Contacts in dimension 2, A penalization method

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Abstract

We address the problem of the numerical treatment of frictionless contacts and selfcontacts between nonlinear elastic bodies, moving in $\mathbb{R}^2$, in the presence of large deformations. We propose a new penalization approach, based on a modelling we have introduced in former works, to take into account the noninterpenetration constraint.

1. Introduction

The simulation of mechanical contacts between deformable bodies, in spite of it has been the topic of many articles over recent years, remains a most challenging task. Realistic simulations require not only to take into account the noninterpenetration constraint but also friction and shocks, in the dynamic case, to which other phenomena can be incorporated like wear, damage, formation of fractures, or even dislocation. In this article, we only address the treatment of the noninterpenetration constraint between deformable solids undergoing large deformations. A large majority of work relating to this subject is based on the master-slave approach, in order to formulate the noninterpenetration condition into a mathematical setting. Let us recall briefly what this formulation consists in.

Let us consider two bodies, whose reference configurations Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$ ($n = 2$, or 3). The deformation of each body is denoted by a map $\varphi_i : \Omega_i \to \mathbb{R}^n$ ($i = 1, 2$). One of the solids, say Ω_1 is designated as the master body, and the other one (i.e. Ω_2) as the slave. For each element $x \in \partial\Omega_2$ of the slave surface, we define $g_\varphi(x)$ as the signed distance of $\varphi_2(x)$ to the master surface $\partial\Omega_1$, that is,

$$g_\varphi(x) = \begin{cases} \operatorname{dist}(\varphi_2(x), \varphi_1(\partial\Omega_1)), & \text{if } \varphi_2(x) \notin \varphi_1(\Omega_1), \\ -\operatorname{dist}(\varphi_2(x), \varphi_1(\partial\Omega_1)), & \text{if } \varphi_2(x) \in \varphi_1(\Omega_1). \end{cases}$$

In order to prevent interpenetration of the two bodies, a constraint is applied to each element of the slave surface. More precisely, it consists in imposing that the signed distance to $\varphi_1(\partial\Omega_1)$ of any element of $\varphi(\partial\Omega_2)$ remains nonnegative:

$$g_\varphi(x) \geq 0, \quad \text{for all } x \in \partial\Omega_2. \tag{1}$$

A complete exposition of this formulation can be found in [1] and [2]. This approach presents two main drawbacks. First, it can neither be applied to the treatment of selfcontacts ($\Omega_1 = \Omega_2$ and $\varphi_1 = \varphi_2$), nor to the case of thin structures ($\varphi_i : \Omega_i \to \mathbb{R}^n$, where Ω_i is an open set of $\mathbb{R}^p$ with $p < n$). In such cases, the constraint (1) is verified for any deformation. Moreover, even if one considers two different n dimensional bodies moving in $\mathbb{R}^n$, some deformations that verify the constraint (1) are not intersection-free, even if $\det(\nabla\varphi_i) > 0$ almost everywhere. Finally, the non differentiability of the gap function g with respect to φ_1 and φ_2 can yield to the nonconvergence of the numerical algorithms, known as the 'chatter' problem. Numerous strategies have been developed in order to circumvent those difficulties. Puso and Laursen proposed a regularisation of the gap function, whereas Heinstein et al. [3] and Benson and Hallquist [4] have adapted the master-slave approach to the case of selfcontacts and thin structures. Their algorithm seems efficient in most practical cases, even if it might fail to treat some complex situations. A large overview of the different strategies developed in this context, is provided by Hallquist et al. [5], which includes many references (see also [6,7]) and monographs of Kikuchi and Oden [8], or [9] and [10]. A different approach, called 'material depth', has been introduced by Hirota et al. [11] to treat the selfcontacts case. Nevertheless, it seems difficult to extend to the study of thin structures. Other researchers have investigated the case of rigid bodies. Their main motivation is either the study of granular media or applications to computer graphics. The NSCD method (Non Smooth Contact Dynamics) has been introduced by M. Jean [12,13] and J.-J. Moreau [14] for granular media. A slightly different approach, called OBA (Optimization Based Animation) has been proposed (it seems independently) by Schmidl and Milenkovic [15]

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(see also Baraff [16,17], and Baraff and Witkin [18,19]).

If the master-slave approach seems natural, it has no plainly satisfying theoretical justification. The study of frictionless contacts raises several problems that have to be addressed: Definition of the set of admissible deformations, existence of minimizers of the energy over this set, derivation of the Euler Lagrange equations satisfied by the minimizers. Ball [20] considers the almost injectivity of hyperelastic bodies with Dirichlet boundary conditions. Ciarlet and Nečas [21] extended its study to mixed boundary conditions; see also Tang Qi [22], Giaguinta and al [23,24]. Nevertheless, none of this approach can be applied to the thin structures case. Gonzalez and al. [25], Schuricht and al. [26] have studied contacts and selfcontacts between one dimensional structures (in $\mathbb{R}^2$ or $\mathbb{R}^3$) ; see also [27–29].

In a recent work, we have proposed a new modeling of frictionless contacts and selfcontacts between deformable bodies [30], that seems physically relevant, in the static case since $2m \geq n$ (where m is the dimension of the body, and n of the space in which it moves). A topological constraint is introduced in order to prevent transversal (self)intersections.

The aim of this article is to present a numerical implementation of the modelling we introduced in [30] by a penalization method in dimension $n = 2$. Let us underline that this approach is radically different from the classical master-slave one. It provides a robust way to treat contacts and selfcontacts for thin structures. Nevertheless, the robustness has a cost, and our algorithm does not pretend to compete with standard contact algorithms. The purpose of this article is more to bring new insights to the nature of frictionless contacts, rather than propose an efficient and fast contact algorithm. The mathematical setting of the problem is presented in section 2, where the set of admissible deformations is defined (section 2.1), and its main properties are discussed (section 2.2). In a second step, we define several penalization functions of the self-intersecting deformations (section 3). The energy minimization problem of an hyperelastic body over the admissible set is studied in section 4. Finally, two numerical examples are presented (section 5). In the first one, we applied our penalization method in order to project any deformation onto the set of admissible deformations. The second one focuses on the study of a physical system, made of two inflated elastic balloons.

2. The set of admissible deformations

Let M be a disjoint union of sets, diffeomorphic either to the closed interval $[0, 1]$, or to the unit circle S^1 of $\mathbb{R}^2$. Any deformation of M, that is any map from M into $\mathbb{R}^2$, is not physically reasonable, as the one represented in Figure 1 (where M is the union of two intervals), that has two intersections. A difficult problem relies on the characterization of the physically admissible deformations, and is particularly challenging when thin structures are involved as in the present case. In a former work [30], we have provided a partial answer to this question. A topological constraint is introduced in order to define a set of admissible deformations. The set thus defined satisfies several interesting properties that make it a set of physically acceptable deformations. In section 2.1, we recall the definition of this set of admissible deformations, whereas its main properties are recalled in section 2.2.

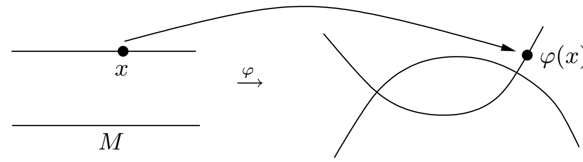


Fig. 1. nonphysically reasonable deformation

2.1. *Definition of the set of admissible deformations*

We first introduce the vector field Φ of $\mathbb{R}^2_* = \mathbb{R}^2 \setminus (0, 0)$ defined by

$$\Phi(x_1, x_2) = \frac{1}{(x_1^2 + x_2^2)} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

The vector field Φ is irrotational. Nevertheless, it is not the gradient of a map. Indeed, if n is the outward unit normal to the boundary of the unit ball, $\int_{S^1} n \wedge \Phi ds$ is not equal to zero, but to 2π. Note that if w is a differentiable map with values in $\mathbb{R}^2_*$ of the form $w(t) = r(t)(\cos(\theta(t)), \sin(\theta(t)))$, we have $\theta'(t) = \Phi(w(t)) \cdot w'(t)$.

Let $j_{_M} : M \to \mathbb{R}^2$ be the reference injection of M into $\mathbb{R}^2$. We assume that $j_{_M} : M \to \mathbb{R}^2$ is an embedding, that is, a $\mathcal{C}^1$ one-to-one mapping with nonzero gradient everywhere.

Let φ be a regular deformation of M. We introduce the map from $M \times M$ into $\mathbb{R}^2$ defined by

$$r_\varphi(x_1, x_2) = \varphi(x_1) - \varphi(x_2),$$

for any (x_1, x_2) in $M \times M$. We say that a deformation is admissible if and only if for any open set U of $\mathrm{int}(M) \times \mathrm{int}(M)$ satisfying

$$\inf_{x \in U} |r_\varphi(x)| > 0, \tag{2}$$

there exists a map $u : U \to \mathbb{R}$ such that

$$\Phi_\varphi(x) = \nabla u(x), \text{ for all } x \in U, \tag{3}$$

where Φ_φ is the vector field of $\mathbb{R}^2$ defined on U by

$$\Phi_\varphi = \nabla r_\varphi(\Phi(r_\varphi)) - \nabla r_{j_M}(\Phi(r_{j_M})).$$

We denote by $\mathcal{A}(j_{_M})$ the set of admissible deformations.

2.2. *Main properties of the set of admissible deformations*

At first glance, it is far from being obvious that criterion (3) defines a reasonable set of deformations. In the following sections, we establish or state several properties of the set $\mathcal{A}(j_{_M})$, which allow us to answer positively to this question. In Section 2.2.1, we prove that any deformation, isotopic to the reference injection, belongs to the set of admissible deformations, and in Section 2.2.2 that any deformation that has transversal selfintersections is not admissible. Section 2.2.3 is devoted to an extension of criterion (3) to any continuous deformation.

2.2.1. *Deformations isotopic to the reference injection*

A deformation φ is said to be isotopic to $j_{_M}$, if there exists a map $F \in \mathcal{C}^1([0,1] \times M; \mathbb{R}^2)$ such that

$$F(0,x) = j(x), \qquad F(1,x) = \varphi(x), \text{ for all } x \in M,$$

and such that the map $x \mapsto F(t,x)$ is an embedding for any $t \in [0,1]$.

Proposition 1 *Any deformation isotopic to the reference embedding $j_{_M}$ belongs to the set of admissible deformations $\mathcal{A}(j_{_M})$.*

PROOF. Let φ be a deformation isotopic to the reference injection.

Let U be an open set of $\mathrm{int}(M) \times \mathrm{int}(M)$ such that

$$\inf_{x \in U} |r_\varphi(x)| > 0.$$

For any regular map G:

$$\begin{aligned} G : [0,1] \times U &\rightarrow \mathbb{R}^2_* \\ (t, x_1, x_2) &\mapsto G(t, x_1, x_2), \end{aligned}$$

we have

$$\frac{d}{dt}(\nabla G(\Phi(G))) = \nabla\left(\frac{dG}{dt} \cdot \phi(G)\right) + \nabla G((\nabla \Phi^T - \nabla \Phi) \circ G) \frac{dG}{dt}.$$

Since the curl of Φ is equal to zero, we deduce, applying the former equation to $G(t, x_1, x_2) = F(t, x_1) - F(t, x_2)$, that

$$\begin{aligned} \nabla r_\varphi(\Phi(r_\varphi)) - \nabla r_j(\Phi(r_j)) &= \int_0^1 \frac{d}{dt}(\nabla G \Phi(G)) dt \\ &= \int_0^1 \nabla\left(\frac{dG}{dt} \cdot \Phi(G)\right) dt \\ &= \nabla\left(\int_0^1 \frac{dG}{dt} \cdot \Phi(G) dt\right). \end{aligned} \qquad (4)$$

2.2.2. *Deformations with transversal selfintersections*

Let us first remark that for any deformation φ, the vector field Φ_φ is irrotational on the set of elements $x \in M \times M$, $r_\varphi(x) \neq 0$. Indeed, we have

$$\nabla \wedge (\nabla r_\varphi \Phi(r_\varphi)) = \frac{\partial r_\varphi}{\partial x_2} \cdot (\nabla \Phi - \nabla \Phi^T)(r_\varphi) \frac{\partial r_\varphi}{\partial x_1}.$$

As $\nabla \wedge \Phi = 0$, we have $\nabla \Phi - \nabla \Phi^T = 0$ and $\nabla \wedge (\nabla r_\varphi \Phi(r_\varphi)) = 0$. Given this result, we cannot deduce that Φ_φ is a gradient, since the set of elements x such that $r_\varphi(x) \neq 0$ is not necessary simply connected. Criterion (3) is not void, as we will prove now.

We say that a deformation φ has a transversal selfintersection at $(x_1, x_2) \in M \times M$, if $\varphi(x_1) = \varphi(x_2)$, and if $(\dot\varphi(x_1), \dot\varphi(x_2))$ is a basis of $\mathbb{R}^2$. Such a deformation (as in Figure 1) is not physically reasonable and od not belong to the set $\mathcal{A}(j_{_M})$.

Proposition 2 *If φ is a regular deformation that has a transversal selfintersection, then it does not belong to the set of admissible deformations $\mathcal{A}(j_{_M})$.*

PROOF. Let φ be a deformation that has a transversal selfintersection at (x_1, x_2). The map r_φ is such that $\nabla r_\varphi(x_1, x_2)$ is of maximal rank. From the Implicit Map Theorem, it follows that there exists a neighborhood V of (x_1, x_2), and a small positive real r such that r_φ defines a diffeomorphism from V into the ball of $\mathbb{R}^2$ centered at 0 of radius r. A simple change of variables leads to

$$\int_{\partial V} \nabla r_\varphi(\Phi(r_\varphi)) \cdot ds = \int_{|x|=r} \Phi(s) \cdot ds = \mathrm{sign}(\det(\nabla r_\varphi)).$$

Moreover, note that for any deformation φ, the field $\nabla r_\varphi(\Phi(r_\varphi))$ is irrotational. In particular,

$$\int_{\partial V} \nabla r_j(\Phi(r_j)) \cdot ds = -\int_V \nabla \wedge (\nabla r_j(\Phi(r_j))) dx = 0.$$

On the other hand, for any map u of $V \setminus (x_1, x_2)$, we have

$$\int_{\partial V} \nabla u \cdot ds = 0.$$

Thus,

$$\begin{aligned} \int_{\partial V} \Phi_\varphi \cdot ds &= \int_{\partial V} (\nabla r_\varphi(\Phi(r_\varphi)) - \nabla r_j(\Phi(r_j)) \cdot ds \\ &\neq \int_{\partial V} \nabla u \cdot ds, \end{aligned}$$

and φ does not belong to the set of admissible deformations.

Let us remark that if all intersections of φ are transversal, then $r_\varphi^{-1}(0) \setminus \Delta(M)$ is a finite set of points, where

$$\Delta(M) = \{(x_1, x_2) \in M \times M \ : \ x_1 = x_2\}. \qquad (5)$$

Furthermore, any element of $r_\varphi^{-1}(0) \setminus \Delta(M)$ can be endowed with a sign determined by the determinant of the gradient of r_φ.

2.2.3. *Extension of the definition to continuous deformations*

The definition of the set of admissible deformations $\mathcal{A}(j_{_M})$ can be extended to continuous deformations, using the following stability property.

Lemma 3 *Let φ be a regular deformation of M. Let U be an open set of* $\mathrm{int}(M) \times \mathrm{int}(M)$ *such that*

$$\inf_{x \in U} |r_\varphi(x)| > \delta > 0.$$

Then for any regular deformation $\widetilde{\varphi}$ satisfying

$$\|\varphi - \widetilde{\varphi}\|_\infty < \delta,$$

there exists u such that

$$\Phi_\varphi - \Phi_{\widetilde{\varphi}} = \nabla u.$$

PROOF. One only has to set $\varphi_t = (1-t)\varphi + t\widetilde{\varphi}$. From (4), we have

$$\Phi_\varphi - \Phi_{\widetilde{\varphi}} = \nabla \left(\int_0^1 \frac{d\varphi_t}{dt} \cdot \Phi \circ \varphi_t dt \right).$$

We deduce from this lemma that, for a given U, the criterion (3) is stable under $\mathcal{C}^0$ perturbations of φ. Therefore, a continuous deformation will be said admissible if and only if, for any open set U of $\mathrm{int}(M) \times \mathrm{int}(M)$ satisfying

$$\inf_{x \in U} |r_\varphi(x)| > \delta > 0,$$

and any deformation $\tilde{\varphi} \in \mathcal{C}^1(M; \mathbb{R}^2)$ close enough to φ, there exists a map u from U into $\mathbb{R}$ such that

$$\Phi_{\tilde{\varphi}}(x) = \nabla u(x) \text{ for all } x \in U.$$

From the stability property, it suffices to verify this condition for any deformation $\tilde{\varphi}$ provided that $\|\varphi - \tilde{\varphi}\|_{L^\infty} < \delta/2$.

2.2.4. *Closure and density*

Let us point out two important properties, whose proofs can be found in [30]. One is a closure property, the other a partial reciprocal to Proposition 1.

Proposition 4 *The set of admissible deformations $\mathcal{A}(j_M)$ is closed for the $\mathcal{C}^0$ topology.*

Proposition 5 *Every regular immersion belonging to $\mathcal{A}(j_M)$ belongs to the $\mathcal{C}^1$-closure of the set of embeddings isotopic to the reference injection j_M.*

3. Penalization

The purpose of this section is to introduce several penalizations of the nonadmissible deformations. In a first step, we perform a heuristic study in order to estimate the distance of a deformation exhibiting an elementary intersection, as the one represented in Figure 1, to the set of admissible deformations. This study is extended to the general case in Section 3.2. Nevertheless, the penalization function thus introduced presents several drawbacks. In particular, it is not differentiable, forbidding the use of classical gradient methods to minimize it. This leads us to introduce two families of penalization functions, whose definitions, however, rely on the same principle (Section 3.4). In the following any deformation considered is assumed to belong at least to $W^{1,1}(M; \mathbb{R}^2)$.

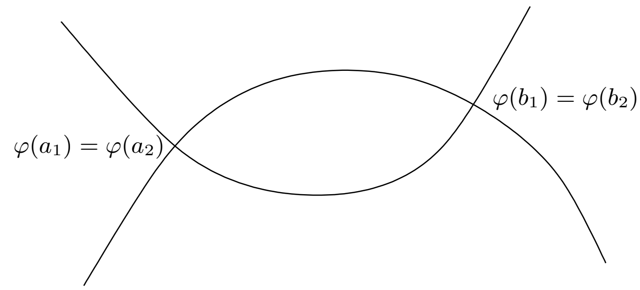


Fig. 2. An elementary intersection

3.1. *A heuristic study*

We consider the case of a system M made of two disjoint intervals, I_1 and I_2, both diffeomorphic to the closed interval $[0,1]$, and the deformation φ represented in Figure 2. This last deformation has two transversal intersections. One at $a = (a_1, a_2) \in I_1 \times I_2$, the other at $b = (b_1, b_2) \in I_1 \times I_2$. In other words,

$$\varphi(a_1) = \varphi(a_2), \qquad \varphi(b_1) = \varphi(b_2),$$

and both families $(\dot{\varphi}(a_1), \dot{\varphi}(a_2))$ and $(\dot{\varphi}(b_1), \dot{\varphi}(b_2))$ are basis of $\mathbb{R}^2$.

We try to attemp an admissible deformation ψ that is close to φ. To this end, we are going to define a family of admissible deformations $\psi_{u,\gamma}$, that depend on two parameters u and γ. The deformation we are looking for ψ will be obtained by minimizing the distance between $\psi_{u,\gamma}$ and φ with respect to γ and u.

Without loss of generality, we can assume that $a_1 < b_1$, and $a_2 < b_2$. The deformation $\psi_{u,\gamma}$ is obtained by gluing $\varphi([a_1, a_2])$, and $\varphi([b_1, b_2])$ along the same curve $u : [0,1] \to \mathbb{R}^2$. Thus $\psi_{u,\gamma}$ is such that $\psi_{u,\gamma}(x) = \varphi(x)$ for all $x \notin [a_1, a_2] \cup [b_1, b_2]$, and

$$\psi_{u,\gamma}([a_1, a_2]) = \psi_{u,\gamma}([b_1, b_2]) = u([0,1]).$$

Moreover u is chosen to satisfy

$$u(0) = \varphi(a_1) = \varphi(a_2), \quad \text{and} \quad u(1) = \varphi(b_1) = \varphi(b_2),$$

so as to ensure the continuity of $\psi_{u,\gamma}$. Now, we have yet to specify the definition of $\psi_{u,\gamma}$ on $[a_1, a_2] \cup [b_1, b_2]$, that is, to describe how $\varphi([a_1, a_2])$, and $\varphi([b_1, b_2])$ are stuck back along u. To this end, we introduce a map $\gamma = (\gamma_1, \gamma_2)$ from $[0,1]$ into $M \times M$. We assume that for $i = 1, 2$ each map γ_i is a diffeomorphism from $[0,1]$ into $[a_i, b_i]$. For any $t \in [0,1]$, we set

$$\psi_{u,\gamma}(\gamma_1(t)) = u(t),$$

and

$$\psi_{u,\gamma}(\gamma_2(t)) = u(t).$$

If the deformation u has non intersection, the deformation $\psi_{u,\gamma}$ is admissible (see Figure 3). It remains to minimize the distance between $\psi_{u,\gamma}$ and φ, in order to obtain an estimation of the distance between φ and the set of admissible deformations. We choose to measure this distance with the $W_0^{1,1}$-norm. In a first step, we minimize this distance with respect to u and set:

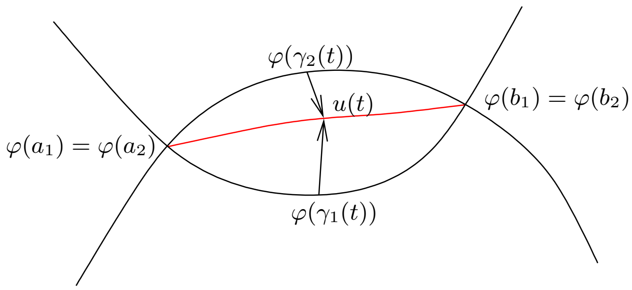


Fig. 3. Elimination of an intersection

$$\mathcal{L}_\varphi(\gamma) = \inf_u \|\dot{\psi}_{u,\gamma} - \dot{\varphi}\|_{L^1}.$$

Remark 6 *Other norms than $W_0^{1,1}$ could be considered. Yet, it seems to be the only choice that leads to an exploitable result.*
The minimum of $\|\dot{\psi}_{u,\gamma} - \dot{\varphi}\|_{L^1}$ with respect to u is obtained for any map u of the form

$$u = \theta\varphi \circ \gamma_1 + (1-\theta)\varphi \circ \gamma_2,$$

with $\theta \in [0,1]$. Hence, we have

$$\mathcal{L}_\varphi(\gamma) = \int_0^1 \left| \frac{\mathrm{d}(\varphi \circ \gamma_1 - \varphi \circ \gamma_2)}{\mathrm{d}\, t} \right| dt.$$

Let us note that $\mathcal{L}_\varphi$ is nothing but the length of the curve $r_\varphi \circ \gamma$, and can therefore be defined for any continuous deformation (in such a case, $\mathcal{L}_\varphi$ is not necessarily finite). We obtain an estimation of the distance of φ to the set of admissible deformation by setting

$$J(\varphi) = \inf_\gamma \mathcal{L}_\varphi(\gamma),$$

where the infimum is take over the set of maps $\gamma : [0,1] \to M \times M$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Thus, $J(\varphi)$ is the distance in $I_1 \times I_2$ between the two points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ for the metric whose the infinitesimal length dl of the vector (dx_1, dx_2) based at (x_1, x_2) is

$$dl = |\dot{\varphi}(x_1)dx_1 - \dot{\varphi}(x_2)dx_2|.$$

It remains to extend this definition to the general case.

3.2. *Definition of a penalization function*

We denote by $C_1(M \times M)$ the set of families each element of which is a continuous map from $[0,1]$ or S^1 into $M \times M$. Let φ be a regular deformation of M (where M is a union of circles and closed intervals) belonging to $W^{1,1}(M; \mathbb{R}^2)$. Let U be an open subset of $\mathrm{int}(M) \times \mathrm{int}(M)$ that satisfies condition (2). We introduce the subset $\Gamma_U(\varphi)$ of $C_1(M \times M)$ defined by

$$\begin{aligned} \Gamma_U(\varphi) = \Big\{ &\gamma \in C_1(M \times M) \ : \ \partial\gamma \subset \complement U, \\ &\int_\gamma (\omega \wedge n) ds = \int_{M \times M} (\omega \wedge \Phi_\varphi) dx, \text{ for all vector fields} \\ &\omega \text{ of } M \times M \text{such that } \nabla \wedge \omega = 0 \\ &\text{whose support is included in } U \Big\}. \end{aligned} \quad (6)$$

We define the function J_U by

$$J_U(\varphi) = \inf_{\gamma \in \Gamma_U(\varphi)} \mathcal{L}_\varphi(\gamma).$$

Finally, the penalization function J is obtained by taking the supremum of J_U over the open set U that fulfils (2):

$$J(\varphi) = \sup_U J_U(\varphi).$$

Let us mention that if U and V are two open set that satisfy (2) with $U \subset V$, then $J_U(\varphi) \le J_V(\varphi)$. It follows that J can also be defined by

$$J(\varphi) = \lim_{\delta \to 0^+} J_{U_\delta(\varphi)}(\varphi),$$

where

$$U_\delta(\varphi) = \big\{ (x_1, x_2) \in \mathrm{int}(M) \times \mathrm{int}(M) \ : \ |\varphi(x_1) - \varphi(x_2)| > \delta \big\}. \quad (7)$$

The definition of J can be extended to continuous deformations, using the following lemma.
Lemma 7 *Let φ be a deformation of M. Let U be an open set of $\mathrm{int}(M) \times \mathrm{int}(M)$ such that*

$$\inf_{x \in U} |r_\varphi(x)| > \delta > 0.$$

Then for any deformation $\widetilde{\varphi}$ of M such that

$$\|\varphi - \widetilde{\varphi}\| < \delta,$$

we have $\Gamma_U(\varphi) = \Gamma_U(\widetilde{\varphi})$.

PROOF. Arguing as in (4), we get that for any $x \in U$,

$$(\Phi_{\tilde{\varphi}} - \Phi_\varphi)(x) = \nabla v(x),$$

where

$$v = \int_0^1 \frac{dr_{\varphi_t}}{dt} \cdot \Phi(r_{\varphi_t}) dt.$$

Thus, for any irrotational vector field ω with support included in U, we have

$$\begin{aligned} \int_{M \times M} (\omega \wedge \Phi_{\tilde{\varphi}}) - (\omega \wedge \Phi_\varphi) dx &= \int_U (\omega \wedge \nabla v) dx \\ &= -\int_U \nabla \wedge (v\omega) dx \\ &= \int_{\partial U} v(n \wedge \omega) ds = 0. \end{aligned}$$

For any continuous deformation φ, and for any open set U that satisfies condition (2), it suffices to set

$$J_U(\varphi) = \inf_{\gamma \in \Gamma_U(\widetilde{\varphi})} \mathcal{L}_\varphi(\gamma),$$

with $\widetilde{\varphi}$ close enough to φ. From the previous stability properties, $\Gamma_U(\widetilde{\varphi})$ is independent from the choice of $\widetilde{\varphi}$, and $J_U(\varphi)$ is correctly defined.

In order to evaluate $J(\varphi)$, we have to explicitly characterize the set $\Gamma_U(\varphi)$. It will allow us to check that the given penalization is indeed a generalization of the one introduced during our heuristic study.

3.3. *Computation of the penalization function J*

The purpose of this section is to compute $J(\varphi)$. In a first step, we consider the case of admissible deformations, for which we prove that $J(\varphi)$ is equal to zero. Then, we consider the case of any deformation, but we restrict our analysis to a system M strictly composed of closed intervals or $M = S^1$. The general case, where M is composed of circles and closed intervals is not addressed, as the result is quite drawn-out and do not bring any new elements.

3.3.1. *Case of an admissible deformation*

Let us consider an admissible deformation $\varphi \in W^{1,1}(M;\mathbb{R}^2)$. For any open set U of $\mathrm{int}(M) \times \mathrm{int}(M)$ satisfying (2), there exists a map $u : U \to \mathbb{R}$ such that

$$\Phi_\varphi(x) = \nabla u(x) \text{ for all } x \in U.$$

If ω is a irrotational vector field, with support included in U, we have

$$\begin{aligned}\int_{M\times M} \omega \wedge \Phi_\varphi dx &= \int_U \omega \wedge \nabla u dx = -\int_U \nabla \wedge (u\omega) dx \\ &= \int_{\partial U} u(n \wedge \omega) ds = 0.\end{aligned}$$

The empty family belongs to $\Gamma_U(\varphi)$ and $J(\varphi) = J_U(\varphi) = 0$. We have proved that

Proposition 8 *If φ belongs to the set of admissible deformations $\mathcal{A}(j)$, then $J(\varphi) = 0$.*

3.3.2. *Case of a solid M with trivial topology*

In order to evaluate $J(\varphi)$, we have to determine explicitly $\Gamma_U(\varphi)$. This depends strongly on the topology of M. The simpler case consists in considering a system M whose each connected component is simple, that is, M is the union of closed intervals. The case $M = S^1$ will be addressed in the next section.

Since the set of deformations that have transversal selfintersections is dense, by Lemma 7, we can assume, without loss of generality, that the selfintersections of M are transversal. If M is the union of closed intervals, then $\Gamma_U(\varphi)$ is determined by the following proposition.

Proposition 9 *Let us assume that M is the disjoint union of closed intervals. Let φ be a regular deformation of M whose intersections are transversal, and let U be an open subset of $\mathrm{int}(M) \times \mathrm{int}(M)$ such that*

$$\inf_{x\in U} |r_\varphi(x)| > 0.$$

Then $\gamma \in \Gamma_U(\varphi)$ if and only if $\partial\gamma \subset \complement U$, and if for any connected component C of the complementary set of U whose intersection with $\partial(M \times M) \cup \Delta(M)$ is empty, we have

$$\sum_{x\in r_\varphi^{-1}(0)\cap C} s(x) = \mathrm{Card}(\gamma(1) \cap C) - \mathrm{Card}(\gamma(0) \cap C),$$

where $s(x) = \mathrm{sign}(\det(\dot\varphi(x_1), \dot\varphi(x_2)))$.

PROOF. Since φ has only transversal selfintersections, the set $r_\varphi^{-1}(0) \setminus \Delta(M)$ is a finite union of points endowed with a sign (see Section 2.2.2). In order to determine $\Gamma_U(\varphi)$, we first compute for any irrotational field ω, with support included in U, the integral $\int_{M\times M}(\omega \wedge \Phi_\varphi)dx$. In a second step, we compute $\int_\gamma(\omega \wedge n)ds$ for any element γ of $C_1(M \times M)$ whose boundary is included in the complementary set of U. Finally, the two results are compared in order to identify $\Gamma_U(\varphi)$.

For any element x of $r_\varphi^{-1}(0) \setminus \Delta(M)$, there exists a small ball $B(x)$ centered at x included in the complementary set of U. We set

$$V = M \times M \setminus \left(\Delta(M) \cup \bigcup_{x\in r_\varphi^{-1}(0)\setminus\Delta(M)} B(x) \right).$$

Let ω be a irrotational vector field of $M \times M$ whose support is included in U. Since $M \times M$ is the union of simply connected sets, the vector field ω is a gradient. Hence, there exists a map $\alpha : M \times M \to \mathbb{R}$ such that $\omega = \nabla\alpha$. Moreover, α can be chosen such that α is zero on $\partial(M \times M) \cup \Delta(M)$. We infer that

$$\begin{aligned}\int_{M\times M}(\omega \wedge \Phi_\varphi)dx &= \int_V \nabla\alpha \wedge \Phi_\varphi dx = \int_V \nabla \wedge (\alpha\Phi_\varphi)dx \\ &= \int_{\partial V} \alpha(\Phi_\varphi \wedge n)ds = \int_{\partial V\setminus\Delta(M)} \alpha(\Phi_\varphi \wedge n)ds.\end{aligned}$$

On each ball $B(x)$, we have $\nabla\alpha = \omega = 0$. Thus, α is constant on each of these balls, and

$$\int_{M\times M}(\omega \wedge \Phi_\varphi)dx = \sum_{x\in r_\varphi^{-1}\setminus\Delta(M)} \alpha(x) \int_{\partial B(x)} (\Phi_\varphi \wedge n)ds.$$

As shown in Section 2.2.2, we have

$$\int_{\partial B(x)} (\Phi_\varphi \wedge n)ds = s(x),$$

where $s(x) = \mathrm{sign}(\det(\nabla r_\varphi(x_1, x_2)))$. We have proved that

$$\int_{M\times M}(\omega \wedge \Phi_\varphi)dx = \sum_{x\in r_\varphi^{-1}(0)\setminus\Delta(M)} \alpha(x_1, x_2)s(x).$$

Let us consider now an element γ of $C_1(M \times M)$ whose boundary is included in the complementary set of U. We have $\gamma = (\gamma_j)_{j\in J_1\cup J_2}$, where γ_j are continuous maps from $[0, 1]$ into $M \times M$, if $j \in J_1$, and from S^1 into $M \times M$, if $j \in J_2$. We set $\gamma(0) = \cup_{j\in J_1}\gamma_j(0)$, and $\gamma(1) = \cup_{j\in J_1}\gamma_j(1)$. We have

$$\int_\gamma (\omega \wedge n)ds = \int_\gamma \nabla\alpha \cdot ds = \sum_{x\in\gamma(1)} \alpha(x) - \sum_{x\in\gamma(0)} \alpha(x). \quad (8)$$

Thus, γ belongs to $\Gamma_U(\varphi)$ if and only if, for any map α of $M\times M$ with values in $\mathbb{R}$ such that $\nabla\alpha = 0$ on $\complement U$ and $\alpha = 0$ on $\partial(M \times M) \cup \Delta(M)$, we have

$$\sum_{x\in r_\varphi^{-1}(0)\setminus\Delta(M)} \alpha(x)s(x) = \sum_{x\in\gamma(1)} \alpha(x) - \sum_{x\in\gamma(0)} \alpha(x).$$

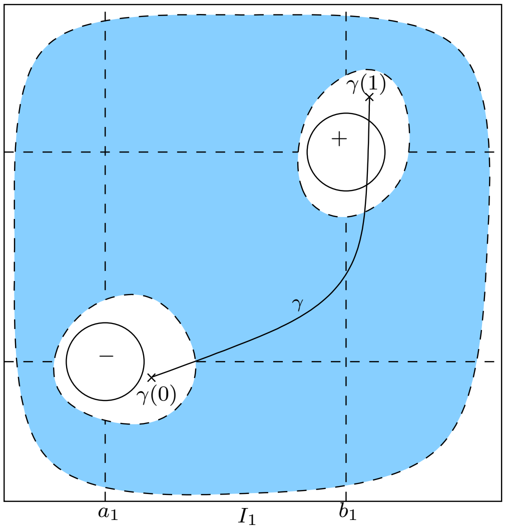


Fig. 4. An element of $\Gamma_U(\varphi)$

In other words, γ belongs to $\Gamma_U(\varphi)$ if and only if

$$\sum_{x \in r_\varphi^{-1}(0) \cap C} s(x) = \mathrm{Card}(\gamma(1) \cap C) - \mathrm{Card}(\gamma(0) \cap C),$$

for any connected component C of the complementary set of U whose intersection with $\partial(M \times M) \cup \Delta(M)$ is empty. Figure 4 represents an element of $\Gamma_U(\varphi)$ where φ is the deformation represented in Figure 2. The balls $B(x)$ and the open set U have also been drawn. We have represented only the connected component $I_1 \times I_2$ of $M \times M$ among the four it has.

The determination of $\Gamma_U(\varphi)$ allows us to give a simpler expression of $J(\varphi)$, at least when φ has only transversal selfintersections. In such a case,

$$J(\varphi) = \inf_{\gamma \in \Gamma(\varphi)} \mathcal{L}_\varphi(\gamma),$$

where

$$\Gamma(\varphi) = \Big\{ \gamma \in C_1(M \times M) \ : \\ \partial\gamma \cap \mathrm{int}(M \times M \setminus \Delta(M)) = r_\varphi^{-1}(0) \setminus \Delta(M) \Big\}. \quad (9)$$

Let us remark that $\partial\gamma$ is a finite set of points which can be endowed with an orientation: positive for the elements of $\gamma(1)$, negative for the elements of $\gamma(0)$. In the previous definition, the equation $\partial\gamma \cap \mathrm{int}(M \times M \setminus \Delta(M)) = r_\varphi^{-1}(0) \setminus \Delta(M)$ means equality of both sets, but also equality of the signs affected to each element.

The derived expression coincides with the one introduced in our heuristic study. Moreover, it ensures that if φ is regular and has only transversal selfintersections, then $J(\varphi)$ is finite. Thus, the penalization J is not trivial (zero on admissible deformations, nonfinite on the nonadmissible ones), and is continuous for the $W^{1,1}-$topology on the set of deformations with transversal selfintersections.

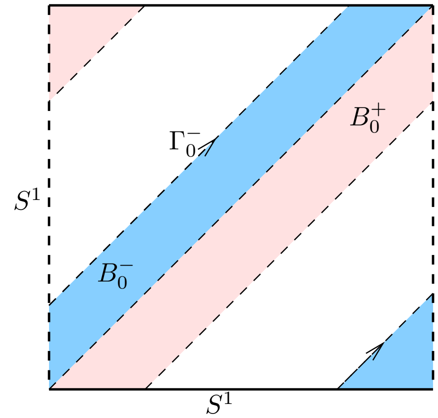


Fig. 5. Subsets B_0^- and B_0^+ of $S^1 \times S^1$

3.3.3. *Case of the circle*

In the case $M = S^1$, a reasoning similar to the one performed in the previous section can be followed. However, the characterization of $\Gamma_U(\varphi)$ is a little more delicate, and requires the introduction of further definitions. Let us first remark, that from the stability property of $\Gamma_U(\varphi)$ with respect to φ (Lemma 7) it suffices to consider the immersion φ with transversal selfintersections. Indeed, their constitute a dense subset of continuous maps. For such deformations, there exists a small positive real ε such that for any $s \in S^1$, the restriction of φ to $]s-\varepsilon, s+\varepsilon[$ is injective. We denote by B_0 the compact neighborhood of the diagonal of $S^1 \times S^1$ defined by

$$B_0 = \{(x_1, x_2) \in S^1 \times S^1 \ : \ |x_1 - x_2| \leq \varepsilon/2\}.$$

The set $B_0 \setminus \Delta(S^1)$ has two connected components denoted by

$$B_0^+ = \{(x_1, x_2) \in S^1 \times S^1 \ : \ \varepsilon/2 \leq x_1 - x_2 < 0\},$$

and

$$B_0^- = \{(x_1, x_2) \in S^1 \times S^1 \ : \ \varepsilon/2 \leq x_2 - x_1 < 0\}.$$

Let Γ_0^- be the connected component of the boundary of B_0^- not equal to $\Delta(S^1)$, that is

$$\Gamma_0^- = \{(x_1, x_2) \in S^1 \times S^1 \ : \ \varepsilon/2 = x_2 - x_1\}.$$

The sets $B_0^\pm$ are represented on Figure 5. We call the turning number of φ the integer

$$\sharp\varphi = (2\pi)^{-1} \int_{\Gamma_0^-} (\nabla r_\varphi(\Phi(r_\varphi)) \wedge n) ds.$$

Remark 10 *The integer $\sharp\varphi$ is nothing else but the number of turns performed by the tangent to φ and does not depend on ε (provided ε is small enough).*

Figure 6 represents immersions of S^1 with turning numbers equal to one or two.

Proposition 11 *Let φ be a regular deformation of S^1 that has only transversal selfintersections, and U an open subset of $S^1 \times S^1$ such that*

$$\inf_{x \in U} |r_\varphi(x)| > 0.$$

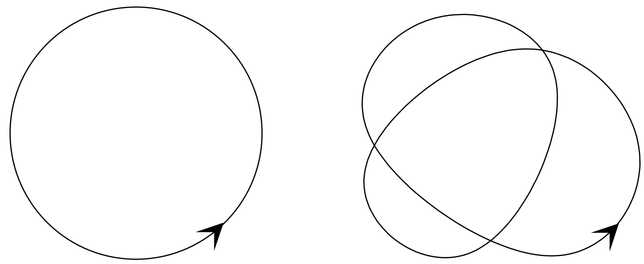


Fig. 6. Immersions of the circle with turning number equal to one and two

Let γ be an element of $C_1(M \times M)$ whose intersections with the diagonal $\Delta(S^1)$ are transversal, then γ belongs to $\Gamma_U(\varphi)$ if and only if

– $\partial\gamma \subset \complement U$,

– *for each connected component C of the complementary set of U having no intersection with $\Delta(S^1)$, we have*

$$\sum_{x \in r_\varphi^{-1}(0) \cap C} s(x) = \mathrm{Card}(\gamma(1) \cap C) - \mathrm{Card}(\gamma(0) \cap C),$$

where $s(x) = \mathrm{sign}(\det(\dot\varphi(x_1), \dot\varphi(x_2)))$,

– *in the case the connected component C_0 of $\complement U \setminus \Delta(S^1)$ containing B_0^- does not contain B_0^+, we have*

$$\sharp\varphi - \sharp j_M + \sum_{x \in r_\varphi^{-1}(0) \cap C_0} s(x) = \mathrm{Card}(\gamma(1) \cap C_0) - \mathrm{Card}(\gamma(0) \cap C_0) + \sum_{\gamma(t) \in \Delta(S^1)} \mathrm{sign}\left(\det\left(\dot\gamma(t), \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\right).$$

Remark 12 *The assumption that the intersections between γ and $\Delta(S^1)$ are transversal is not an important restriction. In fact, every element of $\Gamma_U(\varphi)$ can be approximate for the $\mathcal{C}^0-$topology by regular maps having transversal intersection with the diagonal $\Delta(S^1)$. Thus, Proposition 11 can be applied to any element of $C_1(S^1 \times S^1)$ after a small perturbation.*

PROOF. To each element x of $K = r_\varphi^{-1}(0) \setminus \Delta(S^1)$ we associate a small ball $B(x)$ centred at x and included in the complementary set of U. We set

$$V = S^1 \times S^1 \setminus \left(B_0 \cup \bigcup_{x \in K} B(x)\right).$$

Let ω be a irrotational vector field of $S^1 \times S^1$ whose support is included in U. As ω is zero on $\Delta(S^1)$ and irrotational, it follows that there exists a function α from $S^1 \times S^1 \setminus \Delta(S^1)$ into $\mathbb{R}$ such that $\omega = \nabla\alpha$. Moreover, we can choose α so that the restriction of α to B_0^+ is zero. We have

$$\int_{S^1 \times S^1} (\omega \wedge \Phi_\varphi) dx = \int_{\partial V} \alpha(\Phi_\varphi \wedge n) ds$$
$$= \alpha(B_0^-) \int_{\Gamma_0^-} (\Phi_\varphi \wedge n) ds + \sum_{x \in K} \alpha(x) \int_{\partial B(x)} (\Phi_\varphi \wedge n) ds$$
$$= \alpha(B_0^-)(\sharp\varphi - \sharp j_M) + \sum_{x \in K} \alpha(x) s(x).$$

If γ is an element of $C_1(S^1 \times S^1)$ such that $\partial\gamma \in \complement U$, valued in $S^1 \times S^1 \setminus \Delta(S^1)$, we have

$$\int_\gamma (\omega \wedge n) ds = \sum_{x \in \gamma(1)} \alpha(x) - \sum_{x \in \gamma(0)} \alpha(x).$$

It follows that an element of $C_1(S^1 \times S^1)$ belongs to $\Gamma_U(\varphi)$ if and only, for any connected component C of the complementary set of U that does not contain $\Delta(S^1)$, we have

$$\sum_{x \in r_\varphi^{-1}(0) \cap C} s(x) = \mathrm{Card}(\gamma(1) \cap C) - \mathrm{Card}(\gamma(0) \cap C),$$

and if the connected component C_0 of $\complement U \setminus \Delta(S^1)$ containing B_0^- does not contain B_0^+, we have

$$\sharp\varphi - \sharp j_M + \sum_{x \in K_0} s(x) = \mathrm{Card}(\gamma(1) \cap C_0) - \mathrm{Card}(\gamma(0) \cap C_0),$$

where $K_0 = r_\varphi^{-1}(0) \cap C_0$. It remains to consider the case of an element γ of $C_1(S^1 \times S^1)$ such that $\partial\gamma \in \complement U$ that intersects the diagonal $\Delta(S^1)$ of $S^1 \times S^1$. In order to determine $\int_\gamma (\omega \wedge n) ds$, we can come down to the previous case. First of all, we can assume the intersection between γ and $\Delta(S^1)$ to be transversal, even if it implies perturbing slightly γ. Every curve γ_i that intersects the diagonal $\Delta(S^1)$ can be split into several curves $\widetilde{\gamma}_j : [0,1] \to S^1 \times S^1$ such that $\widetilde{\gamma}_j(]0,1[)$ is included in $S^1 \times S^1 \setminus \Delta(S^1)$. Eroding the end of these curves, we obtain a family $\widetilde{\gamma}$ of $C_1(S^1 \times S^1)$ valued in $S^1 \times S^1 \setminus \Delta(S^1)$ such that $\partial\widetilde{\gamma} \in \complement U$. Moreover,

$$\int_\gamma (\omega \wedge n) ds = \int_{\widetilde{\gamma}} (\omega \wedge n) ds.$$

Hence, γ belongs to $\Gamma_U(\varphi)$ if and only if $\widetilde{\gamma}$ belongs to $\Gamma_U(\varphi)$. Finally, for each connected component C of the complementary set of U that does not contain $\Delta(S^1)$, we have

$$\mathrm{Card}(\widetilde{\gamma}(1) \cap C) - \mathrm{Card}(\widetilde{\gamma}(0) \cap C) = \mathrm{Card}(\gamma(1) \cap C) - \mathrm{Card}(\gamma(0) \cap C),$$

and if C_0 is the connected component of $\complement U \setminus \Delta(S^1)$ containing B_0^-, we have

$$\mathrm{Card}(\widetilde{\gamma}(1) \cap C_0) - \mathrm{Card}(\widetilde{\gamma}(0) \cap C_0) = \mathrm{Card}(\gamma(1) \cap C_0) - \mathrm{Card}(\gamma(0) \cap C_0) + \sum_{\gamma(t) \in \Delta(S^1)} \mathrm{sign}\left(\det\left(\dot\gamma(t), \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\right).$$

We deduce that an element γ of $C_1(S^1 \times S^1)$ such that $\partial\gamma \in \complement U$ belongs to $\Gamma_U(\varphi)$ if and only if

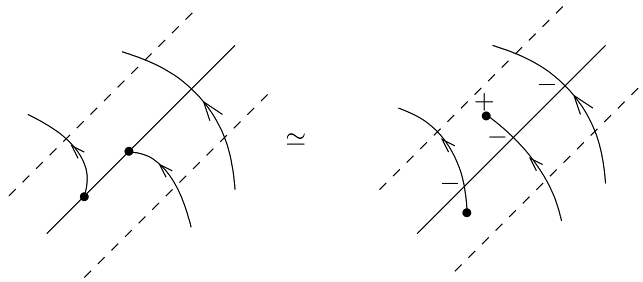


Fig. 7. Computation of $N(\gamma)$

– for each connected component C of the complementary set of U that does not contain $\Delta(S^1)$, we have

$$\sum_{x\in r_\varphi^{-1}(0)\cap C} s(x) = \mathrm{Card}(\gamma(1)\cap C) - \mathrm{Card}(\gamma(0)\cap C)$$

– if the connected component C_0 of $\complement U \setminus \Delta(S^1)$ containing B_0^- does not contain B_0^+, we have

$$\begin{aligned}\sharp\varphi - \sharp j_{_M} + &\sum_{x\in r_\varphi^{-1}(0)\cap C_0} s(x)\\ &= \mathrm{Card}(\gamma(1)\cap C_0) - \mathrm{Card}(\gamma(0)\cap C_0)\\ &\quad + \sum_{\gamma(t)\in\Delta(S^1)} \mathrm{sign}\left(\det\left(\dot\gamma(t), \begin{pmatrix}1\\1\end{pmatrix}\right)\right).\end{aligned}$$

As in the previous section, if φ has only transversal selfintersections, we infer the following expression of $J(\varphi)$

$$J(\varphi) = \inf_{\gamma\in\Gamma(\varphi)} \mathcal{L}_\varphi(\gamma),$$

where

$$\Gamma(\varphi) = \Big\{\gamma \in C_1(S^1\times S^1) \ : \ \sharp\varphi - \sharp j = N(\gamma)$$
$$\text{and } \partial\gamma \cap \mathrm{int}(S^1\times S^1\setminus\Delta(S^1)) = r_\varphi^{-1}(0)\setminus\Delta(S^1)\Big\}, \quad (10)$$

and

$$\begin{aligned}N(\gamma) = \mathrm{Card}(\gamma(1)\cap B_0^-) &- \mathrm{Card}(\gamma(0)\cap B_0^-)\\ &+ \sum_{\gamma(t)\in\Delta(S^1)} \mathrm{sign}\left(\det\left(\dot\gamma(t), \begin{pmatrix}1\\1\end{pmatrix}\right)\right).\end{aligned}$$

Remark 13 *The definition of $N(\gamma)$ is correct only if the intersections between γ and $\Delta(S^1)$ are transversal. Nevertheless, its value is stable under small $\mathcal{C}^0-$perturbations of γ. Thus, the map $N(.)$ can be uniquely extended to any element γ of $C_1(S^1\times S^1)$ such that $\partial\gamma \subset r_\varphi^{-1}(0)$. The computation of $N(\gamma)$ is illustrated by Figure 7, where we have represented an element γ and a perturbation of it having transversal intersections with $\Delta(S^1)$. We have also indicated the sign of each contribution to $N(\gamma)$. In this example, $N(\gamma) = -2$. Only a part of $\Delta(S^1)$ has been drawn.*

3.3.4. *The other cases...*

If the introduction of maps from S^1 into $M\times M$ in the definition of $C_1(M\times M)$ is useless in the cases studied up to now, it turns out to be necessary in more complex situations as, for instance, when M is made of two disjoint circles. The complete computation of $\Gamma_U(\varphi)$ in the general case will illustrate the particular part played by maps from S^1 into $M\times M$ in the definition of $C_1(M\times M)$. Unfortunately, the result is rather technical to state, and does not bring any enlightenment to our penalization method. We prefer to illustrate in one example the necessity of the introduction of maps from S^1 into $M\times M$ in the definition of $C_1(M\times M)$ for deriving a correct penalization of the non admissible deformations.

Assume that M is made of the two disjoint circles $M_1 = S^1$ and $M_2 = S^1 + (0,2)$. Let φ be the deformation of M whose restriction to M_1 is an homothecy of coefficient $1/2$, and whose restriction to M_2 is the translation by vector $-(0,2)$. Note that $U = M_1\times M_2$ is an open that satisfies condition (2). In addition, any family of curves γ from $[0,1]$ into $M_1\times M_2$ such that $\partial\gamma\subset\partial U$ is empty. If we restrict the definition of $C_1(M\times M)$ to this kind of set then $J(\varphi) = +\infty$, which is not satisfactory. On the other hand, one can check that for any element a of M_2, the curve $\gamma = (-M_1)\times a$ belongs to $\Gamma_U(\varphi)$, so that $J(\varphi)$ is actually finite.

3.4. *Two penalization families*

The penalization function J introduced in the previous sections bears several drawbacks. In particular, even if it is continuous on the set of regular deformations with transversal selfintersections, it is not differentiable. Obviously, it is bit of an handicap from the numerical point of view. Hence, it can be interesting to seek for other penalization functions, that does not suffer from this limitation. Ideally, a penalization function should verify the following properties:

(i) $T(\varphi) = 0$ if and only if $\varphi \in \mathcal{A}(j_{_M})$,
(ii) T is lower semi-continuous for the $\mathcal{C}^0-$topology,
(iii) T is differentiable,
(iv) $T(\varphi)$ is independent under reparametrization of φ.

The initial motivation for our work was to solve minimization problems on the set $\mathcal{A}(j_{_M})$ of admissible deformations. The penalization method consists in solving a sequel of unconstrained minimization problems by adding to the cost function a penalization of the non admissible deformations weighted by a increasing coefficient (see Section4). The first property ensures that the minimizers of the penalized problems converge toward the initial problem as the penalization coefficient goes to infinity. The second one enables us to prove that the minimization problems admit at least on solution. The third property allows us to make use of the usual gradient methods to solve the minimization problems. The last one is more subtle. If we assume that the differential of T belongs to $L^2(M;\mathbb{R}^2)$, this property implies that

$$T'(\varphi)\cdot\psi = \int_M f(s)n(s)\cdot\psi(s)ds, \quad (11)$$

where f is an element of $L^2(M)$, and $n(s)$ is the unitary normal to $\dot{\varphi}(s)$. In other words, the penalization function does not add artificial friction terms. Such a property is of main importance from the practical point of view. Indeed, the numerical resolution of minimization problems by gradient methods only leads to local minima. Moreover, it is not granted that those local minima converge toward a local minimum of the initial problem as the penalization coefficient increases. The least we can ask for is that those local minima converge toward a critical point of the initial problem (that is, the limit satisfies the associated Euler-Lagrange equations). This is ensured, at least formally, by the particular form (11) of the gradient of the penalization.

In the following sections, we introduce two families of penalization functions. The first one is invariant under reparametrization and verifies the first property only for particular configurations of M. If M is neither an union of disjoint closed intervals nor the single circle S^1, there exists nonadmissible deformations cancelling the penalization function. Thus, minimizers of the penalized problems could converge toward a non admissible deformation. Nevertheless, it does not happen in practice, as long as some precautions are taken. In all the numerical tests we performed, we used this type of penalization functions without being confronted with such problems. The second family of penalization functions, called non-degenerate, has precisely been conceived to stave off this drawback. Unfortunately, elements of this family are not invariant under reparametrization. Thus, local minimizers do not, in general, converge toward critical points of the initial problem. Both families are built on the same principle as the function J heuristically obtained. Their definitions only differ in the metric used on $M \times M$ to compute the length of the elements γ of $\Gamma_U(\varphi)$.

In sections 3.4.1 and 3.4.2, we define the families of penalization functions, invariant under reparametrization, and non-degenerate respectively. Finally, we study the different properties of the functional introduced:
The equivalence between cancellation of the penalization and admissibility (section 3.4.3), the invariance under reparametrization (section 3.4.4), the lower semi-continuity (section 3.4.5), the differentiability (section 3.4.6). Complete proofs are only given for the first penalization family. Corresponding propositions for the second family can be proved with similar arguments, and proofs can usually be simplified as some technical passages become unnecessary. Sections 3.4.3 to 3.4.6 are the most technical parts of the article and can be skipped on a first reading.

3.4.1. *Penalization invariant under reparametrization*

In this section, we introduce a family of functions invariant under reparametrization of M. Each of these penalization functions is characterized by a regular map G from $\mathbb{R}^2 \times \mathbb{R}^2$ into the set of 4×4 symmetric positive matrices. Moreover, we assume that there exists a continuous map g from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $g(s) = 0$ if and only if $s = 0$, and

$$G(y_1, y_2) \geq g(|y_1 - y_2|^2)\,\mathrm{Id}, \text{ for all } (y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{12}$$

The map G defines a metric d^G on the space $\mathbb{R}^2 \times \mathbb{R}^2$ where the distance between two elements X and Y of $\mathbb{R}^2 \times \mathbb{R}^2$ is

$$d^G(X, Y) = \inf_\gamma \int_0^1 |\dot{\gamma}^T (G \circ \gamma)\dot{\gamma}|^{1/2} ds,$$

where γ is any regular curve such that $\gamma(0) = X$ and $\gamma(1) = Y$. Note that we authorize G to be equal to zero on $\Delta(\mathbb{R}^2)$. In this case, the metric d^G is degenerate on $\Delta(\mathbb{R}^2)$ as for any y_1 and y_2 in $\mathbb{R}^2$, we have $d^G((y_1, y_1), (y_2, y_2)) = 0$. Let γ be a continuous map from $[0, 1]$ (or S^1) into $\mathbb{R}^2 \times \mathbb{R}^2$, the length of γ for the metric G is defined by

$$\mathcal{L}^G(\gamma) = \sup_{(t)} \sum_i d^G(\gamma(t_{i+1}), \gamma(t_i)),$$

where the supremum is computed over the set of subdivisions $t_0 = 0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$. If γ is a regular map, we have

$$\mathcal{L}^G(\gamma) = \int_0^1 |\dot{\gamma}^T (G \circ \gamma)\dot{\gamma}|^{1/2} ds.$$

For any deformation φ of M, we note $\varphi \times \varphi$ the map

$$\begin{aligned} \varphi \times \varphi : M \times M &\to \mathbb{R}^2 \times \mathbb{R}^2 \\ (x, y) &\mapsto (\varphi(x), \varphi(y)). \end{aligned}$$

For any curve γ of $\mathrm{int}(M) \times \mathrm{int}(M)$, we call $\mathcal{L}^G_\varphi(\gamma)$ the length of the curve $(\varphi \times \varphi) \circ \gamma$ for the metric G.

$$\mathcal{L}^G_\varphi(\gamma) = \mathcal{L}^G((\varphi \times \varphi) \circ \gamma).$$

We obtain a new penalization function simply by replacing $\mathcal{L}_\varphi$ by $\mathcal{L}^G_\varphi$ in the definitions of J_U and J. For any open subset U of $M \times M$ that verifies condition (2) we set

$$J^G_U(\varphi) = \inf_{\gamma \in \Gamma_U(\varphi)} \mathcal{L}^G_\varphi(\gamma),$$

and

$$J^G(\varphi) = \sup_U J^G_U(\varphi),$$

where U is any open subset of $\mathrm{int}(M) \times \mathrm{int}(M)$ that verifies (2).

Example 14 *In the numerical simulations presented in Sections 5.2 and 5.3, hte penalization used is the functional J^G defined above with*

$$G(y_1, y_2) = |y_1 - y_2|^4\,\mathrm{Id},$$

for all $(y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2$.

3.4.2. *Non-degenerate penalization*

In the definition of J^G, the metric used to define the length of an element of $\Gamma_U(\varphi)$ is degenerate. In particular cases, there exists $\gamma \in \Gamma_U(\varphi)$ such that $\mathcal{H}^1(\gamma \cap U) \neq 0$, and $\mathcal{L}^G_\varphi(\gamma) = 0$. It follows (see Section 3.4.3) that for certain choices of M, there exists non admissible deformations for which $J^G(\varphi) = 0$. In order to circumvent this problem, we propose another penalization family of penalization based on non-degenerate metrics. Nevertheless, the penalization functions thus obtained are not invariant under reparametrization. Let us mention that if both types of penalization could be combined in order to take advantage of each, we will not investigate this possibility in the present article.

Let g be a regular map form $\mathbb{R}^+$ into $\mathbb{R}^+$ such that

$$g(s) = 0 \Longleftrightarrow s = 0.$$

For any curve γ of $M \times M$, we set

$$\mathcal{L}^g_\varphi(\gamma) = \int_\gamma g(|r_\varphi(s)|^2) ds,$$

where ds is the Hausdorff one-dimensional measure of γ. Our new family of penalization functions is obtained replacing $\mathcal{L}_\varphi$ by $\mathcal{L}^g_\varphi$ in the definition of J. More precisely, if U is an open subset of $\mathrm{int}(M) \times \mathrm{int}(M)$ satisfying (2), we set

$$J^g_U(\varphi) = \inf_{\gamma \in \Gamma_U(\varphi)} \mathcal{L}^g_\varphi(\gamma).$$

Finally, the penalization is obtained setting

$$J^g(\varphi) = \sup_U J^g_U(\varphi),$$

where U is any open subset of $\mathrm{int}(M) \times \mathrm{int}(M)$ verifying (2).

3.4.3. *Penalization and admissible deformations*

Proposition 15 *If M is either the circle S^1, or a union of closed disjoint intervals, then $J^G(\varphi) = 0$ if and only if φ belongs to the set of admissible deformations $\mathcal{A}(j_M)$.*

PROOF. First of all, if φ belongs to the set of admissible deformations, for all open set U fulfilling condition (2), the empty family belongs to $\Gamma_U(\varphi)$. Thus, $J^G_U(\varphi) = 0$ and $J^G(\varphi) = 0$.

On the other hand, let φ be a deformation such that $J^G(\varphi) = 0$. For any subset U verifying condition (2), there exists $\delta > 0$ such that

$$U \subset U_\delta = (\varphi \times \varphi)^{-1}(\complement B_\delta) \cap \mathrm{int}(M) \times \mathrm{int}(M),$$

where

$$B_\delta = \{y = (y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \;:\; |y_1 - y_2| \leq \delta)\}.$$

Let δ' be a positive real strictly lower than δ. We define $U_{\delta'}$ and $B_{\delta'}$ as U_δ and B_δ replacing δ by δ'. Moreover, we set

$$m = \inf_{X \in B_{\delta'}, Y \in \complement B_\delta} d^G(X, Y).$$

From inequality (12), the distance m (for the metric G) between $B_{\delta'}$ and $\complement B_\delta$ is strictly positive. Since $J^G_{U_{\delta'}}(\varphi) = 0$, there exists $\gamma = (\gamma_j) \in \Gamma_{U'_\delta}(\varphi)$, where γ_j are maps from $[0,1]$ into $\mathbb{R}^2$ (from the characterization of $\Gamma_U(\varphi)$ given in Sections 3.3.2) and 3.3.3 such that

$$\mathcal{L}^G(\gamma) < m.$$

In particular, for all $j \in J$, we have $\mathcal{L}^g(\gamma_j) < m$. Moreover, $\gamma_j(0)$ belongs to the complementary set of $U_{\delta'}$, and for all $t \in [0,1]$ we have

$$\begin{aligned} &d^G(B_{\delta'}, (\varphi \times \varphi) \circ \gamma_j(t)) \\ &\leq d^G((\varphi \times \varphi) \circ \gamma_j(0), (\varphi \times \varphi) \circ \gamma_j(t)) \leq \mathcal{L}^G(\gamma_j) \leq m. \end{aligned}$$

From the definition of m, we deduce that for all t, $(\varphi \times \varphi) \circ \gamma_j(t)$ belongs to B_δ. Hence, $\gamma_j(t)$ does not belong to U_δ, and γ is included on the complementary set of U. It follows that the empty family belongs to $\Gamma_\varphi(U)$ and that for any irrotational vector field ω, whose support is included in U, we have

$$\int_U \omega \wedge \Phi_\varphi dx = 0. \tag{13}$$

From Lemma 16 below, we deduce that φ belongs to the set of admissible deformations.

Lemma 16 *Let φ be a deformation of M, and U be an open subset of $\mathrm{int}(M) \times \mathrm{int}(M)$ such that $\inf_{x \in U} |r_\varphi(x)| > 0$. If for all irrotational vector field ω, whose support is included in U we have*

$$\int_U \omega \wedge \Phi_\varphi dx = 0,$$

then there exists a map u from U into $\mathbb{R}$ such that $\Phi_\varphi = \nabla u$.

PROOF. Let γ be a loop in U, that is a regular map from S^1 into U. Let ρ be a regular map form $\mathbb{R}^2$ into $\mathbb{R}^+$ with compact support such that $\int_{\mathbb{R}^2} \rho dx = 1$. We introduce the regular sequence $\rho_\varepsilon(x) = \varepsilon^{-2}\rho(x/\varepsilon)$. The sequence ρ_ε converges toward a Dirac at zero as ε goes to zero. For ε small enough, the support of $\rho_\varepsilon(. - s)$ is included in U for all $s \in \gamma$. We set

$$\alpha(x) = \int_\gamma \rho_\varepsilon(x - s) n(s) ds.$$

The support of α is included in U. Besides, α is a irrotational vector field. Indeed,

$$\nabla \wedge \alpha(x) = \int_\gamma \nabla \rho(x - s) \wedge n(s) ds = 0.$$

From (13), we deduce that

$$\int_U \alpha \wedge \Phi_\varphi dx = 0.$$

Yet

$$\int_U \alpha \wedge \Phi_\varphi dx = \int_\gamma n(s) \wedge (\rho_\varepsilon \star \Phi_\varphi) ds,$$

where $\star$ denotes the convolution. Hence, for all ε small enough, we have

$$\int_\gamma (\rho_\varepsilon \star \Phi_\varphi) \cdot \tau ds = 0.$$

Letting ε goes to zero, we get

$$\int_\gamma \Phi_\varphi \cdot \tau ds = 0. \tag{14}$$

As the integral of Φ_φ over a loop included in U is zero, it is easy to construct a map $u : U \to \mathbb{R}$ such that $\nabla u =$

Φ_φ. To this end, it suffices to choose an element x_0 in each connected component of U, and to set

$$u(x) = \int_{\tilde{\gamma}} \Phi_\varphi \cdot \tau ds,$$

for all x belonging to the same connected component as x_0, where $\tilde{\gamma}$ is any curve in U joining x_0 to x. The map u is uniquely define, as (14) ensures that $u(x)$ does not depend on the choice $\tilde{\gamma}$ made.

In the case $M = S^1 \cup [2,3] \times \{0\}$, there exists non admissible deformations cancelling the penalization J^G, like the deformation φ of M whose restriction to S^1 is the identity and whose restriction to $[2,3] \times \{0\}$ is zero. The family of penalization functions J^g does not have this drawback.

Proposition 17 *For any M, union of disjoint circles and closed intervals, and for each deformation φ of M, $J^g(\varphi) = 0$ if and only if φ belongs to the set of admissible deformations $\mathcal{A}(j_M)$.*

PROOF. First of all, if φ is an admissible deformation, the empty family belongs to $\Gamma_U(\varphi)$ for any open set U of $M \times M$ verifying (2), and $J^g(\varphi) = 0$. On the other hand, let φ be a deformation such that $J^g(\varphi) = 0$. For all open subset of $\mathrm{int}(M) \times \mathrm{int}(M)$ meeting (2) and positive real $m > 0$, there exists $\gamma \in \Gamma_U(\varphi)$ such that

$$\mathcal{L}^g(\gamma) < m.$$

If ω is a irrotational vector field, whose support is included in U, we have

$$\int_U \omega \wedge \Phi_\varphi dx = \int_\gamma (\omega \wedge n) ds.$$

Since the metric $\mathcal{L}^g$ is non-degenerate on U, it follows that there exists a constant C independent of γ and U such that

$$\left| \int_U \omega \wedge \Phi_\varphi dx \right| \leq C \mathcal{L}^g(\gamma) \sup_{x \in U} |\omega(x)| \leq Cm \sup_{x \in U} |\omega(x)|.$$

This inequality is valid for any positive real m, therefore,

$$\int_U \omega \wedge \Phi_\varphi dx = 0.$$

Applying Lemma 16, we infer that φ belongs to the set of admissible deformations.

3.4.4. *Invariance under reparametrization*

Let φ be a deformation of M belonging to $W^{1,1}$. We can reparametrize M so that, in the new configuration, the norm of the gradient of the deformation is constant. We denote by $\tilde{\varphi}$ this deformation. More precisely, in the case $M = [0,1]$, we introduce the map p from M into M defined by

$$p(t) = \int_0^t |\dot{\varphi}(s)| ds \Big/ \int_0^1 |\dot{\varphi}(s)| ds. \tag{15}$$

The deformation $\tilde{\varphi}$ is the unique map from M into $\mathbb{R}^2$ such that

$$\tilde{\varphi} \circ p = \varphi. \tag{16}$$

Note that $\tilde{\varphi}$ is correctly defined by this equation (if $p(s) = p(t)$, we have $\varphi(s) = \varphi(t)$). Next, $\tilde{\varphi}$ is Lipschitizian, and

$$|\tilde{\varphi}'| = \left(\int_0^1 |\dot{\varphi}(s)| ds \right)^{-1}, \text{ a.e.}$$

Proposition 18 *Let $\varphi \in W^{1,1}(M;\mathbb{R}^2)$ be a deformation of M. The value of the penalization $J^G(\varphi)$ is invariant under reparametrization. In other words,*

$$J^G(\varphi) = J^G(\tilde{\varphi}),$$

where $\tilde{\varphi}$ is defined by (16).

The proof of this proposition relies on a set of Lemmas.

Lemma 19 *Let $\varphi \in W^{1,1}(M;\mathbb{R}^2)$ be a deformation of M, and p the reparametrization defined by (15). For all $\delta > 0$, we have*

$$(p \times p)(\Gamma_U(\varphi)) \subset \Gamma_V(\tilde{\varphi}),$$

where $U = U_\delta(\varphi)$ et $V = U_\delta(\tilde{\varphi})$.

PROOF. Let γ be an element of $\Gamma_U(\varphi)$. We set $\widetilde{\gamma} = (p \times p) \circ \gamma$. We want to prove that $\widetilde{\gamma}$ belongs to $\Gamma_V(\widetilde{\varphi})$. Firstly, it is obvious that $\partial\widetilde{\gamma} \subset \complement V$. Secondly, let $\widetilde{\omega}$ be a irrotational vector field, whose support is included in V. We set $\omega = (\nabla(p \times p))\widetilde{\omega} \circ (p \times p)$. Since the curl of $\widetilde{\omega}$ is zero, we deduce that the curl of ω is zero too. Performing a change of variables, we can establish that

$$\int_{\widetilde{\gamma}} \widetilde{\omega} \wedge n ds = \int_\gamma \omega \wedge n ds \tag{17}$$

and

$$\int_{M \times M} \omega \wedge \nabla r_\varphi(\Phi(r_\varphi)) dx = \int_{M \times M} \widetilde{\omega} \wedge \nabla r_{\widetilde{\varphi}}(\Phi(r_{\widetilde{\varphi}})) dx. \tag{18}$$

As the curl of $\nabla r_j(\Phi(r_j)$ is zero, and $(p \times p)$ is homotopic to the identity, we have

$$\begin{aligned} &\int_{M \times M} \widetilde{\omega} \wedge \nabla r_j(\Phi(r_j)) dx \\ &\quad = \int_{M \times M} (\nabla(p \times p)\widetilde{\omega} \circ (p \times p)) \wedge (\nabla r_j \Phi(r_j)) dx \\ &\quad = \int_{M \times M} \omega \wedge (\nabla r_j(\Phi(r_j)) dx. \end{aligned} \tag{19}$$

From the definitions of Φ_φ and $\Phi_{\widetilde{\varphi}}$ and equations (18) and (19), we infer

$$\int_{M \times M} \omega \wedge \Phi_\varphi dx = \int_{M \times M} \widetilde{\omega} \wedge \Phi_{\widetilde{\varphi}} dx. \tag{20}$$

Furthermore, since ω is irrotational, and γ belongs to $\Gamma_U(\varphi)$, we have

$$\int_\gamma \omega \wedge n ds = \int_{M \times M} \omega \wedge \Phi_\varphi dx. \tag{21}$$

Finally, we deduce from (17), (20) and (21) that

$$\int_{\widetilde{\gamma}} \widetilde{\omega} \wedge n ds = \int_{M \times M} \widetilde{\omega} \wedge \Phi_{\widetilde{\varphi}} dx.$$

As this equality is satisfied for any irrotational field $\widetilde{\omega}$ whose support is included in V, $\widetilde{\gamma}$ belongs to $\Gamma_V(\widetilde{\varphi})$.

Lemma 20 *Let φ be a deformation of M, and p the reparametrization defined by (15). For any map $\gamma \in W^{1,1}([0,1];M)$, there exists a sequence of continuous maps γ_n such that $p \circ \gamma_n$ converges toward γ for the strong $W^{1,1}-$topology.*

PROOF. If p is bijective, it suffices to choose γ_n constant equal to $p^{-1} \circ \gamma$. Nevertheless, in the general case, p is not bijective. In order to obtain the desired sequence γ_n, we construct a sequence of approximations p_n of p by bijective maps. Then, we set $\gamma_n = p_n^{-1} \circ \gamma$, and we prove the convergence of $p \circ \gamma_n$ toward γ. For all n, we introduce the following approximation of p

$$p_n = \frac{np + s}{n+1}.$$

The map p_n is a diffeomorphism of M and we have

$$p'_n = \frac{np' + 1}{n+1}.$$

As piecewise affine maps are dense in $W^{1,1}$, we can assume, without loss of generality, that γ is piecewise affine. As claimed, we set $\gamma_n = p_n^{-1} \circ \gamma$. It remains to prove that $(p \circ \gamma_n)'$ converges toward γ' strongly in L^1, that is,

$$\begin{aligned} &R_n = \|(p \circ \gamma_n)' - \gamma'\|_{L^1} \\ &= \int_0^1 (1 + 1/n) \left| \left(\frac{1}{np' + 1} \right) \circ (p_n^{-1} \circ \gamma) \right| |\gamma'| ds \xrightarrow{n \to \infty} 0. \end{aligned}$$

This integral can be split on each interval $[s_i, s_{i+1}]$ where γ is affine. Thus, in order to prove the convergence of the integral toward zero, we can assume that γ is affine. If $\gamma' = 0$, there is nothing to prove. Otherwise, we perform the change of variable $t = p_n^{-1} \circ \gamma(s)$, which leads us to

$$R_n = \int (1 + 1/n) \left| \frac{1}{np'(t) + 1} \right| |p'_n| dt.$$

From the expression of p'_n, we deduce that

$$R_n = \int (1 + 1/n) \left| \frac{1}{np'(t) + 1} \right| \left| \frac{np' + 1}{n + 1} \right| dt = \int dt/n,$$

which converges toward zero as n goes to infinity.

Lemma 21 *Let $\varphi \in W^{1,1}(M;\mathbb{R}^2)$ be a deformation of M. For all $\delta > 0$, and all $\widetilde{\gamma} \in \Gamma_V(\widetilde{\varphi})$ belonging $W^{1,1}$, there exists $\gamma_n \in \Gamma_U(\varphi)$ such that*

$$(p \times p) \circ \gamma_n \to \widetilde{\gamma}$$

for the strong $W^{1,1}$ topology, where $U = U_\delta(\varphi)$ and $V = U_\delta(\widetilde{\varphi})$.

PROOF. First of all, every element of $\Gamma_V(\widetilde{\varphi})$ can be approximated by a piecewise affine element of $\Gamma_V(\widetilde{\varphi})$ such that $\partial\widetilde{\gamma}$ is included in the interior of the complementary set of V. We resume the construction of the approximations sequence p_n of p introduced in the proof of Lemma 20, and we set $\gamma_n = (p_n \times p_n)^{-1}(\widetilde{\gamma})$. From Lemma 20, $(p \times p) \circ \gamma_n$ converges toward $\widetilde{\gamma}$ strongly in $W^{1,1}$. This entails that for n big enough, $\partial\gamma_n$ is included in the complementary set of U. In order to establish that γ_n belongs to $\Gamma_U(\varphi)$, it remains to prove that for any irrotational vector field ω, whose support is included in U, we have

$$\int_{\gamma_n} \omega \wedge n ds = \int_{M \times M} \omega \wedge \Phi_\varphi ds. \tag{22}$$

By a simple change of variable, we get

$$\int_{\gamma_n} \omega \wedge n ds = \int_{\widetilde{\gamma}} \widetilde{\omega}_n \wedge n ds, \tag{23}$$

where

$$\nabla(p_n \times p_n)\widetilde{\omega}_n \circ (p_n \times p_n) = \omega. \tag{24}$$

The field $\widetilde{\omega}_n$ is irrotational and its support is included in V. As $\widetilde{\gamma}$ belongs to $\Gamma_V(\widetilde{\varphi})$, we have

$$\int_{\widetilde{\gamma}} \widetilde{\omega}_n \wedge n ds = \int_{M \times M} \widetilde{\omega}_n \wedge \Phi_{\widetilde{\varphi}} dx. \tag{25}$$

Let $\varphi_n = \varphi \circ p_n^{-1}$. As $r_{\widetilde{\varphi}}$ and $r_{\widetilde{\varphi}_n}$ are homotopic maps from V into $\mathbb{R}^2$, we have

$$\int_{M \times M} \widetilde{\omega}_n \wedge \Phi_{\widetilde{\varphi}} dx = \int_{M \times M} \widetilde{\omega}_n \wedge \Phi_{\widetilde{\varphi}_n} dx \tag{26}$$

Another change of variable, enables us to establish that

$$\int_{M \times M} \widetilde{\omega}_n \wedge \Phi_{\widetilde{\varphi}_n} dx = \int_{M \times M} \omega \wedge \Phi_\varphi dx. \tag{27}$$

Finally, gathering equations (23-27), we obtain (22), which concludes the proof.

Lemma 22 *Let $\varphi \in W^{1,1}(M;\mathbb{R}^2)$. For all open set U of $\mathrm{int}(M) \times \mathrm{int}(M)$ fulfilling condition (2), there exists $\widetilde{\gamma} \in \Gamma_V(\widetilde{\varphi})$ such that*

$$J_V(\widetilde{\varphi}) = \mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma}),$$

where $V = (p \times p)(U)$.

PROOF. Let U be an open subset of $\mathrm{int}(M) \times \mathrm{int}(M)$ meeting condition (2) and $V = (p \times p)(U)$. We recall that

$$J_V(\widetilde{\varphi}) = \inf_{\widetilde{\gamma} \in \Gamma_V(\widetilde{\varphi})} \mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma}).$$

Let $\widetilde{\gamma}_n$ be a minimization sequence of $\mathcal{L}^G_{\widetilde{\varphi}}$ in $\Gamma_V(\widetilde{\varphi})$. Using the characterization of $\Gamma_V(\widetilde{\varphi})$ given in Section 3.3, one can prove that the sequence $\widetilde{\gamma}_n$ can be chosen so that, for each n, the number of elements composing the family $\widetilde{\gamma}_n$ is finite independently of n. For the sake of simplicity, we assume that $\widetilde{\gamma}_n$ has a unique element, for instance an application from $[0,1]$ valued in $M \times M$. As $\widetilde{\gamma}_n$ is a minimization sequence of $\mathcal{L}^G_{\widetilde{\varphi}}$, there exists a constant C such that

$$\mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma}_n) < C.$$

Moreover, we can assume that for any element $t \in [0,1]$, we have $r_\varphi(\gamma_n(t)) > \delta$, where δ is a strictly positive constant. From the ellipticity property (12), we have

$$\mathcal{L}((\widetilde{\varphi} \times \widetilde{\varphi}) \circ \widetilde{\gamma}_n) < C \left(\inf_{t > \delta/2} g(t) \right)^{-1} \mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma}_n),$$

where $\mathcal{L}((\widetilde{\varphi}\times\widetilde{\varphi})\circ\widetilde{\gamma}_n)$ stands for the usual length of the curve $(\widetilde{\varphi}\times\widetilde{\varphi})\circ\widetilde{\gamma}_n$. Even if it means to reparametrize $\widetilde{\gamma}_n$, we can assume that $(\widetilde{\varphi}\times\widetilde{\varphi})\circ\widetilde{\gamma}_n$ belongs to $W^{1,\infty}$, and that the norm of its gradient is constant. As $\widetilde{\varphi}$ belongs himself to $W^{1,\infty}$ and the norm of its gradient is constant, it follows that $\widetilde{\gamma}_n$ belongs to $W^{1,\infty}$ and that the sequence $\widetilde{\gamma}_n$ is equicontinuous. Moreover, as $M\times M$ is compact, we deduce from the Ascoli Theorem that there exists a subsequence of $\widetilde{\gamma}_n$ (which we also denote $\widetilde{\gamma}_n$) which converges toward an element $\widetilde{\gamma}\in\Gamma_V(\widetilde{\varphi})$ for the $\mathcal{C}^0-$topology. Hence,

$$\begin{aligned}\mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma}) &= \mathcal{L}^G((\widetilde{\varphi}\times\widetilde{\varphi})\circ\widetilde{\gamma})\\ &\leq \liminf_n \mathcal{L}^G((\widetilde{\varphi}\times\widetilde{\varphi})\circ\widetilde{\gamma}_n)\\ &= \liminf_n \mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma}_n) = J^G_V(\widetilde{\varphi}).\end{aligned}$$

PROOF. [Proposition 18] For all $\delta>0$, we have

$$\begin{aligned}J_{U_\delta}(\varphi) &= \inf_{\gamma\in\Gamma_{U_\delta}(\varphi)} \mathcal{L}^G_{\varphi}(\gamma)\\ &= \inf_{\gamma\in\Gamma_{U_\delta}(\varphi)} \mathcal{L}^G_{\widetilde{\varphi}}((p\times p)\circ\gamma).\end{aligned}$$

Yet, $(p\times p)(\Gamma_{U_\delta}(\varphi))\subset\Gamma_{V_\delta}(\widetilde{\varphi})$. Thus,

$$J^G_{U_\delta}(\varphi)\geq \inf_{\widetilde{\gamma}\in\Gamma_{V_\delta}(\widetilde{\varphi})} \mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma}) = J^G_{V_\delta}(\widetilde{\varphi}). \tag{28}$$

On the other hand, from Lemma 22, there exists $\widetilde{\gamma}\in\Gamma_{V_\delta}(\widetilde{\varphi})\cap W^{1,\infty}$ such that

$$J^G_{V_\delta}(\widetilde{\varphi}) = \mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma}).$$

From Lemma 21, there exists a sequence γ_n of elements of $\Gamma_{U_\delta}(\varphi)$ such that $(p\times p)\circ\gamma_n$ converges toward $\widetilde{\gamma}$ for the $W^{1,1}$-topology. Therefore,

$$\mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma}) = \lim_n \mathcal{L}^G_{\widetilde{\varphi}}((p\times p)\circ\gamma_n) = \lim_n \mathcal{L}^G_{\varphi}(\gamma_n),$$

and

$$J^G_{V_\delta}(\widetilde{\varphi}) = \lim_n \mathcal{L}^G_{\varphi}(\gamma_n)\geq J^G_{U_\delta}(\varphi)$$

We infer form this inequality and from (28) that

$$J^G_{U_\delta}(\varphi) = J^G_{V_\delta}(\widetilde{\varphi}),$$

from which the conclusion of the proposition follows.

3.4.5. *Lower semicontinuity*

Proposition 23 *The function J^G is lower semicontinuous for the $\mathcal{C}^0$ topology.*

PROOF. As the supremum of lower semicontinuous functions is lower semicontinuous, it suffices to establish the lower semicontinuity of the functions J^G_U. Let φ be a deformation of M and U be an open subset of $M\times M$ satisfying (2). Let φ_n be a sequence of deformations converging toward φ for the $\mathcal{C}^0-$topology. For n big enough, condition (2) is also met by the couple (φ_n, U), so $J^G_U(\varphi_n)$ is correctly defined. For all n, there exists $\gamma_n\in\Gamma_U(\varphi)$ such that

$$J^G_U(\varphi_n)\geq\mathcal{L}^G_{\varphi_n}(\gamma_n)-1/n.$$

We have

$$\liminf J^G_U(\varphi_n) = \liminf \mathcal{L}^G_{\varphi_n}(\gamma_n). \tag{29}$$

Without loss of generality, we can assume that this limit is finite (otherwise, there is nothing to prove). We can choose γ_n such that $|r_{\varphi_n}(\gamma_n(t)|>\delta/2$ for all t, where δ is a positive constant independent of n. Moreover, from (29), and even if we have to extract a subsequence, we can assume that $\mathcal{L}^G_{\varphi_n}(\gamma_n)$ is bounded independently of n. Hence, the maps γ_n can be reparametrerized so that for any s and t of $[0,1]$, we have

$$\mathcal{L}((\varphi_n\times\varphi_n)\circ\gamma_n|[s,t])\leq C|s-t|+1/n,$$

where C is a constant independent of n. In particular,

$$\text{diameter}((\varphi_n\times\varphi_n)\circ\gamma_n([s,t]))\leq C|s-t|+1/n,$$

and since φ_n converges toward φ for the $\mathcal{C}^0-$topology , we deduce that

$$\text{diameter}((\varphi\times\varphi)\circ\gamma_n([s,t]))\leq C|s-t|+1/n+2\|\varphi-\varphi_n\|_\infty.$$

We set $\widetilde{\gamma}=(p\times p)\circ\gamma_n$, where p is the reparametrization function associated to φ. The previous equation implies that

$$\text{diameter}((\widetilde{\varphi}\times\widetilde{\varphi})\circ\widetilde{\gamma}_n([s,t]))\leq C|s-t|+1/n+2\|\varphi-\varphi_n\|_\infty.$$

Using the fact that the norm of the gradient of $\widetilde{\varphi}$ is constant, strictly positive, we deduce that for all $\alpha>0$, there exists $\delta(\alpha)>0$ such that if $C|s-t|+1/n+2\|\varphi-\varphi_n\|_\infty<\delta(\alpha)$, then

$$|\widetilde{\gamma}_n(s)-\widetilde{\gamma}_n(t)|<\alpha.$$

From Ascoli's Theorem, it follows that there exists a subsequence of $\widetilde{\gamma}_n$ (denoted $\widetilde{\gamma}_n$ for simplicity) that converges toward an element $\widetilde{\gamma}$ for the $\mathcal{C}^0-$topology. For all n, $\widetilde{\gamma}_n$ belongs to $\Gamma_V(\widetilde{\varphi})$, where $V=(p\times p)(U)$. As $\Gamma_V(\widetilde{\varphi})$ is closed for the $\mathcal{C}^0$ topology, $\widetilde{\gamma}$ belongs to $\Gamma_V(\widetilde{\varphi})$. Moreover, $(\varphi_n\times\varphi_n)\circ\gamma_n$ converges toward $(\widetilde{\varphi}\times\widetilde{\varphi})\circ\widetilde{\gamma}$ for the $\mathcal{C}^0$ topology. Thence,

$$\begin{aligned}J_U(\varphi) = J_V(\widetilde{\varphi}) &\leq \mathcal{L}^G_{\widetilde{\varphi}}(\widetilde{\gamma})\\ &= \mathcal{L}^G((\widetilde{\varphi}\times\widetilde{\varphi})\circ\widetilde{\gamma})\\ &\leq \liminf \mathcal{L}^G((\varphi_n\times\varphi_n)\circ\gamma_n) = \liminf J_U(\varphi_n),\end{aligned}$$

which completes the proof.

Using similar arguments, we can establish that

Proposition 24 *The function J^g is lower semicontinuous for the $\mathcal{C}^0$ topology.*

3.4.6. *Differentiability of the penalizations*

The penalization functions J^G and J^g are usually not differentiable. Nevertheless, a formal computation of their derivatives can be performed when the deformation φ is an immersion with transversal selfintersections. We recall that, in such a case,

$$J^G(\varphi) = \inf_{\gamma\in\Gamma(\varphi)} \mathcal{L}^G_{\varphi}(\gamma)$$

$$\text{and } J^g(\varphi) = \inf_{\gamma\in\Gamma(\varphi)} \mathcal{L}^g_{\varphi}(\gamma),$$

where $\Gamma(\varphi)$ is given by (9) if M is the union of disjoint intervals. Moreover, there exists $\gamma^G(\varphi)$ and $\gamma^g(\varphi) \in \Gamma(\varphi)$ such that

$$J^G(\varphi) = \mathcal{L}^G_\varphi(\gamma^G(\varphi))$$
$$\text{and } J^g(\varphi) = \mathcal{L}^g_\varphi(\gamma^g(\varphi)).$$

Assuming that $\gamma^G(\varphi)$ and $\gamma^g(\varphi)$ are differentiable with respect to φ, we can establish that J^G and J^g are differentiable. More precisely,

Proposition 25 *Let φ be an immersion whose selfintersections are transversal. If $\gamma^G(\varphi)$ is differentiable with respect to φ, then J^G is differentiable, and*

$$\langle (J^G)'(\varphi)\psi \rangle = \int_\gamma \frac{F\circ(\varphi\times\varphi)\circ\gamma}{|F\circ(\varphi\times\varphi)\circ\gamma|} \cdot \left((F\circ(\varphi\times\varphi)\circ\gamma)\frac{d((\psi\times\psi)\circ\gamma)}{ds} + \left(D_{(\varphi\times\varphi)\circ\gamma}F\right)(\psi\times\psi)\circ\gamma\frac{d((\varphi\times\varphi)\circ\gamma)}{ds}\right)ds,$$

where F is the symmetric positive matrix such that $F^2 = G$. Furthermore,

$$\langle (J^G)'(\varphi), \psi\rangle = \langle (J^G)'(\varphi), (\psi\cdot n_\varphi)n_\varphi\rangle, \tag{30}$$

where n_φ is the normal to φ.

Remark 26 *We can extend (at least formally) this derivation formula to any deformation (with non transversal selfintersections, for instance) if $G = 0$ on $\Delta(\mathbb{R}^2)$.*

A similar proposition can be stated for J^g.

Proposition 27 *Let φ be an immersion whose selfintersections are transversal. If $\gamma^g(\varphi)$ is differentiable with respect to φ, then J^g is differentiable and*

$$\langle (J^g)'(\varphi), \psi\rangle = 2\int_{\gamma(\varphi)} g'(|\varphi(s_1)-\varphi(s_2)|^2)(\varphi(s_1)-\varphi(s_2)) \cdot (\psi(s_1)-\psi(s_2))ds.$$

PROOF. [Propositions 27 and 25] Let

$$j^g(\varphi,\gamma) = \mathcal{L}^g_\varphi(\gamma) = \int_\gamma g(|\varphi(s_1)-\varphi(s_2)|^2)ds.$$

We have $J^g(\varphi) = j^g(\varphi,\gamma^g(\varphi))$. Using the fact that $\gamma^g(\varphi)$ minimizes $j^g(\varphi,\gamma)$ on the set of elements γ of $\Gamma(\varphi)$ and that $g(0) = 0$, we establish that $\partial j^g/\partial\gamma = 0$. Consequently, $(J^g)'(\varphi) = \partial j^g/\partial\varphi(\varphi,\gamma^g(\varphi))$, which leads to the announced expression of the derivative of J^g. We proceed in a similar way to determine the derivative if J^G. In this case, as J^G is invariant under reparametrization, $J^G(\varphi)$ is independent of variations of φ along its tangent. Accordingly, $\langle (J^g)'(\varphi),\psi\rangle$ depends only on the normal component of ψ.

4. An example in nonlinear elasticity

In this section, we present an application of our penalization method to nonlinear elasticity.

4.1. *Setting of the problem*

We consider a one dimensional nonlinear hyperelastic body M moving in $\mathbb{R}^2$, submitted to dead loads $f : M \to \mathbb{R}^2$, fixed on a nonempty set $N \subset M$. The energy $I(\psi)$ associated with a deformation $\psi : M \to M$ is the difference between the internal energy and the work of the external forces:

$$I(\psi) = \int_M W(x,\dot\psi(x))dx - \int f(x)\cdot\psi(x)dx,$$

where W is the stored energy function; it depends on the nature of the material with which M is made (and of other parameters, such as the section of M). Critical points of the energy are equilibrium states. In particular, any minimizer of I is a stable equilibrium state. We assume that $W(x,.)$ is convex and that there exists $p > 1$ and positives constants C_1 and C_2 such that for all $a \in \mathbb{R}^2$ and all $x \in M$,

$$W(x,a) \leq C_1(|a|^p + 1), \tag{31}$$

and

$$W(x,a) \geq C_2(|a|^p - 1). \tag{32}$$

The set of admissible deformations with finite energy is

$$\Phi(j_M) = \Big\{\varphi \in W^{1,p}(M;\mathbb{R}^2) \ : \ \varphi\in\mathcal{A}(j_M) \text{ and } \varphi(x) = j_M(x) \text{ for all } x \in N\Big\}.$$

Proposition 28 *There exists a deformation $\varphi \in \Phi(j_M)$ that minimizes I on $\Phi(j_M)$:*

$$I(\varphi) = \inf_{\psi\in\Phi(j_M)} I(\psi). \tag{33}$$

PROOF. Let φ_n be a minimization sequence. From the coercivity property (32), the sequence is bounded in $W^{1,p}(M;\mathbb{R}^2)$. Therefore, after having extracted a subsequence, we may assume that the sequence φ_n is convergent for the weak-* topology of $W^{1,p}(M;\mathbb{R}^2)$ toward a deformation φ. Since $\mathcal{A}(j_M)$ is closed for the $\mathcal{C}^0-$topology and the injection from $W^{1,p}$ toward $\mathcal{C}^0$ is compact, φ belong to $\Phi(j_M)$. Finally, the convexity of W and the growing property (31) imply that I is sequentially lower semicontinuous for the weak-* $W^{1,p}-$topology, and φ is a minimizer of I over $\Phi(j_M)$.

4.2. *Penalized formulation*

Let T be a penalization function of the selfintersections, lower semicontinuous for the $\mathcal{C}^0-$topology such that for any deformation ψ of M, $T(\psi) = 0$ if and only if $\psi \in \mathcal{A}(j_M)$. In particular, we can choose $T = J^g$ for any M or $T = J^G$ if M is either a circle or a disjoint union of closed curves. For any real $\delta > 0$, we denote I_δ the sum between the energy I and the penalization weighted by δ^{-1}

$$I_\delta(\varphi) = I(\varphi) + \delta^{-1}T(\varphi),$$

and set

$$W(j_{_M}) = \left\{ \varphi \in W^{1,p}(M;\mathbb{R}^2) \ : \ \varphi(x) = j_{_M}(x) \text{ for all } x \in N \right\}.$$

Proposition 29 *The minimization problem of I_δ over $W(j_{_M})$ admits at least one solution. Moreover, any family of solutions φ_δ is relatively compact for the weak* $W^{1,p}$ topology and its closure values are solutions of the minimization problem (33).*

PROOF. The same argument as the one used in the proof of Proposition 33 enables us to prove the existence of solutions to the penalized problems. Moreover, any sequence of solutions φ_δ is bounded in $W^{1,p}(M;\mathbb{R}^2)$ and admits a converging subsequence (which will be denoted φ_δ). We have

$$T(\lim \varphi_\delta) \leq \liminf T(\varphi_\delta) = 0.$$

Thus, $\lim \varphi_\delta \in \mathcal{A}(j_{_M})$ and for all $\varphi \in \mathcal{A}(j_{_M})$,

$$\begin{aligned} I(\varphi) &= \liminf I_\delta(\varphi) \geq \liminf I_\delta(\varphi_\delta) \\ &\geq \liminf I(\varphi_\delta) \geq I(\lim \varphi_\delta). \end{aligned}$$

As I_δ is differentiable, we can compute local minima of I_δ using classical gradient methods. It remains to precise how to compute, from the practical viewpoint, the penalization terms and their derivatives.

5. Numerical Applications

This section is devoted to the numerical applications of the penalization method introduced. In a first step, we present two different methods in order to compute the penalization terms J^g and J^G. In a second step, we present several numerical results obtained using the second method. Firstly, we consider the projection problem of a deformation onto the set of admissible deformations. Secondly we study a physical problem of interaction between elastic balloons inflated with a perfect gas.

5.1. *Evaluation of the penalization terms*

5.1.1. *A direct method*

Let us recall, that for any deformation φ whose selfintersections are transversal, the penalizations $J^g(\varphi)$ and $J^G(\varphi)$ are defined as the minimal length, for a metric depending on φ, of the curves belonging to $\Gamma(\varphi)$:

$$J^G(\varphi) = \inf_{\gamma\in\Gamma(\varphi)} \mathcal{L}^G_\varphi(\gamma) \quad \text{and} \quad J^g(\varphi) = \inf_{\gamma\in\Gamma(\varphi)} \mathcal{L}^g_\varphi(\gamma)$$

In order to simplify the notation, we will merely note J in place of J^g or J^G, and $\mathcal{L}_\varphi$ in place of $\mathcal{L}^G_\varphi$ or $\mathcal{L}^g_\varphi$. Moreover, we shall only consider the case of a body M made of a disjoint union of closed intervals. In this case, we recall that

$$\Gamma(\varphi) = \Big\{ \gamma \in C_1(M \times M) \ : \ \partial\gamma \cap \operatorname{int}(M \times M \setminus \Delta(M)) = r_\varphi^{-1}(0) \setminus \Delta(M) \Big\}.$$

We denote by $(y_i^+)_{i=1,\cdots,n^+}$ and $(y_j^-)_{i=1,\cdots,n^-}$ the families of elements of $r_\varphi^{-1}(0) \setminus \Delta(M)$ such that $s(y_i^+) = +1$ and $s(y_j^-) = -1$. For any couple (i,j) such that $1 \leq i \leq n^+$ and $1 \leq j \leq n^-$, we denote A_{ij} the distance between y_i^+ and y_j^- for the metric associated with $\mathcal{L}_\varphi$. Moreover, we denote A_{i0} the distance between y_i^+ and $\partial(M \times M \setminus \Delta(M))$ and A_{0j} the distance between y_j^- and $\partial(M \times M \setminus \Delta(M))$ for the same metric. Finally, we set $A_{00} = 0$. One can easily check that

$$J(\varphi) = \inf_{v\in V} \sum_{\substack{i=0,\cdots n^+ \\ j=0,\cdots,n^-}} v_{ij} A_{ij}, \tag{34}$$

where V is the set of matrices $(n^+ + 1) \times (n^- + 1)$ with coefficients in $\{0, 1\}$ such that

$$\sum_{k=0,\cdots,n^-} v_{ik} = 1 \text{ for all } i = 1, \cdots, n^-, \tag{35}$$

and

$$\sum_{k=0,\cdots,n^+} v_{kj} = 1 \text{ for all } j = 1, \cdots, n^+. \tag{36}$$

In order to evaluate $J(\varphi)$, it suffices to compute the matrix A and to solve the linear program (34). Let us remark that the set of deformations with transversal selfintersections is generic. As a consequence, the method proposed can be applied to any deformation, after small perturbations of it. Still, from the practical viewpoint, this method is difficult to implement. The computation time rapidly increases with the number of selfintersections. The determination of A requires $(1 + n^+)(1 + n^-)$ geodesics computation, whereas the number of elements V satisfying (35) and (36) increases exponentially with respect to the number of intersection points.

5.1.2. *A phasefield-like method*

The main drawback of the previous method lies in the fast growing of running time with respect to the number of selfintersections. In order to circumvent this problem, we propose another method founded on phase field theory. Let us recall some basic results on this topic (see [31–33]).

Modica-Mortola Theorem

Let Ω be an open bounded subset of $\mathbb{R}^2$ and R a map from $\mathbb{R}$ into $\mathbb{R}^+$ with exactly two wells at 1 and 0, such that $R(0) = R(1) = 0$. For any real $\varepsilon > 0$, we introduce the functional

$$F_\varepsilon(u) = \begin{cases} \varepsilon \displaystyle\int_\Omega |\nabla u|^2 dx + \varepsilon^{-1} \int_\Omega R(u) dx, & \text{if } u \in H^1(\Omega), \\ +\infty, & \text{if } u \notin H^1(\Omega), \end{cases}$$

defined for all map $u \in L^1(\Omega)$. The functional F_ε is nothing but the interface energy of a system made of two different fluids. With respect to the minimization problem of F_ε, the second term favors values 0 and 1 of u, while the first one penalized jumps of the phase. The functional F_ε Γ-converges (for the L^1 topology) as ε goes to zero to the length functional $\mathcal{L}$ of the jumps of u between 0 and 1 defined by

$$\mathcal{L}(u) = \begin{cases} \alpha\mathcal{H}^1(Su), & \text{if } u \in SBV(\Omega;\{0,1\}), \\ +\infty, & \text{if } u \notin SBV(\Omega;\{0,1\}), \end{cases},$$

where Su is the set of essentially discontinuous points of u, $SBV(\mathbb{R}^2,\{0,1\})$ is the set of special functions of bounded variations of $\mathbb{R}^2$ valued in $\{0,1\}$, and α is a strictly positive real depending on R.

Remark 30 *For any map $u \in SBV(\mathbb{R}^2,\{0,1\})$, $\mathcal{L}(u)$ is nothing else but the perimeter of the set of points such that $u = 1$.*

Any sequence u_ε, such that $F_\varepsilon(u_\varepsilon)$ is bounded, is relatively compact in $L^1(\Omega)$. Together, these properties allows us, through Γ-convergence theory, to deduce convergence results of minimizers of F_ε toward minimizers of $\mathcal{L}$. These results can be extended to metrics other than the identity. Via a simple change of variable, we can prove that if A is a symmetric definite positive matrix and $a = \det(A)$, then the functional definied for all $u \in H^1(\Omega)$

$$F_\varepsilon^A(u) = \int_\Omega \left(\varepsilon A^{-1}\nabla u \cdot \nabla u + \varepsilon^{-1}R(u)\right)\sqrt{a}dx,$$

and $F_\varepsilon^A(u) = +\infty$ if $u \notin H^1(\Omega)$ $\Gamma-$converges, as ε goes to zero, towards the functional

$$\mathcal{L}^A(u) = \begin{cases} \alpha\mathcal{H}^1_A(Su), & \text{if } u \in SBV(\Omega;0,1), \\ +\infty, & \text{if } u \notin SBV(\Omega;0,1), \end{cases}$$

where $\mathcal{H}^1_A$, is the onedimensional Hausdorff measure in $\mathbb{R}^2$ endowed with the norm $\|x\|_A^2 = Ax\cdot x$. Moreover this result remains true for the non homogeneous case, that is when A is a field of symmetric positive definite matrices.

Application to the computation of the penalization terms

Let us recall that a deformation φ is admissible if for any open subset U of $\mathrm{int}(M) \times \mathrm{int}(M)$, such that $\inf_{x\in U}|r_\varphi(x)| > 0$, there exists $u : U \to \mathbb{R}$ such that

$$\Phi_\varphi(x) = \nabla u(x), \text{ for all } x \in U, \tag{37}$$

where

$$\Phi_\varphi = \nabla r_\varphi(\Phi(r_\varphi)) - \nabla r_{j_M}(\Phi(r_{j_M}))$$

and

$$r_\varphi(x_1,x_2) = \varphi(x_1) - \varphi(x_2).$$

If θ_φ is the application from U into $\mathbb{R}/2\pi\mathbb{Z}$ that maps any couple (x_1,x_2) to the angle the vector $\varphi(x_1)-\varphi(x_2)$ makes with the abscissa axis, that is,

$$r_\varphi(x) = |r_\varphi(x)|(\cos(\theta_\varphi),\sin(\theta_\varphi)),$$

we have

$$\Phi_\varphi = \nabla\theta_\varphi - \nabla\theta_{j_M}.$$

Therefore, the admissibility condition (37) is equivalent to the existence of a map u from U into $\mathbb{R}$ such that

$$\begin{cases} \nabla u(x) = (\nabla\theta_\varphi - \nabla\theta_{j_M})(x), \\ u(x) \equiv (\theta_\varphi - \theta_{j_M})(x)[2\pi], \end{cases}$$

for all $x \in U$. Let $\mathcal{R}$ be a regular 2π-periodic map valued in $\mathbb{R}^+$, such that $\mathcal{R}(x) = 0$ if and only if $x \equiv 0[2\pi]$. For any real $\varepsilon > 0$, we introduce the functional defined on $L^1(U)$ by

$$\mathcal{F}_{\varphi,\varepsilon}(u) = \int_U \Big(\varepsilon|\nabla u - \nabla(\theta_\varphi - \theta_{j_M})|^2 + \varepsilon^{-1}\mathcal{R}\left(u - (\theta_\varphi - \theta_{j_M})\right)\Big)dx,$$

for any $u \in H^1(U)\cap L^1(U)$, and by $\mathcal{F}_{\varphi,\varepsilon}(u) = +\infty$ for any $u \in L^1(U)$ that does not belong to $H^1(U)$. The functional $\mathcal{F}_{\varphi,\varepsilon}$ has to be compared to F_ε introduced in the previous section. The main differences between them lie on the phase shift $\theta_\varphi - \theta_{j_M}$ introduced and in the function R, which has been replaced by a periodic map with an infinite numbers of wells (and not only two). Let us remark that if $u-(\theta_\varphi-\theta_{j_M})$ is only defined up to a multiple of 2π, $\mathcal{R}(u-(\theta_\varphi-\theta_{j_M}))$ is correctly defined as $\mathcal{R}$ is periodic.

We can state a Γ-convergence result for the functionals $\mathcal{F}_{\varphi,\varepsilon}(u)$ similar to the classical Modica-Mortola result. Indeed, $\mathcal{F}_{\varphi,\varepsilon}$ Γ-converges (for the L^1-topology) toward the functional $\widetilde{\mathcal{L}_\varphi}$ defined for all $u \in SBV(\Omega)$ such that $u \equiv \theta_\varphi - \theta_{j_M}[2\pi]$ almost everywhere by

$$\widetilde{\mathcal{L}_\varphi}(u) = \beta\int_U |[u]|\mathcal{H}^1\llcorner,$$

and for all $u \in L^1(\Omega)$ such that $u \not\in SBV(\Omega)$ or $u \not\equiv \theta_\varphi - \theta_{j_M}[2\pi]$ by $\widetilde{\mathcal{L}_\varphi}(u) = +\infty$. In this expression, Su denotes the set of essentially discontinuous points of u and $[u]$ the jump of u at these points, whereas β is a positive real depending on $\mathcal{R}$. Let us remark that if $\widetilde{\mathcal{L}_\varphi}(u)$ is finite, then the gradient of u has the following form

$$\nabla u = \nabla\theta_\varphi - \nabla\theta_{j_M} + [u]\nu\mathcal{H}^1\llcorner Su,$$

where ν is the unitary normal to Su. Finally, for any u such that $\widetilde{\mathcal{L}_\varphi}(u)$ is finite, if the set of singular points Su is smooth enough, and if u has only $\pm 2\pi$ jumps, then Su is an element of $\Gamma_U(\varphi)$ (we do not prove this assertion in the present article). Accordingly, the minimization of $\widetilde{\mathcal{L}_\varphi}$ is equivalent to finding the element of $\Gamma_U(\varphi)$ of minimal length (for the Euclidean metric). In other words, we have

$$\inf_u \widetilde{\mathcal{L}_\varphi}(u) = \inf_{\gamma\in\Gamma_U(\varphi)} \mathcal{L}(\gamma).$$

The penalization functions J_U^g and J_U^G are precisely defined as the minimal length of the elements of $\Gamma_U(\varphi)$, only the metrics used to compute those lengths are not the Euclidean one, but metrics depending on φ. In order to determine the value of the maps J^g and J^G, we only have to adapt the definition of $\widetilde{\mathcal{L}_\varphi}$.

Let A^g_φ and A^G_φ be the metrics of $M\times M$ associated to the computation of J^g and J^G respectively, that is

$$A^G_\varphi(x)=\nabla_x(\varphi\times\varphi(x))G(\varphi\times\varphi))\nabla_x(\varphi\times\varphi(x))^T,$$

and

$$A^g_\varphi(x)=g(|r_\varphi(x)|^2)\nabla_x(\varphi\times\varphi(x))\nabla_x(\varphi\times\varphi(x))^T.$$

We introduce the sequel of functionals $\mathcal{L}^G_{\varphi,\varepsilon}$ and $\mathcal{L}^g_{\varphi,\varepsilon}$ defined by

$$\mathcal{L}^G_{\varphi,\varepsilon}(u)=\int_{M\times M\setminus\Delta M}\Big(\varepsilon(A^G_\varphi)^{-1}(\Phi_\varphi-\nabla u)\cdot(\Phi_\varphi-\nabla u)+\varepsilon^{-1}\mathcal{R}(u-\theta_\varphi+\theta_{j_M})\Big)(\det A^G_\varphi)^{1/2}dx$$

and

$$\mathcal{L}^g_{\varphi,\varepsilon}(u)=\int_{M\times M\setminus\Delta M}\Big(\varepsilon(A^g_\varphi)^{-1}(\Phi_\varphi-\nabla u)\cdot(\Phi_\varphi-\nabla u)+\varepsilon^{-1}\mathcal{R}(u-\theta_\varphi+\theta_{j_M})\Big)(\det A^g_\varphi)^{1/2}dx.$$

We have

Proposition 31 *If φ is a deformation that has only transversal selfintersections, then the functionals $\mathcal{L}^G_{\varphi,\varepsilon}$ $\Gamma-$converges, as ε goes to zero, toward $\widetilde{\mathcal{L}^G_\varphi}$ defined for any $u\in SBV(\Omega)$ such that $u\equiv\theta_\varphi-\theta_{j_M}[2\pi]$ a.e. by*

$$\widetilde{\mathcal{L}^G_\varphi}(u)=\beta\int_U|[u]|\mathcal{H}^1_{A^G_\varphi}\llcorner Su,$$

and for any $u\in L^1(\Omega)$ such that $u\notin SBV(\Omega)$ or $\not\equiv\theta_\varphi-\theta_{j_M}[2\pi]$ on a nonnegligeable set by $\widetilde{\mathcal{L}^G_\varphi}(u)=+\infty$. We also have

$$\inf_u\widetilde{\mathcal{L}^G_\varphi}(u)=\inf_{\gamma\in\Gamma(\varphi)}\mathcal{L}^G(\gamma)=J^G(\varphi).$$

Furthermore, if u_ε is a minimizer of $\mathcal{L}^G_{\varphi,\varepsilon}$ and if

$$\gamma_\varepsilon=(u_\varepsilon-(\theta_\varphi-\theta_{j_M}))^{-1}(\pi\ [2\pi]),$$

then γ_ε converges (even if we have to extract a subsequence) toward a minimizer of $\mathcal{L}^G$ on $\Gamma(\varphi)$.

The same result remains true if we replace G by g. It follows that if

$$J^G_\varepsilon(\varphi)=\inf_u\mathcal{L}^G_{\varphi,\varepsilon}(u)\text{ and }J^g_\varepsilon(\varphi)=\inf_u\mathcal{L}^g_{\varphi,\varepsilon}(u),$$

then $J^G_\varepsilon(\varphi)$ (resp. $J^g_\varepsilon(\varphi)$) converges toward $J^G(\varphi)$ (resp. $J^g(\varphi)$). A correct approximation of the penalization terms can be derived by computing, according to the penalization used, J^G_ε or J^g_ε instead of J^G and J^g.

Even if we have replaced a onedimensional problem (computation of geodesics $\gamma\in\Gamma(\varphi)$ that minimize $\mathcal{L}^G_\varphi$ or $\mathcal{L}^g_\varphi$) with a twodimensional one (computation of the phase u which minimizes $\mathcal{L}^G_{\varphi,\varepsilon}$ or $\mathcal{L}^g_{\varphi,\varepsilon}$), the computational time of J^G_ε or J^g_ε is more or less independent of the selfintersections number of the deformation contrary to the direct computation of J^G and J^g. What is more, this method does not require an explicit description of the set $\Gamma(\varphi)$, which depends on M.

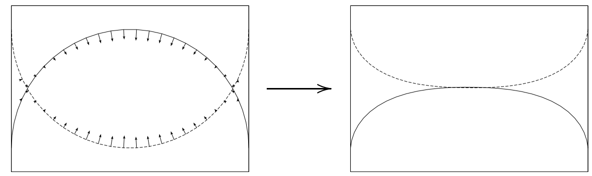


Fig. 8. Projection of a deformation which has a simple intersection

5.2. *Projection on the set of admissible deformations*

In the next section, we are going to apply our penalization method to a physical case. Before this, we address the projection problem on the set of admissible deformation. In other words, for a given deformation, to compute the "closest" admissible one. To this end, we propose to apply a gradient type algorithm, initialized with φ, to the penalization function J^G. In the following examples, we have used a constant step descent algorithm, endowed the set of deformations with the H^1 scalar product and used the metric $G(x,y)=|x-y|^4\,\mathrm{Id}$. The sequence of deformations φ_n are defined as follows

(i) Initialization. $\varphi_0=\varphi$
(ii) For all $n\geq 0$
  (a) Compute $J^G(\varphi_n)$ using the phasefield-like method (see Section 5.1.2) and its associated phase u_n.
  (b) Determination of the family of geodesics $\gamma_n\in\Gamma(\varphi_n)$, solution of the minimization problem

$$\mathcal{L}^G(\gamma_n)=\inf_{\gamma\in\Gamma(\varphi_n)}\mathcal{L}^G_{\varphi_n}(\gamma).$$

  The family γ_n is derived as the level sets of u_n whose values are equal to $\theta_{\varphi_n}-\theta_{j_M}+\pi\ [2\pi]$.
  (c) Computation of the gradient of J (see Proposition 25).
  (d) Gradient descent: Resolution of the variational problem

$$(\varphi_{n+1},\psi)_{H^1}=(\varphi_n,\psi)_{H^1}-h\langle J',\psi\rangle,$$

  for all test function ψ, where $h>0$ is the descent step.

The algorithm stops when the gradient of J is small enough.

Figure 8 illustrates the action of this algorithm on a deformation that has a simple intersection. The manifold M is made of two intervals, each of them are fixed at their ends. Note that, we have represented by a vector field the gradient of J on the initial configuration.

Figure 9 was obtained with other initializations.

Our algorithm can be applied to the particular case of selfintersections. We present several results produced for $M=S^1$ identified to $\mathbb{R}/2\pi\mathbb{Z}$. The chosen reference injection is $j_M(t)=(\cos(t),\sin(t))$ whereas the initialization deformation φ_0 is chosen either equal to $(\cos(2t),\sin(2t))$ (see Figures 11), to $(\cos(-t),\sin(-t))$ (see Figures 12), or to $(\cos(3t),\sin(3t))$ (see Figures 12). None of these deforma-

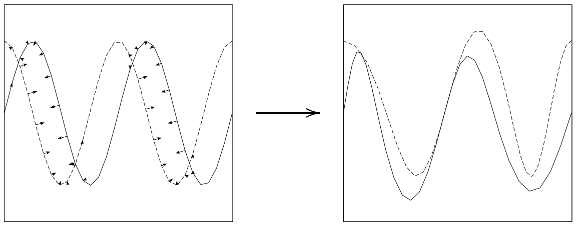


Fig. 9. Projection of a deformation with several intersections

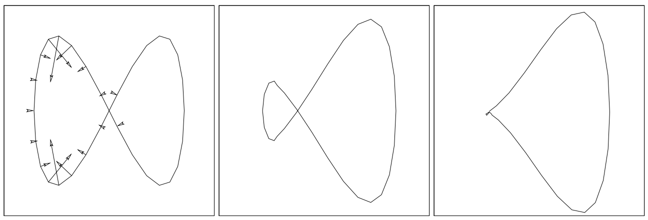


Fig. 10. Projection of the deformation $\varphi(t) = (\cos(t), \sin(2t))$

tions belongs to the admissible set $\mathcal{A}(j_M)$ as their turning numbers are not equal to one.

5.3. *Application to a physical case*

In this section, we give an example of application of our penalization method to a physical nonlinear problem. We consider two membranes M_1 and M_2 diffeomorphic to the interval $[0, \pi]$. Each one is made of a nonlinear elastic material, whose stored energy function is

$$W_\alpha(F) = \mu_\alpha \begin{cases} (|F|^2 - 1)^2, & \text{if } |F| \geq 1, \\ 0, & \text{if } |F| < 1. \end{cases},$$

where $\mu_\alpha > 0$ is an elasticity coefficient associated to the membrane M_α ($\alpha = 1, 2$). The internal energy associated to a deformation ψ_α of the membrane M_α is

$$E_\alpha(\psi_\alpha) = \int_{M_\alpha} W_\alpha(\dot{\psi}_\alpha) dx.$$

We assume that each membrane is fixed to a plane support

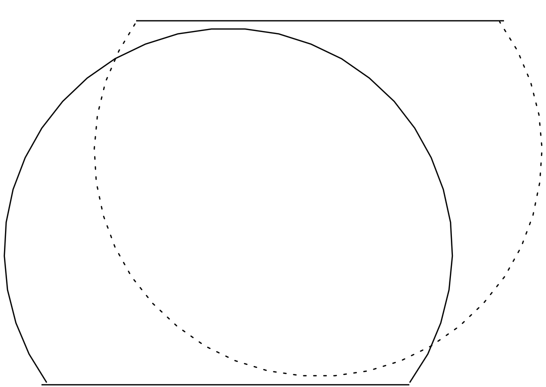


Fig. 14. Equilibrium state of the membranes without penalization

and that the space contained between the membranes and the supports is filled with a rare gas which exerts on each membrane a uniform pressure inversely proportional to the area V_α it occupies. Thus, the total energy associated to a couple $\psi = (\psi_1, \psi_2)$ of deformations is

$$I(\psi) = \sum_{\alpha=1,2} \int_{M_\alpha} W_\alpha(\dot{\psi}_\alpha) dx - C_\alpha \ln(V_\alpha).$$

Every equilibrium position of the membrane is a critical point of the energy on the set of admissible deformations. Since functional W_α is convex, there exists at least one minimizer φ on the set of admissible deformations:

$$I(\varphi) = \inf_{\psi \in \Phi(j_M)} I(\psi). \tag{38}$$

We have solved this problem by the penalization method we have presented. The minimizers of

$$I_\delta(\varphi^\delta) = \inf_\psi \left\{ I_\delta(\psi) := I(\psi) + \delta^{-1} J^G(\psi) \right\} \tag{39}$$

converge (even if we have to extract a subsequence) toward a solution φ of the initial problem (38) as $\delta > 0$ goes to zero.

On the numerical examples presented, we have chosen $C_1 = C_2 = 140/\pi$, $\mu_1 = \mu_2 = 1$; the reference injections are defined by $j_M|_{M_1}(t) = (-\cos(t), \sin(t))$ and $j_M|_{M_2}(t) = (0.5 - \cos(t), 2 - \sin(t))$, where $M = M_1 \cup M_2$ and M_1, M_2 are identified to the interval $[0, \pi]$. Figure 14 represents the equilibrium state of the membranes without taking into account the constraint of nonintersection. Figure 15 represents solutions φ^δ of the problem (39) obtained by a gradient method with constant step for different values of δ.

References

[1] L. T.A., Formulation and treatment of frictional contact problems using finite elements, SUDAM Report 92 (6).

[2] J. Laursen, T.A.; Simo, A continuum-based finite element formulation for the implicit solution of multibody, large deformation frictional contact problems., Internat. J. Numer. Methods Engrg. 36 (20) (1993) 3451–3485.

[3] H. M. A. S. S. J.W., M. F.J., A general-purpose contact detection algorithm for nonlinear structural analysis code, Sandia Report.

[4] D. J. Benson, Hallquist, A single surface contact algorithm for the post-buckling analysis of shell structures, Comput. Methods Appl. Mech. Engrg. 78 (2) (1990) 141–163.

[5] J. O. Hallquist, G. L. Goudreau, D. J. Benson, Sliding interfaces with contact-impact in large-scale Lagrangian computations, Comput. Methods Appl. Mech. Engrg. 51 (1-3) (1985) 107–137, fENOMECH '84, Part I, II (Stuttgart, 1984).

[6] P. Wriggers, Finite element algorithms for contact problems, Arch. Comput. Methods Engrg. 2 (4) (1995) 1–49.

[7] A. Klarbring, Large displacement frictional contact: a continuum framework for finite element discretization, European J. Mech. A Solids 14 (2) (1995) 237–253.

[8] N. Kikuchi, J. T. Oden, Contact problems in elasticity: a study of variational inequalities and finite element methods, Vol. 8 of SIAM Studies in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988.

[9] W. P., Computational Contact Mechanics, J. Wiley & Sons, New York, 2002.

[10] U. Wriggers, Peter; Nackenhorst, Analysis and Simulation of Contact Problems, Vol. 27 of Lecture Notes in Applied and Computational Mechanics, Springer-Verlag, Berlin, 2006.

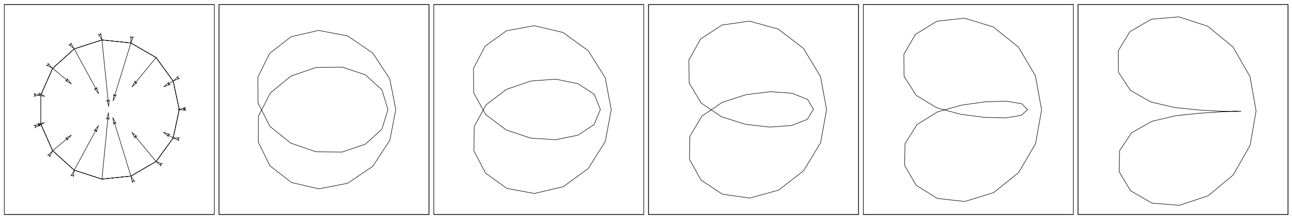


Fig. 11. Projection of the deformation $\varphi(t) = (\cos(2t), \sin(2t))$

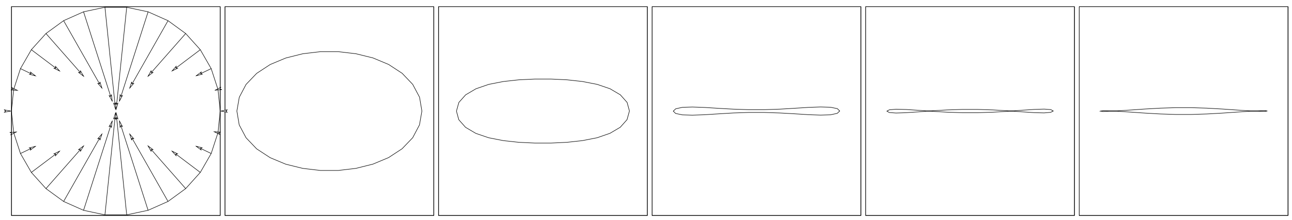


Fig. 12. Projection of the deformation $\varphi(t) = (\cos(-t), \sin(-t))$

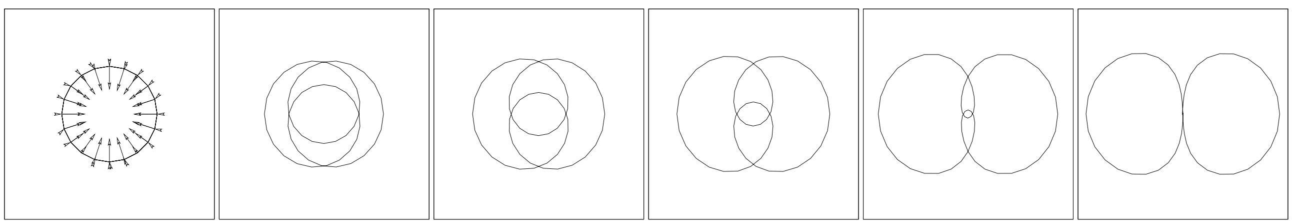
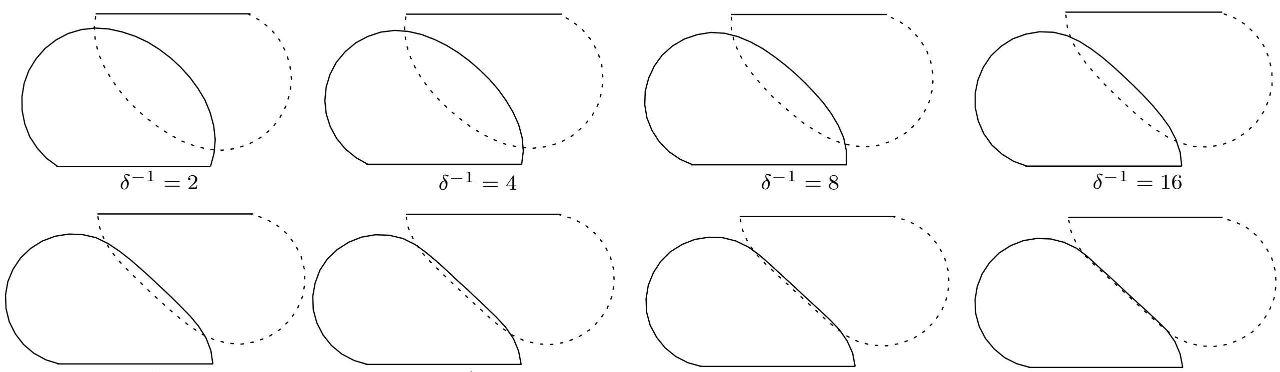


Fig. 13. Projection of the deformation $\varphi(t) = (\cos(3t), \sin(3t))$



$\delta^{-1} = 2$ $\delta^{-1} = 4$ $\delta^{-1} = 8$ $\delta^{-1} = 16$

$\delta^{-1} = 32$ $\delta^{-1} = 64$ $\delta^{-1} = 128$ $\delta^{-1} = 256$

Fig. 15. Equilibrium state of the membranes for different penalizations

[11] G. Hirota, F. S., S. A., An improved finite-element contact model for anatomical simulations, The Visual Computer 19 (5) (2003) 291–309.

[12] M. Jean, The non-smooth contact dynamics method, Comput. Methods Appl. Mech. Engrg. 177 (3-4) (1999) 235–257, computational modeling of contact and friction.

[13] M. Jean, V. Acary, Y. Monerie, Non-smooth contact dynamics approach of cohesive materials, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. 359 (1789) (2001) 2497–2518, non-smooth mechanics.

[14] J. J. Moreau, An introduction to unilateral dynamics, Lect. Notes Appl. Comput. Mech. 14 (2004) 1–26.

[15] V. J. Milenkovic, H. Schmidl, Optimization-based animation, in: SIGGRAPH '01: Proceedings of the 28th annual conference on Computer graphics and interactive techniques, ACM Press, New York, NY, USA, 2001, pp. 37–46.

[16] D. Baraff, Analytical methods for dynamic simulation of non-penetrating rigid bodies, in: SIGGRAPH '89: Proceedings of the 16th annual conference on Computer graphics and interactive techniques, ACM Press, New York, NY, USA, 1989, pp. 223–232.

[17] D. Baraff, Fast contact force computation for nonpenetrating rigid bodies, in: SIGGRAPH '94: Proceedings of the 21st annual conference on Computer graphics and interactive techniques, ACM Press, New York, NY, USA, 1994, pp. 23–34.

[18] D. Baraff, A. Witkin, Dynamic simulation of non-penetrating flexible bodies, in: SIGGRAPH '92: Proceedings of the 19th annual conference on Computer graphics and interactive techniques, ACM Press, New York, NY, USA, 1992, pp. 303–308.

[19] D. Baraff, A. Witkin, Large steps in cloth simulation, in: SIGGRAPH '98: Proceedings of the 25th annual conference on Computer graphics and interactive techniques, ACM Press, New York, NY, USA, 1998, pp. 43–54.

[20] J. M. Ball, Global invertibility of Sobolev functions and the interpenetration of matter, Proc. Roy. Soc. Edinburgh Sect. A 88 (3-4) (1981) 315–328.

[21] P. G. Ciarlet, J. Nečas, Injectivity and self-contact in nonlinear

elasticity, Arch. Rational Mech. Anal. 97 (3) (1987) 171–188.

[22] Q. Tang, Almost-everywhere injectivity in nonlinear elasticity, Proc. Roy. Soc. Edinburgh Sect. A 109 (1-2) (1988) 79–95.

[23] M. Giaquinta, G. Modica, J. Souček, Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity, Arch. Rational Mech. Anal. 106 (2) (1989) 97–159.

[24] M. Giaquinta, G. Modica, J. Souček, Erratum and addendum to: "Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity" [Arch. Rational Mech. Anal. **106** (1989), no. 2, 97–159; MR 90c:58044], Arch. Rational Mech. Anal. 109 (4) (1990) 385–392.

[25] O. Gonzalez, J. H. Maddocks, F. Schuricht, H. von der Mosel, Global curvature and self-contact of nonlinearly elastic curves and rods, Calc. Var. Partial Differential Equations 14 (1) (2002) 29–68.

[26] F. Schuricht, von der Mosel H., Euler-lagrange equations for nonlinearly elastic rods with self-contact, Arch. Rational Mech. Anal. 168 (1) (2003) 35–82.

[27] F. Schuricht, Regularity for shearable nonlinearly elastic rods in obstacle problems, Arch. Rational Mech. Anal. 145 (1) (1998) 23–49.

[28] F. Schuricht, A variational approach to obstacle problems for shearable nonlinearly elastic rods, Arch. Rational Mech. Anal. 140 (2) (1997) 103–159.

[29] F. Schuricht, Variational approach to contact problems in nonlinear elasticity, Calc. Var. Partial Differential Equations 15 (4) (2002) 433–449.

[30] O. Pantz, The modeling of deformable bodies with frictionless (self-)contacts, R.I. CMAP. Ecole Polytechnique, Palaiseau 585.

[31] L. Modica, S. Mortola, Un esempio di Γ^--convergenza, Boll. Un. Mat. Ital. B (5) 14 (1) (1977) 285–299.

[32] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, Arch. Rational Mech. Anal. 98 (2) (1987) 123–142.

[33] G. Alberti, Variational models for phase transitions, an approach via Γ-convergence, in: Calculus of variations and partial differential equations (Pisa, 1996), Springer, Berlin, 2000, pp. 95–114.