hooke law in a relatively rough fashion which is nonetheless quite accurate in practice as least for the compliance optimization problem studied in the next paragraph.

The main difficulty here is the lack of analytical formulae for the computation of the Hooke laws in terms of the parameters v^* and ω. Nevertheless, the Hooke laws of rectangular cells in which rectangular holes have been carved – each cell being oriented along the same direction than the hole it contains – are well approximate by appropriate rank 2 laminates with orthogonal directions of lamination parallel to the orientation of the cell. Note that such laminates are optimal for compliance optimization probleme (see [1]).

Another alternative consists in carrying out a pre-computation for a certain pack of parameters then use some interpolation. However, this procedure seems quite heavy and has not been attempted yet. At any rate, our method considers rectangular cells, that is $a_2 = 0$. This greatly simplified the integrability condition of Proposition 4.

Proposition 5: Let v^* be a regular field from D into $\mathrm{SL}(2,\mathbb{R})$, defined by the variables β and a. Assume that $a_2 = 0$. Then v^* is integrable iff there exists a map r from D into $\mathbb{R}$ such that

$$\nabla r = Q_\beta\big((\partial_1 a_1, -\partial_2 a_1)^T - 1/2\operatorname{div}(Q_\beta^T)\big) \quad (16)$$

Moreover, the approximation of the Hooke laws by rank 2 laminates solely depends on the orientation of the cells given by β, the density of material and the proportions of lamination, thus are independent of a_1.

C. Optimization Process

We proceed in two steps. First, we solve the compliance minimization problem (15). To this end, we use a gradient type optimization method, the initial guess being chosen as the solution of the standard homogenization method (see [1]). Moreover, the integrability constraint is enforced gradually during the iterations. The solution obtained is usually very close to the initial guess, the gradient optimization process mainly provides us the value of a_1 compatible with the integrability constraint (16). Moreover, the optimization with respect to ω (that is w.r.t the density and the proportions of lamination) could be made pointwise and computed exactly for a given orientation β at each step of the minimization process.

Once a solution of (15) is computed, we remove from the optimization domain D_0 small neighborhoods of the points where the field β vanished, to obtain a new optimization domain D defined as in (10). We denote by $(B_i)_{i\in I}$ the removed holes.

Then, we build the singular functions $(\psi^i)_{i\in I}$ associated to the holes and compute a function φ and the coefficients $(c^i)_{i\in I}$ given by Proposition 3. Finally, the shape sequence is computed using (12). Note that in [7], a partial relaxation of the problem similar to ours is performed albeit for a more restrictive set of functions φ. Also, in [8], a different construction based on the deformation of a periodic lattice is proposed.

IV. Numerical Example: Compliance Optimization

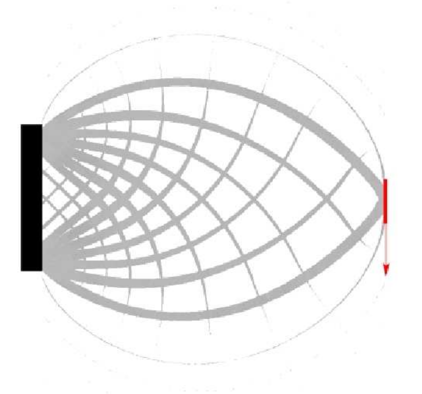

Fig. 5. An optimal cantilever

We consider the minimization problem (3) of the compliance of a two dimensional elastic structure made of an isotropic material and of given weight V. Two examples are presented here. The first one consists in the computation of an optimal cantilever, while the second one concerns the optimization of a bridge. The compliances of the obtained shapes are very close to the ones obtained from the classical homogenization method (and are thus thought to be almost optimal).

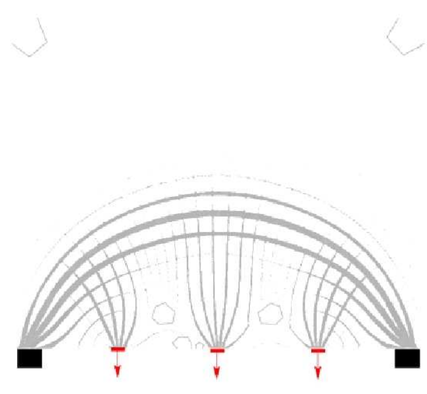

Fig. 6. An optimal triple bridge

In Figures 5-6, we display results obtained for two different shapes namely a cantilever and a bridge. The level of details has been chosen arbitrarily. It could have been increased, leading to even more optimal shapes with no extra cost of computation. Note that the black rectangles denote clamping conditions and the red bars and arrows show the loads and their directions. The cantilever exhibits no singularity whereas the bridge has essentially two singularities (each one between two loadings). The holes that were nucleated to get rid of the singularities are also displayed. Finally, let us remark that the shape computed for the cantilever, looks like the well known Michell truss, that are optimal in the case of low density.

V. Conclusion

In this work, we have set free from the obstruction caused by non regular lattices. Indeed, we have improved our initial method presented in [5]. It is clear that one may carry on with the optimization procedure using a geometrical optimization method in order to sharpen the contours of the shapes we have obtained. This may be done using a level set approach for instance as it uses a fixed mesh (see [9], [10] and [6]).

Our next move is to generalize the procedure to the compliance optimization in $\mathbb{R}^3$, and to other objective functions that can be tackled by the homogenization method.

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