Noname manuscript No.
(will be inserted by the editor)

0livier Pantz

A frictionless contact algorithm for deformable bodies

Avril 2006

Abstract We present a new contact algorithm for deformable bodies undergoing finite deformations. In this article, we focus our attention on the numerical treatment of the non-intersection constraint. In consequence, we do not consider friction, adhesion or wear, and restrict our analysis to the simplest frictionless case. Our method allows us to treat contacts and self-contacts, thin or not thin structures in a single setting.

1 Introduction

Industrial needs have stimulated the numerical simulation of mechanical contacts which has drawn a lot of attention over the last years. Many algorithms have been proposed in order to solve problems of growing complexity: contacts between rigid or (visco-)elastic bodies, submitted to small or finite deformations, with or without friction, adhesion or wear. In this article, we restrict ourself to the study of frictionless contact between deformable bodies undergoing large deformations. Such a goal may sound limited, but as it constitutes the elementary component of any other contact algorithm, it seems justified to study it for itself. If friction, wear or adhesion are mechanical properties that depend on the nature of the surfaces in contact, this is not the case for the kinematic condition of non-intersection.

The most common approach is known as the master/slave formulation. Let us briefly recall what it consists in. Consider two deformable bodies whose reference configurations are two open subsets of $\mathbb{R}^n$ ($n = 2$ or 3), M_1 and M_2. Let $\varphi_i : M_i \to \mathbb{R}^n$ ($i = 1, 2$) be their deformations. One of the two solids, say M_1, is called the master body, whereas the other one, i.e. M_2, is called the slave. We introduce the gap function g_φ, which maps every element x of

Address(es) of author(s) should be given

M_2 to the signed distance between $\varphi_2(x)$ and $\varphi_1(\partial M_1)$, defined by

$$g_\varphi(x) = \begin{cases} \operatorname{dist}(\varphi_2(x), \varphi_1(\partial M_1)) & \text{si } \varphi_2(x) \notin \varphi_1(M_1) \\ -\operatorname{dist}(\varphi_2(x), \varphi_1(\partial M_1)) & \text{si } \varphi_2(x) \in \varphi_1(M_1) \end{cases}$$

In order to prevent any overlapping between the two deformed solids, a constraint is applied to any element of the boundary of the slave body M_2. More precisely, it consists to impose that the signed distance between any element of $\varphi(\partial M_2)$ and $\varphi(\partial M_1)$ is positive:

$$g_\varphi(x) \geq 0 \text{ for all } x \in \partial M_2. \tag{1}$$

A breakdown of this formulation can be found in Laursen's thesis [30], [18] (see also [17]). This approach is confronted with two important difficulties. Firstly, it can not be directly applied neither to the case of self-contacts (where $M_1 = M_2$ and $\varphi_1 = \varphi_2$) nor to thin-structures ($\varphi_i : M_i \to \mathbb{R}^n$ and M_i is a submanifold of dimension $m < n$). For instance, in the self-contact case, the constraint (1) is empty: The signed distance between an element of $\varphi_1(\partial M_1)$ and an element of $\varphi_1(\partial M_1)$ is always zero. Moreover, even if we consider contact between two distinct bodies such that $\det(\nabla\varphi_i) > 0$, condition (1) is not sufficient to ensure non-intersection (see [23]). Secondly, the gap function is not everywhere differentiable with respect to φ_1 and φ_2. This lack of differentiability seems to be the main cause of the well known chatter phenomena, a loss of convergence in the numerical schemes (the computed solutions chatter around the equilibrium state). Several strategies have been developed in order to overcome those reefs. Puso and Laursen [25] notably proposed a regularization of the gap function, whereas Heinstein et al. [14] have adapted the master/slave approach to the study of thin structures and self-contacts. Their algorithm seems to be efficient in most common cases, but there is no guarantee it could correctly treat every situation. For a detailed presentation of the different strategies developed in this context, we refer to the Hallquist et al. article [10], which contains many references, and to the Kikuchi and Oden [15] or Wriggers [21] monographs (see also [32,16]). A different approach, called "material depth", has been proposed by Hirota, Fisher and State [11] to treat self-contacts, but it seems difficult to adapt to the thin-structure case. Other methods have been designed to treat the particular case of rigid bodies. The Non Smooth Contact Dynamic (NSCD) formulation has been introduced by M. Jean [12],[13] and J.-J. Moreau [20] for the study of granular media. For computing graphics purposes (see also Baraff [2,3], Baraff and Witkin [4,5]), Schmidl and Milenkovic [19] proposed an "Optimization Based Animation" (OBA) approach. However similar the NSCD and OBA methods are, they seem to have been developed independently.

If the master/slave formulation consists in a natural extension of the contact treatment between a deformable body and a rigid ground, it has no complete theoretical justification. The main theoretical issues linked to the study of frictionless contacts deal with the definition of the set of admissible deformations, the existence of minimizers of the energy over this set and the derivation of the Euler-Lagrange equations. Ball [1] considered the case of

hyperelastic bodies fixed on all of their boundaries, and whose stored energy grows to infinity as the determinant of the gradient of the deformation goes to zero. Assuming the polyconvexity of the stored energy function, he states an existence result to the minimization problem of the energy and proves that minimizers of the energy are injective almost everywhere : local injectivity implies global injectivity. Ciarlet and Nečas [6] extended his works to mixed boundary conditions (see also Tang Qi [31], Giaguinta et al [7],[8]). Nevertheless their analysis does not apply to the study of thin structures. Gonzalez et al. [9] and Schuricht et al. [29] introduced the notion of global curvature in order to consider contacts and self-contacts between unidimensional structures (see also [27],[26] and [28]). In a recent work [23], we have proposed a new modeling of the contacts and self-contacts between deformable bodies, which seems physically relevant in the static case when $\dim(M_i) = m$ and $2m \geq n$ (where n is the dimension of the space where M_i is injected, that is φ_i is a mapping from M_i into $\mathbb{R}^n$). This modeling relies on the introduction of a new set of admissible deformations. A topological constraint is introduced to ensure that no admissible deformation exhibits transversal (self)intersections.

In this article, we only address the numerical aspect of the problem, excluding any theoretical considerations. As we previously underline, the master/slave approach is not completely consistent when selfintersections or thin-structures are involved. Even if it could be adapted in order to correctly manage such cases in most situations, this is done only to a costly development price: Many subtle tricks have to be introduced to this end. Instead of trying to adapt the master/slave approach, we adopt a totally different point of view, which allows us to consider contacts and self-contacts as well as contact between thin-structures in a single setting.

2 Setting of the problem

Let us consider a family of connected deformable bodies $M = (M_i)$ moving in $\mathbb{R}^n$ ($n = 2$ or 3), such that for each index i, we have $n \geq \dim(M_i) \geq 1$. We denote by $\varphi_i : M_i \to \mathbb{R}^n$ the deformation of the body M_i, which maps each point x of M_i onto its position $\varphi_i(x)$ in $\mathbb{R}^n$. In the static case, the state of the system is completely described by the mapping φ from $\cup_i M_i$ into $\mathbb{R}^n$ whose restriction to each body M_i is its deformation φ_i.

We assume that we can associate to each state of the system an energy, and denote by J the functional that maps every deformation φ onto its energy. Our goal is to determine the minimum energy state, that is to find $\varphi \in \mathcal{A}(M)$ such that

$$J(\varphi) := \inf_{\psi \in \mathcal{A}(M)} J(\psi), \tag{2}$$

where $\mathcal{A}(M)$ is the set of deformations without self-intersection. It remains to give a mathematical definition to this set.

Remark 1 The family of bodies M could not contain single points as we assume for each index i the dimension of M_i to be positive. The results we present in the following could be extended in order to take into account this

particular class of bodies. As it is difficult to consider single material points as "deformable" bodies, we have chosen to exclude them from our analysis. It allows us to slightly simplify the presentation.

The master/slave approach consists in imposing constraint (1) to any couple of bodies of the collection M. Unfortunately, this constraint is empty if M is made of a single body. Moreover, it can not be applied whenever M contains more than one thin-structure (it is still reasonable for one thin-structure, chosen as the slave, and one solid of dimension equal to n, chosen as the master). In order to take these cases into account, we have to look for another definition. Firstly, let us recall that an embedding of M into $\mathbb{R}^n$ is a regular one-to-one mapping from M with values in $\mathbb{R}^n$, whose gradient $D\varphi$ is everywhere of maximal rank (that is equal to m). We denote by $\mathrm{Emb}(M;\mathbb{R}^n)$ the set of embeddings of M into $\mathbb{R}^n$, and define the admissible set $\mathcal{A}(M)$ as the closure of $\mathrm{Emb}(M;\mathbb{R}^n)$ (for a topology depending on the energy J):

$$\mathcal{A}(M) = \overline{\mathrm{Emb}(M;\mathbb{R}^n)}.$$

We proved several properties of this set in [22,23]. In particular, it does not contain any deformation with transversal selfintersections. This definition the treatment in a single setting the case of contacts, self-contacts, thin or not thin structures, however, it is implicit. Contrarily to the definition arising from the master/slave approach, it is not obvious to find out whether or not a given deformation belongs to the admissible set. In [22,23], we prove that any deformation of $\mathcal{A}(M)$ satisfies an explicit criterion. Moreover, we conjecture that this criterion is only fulfilled by the elements of $\mathcal{A}(M)$ in dimension $n = 2$. This allows to solve problem (2) by a penalization method (see [24]). Nevertheless, to our knowledge, there is currently no explicit definition of $\mathcal{A}(M)$ in the general case.

3 An optimization algorithm under nonconvex constraints

A major difficulty in the resolution of problem (2) is due to the strong nonlinearity introduced by the nonconvex constraint of non-interpenetration. There is no general method to directly solve such problems. On the other hand, efficient algorithms are available to minimize convex functions under convex constraints. Thus, a classical method for tackling this sort of problem consists to transform it into a sequel of convex problems. With this in mind, two radically different options are conceivable. One consists in removing any constraint: We minimize on the set of all deformations and penalize the non-admissible ones. We have developed this strategy in [24]. Another approach consists in minimizing on a convex subset of the admissible set. A local minimizer could thus be achieved by a recursive procedure. At each step, the energy is minimized over a convex "neighborhood" of the previous solution, included in the admissible set, until a fixed point is reached. This option has two advantages. Firstly, it requires neither the introduction of a penalization function nor the definition of an explicit characterization of the admissible set, which is to our knowledge an open problem in the general

case. Secondly, no undesirable intersections have to be removed during the minimization process, which seems to be a difficult issue. To be more precise, our algorithm consists in

1. Initialization of φ_0 by an admissible deformation.
2. For all $n \geq 0$, we denote $\varphi_{n+1} \in T(\varphi_n)$, the solution of the minimization problem
$$J(\varphi_{n+1}) := \inf_{\psi \in T(\varphi_n)} J(\psi),$$
3. STOP when $J(\varphi_{n+1}) \simeq J(\varphi_n)$,

where $T(.)$ maps any admissible deformation ψ to a convex subset $T(\psi)$ of the admissible deformations that contains the element ψ. Note that the stopping criterion is always reached as $J(\varphi_n)$ is a decreasing sequence, bounded from below (assuming the inifimum is finite). If ψ belongs to the interior of the admissible set, $T(\psi)$ will always be chosen as a (closed) neighborhood of ψ. On the contrary, if ψ belongs to the boundary of the admissible set, it is not granted that any convex neighborhood of ψ in $\mathcal{A}(M)$ does actually exist, since $\mathcal{A}(M)$ is not necessarily locally convex. Nevertheless, we will often refer to $T(\psi)$ as a "neighborhood" of ψ by language abuse.

We will not apply this algorithm directly to the initial problem (2), but to a discretized version for which we have an explicit definition of the admissible set.

4 Discretization

We assume that the space of deformations is discretized with P_1 Lagrangian elements. Let $\mathcal{T}_h$ be a regular mesh of M — the parameter h is the mesh size — and X_h the space of P_1 Lagrange elements over this mesh,

$$X_h = \left\{ \psi \in \mathcal{C}^0(M; \mathbb{R}^n) \ : \ \psi|_T \in P_1^N, \text{ for all } T \in \mathcal{T}_h \right\},$$

where P_1 is the set of polynomials of degree lower or equal to one. We introduce another parameter ε, the minimal distance we will impose between any disjoint elements of the mesh. Finally, the discretized set of admissible deformations is defined by

$$\mathcal{A}_{h,\varepsilon} = \Big\{ \psi_h \in X_h \ : \ \operatorname{dist}(\psi_h(T_1), \psi_h(T_2)) \geq \varepsilon, \\ \text{for all } T_1 \text{ and } T_2 \in \mathcal{T}_h \text{ such that } T_1 \cap T_2 = \emptyset \Big\}.$$

We set to solve the discretized minimization problem of J over $\mathcal{A}_{h,\varepsilon}$

$$\min_{v_h \in \mathcal{A}_{h,\varepsilon}} J(v_h) \tag{3}$$

using the procedure introduced in the previous section. To this end, it remains to define the map T that maps every element ψ_h of $\mathcal{A}_{h,\varepsilon}$ onto a convex "neighborhood" $T(\psi_h)$ included in $\mathcal{A}_{h,\varepsilon}$.

Remark 2 The definition of $\mathcal{A}_{h,\varepsilon}$ can be generalized by choosing a parameter ε depending on the elements T_1 and T_2 of the mesh considered. Note that, in the definition of $\mathcal{A}_{h,\varepsilon}$, T_1 and T_2 can be any elements of the mesh $\mathcal{T}_h$, that is vertices, edges, triangles or tetrahedrons.

4.1 Definition of a convex "neighborhood"

In this section, we define the map $T(.)$ that maps every admissible deformation $\psi_h \in \mathcal{A}_{h,\varepsilon}$ to a closed convex subset of $\mathcal{A}_{h,\varepsilon}$ containing ψ_h. Let us underline that there is a margin of freedom in the definition of T. Our choice is mostly guided by a wish for simplicity, others are conceivable. In order to avoid an unnecessarily complex formalism, we distinguish the two-dimensional case from the tridimensional one.

4.1.1 The two-dimensional case

In the two-dimensional case, that is $n = 2$, we define the map $T(\cdot)$ by

$$T(\psi_h) = \Big\{ \varphi_h \in X_h \ : \ \min_{x_a \in a} n_{a,x}(\psi_h) \cdot (\varphi_h(x_a) - \varphi_h(x)) \geq \varepsilon, \\ \text{for all edge } a \text{ and all vertex } x \text{ of the mesh such that } x \notin a \Big\}, \quad (4)$$

where $n_{a,x}(\psi_h)$ is defined for all edge a and vertex x of the mesh such that $x \notin a$ as the only unitary element of $\mathbb{R}^n$ such that

$$\min_{x_a \in a} n_{a,x}(\psi_h) \cdot (\psi_h(x_a) - \psi_h(x)) = \mathrm{dist}(\psi_h(a), \psi_h(x)) \geq \varepsilon. \quad (5)$$

Lemma 1 *For all element ψ_h of $\mathcal{A}_{h,\varepsilon}$, the convex set $T(\psi_h)$ is included in $\mathcal{A}_{h,\varepsilon}$.*

Proof Let φ_h be a deformation that does not belong to $\mathcal{A}_{h,\varepsilon}$. There exists T_1 and T_2 elements of $\mathcal{T}_h$ such that $\mathrm{dist}(\varphi_h(T_1), \varphi_h(T_2)) < \varepsilon$. We introduce the deformation ψ_s parametrized by $s \in [0, 1]$ and defined by

$$\psi_s = (1 - s)\psi_h + s\varphi_h.$$

We also denote by f the function that maps any real $s \in [0, 1]$ onto the distance between $\psi_s(T_1)$ and $\psi_s(T_2)$,

$$f(s) = \mathrm{dist}(\psi_s(T_1), \psi_s(T_2)).$$

The map f is continuous, $f(0) \geq \varepsilon$ and $f(1) < \varepsilon$. Thus, there exists a real $s \in [0, 1]$ such that $0 < f(s) < \varepsilon$ (let us remark that $f(1)$ could be equal to zero, hence $s = 1$ is not necessarily suitable). As the minimal distance between $\psi_s(T_1)$ and $\psi_s(T_2)$ is not zero, it is reached, up to a permutation of

T_1 and T_2, for a couple of points (x, x_a) of $T_1 \times T_2$ where x is a vertex of the mesh, and x_a belongs to an edge a of $\mathcal{T}_h$. We have

$$\begin{aligned} n_{a,x}(\psi_h) \cdot (\psi_s(x_a) - \psi_s(x)) &\leq |\psi_s(x_a) - \psi_s(x)| \\ &= \mathrm{dist}(\psi_s(T_1), \psi_s(T_2)) < \varepsilon. \end{aligned}$$

We deduce from this relation that ψ_s does not belong to $T(\psi_h)$. As ψ_s is a convex combination of ψ_h and φ_h, they could not both belong to $T(\psi_h)$.

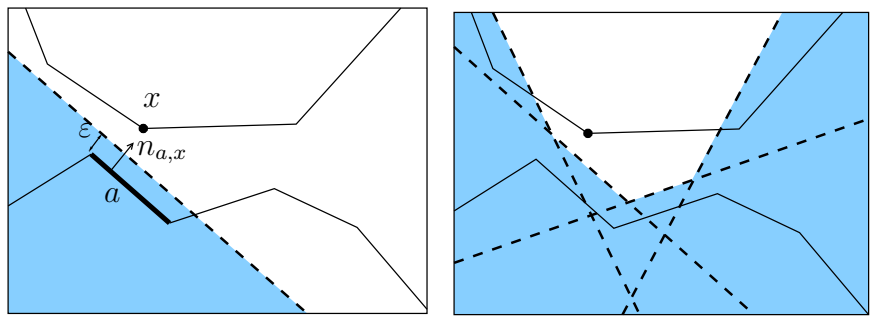

Fig. 1 Constraints associated with a vertex

Figures 1 represent the constraints imposed to a vertex x of the mesh, assuming the rest of the structure remains fixed. On the left-hand side, only the constraint associated with the single edge a is represented, whereas on the right-hand side all the constraints imposed by the lower structure are displayed.

4.1.2 The tridimensional case

In the three-dimensional case, the definition is slightly more complex. Not only do we have to associate a constraint with each couple vertex/triangle but also to each couple edge/edge. Thus,we set

$$\begin{aligned} T(\psi_h) = \Big\{ \varphi_h \in X_h \ : \ & \min_{\substack{x_a \in a \\ x_b \in b}} n_{a,b}(\psi_h) \cdot (\varphi_h(x_a) - \varphi_h(x_b)) \geq \varepsilon, \\ & \text{for all edges } a \text{ and } b \text{ of the mesh } \mathcal{T}_h \text{ such that } a \cap b = \emptyset \\ & \text{and } \min_{x_T \in T} n_{T,x}(\psi_h) \cdot (\varphi_h(x_T) - \varphi_h(x)) \geq \varepsilon, \\ & \text{for all triangle } T \text{ and all vertex } x \text{ of the mesh } \mathcal{T}_h \text{ such that } x \notin T \Big\}, \end{aligned} \tag{6}$$

where $n_{a,b}(\psi_h)$ and $n_{T,x}(\psi_h)$ are the unitary elements of $\mathbb{R}^n$ defined by

$$\min_{\substack{x_a \in a \\ x_b \in b}} n_{a,b} \cdot (\psi_h(x_a) - \psi_h(x_b)) = \mathrm{dist}(\psi_h(a), \psi_h(b))$$

and

$$\min_{x_T \in T} n_{T,x} \cdot (\psi_h(x_T) - \psi_h(x)) = \mathrm{dist}(\psi_h(T), \psi_h(x)).$$

Lemma 1 remains true in this case.

Remark 3 In the master/slave approach, only the triangle/vertex constraints are usually considered with the exclusion of the edge/edge type constraints.

5 Optimality conditions

Let us assume that the proposed algorithm converges toward an element φ of $\mathcal{A}_{h,\varepsilon}$. Contrarily to our hopes, φ is not, in general, a solution of the minimization problem of J over $\mathcal{A}_{h,\varepsilon}$. Such a result is not surprising. Since $\mathcal{A}_{h,\varepsilon}$ is not convex, the functional J may possess several local minima, even if J is strongly convex. Because of the sequential nature of our algorithm, we could at most expect that a local minimum is reached. Yet this does not occur. Indeed, a fixed point of our algorithm does not necessary fulfill the optimality conditions: The action/reaction principle is partially violated. At first sight, this result seems disastrous. Fortunately, it is not because the optimality conditions associated to the minimization problem of J over $\mathcal{A}_{h,\varepsilon}$ are almost fulfilled by the fixed points of our algorithm.

5.1 Optimality conditions associated to the discretized problem

5.1.1 two-dimensional case

Let φ be a solution of the minimization problem of J over $\mathcal{A}_{h,\varepsilon}$. If V is a small enough neighborhood of φ in X_h, arguing as in the proof of Lemma 1, it is easy to show that

$$\mathcal{A}_{h,\varepsilon} \cap V = \Big\{ \varphi \in X_h \ : \ \min_{x_a \in a} \mathrm{dist}(\varphi(x_a), \varphi(x)) \geq \varepsilon \\ \text{for all edge } a \text{ and all vertex } x \text{ of } \mathcal{T}_h \text{ such that } x \notin a \Big\} \cap V.$$

For all vertex x and all edge a of $\mathcal{T}_h$, the constraint

$$F_{a,x}(\psi) := \varepsilon - \mathrm{dist}(\psi(a), \psi(x)) \tag{7}$$

is continuously differentiable on V. More precisely, let us denote by a_0 and a_1 the endpoints of the edge a, by $p_{a,x}(\varphi)$ the element of the edge a that is mapped by φ onto the projection of $\varphi(x)$ on $\varphi(a)$ and by $\alpha^0_{a,x}$ and $\alpha^1_{a,x}$ the

homogeneous barycentric coordinates of $p_{a,x}(\varphi)$ on a. We have

$$\begin{aligned} p_{a,x}(\varphi) &\in a \\ p_{a,x}(\varphi) &= \alpha^0_{a,x} a_0 + \alpha^1_{a,x} a_1 \\ \mathrm{dist}(\varphi(a), \varphi(x)) &= |\varphi(p_{a,x}(\varphi)) - \varphi(x)|, \\ \alpha^0_{a,x} + \alpha^1_{a,x} &= 1, \\ \alpha^0_{a,x}, \alpha^1_{a,x} &\geq 0, \end{aligned} \tag{8}$$

and

$$\langle DF_{a,x}(\varphi), \hat{\varphi} \rangle = -n_{a,x}(\varphi) \cdot \big(\hat{\varphi}(x) - \hat{\varphi}(p_{a,x}(\varphi))\big),$$

where $n_{a,x}(\varphi)$ is defined by (5). Since $\langle DF_{a,x}(\varphi), \varphi \rangle < 0$, the constraints are qualified. Hence, the optimality conditions are given by the following proposition.

Proposition 1 *Let $\varphi : M \to \mathbb{R}^2$ be a solution of (3), that is a minimizer of J over $\mathcal{A}_{h,\varepsilon}$. Assume that J is differentiable, then there exists a family of nonnegative reals $\lambda_{a,x}$, where a spans the edges of $\mathcal{T}_h$ and x its vertices, such that for any test function $\hat{\varphi} \in X_h$, we have*

$$\begin{cases} \langle J'(\varphi), \hat{\varphi} \rangle = \displaystyle\sum_{x \notin a} h \lambda_{a,x} n_{a,x} \cdot \big(\hat{\varphi}(x) - \hat{\varphi}(p_{a,x})\big), \\ \lambda_{a,x} F_{a,x}(\varphi) = 0, \\ \varphi \in \mathcal{A}_{h,\varepsilon}, \end{cases} \tag{9}$$

where $p_{a,x}$ is define by (8), $n_{a,x}$ by (5) and $F_{a,x}$ by (7).

Remark 4 The Lagrange multiplier $\lambda_{a,x}$ is the contact force by unit length exerted by the edge a onto the point x (in the reference configuration). Note that at most six edges may be in contact with a given vertex of the mesh.

5.1.2 The three-dimensional case

In order ot obtain the optimality conditions fulfilled by the solutions of (3) in the three-dimensional case, we can proceed exactly as in the two-dimensional case. However, the formulation is slightly more complicated, since two kinds of contacts have to be considered, not only contacts between two edges but also contacts between a vertex and a triangle. For any couple of edges (a, b) of the mesh $\mathcal{T}_h$, we introduce the constraint

$$F_{a,b}(\psi) := \varepsilon - \mathrm{dist}(\psi(a), \psi(b)). \tag{10}$$

Likewise, for all vertex x and all triangle T of the mesh $\mathcal{T}_h$, we set

$$F_{T,x}(\psi) := \varepsilon - \mathrm{dist}(\psi(T), \psi(x)). \tag{11}$$

For all couple of edges (a, b) of $\mathcal{T}_h$, we denote by $\mathcal{P}_{a,b}$ the set of couples of points belonging to $a \times b$ that minimize the distance between $\varphi(a)$ and $\varphi(b)$, that is

$$\begin{aligned} \mathcal{P}_{a,b} := \{(p_{a,b}, p_{b,a}) \in a \times b \text{ such that} \\ \mathrm{dist}(\varphi(a), \varphi(b)) = |\varphi(p_{a,b}) - \varphi(p_{b,a})|\}. \end{aligned} \tag{12}$$

Observe that $\mathcal{P}_{a,b}$ contains a unique element, except when $\varphi(a)$ and $\varphi(b)$ are colinear. Finally, for any couple of vertex x and triangle T, we denote by $p_{T,x}$ the element of T such that

$$\mathrm{dist}(\varphi(T), \varphi(x)) = |\varphi(p_{T,x}) - \varphi(x)| \text{ and } p_{T,x} \in T. \tag{13}$$

The function $F_{T,x}$ is differentiable and for all test function $\hat{\varphi} \in X_h$, we have

$$\langle DF_{T,x}(\varphi), \hat{\varphi}\rangle = -n_{T,x}(\varphi) \cdot (\hat{\varphi}(x) - \hat{\varphi}(p_{T,x})).$$

On the other hand, the function $F_{a,b}$ is not differentiable. Nevertheless, it admits a subdifferential containing the linear forms L for which there exists an element $(p_{a,b}, p_{b,a}) \in \mathcal{P}_{a,b}$ such that

$$L(\hat{\varphi}) = -n_{a,b} \cdot (\hat{\varphi}(p_{b,a}) - \hat{\varphi}(a,b)).$$

A proposition similar to the one obtained in the two-dimensional case can be stated.

Proposition 2 *Assume that J is differentiable. Let $\varphi : M \to \mathbb{R}^3$ be a solution of the minimization problem (3). Then, there exist two families of non negative reals $\lambda_{a,b}$ and $\lambda_{T,x}$, where a and b span the set of edges of $\mathcal{T}_h$, T its triangles, and x its vertices, such that for any test function $\hat{\varphi} \in X_h$, we have*

$$\begin{cases} \langle J'(\varphi), \hat{\varphi}\rangle = \displaystyle\sum_{\substack{a,b \\ a\cap b=\emptyset}} h^2 \lambda_{a,b} n_{a,b} \cdot \big(\hat{\varphi}(p_{b,a}) - \hat{\varphi}(p_{a,b})\big) \\ \qquad + \displaystyle\sum_{\substack{T,x \\ x\notin T}} h^2 \lambda_{T,x} n_{T,x} \cdot \big(\hat{\varphi}(x) - \hat{\varphi}(p_{T,x})\big), \\ \lambda_{T,x} F_{T,x}(\varphi) = 0, \\ \lambda_{a,b} F_{a,b}(\varphi) = 0, \\ \varphi \in \mathcal{A}_{h,\varepsilon}. \end{cases} \tag{14}$$

where $(p_{a,b}, p_{b,a}) \in \mathcal{P}_{a,b}$, and $(p_{T,x})$ is defined by (13).

5.2 Optimality conditions associated with a fixed point

In this section, we derive the optimality conditions fulfilled by a fixed point of our algorithm. However those conditions differ from the one associated with the solutions of the discrete minimization problem (3) obtained in the previous section, they remain close. Every fixed point of the algorithm satisfies the optimality conditions associate with the discretized optimization problem up to a small error, which scales like the size h of the mesh. For the sake of simplicity, we confine to the two-dimensional case. The three-dimensional case may be addressed in a similar way.

Proposition 3 *Let φ be a fixed point of the algorithm presented in section 3 applied to the minimization of J over $\mathcal{A}_{h,\varepsilon}$, where the "neighborhoods" $T(\psi)$ are defined by (4) or equivalently by*

$$T(\psi) = \Big\{ \varphi_h \in X_h \ : \ F^0_{a,x}(\varphi_h) \leq 0 \ \textit{et} \ F^1_{a,x}(\varphi_h) \leq 0, \\ \textit{for any vertex } x \textit{ and any edge } a \textit{ of } \mathcal{T}_h \Big\},$$

where $F^0_{a,x}$ and $F^1_{a,x}$ are defined by

$$F^0_{a,x}(\varphi_h) = \varepsilon - n_{a,x}(\psi) \cdot (\varphi_h(a_0) - \varphi_h(x))$$

and

$$F^1_{a,x}(\varphi_h) = \varepsilon - n_{a,x}(\psi) \cdot (\varphi_h(a_1) - \varphi_h(x)).$$

Then, there exists a family $\lambda^i_{a,x}$ (where $i = 0, 1$, a is any edge of the mesh $\mathcal{T}_h$ and x any of its vertices) of non negative reals, such that for any test function $\hat{\varphi} \in X_h$,

$$\begin{cases} \langle J'(\varphi), \hat{\varphi} \rangle = \displaystyle\sum_{x \notin a} n_{a,x} \cdot \big((\lambda^0_{a,x} + \lambda^1_{a,x}) \hat{\varphi}(x) + \lambda^0_{a,x} \hat{\varphi}(a_0) - \lambda^1_{a,x} \hat{\varphi}(a_1) \big), \\ \lambda^0_{a,x} F^0_{a,x}(\varphi) = 0, \\ \lambda^1_{a,x} F^1_{a,x}(\varphi) = 0, \\ \varphi \in \mathcal{A}_{h,\varepsilon}. \end{cases} \tag{15}$$

Remark 5 The dependence of the functions $F^0_{a,x}$ and $F^1_{a,x}$ with respect to ψ in the definition of $T(\psi)$ is implicit and does not appear in the notations used.

Proof The optimality conditions associated with the minimization problem

$$J(\varphi_{n+1}) = \min_{\psi \in T(\varphi_n)} J(\psi)$$

satisfied by $\varphi_{n+1} \in T(\varphi_n)$ are

$$\langle J'(\varphi_{n+1}), \hat{\varphi} \rangle - \sum_{x \notin a} n_{a,x}(\varphi_n) \cdot \big((\lambda^0_{a,x} + \lambda^1_{a,x}) \hat{\varphi}(x) - \lambda^0_{a,x} \hat{\varphi}(a^0) - \lambda^1_{a,x} \hat{\varphi}(a^1_{a,x}) \big) = 0,$$

for any test function $\hat{\varphi} \in X_h$,

$$\lambda^0_{a,x} F^0_{a,x}(\varphi_{n+1}) = 0, \qquad \lambda^1_{a,x} F^1_{a,x}(\varphi_{n+1}) = 0,$$

and $\varphi_{n+1} \in T(\varphi_n)$ with $\lambda^i_{a,x}$ $(i = 0, 1)$ positive reals. Thence, if φ is a fixed point of our algorithm, we have

$$\langle J'(\varphi), \hat{\varphi} \rangle - \sum_{x \notin a} n_{a,x}(\varphi) \cdot \big((\lambda^1_{a,x} + \lambda^0_{a,x}) \hat{\varphi}(x) - \lambda^0_{a,x} \hat{\varphi}(a^0) - \lambda^1 \hat{\varphi}(a^1) \big) = 0,$$

$$\lambda^0_{a,x} F^0_{a,x}(\varphi) = 0, \qquad \lambda^1_{a,x} F^1_{a,x}(\varphi) = 0,$$

and $\varphi \in \mathcal{A}_h$.

Proposition 4 *Every fixed point of the algorithm of the Section 3 applied to the discrete minimization problem (3), where the map $T(\cdot)$ is defined by (4) satisfies the optimality conditions (9) up to an error of order h. More precisely, there exists a family $\lambda_{a,x}$ of non negative reals, where a spans the edges of $\mathcal{T}_h$, and x its vertices such that*

$$\begin{cases} \|R(\varphi_h,\lambda)\|_{H^{-1}(M;\mathbb{R}^2)} \leq \sqrt{C(\varphi_h)}\|\lambda\|_h h \\ \lambda_{a,x}F_{a,x}(\varphi)=0, \\ \varphi_h \in \mathcal{A}_{h,\varepsilon}, \end{cases}$$

where $R(\varphi_h,\lambda)$ is the residual associated to the minimization problem (3)

$$R(\varphi_h,\lambda) = \langle J'(\varphi),\hat{\varphi}\rangle - \sum_{x\notin a} h\lambda_{a,x}n_{a,x}\cdot\big(\hat{\varphi}(x)-\hat{\varphi}(p_{a,x})\big),$$

$\|.\|_h$ is the norm associated with the Lagrange multipliers λ,

$$\|\lambda\|_h = \left(\sum_{x\notin a}|\lambda_{a,x}|^2 h\right)^{1/2},$$

and $C(\varphi_h)$ is the constant defined by

$$C(\varphi_h) := \max_a \left\{ C_a(\varphi_h) := \mathrm{Card}\Big(\{x\in\mathcal{T}_h \textit{ such that } F_{a,x\neq 0}\}\Big)\right\}.$$

Remark 6 We recall that the Lagrange multipliers $\lambda_{a,x}$ represent the linear force exerted by a part of the solid on another. The sum which defines the norm $\|\lambda\|_h$ contains Ch^2 elements so that,at the first glance, this sum may not remain bounded as the discretization gets finer. Fortunately, the number of edges in contact with one vertex is bounded independently of h. Thus, the sum which defined $\|\lambda\|_h$ only contains a number of nonzero elements of order at most h^{-1}. This justifies the normalisation by h (and not h^2) used in the definition of the norm $\|.\|_h$.

Remark 7 Assume that the sequence φ_h converges toward a regular deformation φ (as h and ε goes to zero) and that a uniform mesh is used, then the number of vertices $C_a(\varphi_h)$ in contact with one edge a is of the order

$$C_a(\varphi_h) \simeq \sum_{\substack{x\in\varphi^{-1}(\varphi(y))\\ x\neq y}} |\dot{\varphi}(y)|/|\dot{\varphi}(x)|,$$

where y is an element of the edge a. In such a situation, $C(\varphi_h)$ remains bounded.

Proof The deformation φ_h verifies the optimality conditions given by Proposition 3. We set $\lambda_{a,x} = \lambda^0_{a,x} + \lambda^1_{a,x}$, then

$$\begin{aligned}
\langle R(\varphi_h, \lambda), \hat{\varphi} \rangle &= \sum_{x \notin a} h n_{a,x} \cdot \big((\alpha^0_{a,x} \lambda_{a,x} - \lambda^0_{a,x}) \hat{\varphi}(a_0) + (\alpha^1_{a,x} \lambda_{a,x} - \lambda^1_{a,x}) \hat{\varphi}(a_1) \big) \\
&= \sum_{x \notin a} h n_{a,x} \cdot (\hat{\varphi}(a_0) - \hat{\varphi}(a_1))(\alpha^0_{a,x} \lambda^1_{a,x} - \alpha^1_{a,x} \lambda^0_{a,x}) \\
&\leq \sum_a \left(\sum_{x \notin a} h |\alpha^0_{a,x} \lambda^1_{a,x} - \alpha^1_{a,x} \lambda^0_{a,x}|^2 \right)^{1/2} \cdots \\
&\qquad \cdots \left(\sum_{\substack{x \notin a \\ \lambda_{a,x} \neq 0}} h (\hat{\varphi}(a_0) - \hat{\varphi}(a_1))^2 \right)^{1/2} \\
&\leq \sum_a \left(\sum_{x \notin a} h |\lambda_{a,x}|^2 \right)^{1/2} \left(C_a(\varphi_h) h (\hat{\varphi}(a_0) - \hat{\varphi}(a_1))^2 \right)^{1/2} \\
&\leq \|\varphi_h\|_h \sqrt{C(\varphi_h)} \left(\sum_a \left| \frac{\hat{\varphi}(a_0) - \hat{\varphi}(a_1)}{h} \right|^2 h \right)^{1/2} h.
\end{aligned}$$

Moreover, since φ_h is a fixed point, we have

$$F_{a,x}(\varphi_h) = \min(F^0_{a,x}(\varphi_h), F^1_{a,x}(\varphi_h)). \tag{16}$$

If $\lambda_{a,x} \neq 0$, we either have $\lambda^0_{a,x} \neq 0$ or $\lambda^1_{a,x} \neq 0$. Then, from the optimality conditions satisfied by φ_h, we infer that either $F^0_{a,x}(\varphi)$ or $F^1_{a,x}(\varphi)$ is equal to zero. From (16), it follows that $F_{a,x}(\varphi_h) = 0$, and that for any couple vertex/edge (x, a), we have

$$F_{a,x}(\varphi_h) \lambda_{a,x} = 0.$$

Remark 8 The error made on the residual due to the contact algorithm is of the same order as the error due to the P_1 finite element discretization. Though it is possible to use an alternate definition of the "neighborhood" $T(.)$ so that any fixed point of our algorithm exactly matched the optimality conditions associated to the discrete minimization problem of J on $\mathcal{A}_{h,\varepsilon}$, it will not increase the precision of the global scheme.

6 The dynamic case

We can extend our analysis to the study of dynamic systems with frictionless contacts and soft impacts. In such a case, the evolution of a family of deformable bodies can be approximated by a sequence of minimization problems defined on the set of admissible deformations. More precisely, once the

problem is discretized in time and space, we have to compute at each time $t = i(\Delta t)$ the solution of the minimization problem

$$\min_{\varphi_{i+1} \in \mathcal{A}_{h,\varepsilon}} \frac{1}{2} \int_M m \left(\frac{v_{i+1} - v_i}{\Delta t} \right)^2 dx + J(\varphi_{i+1}),$$

where $v_{i+1} = (\varphi_{i+1} - \varphi_i)/\Delta t$, and m is the inertial mass per volume unit.

7 A numercial example

In this section, we present an application of our method in the two dimensional static case. We consider a system made of elastic membranes M_i, diffeomorphic to the interval $[0, 1]$ that contains a gas. The stored elastic energy of a membrane M_i is defined by

$$W_i(F) = \mu_i \begin{cases} (|F|^2 - 1)^2 & \text{if } |F| \geq 1 \\ 0 & \text{if } |F| < 1. \end{cases},$$

where $\mu_i > 0$ is an elasticity coefficient. The internal energy of a deformation ψ_i of the membrane M_i is

$$E_i(\psi_i) = \int_{M_i} W_i(\dot{\psi_i}) dx.$$

Moreover, we assume that each membrane is fixed on a plane support and that the space between the membranes and the supports is filled with a perfect gas, which exerts on each membrane a uniform pressure inversely proportional to the area V_i it is occupying. Thus, the total energy associated to the deformations $(\psi_i) = \psi$ of the membranes is

$$I(\psi) = \sum_i \int_{M_i} W_i(\dot{\psi_i}) dx - C_i \ln(V_i),$$

where C_i are positive constants depending on the quantity of gas contained in each membrane M_i. Any equilibrium position of the membranes is a critical points of the energy over the set of admissible deformations. As the functions W_i are convex, there exists a configuration φ that minimizes the energy over the set of admissible deformations,

$$I(\varphi) = \inf_{\psi \in \mathcal{A}(M)} I(\psi).$$

We solve the discretized version of this problem

$$I(\varphi_h) = \inf_{\psi_h \in \mathcal{A}_{h,\varepsilon}(M)} I(\psi_h), \tag{17}$$

using our method. Let us recall the steps of our algorithm

1. Initialization of φ_0 by an admissible deformation.

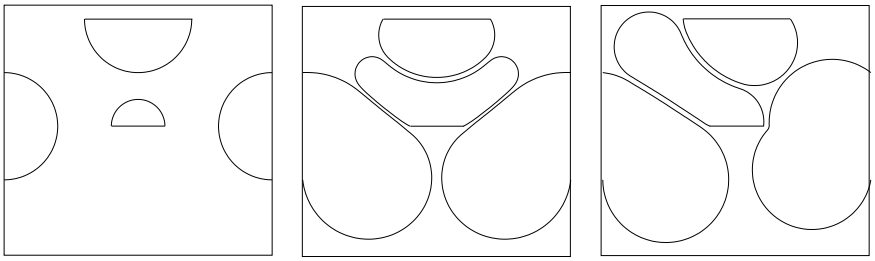

Fig. 2 Different steps of the minimization of the energy of a system made of inflated elastic balloons.

2. For all $n \geq 0$, we denote by $\varphi_{n+1} \in T(\varphi_n)$ the solution of the minimization problem
$$I(\varphi_{n+1}) := \inf_{\psi \in T(\varphi_n)} I(\psi), \tag{18}$$

where $T(.)$ maps any admissible deformation ψ_h onto the convex subset of the admissible set $\mathcal{A}_{h,\varepsilon}(M)$ defined by

$$T(\psi_h) = \Big\{ \varphi_h \in X_h \ : \ n_{a,x}(\psi_h).(\varphi_h(a) - \varphi_h(x)) \geq \varepsilon, \\ \text{for all edge } a \text{ and all vertex } x \text{ of the mesh such that } x \notin a \Big\},$$

where $n_{a,x}(\psi_h)$ is defined for each edge a and each vertex x of the mesh such that $x \notin a$ by

$$\min_{x_a \in a} n_{a,x}(\psi_h).(\psi_h(x_a) - \psi_h(x)) = \mathrm{dist}(\psi_h(a), \psi_h(x)) \geq \varepsilon.$$

3. STOP when $I(\varphi_{n+1}) \simeq I(\varphi_n)$.

We solve each minimization problem (18) with the classical Uzawa's algorithm. Figure 2 shows different steps of the minimization process. Here, four membranes are considered. In a first step, an unstable symmetric equilibrium state is reached, then the symmetry is broken leading to a stable equilibrium state.

8 Conclusion

The main advantage of the method proposed in this article in order to take into account frictionless (self)contacts between deformable bodies is its robustness. During the numerical simulations we have performed, never was the classical chatter problem, that undermines many other algorithms, encountered. The drawback of our method lies in the resolution of a minimization problem with n^2 constraints (where n is the number of elements of our discretization) at each step, which is prohibitive as the number of elements of the mesh becomes important. Moreover, the number of steps also depends

on the size of the discretization, and is of order $\max(h, \varepsilon)^{-1}$. In the case of self-contacts, ε has to be chosen smaller than h, which entails that we have to solve about n minimization problems with n^2 constraints each! Nevertheless, it is possible to drastically reduce the computational time by regrouping the elements in bundles, and imposing non-intersection constraints to their convex hull. Moreover, in the present form, our algorithm does not take into account the rigid case. In thus case, the natural variables are the position of the gravity center, and the orientation of the solid (and not, for polygonal shapes, the position of the vertices). Thus, the definition of the "neighborhoods" $T(\cdot)$ can not be trivially extended since they are not convex sets with respect to those variables. The rigidity constraints introduce a new nonlinearity, which has be treated on its own. Finally, in order to consider realistic applications, mechanical phenomena involved during the contact have also to be taken into account, in particular friction. It seems that classical treatments of friction should be easily adapted to our approach.

References

1. J. M. Ball. Global invertibility of Sobolev functions and the interpenetration of matter. *Proc. Roy. Soc. Edinburgh Sect. A*, 88(3-4):315–328, 1981.
2. D. Baraff. Analytical methods for dynamic simulation of non-penetrating rigid bodies. In *SIGGRAPH '89: Proceedings of the 16th annual conference on Computer graphics and interactive techniques*, pages 223–232, New York, NY, USA, 1989. ACM Press.
3. D. Baraff. Fast contact force computation for nonpenetrating rigid bodies. In *SIGGRAPH '94: Proceedings of the 21st annual conference on Computer graphics and interactive techniques*, pages 23–34, New York, NY, USA, 1994. ACM Press.
4. D. Baraff and A. Witkin. Dynamic simulation of non-penetrating flexible bodies. In *SIGGRAPH '92: Proceedings of the 19th annual conference on Computer graphics and interactive techniques*, pages 303–308, New York, NY, USA, 1992. ACM Press.
5. D. Baraff and A. Witkin. Large steps in cloth simulation. In *SIGGRAPH '98: Proceedings of the 25th annual conference on Computer graphics and interactive techniques*, pages 43–54, New York, NY, USA, 1998. ACM Press.
6. P. G. Ciarlet and J. Nečas. Injectivity and self-contact in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 97(3):171–188, 1987.
7. M. Giaquinta, G. Modica, and J. Souček. Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 106(2):97–159, 1989.
8. M. Giaquinta, G. Modica, and J. Souček. Erratum and addendum to: "Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity" [Arch. Rational Mech. Anal. **106** (1989), no. 2, 97–159; MR 90c:58044]. *Arch. Rational Mech. Anal.*, 109(4):385–392, 1990.
9. O. Gonzalez, J. H. Maddocks, F. Schuricht, and H. von der Mosel. Global curvature and self-contact of nonlinearly elastic curves and rods. *Calc. Var. Partial Differential Equations*, 14(1):29–68, 2002.
10. J. O. Hallquist, G. L. Goudreau, and D. J. Benson. Sliding interfaces with contact-impact in large-scale Lagrangian computations. *Comput. Methods Appl. Mech. Engrg.*, 51(1-3):107–137, 1985. FENOMECH '84, Part I, II (Stuttgart, 1984).
11. G. Hirota, F. S., and S. A. An improved finite-element contact model for anatomical simulations. *The Visual Computer*, 19(5):291–309, 2003.
12. M. Jean. The non-smooth contact dynamics method. *Comput. Methods Appl. Mech. Engrg.*, 177(3-4):235–257, 1999. Computational modeling of contact and friction.

13. M. Jean, V. Acary, and Y. Monerie. Non-smooth contact dynamics approach of cohesive materials. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 359(1789):2497–2518, 2001. Non-smooth mechanics.
14. H. M. A. S. S. J.W. and M. F.J. A general-purpose contact detection algorithm for nonlinear structural analysis code. *Sandia Report*, 1993.
15. N. Kikuchi and J. T. Oden. *Contact problems in elasticity: a study of variational inequalities and finite element methods*, volume 8 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988.
16. A. Klarbring. Large displacement frictional contact: a continuum framework for finite element discretization. *European J. Mech. A Solids*, 14(2):237–253, 1995.
17. J. Laursen, T.A.; Simo. A continuum-based finite element formulation for the implicit solution of multibody, large deformation frictional contact problems. *Internat. J. Numer. Methods Engrg.*, 36(20):3451–3485, 1993.
18. T. A. Laursen. *Computational contact and impact mechanics*. Springer-Verlag, Berlin, 2002. Fundamentals of modeling interfacial phenomena in nonlinear finite element analysis.
19. V. J. Milenkovic and H. Schmidl. Optimization-based animation. In *SIGGRAPH '01: Proceedings of the 28th annual conference on Computer graphics and interactive techniques*, pages 37–46, New York, NY, USA, 2001. ACM Press.
20. J. J. Moreau. An introduction to unilateral dynamics. *Lect. Notes Appl. Comput. Mech.*, 14:1–26, 2004.
21. W. P. *Computational Contact Mechanics*. J. Wiley & Sons, New York, 2002.
22. O. Pantz. The modeling of deformable bodies with frictionless (self-)contacts. *Arch. Rational Mech. Anal.* to appear.
23. O. Pantz. The modeling of deformable bodies with frictionless (self-)contacts. Rapport Interne 585, CMAP, École Polytechnique, Palaiseau France, 2005.
24. O. Pantz. Contacts en dimension 2: Une mthode de pnalisation. Rapport Interne 597, CMAP, École Polytechnique, Palaiseau France, 2006.
25. M. A. Puso and T. A. Laursen. A 3D contact smoothing method using Gregory patches. *Internat. J. Numer. Methods Engrg.*, 54(8):1161–1194, 2002.
26. F. Schuricht. A variational approach to obstacle problems for shearable nonlinearly elastic rods. *Arch. Rational Mech. Anal.*, 140(2):103–159, 1997.
27. F. Schuricht. Regularity for shearable nonlinearly elastic rods in obstacle problems. *Arch. Rational Mech. Anal.*, 145(1):23–49, 1998.
28. F. Schuricht. Variational approach to contact problems in nonlinear elasticity. *Calc. Var. Partial Differential Equations*, 15(4):433–449, 2002.
29. F. Schuricht and von der Mosel H. Euler-lagrange equations for nonlinearly elastic rods with self-contact. *Arch. Rational Mech. Anal.*, 168(1):35–82, 2003.
30. L. T.A. Formulation and treatment of frictional contact problems using finite elements. *SUDAM Report*, 92(6), 1992.
31. Q. Tang. Almost-everywhere injectivity in nonlinear elasticity. *Proc. Roy. Soc. Edinburgh Sect. A*, 109(1-2):79–95, 1988.
32. P. Wriggers. Finite element algorithms for contact problems. *Arch. Comput. Methods Engrg.*, 2(4):1–49, 1995.