Derivation of nonlinear shell models combining shear and flexure. Application to biological membranes.

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Abstract

Biological membranes are often idealized as incompressible elastic surfaces whose strain energy only depends on their mean curvature and possibly on their shear. We show that this type of model can be derived using a formal asymptotic method by considering biological membranes to be thin, strongly anisotropic, elastic homogeneous bodies.

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1. Introduction

Shells, plates and membranes are solid deformable bodies having one characteristic dimension small by comparison with the other two dimensions. Their behavior is fully described by standard three-dimensional laws of continuum mechanics. Nevertheless, it is tempting, at least from the modeling viewpoint, to consider them as two-dimensional structures and to replace the genuine mechanical laws by two-dimensional reduced versions. This immediately raises two questions: (1) What is the correct model? and (2) How can it be mathematically justified? To this end, we consider the thickness ε of the plate/shell/membrane as a parameter and identify the limit behavior of the structure as ε goes to zero. According to the dependence of the elasticity moduli on the thickness of the shell, a full zoology of models may be derived. Membrane, isometric bending and von Kármán theories have been derived (amongst others), first formally (see Fox, Raoult and Simo [16]), then by means of Γ-convergence (see Le Dret and Raoult [25, 26], Pantz [35], Müller, Friesecke and James [18], Friesecke, James and Mora [17], see also [6]). In those works, elasticity coefficients are assumed to scale like a power of the thickness ε of the plate or shell, that is, like $\varepsilon^{-\alpha}$. Membrane theory corresponds to the case $\alpha = 1$, isometric bending to the case $\alpha = 3$ and von Kármán to $\alpha = 4$. Intermediate values of α have also been considered, and an almost exhaustive hierarchy of models has thus been produced (see Müller, Friesecke and James [19]). Some cases remain to be treated, Conti and Maggi [5], for instance, investigate the scaling of the energy corresponding to folds. The initial motivation for this article was the study of the mechanical behavior of Red Blood Cells (RBCs), and our aim was to determine whether the classical RBC model could be derived by the above procedure.

The mature anucleate RBCs[1] are made of two mechanical structures: The cytoskeleton – a two-dimensional network of protein filaments that extends throughout the interior of the cell – and a lipid bilayer. Both are bound together by proteins linking the nodes of the mesh of the cytoskeleton to the lipid bilayer via transmembrane proteins. Lipid bilayers are self-assembled structures of phospholipids which are small molecules containing a negatively charged phosphate group (called the head), and two highly hydrophobic fatty acid chains (called the tails). In an aqueous environment, phospholipids spontaneously form a double layer whose configuration enables to isolate the hydrophobic tails from the watery environment. Modifying the area of such a lipid bilayer is energy-costly because it exposes some of the tails to the environment.

A bilayer that supports no other mechanical structure, which is connected and has no boundary is called a vesicle. Vesicles are massively studied because they are easy to obtain experimentally. Moreover, they partially mimic the behavior of RBCs. Roughly speaking, they are RBCs without cytoskeleton (even if the RBC bilayer does embed a lot of different proteins responsible for different functions of the cell). They similarly resist to bending. However, a vesicle shows no resistance to shear stress, contrarily to RBCs owing to their cytoskeleton.

A widely used model consists in considering that a lipid bilayer may be endowed with an elastic energy depending solely on the mean curvature of the vesicle usually known as the Helfrich functional (named after Willmore in other contexts). It has been introduced, as far as we know independently, by Canham [4] and Helfrich [20] some forty years ago. Evans [14] shows that the Helfrich functional can be derived by assuming a vesicle to be made of two interconnected elastic fluid membranes, each of them resisting to change of local area but not to bending itself. Jenkins[21] has extended the analysis of Helfrich to general two-dimensional liquid crystals [39]. In particular, he derives the Euler-Lagrange equations satisfied by the equilibrium states, and examines the consequences of fluidity on the form of the strain energy (see also [41]). As a means to take into account the various vesicle shapes observed, it is common to presume that the vesicle is endowed with a nonzero spontaneous curvature. The origin of this spontaneous curvature is usually attributed to different compositions of the outer and inner layers. Several refinements to this basic model have since been proposed as the so-called bilayer-couple model [42], that consists in allowing the two lipidic layers to slip on one another, and imposing that the total area of each layer remains constant (see also [38] for a comparison between the two models). Miao, Seifert, Wortis and Döbereiner [34] proposed an intermediate model called area-difference elasticity model, where slight total area changes of each layer are allowed but still penalized.

As previously mentioned, the mechanical structure of the RBC is not only imputable to its bilayers. Their cytoskeleton endows them with resistance to shear stress. In most models, only the deformation of the RBC membrane

[1]Every mention to RBCs in this article will implicitly refer to anucleate mature RBCs without further notice.

is considered (that is of the bilayer). To take into account the presence of the cytoskeleton an additional term is added to the the total energy depending on the change of the metric of the membrane. Krishnaswamy [24] proposed another model for which the deformations of the cytoskeleton and the fluid bilayer may differ.

Finally, the aforementioned models for vesicles and RBCs are backed up by numerous numerical studies that allow to reproduce various shapes observed experimentally. Amongst others, Deuling and Helfrich [8] (see also Jenkins [22] and Luke [29, 30]) compute axisymmetric vesicle shapes of minimum energy with respect to the values of the reduced volume and spontaneous curvature. Seifert, Berndl and Lipowsky [38] compare the axisymmetric solutions obtained using the spontaneous curvature model and the bilayer-couple model, whereas Agrawal and Steigmann [1] include contact conditions between the vesicle and a substrate. Full three-dimensional simulations have been performed by Feng and Klug [15], Bonito, Nochetto and Pauletti [2, 3], Dziuk [13] using a finite element method. Peng et al [36] use a dissipative particle dynamic approach and focus on the interaction between the lipid bilayer and the cytoskeleton. Du, Chun and Xiaoqiang perform numerical computations based on a Phase Field Method [11, 10, 12]. Boundary Integral Methods have been used by Veerapaneni, Gueyffier and Zorin [43], Sohn, Tseng, Li, Voigt and Lowengrub [40]. Another approach based on the Immersed Boundary Method has been investigated by Kim and Lai [23], Liu et al [28, 27] and, together with a Lattice Boltzmann approach by Crowl and Fogleson [7]. Finally, Level Set Methods have also been implemented in this context by Salac and Miksis [37], and Maitre, Milcent, Cottet, Raoult and Usson [31] (see also Doyeux et al. [9]).

We prove in this article that the classical mechanical model of the RBC can be recovered by means of a formal asymptotic analysis assuming that the RBC membrane is made of a homogeneous, albeit strongly anisotropic, non-linearly elastic material. The main difference with previous works on the justification of thin structures is that we assume different scalings for the elastic moduli in the tangential and normal directions to the midsection. Let us underline that our work cannot be considered as a justification of the classical RBC mechanical model. Indeed, the RBC is not a homogeneous elastic membrane. Firstly because it is made of two different structures: The lipid bilayer (responsible for the resistance to bending) and a cytoskeleton (responsible for resistance to shear). Even the lipid bilayer could hardly be considered as made of a homogeneous material, the scale of the phospholipids it contains being of the same order as the thickness of the membrane. The cytoskeleton, being a two-dimensional spectrin network, is no more a homogeneous elastic body. Even if it is not overt at first glance, our work is strongly related to the justification, already mentioned, proposed by Evans [14].

We have chosen to consider a rather general setting (presented in section 2) for which the modeling of the RBCs is obtained as a particular case (see section 6). The asymptotic analysis is performed in section 3. Assuming that the minimizers of the energy admit an asymptotic expansion with respect to the thickness (section 3.2), they converge toward the solutions of a two-dimensional problem (see section 3.3). The limit energy, computed in section 3, contains membrane and flexural terms. In section 4, we prove that under invariance assumptions on the stored energy of the material, the flexural term depends only upon the second fundamental form, or even only upon the mean curvature of the shell. The isometric bending shell, RBC and vesicle models are obtained as particular applications in section 6. The last section is devoted to some general remarks in particular on the relaxation of the formal energy limit.

Finally, let us specify some notations. If M is a differentiable manifold, we denote by TM and T^*M its tangent and cotangent bundles. Moreover, $T^*(M;\mathbb{R}^3)$ will stand for the triple Withney sum $T^*M \oplus T^*M \oplus T^*M$. The tangent spaces of a product of manifolds will be implicitly identified with the product of the tangent spaces: so that if M_1 and M_2 are differentiable manifolds and $M = M_1 \times M_2$, the bundle TM will be implicitly identified with $TM_1 \times TM_2$. If M is an open subset of $\mathbb{R}^N$, TM will be identified with $M \times \mathbb{R}^N$. The corresponding identifications will also be made for T^*M and $T^*(M;\mathbb{R}^3)$. The set of reals $\mathbb{R}$ and its dual $\mathbb{R}'$ will also be often implicitly identified. Sets will always be displayed in capital letter (for instance the set of deformations ψ^ε will be denoted Ψ^ε). Sequences of terms of an asymptotic expansion are denoted using bold letters (for instance $\boldsymbol{\psi} = (\psi_k)_{k\in\mathbb{N}}$ stands for the asymptotic expansion of ψ^ε). Accordingly, the sets of asymptotic expansions used both bold and capitalized letters (for instance $\boldsymbol{\psi} \in \boldsymbol{\Psi}$). Moreover, calligraphic letters will be exclusively used for fiber spaces. Two different reference configurations are used throughout our article, one is qualified to be abstract and the other geometrical. The same notations are used for both configurations, the only distinction being that a tilde is added over variables, sets and functionals defined on the geometric configuration (for instance $\widetilde{\psi}^\varepsilon$ is the deformation defined on the geometric configuration, whereas ψ^ε stands for the deformation over the abstract one). All the notations introduced are recalled at the end of the article for convenience.

2. Elastic shells – Three dimensional modeling

We consider a thin nonlinearly elastic shell of midsurface S' and constant half thickness $\varepsilon > 0$, and choose $S^\varepsilon = S' \times (-\varepsilon, \varepsilon)$ to be the reference configuration of this elastic body. We assume S' to be a regular two-dimensional orientable submanifold of $\mathbb{R}^3$ with or without boundary. In the following, S' is implicitly endowed with the metric induced by the Euclidean metric in $\mathbb{R}^3$. Let ψ^ε be the deformation of the shell, that is, a map from S^ε into $\mathbb{R}^3$. The differential $D\psi^\varepsilon(x^\varepsilon)$ of ψ^ε at x^ε is a linear map from $T_{x^\varepsilon}S^\varepsilon$ into $T_{\psi^\varepsilon(x^\varepsilon)}\mathbb{R}^3$. Since $T_{\psi^\varepsilon(x^\varepsilon)}\mathbb{R}^3$ is canonically isomorphic to $\mathbb{R}^3$, $D\psi^\varepsilon(x^\varepsilon)$ is identified with an element of the Whitney sum $T^*S^\varepsilon \oplus T^*S^\varepsilon \oplus T^*S^\varepsilon$ denoted by $T^*(S^\varepsilon;\mathbb{R}^3)$. We denote by $J_\varepsilon(\psi^\varepsilon)$ the elastic energy of the shell under the deformation ψ^ε. We assume that the elastic energy is local and depends only on the first derivatives of the deformation. In other words, there exists a map W^ε from $T^*(S^\varepsilon;\mathbb{R}^3)$ into $\overline{\mathbb{R}}^+$ such that

$$J_\varepsilon(\psi^\varepsilon) := \int_{S^\varepsilon} W^\varepsilon(D\psi^\varepsilon)\, dx^\varepsilon,$$

where $dx^\varepsilon = dx' \wedge dx_3^\varepsilon$ and dx' is the two-dimensional Hausdorff measure restricted to S', whereas $D\psi^\varepsilon(x^\varepsilon)$ stands for the differential of ψ^ε at $x^\varepsilon \in S^\varepsilon$. Note that this representation enables us to consider inhomogeneous shells. The shell is assumed to be submitted to volumic dead body loads $f_\varepsilon \in L^2(S^\varepsilon;\mathbb{R}^3)$, and we set

$$L_\varepsilon(\psi^\varepsilon) := \int_{S^\varepsilon} f_\varepsilon \cdot \psi^\varepsilon\, dx^\varepsilon.$$

The total energy of the system is accordingly given by

$$I_\varepsilon(\psi^\varepsilon) := J_\varepsilon(\psi^\varepsilon) - L_\varepsilon(\psi^\varepsilon).$$

Finally, boundary conditions may also be added. We set $\Gamma^\varepsilon = \gamma \times (-\varepsilon, \varepsilon)$, where $\gamma \subset \partial S'$ is the – possibly empty – part of the boundary where the shell is clamped, and by ϕ^ε the imposed deformation on this set. Our aim is to determine the behavior of the minimizers φ^ε of I_ε over

$$\Psi^\varepsilon := \left\{\psi^\varepsilon \in W^{1,\infty}(S^\varepsilon)^3 \text{ such that } \psi^\varepsilon(x^\varepsilon) = \phi^\varepsilon(x^\varepsilon),\ x^\varepsilon \in \Gamma^\varepsilon \text{ a.e. } \right\}$$

as ε goes to zero. Note that the minimization problem of I_ε over Ψ^ε without any growth and polyconvex or quasiconvex assumptions on the stored energy function is generally not well posed. Here, we implicitly assume this problem to have a regular solution. Various assumptions have to be made regarding the dependence of the energy on the thickness for the needs of our analysis. These mainly concern the stored energy W^ε (see 2.1), but also the applied loads (see 2.2).

2.1. *Dependence of the stored energy functions with respect to the thickness*

We set $S = S^1$, and assume that the stored energy W^ε to be of the form

$$W^\varepsilon(F) = \varepsilon^{-1}\left(\varepsilon^{-2}W_2(F) + W_0(F)\right), \tag{1}$$

for every $F \in T^*(S^\varepsilon;\mathbb{R}^3)$ and $\varepsilon \leq 1$ where W_0 and W_2 are continuous nonnegative maps from $T^*(S;\mathbb{R}^3)$ into $\overline{\mathbb{R}}^+$. Note that we implicitly use the injection of $T^*(S^\varepsilon;\mathbb{R}^3) = T^*(S' \times (-\varepsilon,\varepsilon);\mathbb{R}^3) = T^*(S';\mathbb{R}^3) \times T^*((-\varepsilon,\varepsilon);\mathbb{R}^3)$ into $T^*(S;\mathbb{R}^3) = T^*(S' \times (-1,1);\mathbb{R}^3) = T^*(S';\mathbb{R}^3) \times T^*((-1,1);\mathbb{R}^3)$ in the definition (1) of W^ε. Standard analysis focuses on the case where only one element of this expansion is not zero. For instance, if $W_2 = 0$, we recover a nonlinear membrane model [26], and if $W_0 = 0$ we obtain the isometric bending one [17].

Behavior of strongly extended fibers. We assume that the stored energy W_2 is bounded from below by a positive constant for strongly extended fibers, namely, there exists $\delta, c > 0$ such that

$$\forall F' \in T^*(S';\mathbb{R}^3),\ \forall F_3 \in T^*((-1,1);\mathbb{R}^3) \text{ such that } |F_3| \geq \delta,\ \text{we have } W_2(F', F_3) \geq c. \tag{2}$$

Note that for every element F of a vector bundle endowed with a Riemann metric, the notation $|F|$ should be understood as the norm of the vectorial part of F. In particular, in the statement (2), $|F_3| = |v|$ if $F_3 = (x_3, v) \in (-1,1) \times \mathbb{R}^3 = T^*((-1,1);\mathbb{R}^3)$.

Zero set of W_2. We assume that W_2 is a C^2-nonnegative function and denote by $\mathcal{M}$ the restriction of its zero set to the midsection, that is,

$$\mathcal{M} := \Big\{F \in T^*(S';\mathbb{R}^3) \times T_0^*((-1,1);\mathbb{R}^3) \text{ such that } W_2(F) = 0\Big\}.$$

Let $\mathcal{M}'$ be the projection of $\mathcal{M}$ onto $T^*(S';\mathbb{R}^3)$, that is,

$$\mathcal{M}' := \Big\{F' \in T^*(S';\mathbb{R}^3) \ : \ \text{ there exists } n_0 \in T_0^*((-1,1);\mathbb{R}^3) \text{ such that } (F', n_0) \in \mathcal{M}\Big\}.$$

We assume that the projection of $\mathcal{M}$ onto $\mathcal{M}'$ is one to one. We denote by $n_0 : \mathcal{M}' \to T_0^*((-1,1);\mathbb{R}^3)$ the function that maps the elements F' of $\mathcal{M}'$ to the corresponding element F_3 of $T_0^*((-1,1);\mathbb{R}^3)$, so that

$$\mathcal{M} = \Big\{(F', n_0(F')) \in T^*(S;\mathbb{R}^3) \ : \ F' \in \mathcal{M}'\Big\}. \tag{3}$$

We recall that $S = S' \times (-1,1)$ and that $T^*(S;\mathbb{R}^3)$ is identified with $T^*(S';\mathbb{R}^3) \times T^*((-1,1);\mathbb{R}^3)$. The vectorial part of $n_0(F') \in T_0^*((-1,1);\mathbb{R}^3) = \{0\} \times \mathbb{R}'^3$ will be noted $n(F')$, so that $n_0(F') = (0, n(F'))$.

Local interpenetration. To avoid local interpenetration of matter, it is geometric to expect $D\psi^\varepsilon$ to be invertible. To this end, we require that $W_0(F) = \infty$ for all $F \in \mathcal{M}$ such that $\det F < 0$ and that

$$W_0(F) \to \infty \quad \text{if} \quad F \in \mathcal{M} \quad \text{and} \quad \det(F) \to 0. \tag{4}$$

2.2. Dependence of the applied loads with respect to the thickness

The volumic loads are assumed to scale as the inverse of the thickness of the shell and more precisely that there exists $f : S \to \mathbb{R}^3$ such that for all $\varepsilon \le 1$,

$$f_\varepsilon(x) = \varepsilon^{-1} f(x) \qquad \text{for every } x \in S^\varepsilon. \tag{5}$$

3. From 3D to 2D – A formal asymptotic analysis

3.1. Rescaling

We set $\psi(\varepsilon)(x', x_3) = \psi^\varepsilon(x', \varepsilon x_3)$. Moreover, we define the rescaled energies

$$J(\varepsilon)(\psi(\varepsilon)) := J_\varepsilon(\psi^\varepsilon) \quad \text{and} \quad I(\varepsilon)(\psi(\varepsilon)) := I_\varepsilon(\psi^\varepsilon).$$

For every map $\psi^\varepsilon : S^\varepsilon \to \mathbb{R}^3$, we denote by $(D'\psi^\varepsilon, D_3\psi^\varepsilon)$ the decomposition of the differential ψ^ε along the sections of the cylinder S^ε and along its fibers respectively. In other words, for all $x^\varepsilon = (x', x_3) \in S^\varepsilon$, $D'\psi^\varepsilon(x^\varepsilon)$ and $D_3\psi^\varepsilon(x^\varepsilon)$ stand for the elements of $T^*_{x'}(S'; \mathbb{R}^3)$ and $T^*_{x_3}((-1,1); \mathbb{R}^3)$ such that $D\psi^\varepsilon(x^\varepsilon) = (D'\psi^\varepsilon(x^\varepsilon), D_3\psi^\varepsilon(x^\varepsilon))$.

For every deformation $\psi(\varepsilon)$ of S, we define its partial derivation $\partial_3\psi(\varepsilon)$ with respect to the normal direction as

$$\partial_3\psi(\varepsilon)(x', x_3) = \lim_{t\to 0} \frac{\psi(\varepsilon)(x', x_3 + t) - \psi(\varepsilon)(x', x_3)}{t}.$$

Performing a simple change of variable, we get

$$\begin{aligned}
J(\varepsilon)(\psi(\varepsilon)) &= \varepsilon^{-1} \int_{S^\varepsilon} (\varepsilon^{-2} W_2 + W_0)\,(D'\psi^\varepsilon(x^\varepsilon), D_3\psi^\varepsilon(x^\varepsilon))\, dx^\varepsilon \\
&= \varepsilon^{-1} \int_{S^\varepsilon} (\varepsilon^{-2} W_2 + W_0)\left(D'\psi^\varepsilon(x^\varepsilon), (x_3^\varepsilon, \partial_3\psi^\varepsilon(x^\varepsilon))\right) dx^\varepsilon \\
&= \int_S (\varepsilon^{-2} W_2 + W_0)\left(D'\psi(\varepsilon)(x), (\varepsilon x_3, \varepsilon^{-1}\partial_3\psi(\varepsilon)(x))\right) dx.
\end{aligned}$$

3.2. Ansatz

In order to perform our formal analysis, we assume that the minimizers $\varphi(\varepsilon)(x', x_3) = \varphi^\varepsilon(x', \varepsilon x_3)$ of the energy admit an asymptotic expansion

$$\varphi(\varepsilon)(x) = \sum_{k\geq 0} \varepsilon^k \varphi_k(x) \text{ for every } x \in S, \tag{6}$$

with $(\varphi_k) \in \ell^1(W^{1,\infty}(S, \mathbb{R}^3))$. Obviously, the same assumption has to be made on the applied Dirichlet boundary conditions and we let $\boldsymbol{\phi} = (\phi_k)$ be the terms of the asymptotic expansion of the deformation $\phi(\varepsilon)(x) = \phi^\varepsilon(x', \varepsilon x_3)$ imposed on $\Gamma := \gamma \times (-1, 1)$, that is

$$\phi(\varepsilon)(x) = \sum_{k\geq 0} \varepsilon^k \phi_k(x) \text{ for every } x \in S. \tag{7}$$

The condition $\varphi^\varepsilon \in \Psi^\varepsilon$ reads as $\varphi_k(x) = \phi_k(x)$, $x \in \Gamma$ almost everywhere. Consequently, we introduce the admissible set

$$\boldsymbol{\Psi} := \left\{\boldsymbol{\psi} = (\psi_k) \in \ell^1(W^{1,\infty}(S, \mathbb{R}^3)) \text{ such that } \psi_k = \phi_k \text{ for a.e. } x \in \Gamma\right\},$$

and the rescaled energies $\mathbf{J}(\varepsilon)$ and $\mathbf{I}(\varepsilon)$ from $\boldsymbol{\Psi}$ into $\overline{\mathbb{R}}$ defined by

$$\mathbf{J}(\varepsilon)(\boldsymbol{\psi}) := J(\varepsilon)\left(\sum_{k\geq 0} \varepsilon^k \psi_k\right) \quad \text{and} \quad \mathbf{I}(\varepsilon)(\boldsymbol{\psi}) := I(\varepsilon)\left(\sum_{k\geq 0} \varepsilon^k \psi_k\right). \tag{8}$$

3.3. Limit of the total energy

The first step of our analysis consists in computing the limit of $\mathbf{J}(\varepsilon)(\boldsymbol{\psi})$ as ε goes to zero for $\boldsymbol{\psi} \in \boldsymbol{\Psi}$. As we shall see in Proposition 1, the limit of $\mathbf{J}(\varepsilon)$ contains two terms. Roughly speaking, one term measures the elastic energy due to the change of the metric of the midsection of the shell. It depends only on W_0. The second term measures the elastic energy due to the variations of the orientation of its fibers. It depends on the second derivative of the stored energy function W_2 through a quadratic form $Q_{D'\psi_0}$.

In order to enhance the readability of the sequel, we introduce a practical notation. We recall that a section F of a vector bundle $\mathcal{F}$ is a map from its base into $\mathcal{F}$ such that $\pi_B(F)$ is the identity, where π_B stands for the projection of $\mathcal{F}$ onto its base B. Given such a section, we define the bundle map

$$\mathcal{F} \to T\mathcal{F}, \quad G \mapsto G_F = \frac{d}{dt}(F(\pi_B(G)) + tG)_{|t=0}. \tag{9}$$

Roughly speaking, G_F is the element G of $T_{F(\pi_B(G))}\mathcal{F}$. Similarly, for every $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$, we will sometimes denote $(x, v) \in T_x\mathbb{R}^N$ by v_x. Let F' be a section of $\mathcal{M}'$, we set for every $(G', s, v) \in T^*(S'; \mathbb{R}^3) \times \mathbb{R} \times (\mathbb{R}')^3$,

$$Q_{F'}(G', s, v) := D^2 W_2[G'_{F'}, s_0, v_{n(F')}]^2, \tag{10}$$

where $(G'_{F'}, s_0, v_{n(F')})$ is the element of $T_{(F',\, n_0(F'))}(T^*(S\,; \mathbb{R}^3))$ defined in (9) based on the decomposition of $T^*(S\,; \mathbb{R}^3) = T^*(S'; \mathbb{R}^3) \times (-1, 1) \times (\mathbb{R}')^3$, while $D^2 W_2$ stands for the Hessian of W_2. Namely, we have

$$(G'_{F'}, s_0, v_{n(F')}) = \frac{d\gamma}{dt}(0) \quad \text{where} \quad \gamma(t) = (F'_{\pi_{S'}(G')} + tG', ts, n(F') + tv). \tag{11}$$

At first glance, the meaning of $D^2 W_2[\dot{\gamma}(0)]^2$ is unclear, considering that the Hessian of a map defined on a manifold is not, in general, intrinsically defined. Nevertheless, it is well known that this is consistent on the set of critical points, which is precisely what is considered here. Indeed, $\gamma(0)$ is equal to the value of the section $(F', n_0(F'))$ at $\pi_{S'}(G)$. Yet, F' is a section of $\mathcal{M}'$, hence $W_2(F', n_0(F)) = 0$, $W_2(\gamma(0)) = 0$ and $DW_2(\gamma(0)) = 0$. As a result, $D^2 W_2[\dot{\gamma}(0)]^2$ is well defined and, accordingly,

$$D^2 W_2[\dot{\gamma}(0)]^2 = 2 \lim_{t\to 0} t^{-2} W_2(\gamma(t)). \tag{12}$$

Note that the right-hand side of (12) only depends on $\dot{\gamma}(0)$, so that the particular choice of the representative $\gamma(t)$ of $\dot{\gamma}(0)$ is irrelevant as already mentioned.

We are now in a position to state the main result of this section.

Proposition 1. *Let $\boldsymbol{\Phi}$ be the subset of the admissible set $\boldsymbol{\Psi}$ defined by*

$$\boldsymbol{\Phi} := \{\boldsymbol{\psi} \in \boldsymbol{\Psi} : \partial_3\psi_0 = 0,\ D'\psi_0(x) \in \mathcal{M}' \text{ and } \partial_3\psi_1(x) = n(D'\psi_0(x)) \text{ for a.e. } x \in S\}. \tag{13}$$

Let $\boldsymbol{\psi} \in \boldsymbol{\Psi}$, then

$$\lim_{\varepsilon\to 0} \mathbf{I}(\varepsilon)(\boldsymbol{\psi}) = \begin{cases} I_0\left(\psi_0, \frac{1}{2}\int_{-1}^{1} \psi_1\, dx_3, \partial_3\psi_2\right) & \text{if } \boldsymbol{\psi} \in \boldsymbol{\Phi}, \\ +\infty & \text{if } \boldsymbol{\psi} \notin \boldsymbol{\Phi}, \end{cases}$$

where

$$I_0(\psi_0, u, v) := J_0(\psi_0, u, v) - 2\int_{S'} f_0 \cdot \psi_0\, dx,$$

$$J_0(\psi_0, u, v) := \frac{1}{2}\int_S Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v)\, dx, +2\int_{S'} W_0(D'\psi_0, n_0)\, dx',$$

where n and n_0 stand for $n(D'\psi_0)$ and $n_0(D'\psi_0)$ for short, $Q_{D'\psi_0}$ is defined by (10) and $f_0(x') = f(x', 0)$.

Proof. We proceed in two steps. First, we prove that every sequence of deformations $\boldsymbol{\psi} \in \boldsymbol{\Psi}$ of finite elastic energy, namely which satisfies

$$\liminf_{\varepsilon\to 0} \mathbf{J}(\varepsilon)(\boldsymbol{\psi}) < +\infty,$$

belongs to $\boldsymbol{\Phi}$. In particular, this implies that $\mathbf{J}(\varepsilon)(\boldsymbol{\psi})$ converges to infinity as ε goes to zero, for every $\boldsymbol{\psi}$ that is not in $\boldsymbol{\Phi}$. In a second step, we compute the limit of $\mathbf{J}(\varepsilon)(\boldsymbol{\psi})$ for every $\boldsymbol{\psi}$ in $\boldsymbol{\Phi}$.

Let $\boldsymbol{\psi} \in \boldsymbol{\Psi}$ be the asymptotic expansion of a deformation of finite elastic energy. From Fatou's lemma, we deduce

$$\int_S \liminf_{\varepsilon\to 0}(\varepsilon^{-2} W_2)\left(\sum_{k\geq 0} \varepsilon^k \left(D'\psi_k, \left(\varepsilon x_3, \varepsilon^{-1}\partial_3\psi_k\right)\right)\right) dx$$

$$\leq \liminf_{\varepsilon\to 0} \int_S (\varepsilon^{-2} W_2 + W_0)\left(\sum_{k\geq 0} \varepsilon^k \left(D'\psi_k, \left(\varepsilon x_3, \varepsilon^{-1}\partial_3\psi_k\right)\right)\right) dx = \liminf_{\varepsilon\to 0} \mathbf{J}(\varepsilon)(\boldsymbol{\psi}) < \infty.$$

Hence, we have

$$\liminf_{\varepsilon\to 0} W_2\left(\sum_{k\geq 0}\varepsilon^k\left(D'\psi_k,\left(\varepsilon x_3,\varepsilon^{-1}\partial_3\psi_k\right)\right)\right) = 0 \text{ a.e.}$$

From the assumption (2) made on the behavior of strongly extended fibers, if follows that for almost every $x \in S$, $\sum_{k\geq 0}\varepsilon^{k-1}\partial_3\psi_k$ remains bounded (up to a subsequence), that is $\partial_3\psi_0 = 0$. Now, since

$$\sum_{k\geq 0}\varepsilon^k\left(D'\psi_k,\left(\varepsilon x_3,\varepsilon^{-1}\partial_3\psi_k\right)\right) \xrightarrow[\varepsilon\to 0]{L^\infty} (D'\psi_0,(0,\partial_3\psi_1)), \tag{14}$$

and W_2 is assumed to be continuous, we have $W_2(D'\psi_0,(0,\partial_3\psi_1)) = 0$ almost everywhere. From the hypothesis (3), we get $D'\psi_0 \in \mathcal{M}'$ and $\partial_3\psi_1 = n(D'\psi_0)$. As a conclusion, every sequence of deformations of finite elastic energy belongs to $\mathbf{\Phi}$ as announced.

We move on to the next step. Let us consider an element $\boldsymbol{\psi} \in \mathbf{\Phi}$ and its associated energy

$$\lim_{\varepsilon\to 0}\mathbf{J}(\varepsilon)(\boldsymbol{\psi}) = \lim_{\varepsilon\to 0}\varepsilon^{-2}\int_S W_2\left(\sum_{k\geq 0}\varepsilon^k\left(D'\psi_k,\left(\varepsilon x_3,\varepsilon^{-1}\partial_3\psi_k\right)\right)\right)dx + \int_S W_0(D'\psi_0,(0,\partial_3\psi_1))\,dx.$$

Considering that W_2 is a C^2-function, and that $W_2(D'\psi_0,(0,\partial_3\psi_1)) = 0$, combined with (12) with $\gamma(t) = (D'\psi_0 + tD'\psi_1, tx_3, \partial_3\psi_1 + t\partial_3\psi_2)$, Lebesgue's theorem implies

$$\begin{aligned}
\lim_{\varepsilon\to 0}\varepsilon^{-2}\int_S W_2\left(\sum_{k\geq 0}\varepsilon^k\left(D'\psi_k,\left(\varepsilon x_3,\varepsilon^{-1}\partial_3\psi_k\right)\right)\right)dx &= \frac{1}{2}\int_S D^2W_2\left[(D'\psi_{1D'\psi_0(x)}, x_{30}, \partial_3\psi_{2\partial_3\psi_1})\right]^2 dx\\
&= \frac{1}{2}\int_S D^2W_2\left[(D'\psi_{1D'\psi_0(x)}, x_{30}, \partial_3\psi_{2n(D'\psi_0)})\right]^2 dx\\
&= \frac{1}{2}\int_S Q_{D'\psi_0}(D'\psi_1, x_3, \partial_3\psi_2)\,dx.
\end{aligned}$$

The limit of the elastic energy $\mathbf{J}(\varepsilon)$ falls out from the fact that ψ_1 may be written as

$$\psi_1 = \frac{1}{2}\int_{-1}^{1}\psi_1\,dx_3 + x_3 n(D'\psi_0)$$

Finally, according to the definition (5) of f, we have

$$\mathbf{I}(\varepsilon)(\boldsymbol{\psi}) = \mathbf{J}(\varepsilon)(\boldsymbol{\psi}) - \int_S f(x',\varepsilon x_3)\cdot\left(\sum_{k\geq 0}\varepsilon^k\psi_k\right)dx.$$

The second term on the right-hand side converges towards $2\int_{S'} f\cdot\psi_0\,dx$ as ε goes to zero. □

Since the limit energy is finite only for elements $\boldsymbol{\psi}$ in $\mathbf{\Phi}$, $\boldsymbol{\phi}$ has to be be equal to an element of $\mathbf{\Phi}$ on the subset Γ of the boundary where clamping conditions are imposed.

Corollary 1. *If the minimizers $\varphi(\varepsilon)$ of the total rescaled energy $I(\varepsilon)$ admit an asymptotic expansion as in (6), and if their total energy $I(\varepsilon)(\varphi(\varepsilon))$ remains bounded, then $\phi_0(x', x_3)$ depends only on $x' \in \Gamma$. In addition, there exists $u_\gamma, n_\gamma \in W^{1,\infty}(\gamma)^3$ such that*

$$\phi_1(x',x_3) = u_\gamma(x') + x_3 n_\gamma(x') \quad \textit{for a.e. } x \in \Gamma.$$

Note: since ϕ_0 depends only on x', we shall write $\phi_0(x')$ instead of $\phi_0(x', x_3)$ henceforth.

3.4. Convergence of the minimizers

Lemma 1. *If the minimizers $\varphi(\varepsilon)$ of the total rescaled energy $I(\varepsilon)$ admit an asymptotic expansion as in (6), and if their total energy $I(\varepsilon)(\varphi(\varepsilon))$ remains bounded, then*

$$I_0\left(\varphi_0, \frac{1}{2}\int_{-1}^{1} \varphi_1\, dx_3, \partial_3\varphi_2\right) \le \inf_{(\psi_0,u,v)\in\Phi_0} I_0(\psi_0, u, v),$$

where

$$\Phi_0 := \Big\{(\psi_0, u, v) \in W^{1,\infty}(S')^3 \times W^{1,\infty}(S')^3 \times W^{1,\infty}(S'; L^\infty(-1,1)^3) \ : \ n(D'\psi_0) \in W^{1,\infty}(S')^3,$$
$$\psi_0(x') = \phi_0(x'),\ u(x') = u_\gamma(x'),\ n(D'\psi_0(x')) = n_\gamma(x') \textit{ for a.e. } x' \in \gamma,\ \textit{and } v(x) = \partial_3\phi_2(x) \textit{ for a.e. } x \in \Gamma\Big\}.$$

Proof. Let (φ_k) be the asymptotic expansion of a minimizer $\varphi(\varepsilon)$ of the total energy $I(\varepsilon)$. For every $(\psi_0, u, v) \in \Phi_0$, we set

$$\psi_1 = u + x_3 n(D'\psi_0), \quad \psi_2(x) = \phi_2(x', 0) + \int_0^{x_3} v(x', s)\, ds, \quad \text{and} \quad \psi_k = \phi_k \quad \text{for every } k \ge 3.$$

It can be easily checked that $\boldsymbol{\psi}$ belongs to $\boldsymbol{\Phi}$. Therefore, we have $\mathbf{I}(\varepsilon)(\boldsymbol{\varphi}) \le \mathbf{I}(\varepsilon)(\boldsymbol{\psi})$, and from Proposition 1, we get

$$I_0\left(\varphi_0, \frac{1}{2}\int_{-1}^{1} \varphi_1\, dx_3, \partial_3\varphi_2\right) \le I_0(\psi_0, u, v). \tag{15}$$

□

Lemma 2. *If the minimizers $\varphi(\varepsilon)$ of the total rescaled energy $I(\varepsilon)$ admit an asymptotic expansion as in (6), and if their total energy $I(\varepsilon)(\varphi(\varepsilon))$ remains bounded, then*

$$\left(\varphi_0, \frac{1}{2}\int_{-1}^{1} \varphi_1\, dx_3\right) = \arg\min_{(\psi_0,u)\in\Phi_1} I_1(\psi_0, u),$$

where

$$\Phi_1 := \Big\{(\psi_0, u) \in W^{1,\infty}(S')^3 \times W^{1,\infty}(S')^3 \ : \ n(D'\psi_0) \in W^{1,\infty}(S')^3,$$
$$\psi_0(x') = \phi_0(x'),\ u(x') = u_\gamma(x'),\ \textit{and } n(D'\psi_0(x')) = n_\gamma(x') \textit{ for a.e. } x' \in \gamma\Big\},$$

$$I_1(\psi_0, u) := \int_{S'} Q^0_{D'\psi_0}(D'u, 0)dx', + \frac{1}{3}\int_{S'} Q^0_{D'\psi_0}(D'n, 1)dx' + 2\int_{S'} W_0(D'\psi_0, n_0(D'\psi_0))dx' - 2\int_{S'} f_0 \cdot \psi_0 dx',$$

$f_0(x') = f(x', 0)$ for every $x' \in S'$, and

$$Q^0_{F'}(G', x_3) = \inf_{v\in\mathbb{R}^3} Q_{F'}(G', x_3, v). \tag{16}$$

Proof. From Lemma 1, we have

$$I_0\left(\varphi_0, \frac{1}{2}\int_{-1}^{1} \varphi_1\, dx_3, \partial_3\varphi_2\right) \le \inf_{(\psi_0,u,v)\in\Phi_0} I_0(\psi_0, u, v).$$

Moreover, from Proposition 1, we have $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ which implies that $\left(\varphi_0, \frac{1}{2}\int_{-1}^{1} \varphi_1\, dx_3\right) \in \Phi_1$. Since $\Phi_0 = \Phi_1 \times V$ with $V := \{v \in W^{1,\infty}(S'; L^\infty(-1,1)^3) \ : \ v(x) = \partial_3\phi_2(x) \text{ for a.e. } x \in \Gamma\}$, it follows that

$$\left(\varphi_0, \frac{1}{2}\int_{-1}^{1} \varphi_1\, dx_3\right) = \arg\min_{(\psi_0,u)\in\Phi_1} \inf_{v\in V} I_0(\psi_0, u, v).$$

To complete the proof, we need to show that for every $(\psi_0, u) \in \Phi_1$, we have

$$\inf_{v\in V} I_0(\psi_0, u, v) = I_1(\psi_0, u). \tag{17}$$

We recall that for every $(\psi_0, u) \in \Phi_1$ and every $v \in V$, we have

$$I_0(\psi_0, u, v) = \frac{1}{2}\int_S Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v)\, dx + 2\int_{S'} W_0(D'\psi_0, n_0(D'\psi_0))dx' - 2\int_{S'} f_0 \cdot \psi_0 dx'.$$

Next, the definition of $Q^0_{D'\psi_0}$ entails that

$$I_0(\psi_0, u, v) \geq \frac{1}{2}\int_S Q^0_{D'\psi_0}((D'u, 0) + x_3(D'n, 1))\, dx + 2\int_S W_0(D'\psi_0, n_0(D'\psi_0))dx' - 2\int_{S'} f_0 \cdot \psi_0 dx'.$$

Furthermore, for every $x' \in S'$ and every $F \in T^*_{(x',0)}(S;\mathbb{R}^3)$, the quadratic form $Q^0_{F'}$ derives from a bilinear form. Hence,

$$\begin{aligned}\int_{-1}^{1} Q^0_{D'\psi_0}((D'u, 0) + x_3(D'n, 1))\, dx_3 &= \int_{-1}^{1}\left[Q^0_{D'\psi_0}(D'u, 0) + x_3^2 Q^0_{D'\psi_0}(D'n, 1)\right] dx_3 \\ &= 2Q^0_{D'\psi_0}(D'u, 0) + \frac{2}{3}Q^0_{D'\psi_0}(D'n, 1).\end{aligned}$$

Accordingly, we obtain that $I_0(\psi_0, u, v) \geq I_1(\psi_0, u)$, so that $\inf_{v\in V} I_0(\psi_0, u, v) \geq I_1(\psi_0, u)$. It remains to prove the converse inequality to establish (17). For every $\delta \geq 0$, we have

$$I_0(\psi_0, u, v) \leq I_0(\psi_0, u, v) + \int_S \delta|v|^2\, dx.$$

As a consequence

$$\inf_{v\in V} I_0(\psi_0, u, v) \leq \inf_{v\in V}\left(I_0(\psi_0, u, v) + \int_S \delta|v|^2\, dx\right).$$

Let $v_\delta : S \to \mathbb{R}^3$ be the map such that $v_\delta = \arg\min_v Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v) + \delta|v|^2$. Since W_2 is assumed to be of class C^2 and $D'\psi_0$ is bounded, the norm of the quadratic form $Q_{D'\psi_0}$ is uniformly bounded. As a result, v_δ is measurable and belongs to $L^\infty(S)^3$. Also, there exists a sequence v^k_δ in V converging towards v_δ in $L^2(S)^3$ as k goes to infinity, due to the density of V in $L^2(S)^3$. For every k, we have

$$\inf_{v\in V} I_0(\psi_0, u, v) \leq \frac{1}{2}\int_S \left[Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v^k_\delta) + \delta|v^k_\delta|^2\right] dx + 2\int_{S'} W_0(D'\psi_0, n_0(D'\psi_0))dx' - 2\int_{S'} f_0 \cdot \psi_0 dx'$$

Taking the limit with respect to k, we infer that

$$\inf_{v\in V} I_0(\psi_0, u, v) \leq \frac{1}{2}\int_S \left[Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v_\delta) + \delta|v_\delta|^2\right] dx + 2\int_{S'} W_0(D'\psi_0, n_0(D'\psi_0))dx' - 2\int_{S'} f_0 \cdot \psi_0 dx'$$

Note that $Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v_\delta) + \delta|v_\delta|^2$ is a decreasing sequence (as δ goes to zero) of nonnegative functions. Therefore, its integral over S converges towards its pointwise limit $Q^0_{D'\psi_0}(D'u + x_3 D'n, x_3)$, and the intended inequality follows

$$\inf_{v\in V} I_0(\psi_0, u, v) \leq \frac{1}{2}\int_S Q^0_{D'\psi_0}(D'u + x_3 D'n, x_3)\, dx + 2\int_{S'} W_0(D'\psi_0, n)dx' - 2\int_{S'} f_0 \cdot \psi_0 dx' = I_1(\psi_0, u).$$

□

3.5. Boundary conditions

An interesting feature of the limit energy is that it depends on both ψ_0 and $u = \frac{1}{2}\int \psi_1 \, dx_3$. For general boundary conditions, it implies a coupling between both quantities through the term $\int_{S'} Q^0_{D'\psi_0}(D'u, 0)\, dx'$ of $I_1(\psi_0, u)$. Hence, small perturbations scaling as the thickness of the shell may have an influence on the deformation ψ_0 of the midsection. In the literature, the boundary conditions are usually chosen to satisfy $u_\gamma = 0$, that is,

$$\phi_0(x) = \phi_0(x') \quad \text{and} \quad \phi_1(x) = x_3 n_\gamma(x') \quad \text{for all } (x', x_3) \in \Gamma^\varepsilon, \tag{18}$$

where n_γ is a unit vector. In this case, the minimization of $I_1(\psi_0, u)$ with respect to u is trivial and the limit energy can be expressed solely in terms of ψ_0.

Proposition 2. *If the minimizers $\varphi(\varepsilon)$ of the total rescaled energy $I(\varepsilon)$ admit an asymptotic expansion as in (6), and if their total energy $I(\varepsilon)(\varphi(\varepsilon))$ remains bounded with $u_\gamma = 0$ on γ, then*

$$\varphi_0 = \arg\min_{\psi_0 \in V_0} I_0(\psi_0),$$

where

$$I_0(\psi_0) := \frac{1}{3}\int_{S'} Q^0_{D'\psi_0}(D'n, 1)dx' + 2\int_{S'} W_0(D'\psi_0, n_0(D'\psi_0))dx' - 2\int_{S'} f_0 \cdot \psi_0 dx', \tag{19}$$

$f_0(x') = f(x', 0)$ for every $x' \in S'$, and

$$\Psi_0 := \left\{\psi_0 \in W^{1,\infty}(S')^3 \ : \ n = n(D'\psi_0) \in W^{1,\infty}(S')^3, D'\psi_0 \in \mathcal{M}', \ \psi_0(x') = \phi_0(x'), \ \text{and } n = n_\gamma(x') \text{ for a.e. } x' \in \gamma\right\}.$$

4. Invariance and flexural energy

Under several assumptions on the stored energy function W_2, the expression of the flexural part

$$I_{flex}(\psi_0) := \frac{1}{3}\int_{S'} Q^0_{D'\psi_0}(D'n, 1)\, dx \tag{20}$$

of the total limit energy $I_0(\psi_0)$ may be reduced. More precisely, we shall consider the implications of homogeneity along the fibers, frame-indifference (left invariance under SO(3)), planar isotropy (right invariance under in-plane rotations), and finally right invariance under the special linear group of TS' of the stored energy.

4.1. Homogeneity along the fibers

We say that the shell is homogeneous along the fibers if for every $(F', x_3, v) \in T^*(S;\mathbb{R}^3) = T^*(S';\mathbb{R}^3) \times (-1,1) \times (\mathbb{R}')^3$, we have $W_2(F', x_3, v) = W_2(F', 0, v)$. In this case, we have for every $(G', s, v) \in T^*(S';\mathbb{R}^3) \times \mathbb{R} \times (\mathbb{R}')^3$ and every section F' of $\mathcal{M}'$, $D^2W_2[G'_{F'}, s_0, v_{n(F')}]^2 = D^2W_2[G'_{F'}, 0, v_{n(F')}]^2$. It follows that $Q^0_{F'}(G', s)$ is independent of s, and is simply denoted by $Q^0_{F'}(G')$ so that

$$I_{flex}(\psi_0) = \frac{1}{3}\int_{S'} Q^0_{D'\psi_0}(D'n)\, dx.$$

4.2. Frame-indifference

The principle of frame-indifference states that the space is invariant under rotation which translates in our case in the following condition on the stored energy function W^ε, $W^\varepsilon(F) = W^\varepsilon(RF)$ for every rotation $R \in \mathrm{SO}(3)$. This is assumed in the sequel. Accordingly, the same property is satisfied by W_2, i.e., $W_2(F) = W_2(RF)$, for every $R \in \mathrm{SO}(3)$.

In the following, we denote by $\mathcal{E}_{S'}$ the set of symmetric bilinear forms on TS', that is, the fiber bundle of base space S' and whose fiber $(\mathcal{E}_{S'})_{x'}$ at $x' \in S'$ is the set of symmetric bilinear forms on $T_{x'}S'$. The fiber bundle $\mathcal{E}_S$ is defined in a similar way and $\mathcal{E}'_S$ stands for its restriction to S'. Moreover, if $F \in T^*_x(S;\mathbb{R}^3)$, F^TF stands for the element of $(\mathcal{E}_{S'})_x$ that maps every element (u,v) of $(T_xS)^2$ to the scalar product between Fu and Fv. A similar notation is used to defined $(F')^TF' \in \mathcal{E}_{S'}$ for every $F' \in T^*(S';\mathbb{R}^3)$.

Lemma 3. *If the stored energy W_2 is frame-indifferent, then for every $F' \in \mathcal{M}'$ of maximum rank and every $R \in \mathrm{SO}(3)$, we have*

$$RF' \in \mathcal{M}' \quad \text{and} \quad n(RF') = Rn(F').$$

Moreover, there exists a bundle map $\tau' : \mathcal{M}' \to TS'$ and a map $\tau_3 : \mathcal{M}' \to \mathbb{R}$ such that for every $F' \in \mathcal{M}'$ of maximal rank,

$$n(F') = F'\tau'(F') + n_{F'}\tau_3(F'),$$

with both $\tau'(F')$ and $\tau_3(F')$ depending only on $C' = (F')^TF'$ and $n_{F'} \in \mathbb{R}^3$ is defined by

$$n_{F'} \cdot w = \det(F', w) \quad \text{for every } w \in \mathbb{R}^3. \tag{21}$$

Lastly, $C = (F', n_0(F'))^T(F', n_0(F'))$ depends only on C'.

Proof. The first part of the proposition is obvious. Next, since F' is of maximum rank, $(F', n_{F'})$ is invertible so we can set $(\tau'(F'), \tau_3(F')) = (F', n_{F'})^{-1}n(F')$. Moreover, we can check that

$$\begin{aligned}(\tau'(RF'), \tau_3(RF')) &= (RF', n_{RF'})^{-1}n(RF) = (RF', R(n_{F'}))^{-1}Rn(F) \\ &= (R(F', n_{F'}))^{-1}Rn(F) = (F', n_{F'})^{-1}n(F') = (\tau'(F'), \tau_3(F')),\end{aligned}$$

whence both τ' and τ_3 only depend on $(F')^TF'$. Finally, it is readily verified that both $n(F')^Tn(F')$ and $n(F')^TF'$ are invariant under rotations of F'. As a result, $C = (F', n_0(F'))^T(F', n_0(F'))$ depends only on F'^TF' as well. □

Since $T_{(x',x_3)}S = T_{x'}S' \times T_{x_3}(-1,1) = T_{x'}S' \times \mathbb{R}$, every element $C \in (\mathcal{E}_S)_x$ can be decomposed univocally into $(C', C_3, C_{33}) \in (\mathcal{E}_{S'})_{x'} \times T^*_{x'}S' \times \mathbb{R}$ such that for every (u', u_3) and every (v', v_3) in $T_{x'}S' \times \mathbb{R} = T_xS$,

$$C((u',u_3),(v',v_3)) = C'(u',v') + u_3C_3(v') + v_3C_3(u') + C_{33}u_3v_3.$$

In addition, we write this decomposition as follows

$$C = \begin{pmatrix} C' & C_3^T \\ C_3 & C_{33} \end{pmatrix}.$$

Let us introduce the fiber bundle $\mathcal{P}$ of base space S', and whose fiber at $x' \in S'$ is the set of polynomials of degree lower than or equal to two on $(\mathcal{E}_{S'})_{x'}$.

Proposition 3. *If the stored energy function W_2 is frame-indifferent, then there exists a bundle map $P : C' \mapsto P_{C'}$ over S' from $\mathcal{E}_{S'}$ into $\mathcal{P}$ such that for every deformation ψ_0 of finite limit energy $I_0(\psi_0)$, we have for all $G' \in T^*(S';\mathbb{R}^3)$,*

$$Q^0_{D'\psi_0}(G',1) = P_{C'}(D'\psi_0{}^T G' + G'^T D'\psi_0),$$

where $C' = D'\psi_0{}^T D'\psi_0$ and n stands for $n(D'\psi_0)$ for short. Moreover, if W_2 is homogeneous along the fibers, then $P_{C'}$ is homogeneous of degree two.

Proof. Let $\mathcal{M}^+ = \{F \in \mathcal{M} : \det F > 0\}$. Since W_2 is assumed to be frame-indifferent, there exists a map $\hat{W} : \mathcal{E}_S \to \mathbb{R}$ such that for every F in a neighborhood of $\mathcal{M}^+$,

$$W_2(F) = \hat{W} \circ m(F), \tag{22}$$

where $m : T^*(S;\mathbb{R}^3) \to \mathcal{E}_S$ is the bundle map defined by $m(G) = G^TG$. Let F' be a section of $\mathcal{M}'$, $s \in \mathbb{R}$ and $G' \in T^*(S';\mathbb{R}^3)$ such that $(F', n_0(F')) \in \mathcal{M}^+$ a.e. Then, definition (10) combined with (22), gives

$$Q^0_{F'}(G',s) = \inf_{v\in\mathbb{R}^3} D^2W_2[G'_{F'}, s_0, v_{n(F')}]^2 = \inf_{v\in\mathbb{R}^3} D^2\hat{W}[Dm(G'_{F'}, s_0, v_{n(F')})]^2.$$

Since $\mathcal{E}_S = (-1,1) \times \mathcal{E}'_S$, we can identify $T\mathcal{E}_S$ with $T(-1,1) \times T\mathcal{E}'_S$. Doing so, we obtain that

$$\begin{aligned} Dm(G'_{F'}, s_0, v_{n(F')}) &= \frac{d}{dt}\left(ts, C + t\begin{pmatrix} (F')^TG' + (G')^TF' & (F')^Tv + (G')^Tn \\ v^TF' + n^TG' & n^Tv + v^Tn \end{pmatrix}\right)(t=0) \\ &= \left(s_0, \begin{pmatrix} (F')^TG' + (G')^TF' & (F')^Tv + (G')^Tn \\ v^TF' + n^TG' & n^Tv + v^Tn \end{pmatrix}_C\right), \end{aligned}$$

where $n(F')$ is denoted n for short, and $C = m(F)$. Since $(F', n(F'))(x')$ is assumed to be invertible for all $x' \in S'$, setting $w' = F'^Tv + G'^Tn$ and $w_3 = n^Tv + v^Tn$, we get

$$Q^0_{F'}(G',1) = P_{C'}((F')^TG' + (G')^TF'), \tag{23}$$

where $P_{C'}$ is the section of $\mathcal{P}$ is defined for every $M \in \mathcal{E}_{S'}$ by

$$P_{C'}(M) = \inf_{w\in\mathbb{R}^3} D^2\hat{W}\left[1_0, \begin{pmatrix} M & w' \\ (w')^T & w_3 \end{pmatrix}_C\right]^2, \tag{24}$$

and $C = (F', n_0(F'))^T(F', n_0(F'))$, which according to Lemma 3 depends only on C'. Finally, if ψ_0 is a deformation satisfying $I_0(\psi_0) < \infty$, owing to the noninterpenetration assumptions made, we know that $D\psi_0 \in \mathcal{M}^+$ a.e. As a result, $Q^0_{D'\psi_0}(D'n,1) = P_{C'}(M)$ a.e. on S' with $M = F^TD'n + D'n^TF'$ as claimed. Moreover, if the shell is homogeneous along its fibers, (24) reduces to

$$P_{C'}(M) = \inf_{w\in\mathbb{R}^3} D^2\hat{W}\left[0, \begin{pmatrix} M & w' \\ (w')^T & w_3 \end{pmatrix}_C\right]^2, \tag{25}$$

which is homogeneous of degree two with respect to M. □

Remark 1. *Note that $D'\psi_0{}^TD'n + D'n^TD'\psi_0$ is not, in general, the second fundamental form of the deformation, except in the case where $n(D'\psi_0)$ is the normal vector to $D'\psi_0$.*

4.3. Planar isotropy

We say that the material is isotropic along the midsection of the shell, if for every planar rotation R in the set $\mathrm{SO}(T_{x'}S')$ of rotations in $T_{x'}S'$, and every $F = (F', F_3) \in T^*_{x'}(S^\varepsilon; \mathbb{R}^3)$, we have

$$W^\varepsilon(F', F_3) = W^\varepsilon(F'R, F_3). \tag{26}$$

As a consequence, for deformations of finite energy, the fibers of the shell remain normal to its section.

Lemma 4. *Assume that the shell is isotropic along its midsection, then there exists a map $\tau_3 : \mathcal{M}' \to \mathbb{R}$ such that for all $F' \in \mathcal{M}'$ of maximal rank,*

$$n(F') = n_{F'}\tau_3(F'),$$

where $n_{F'}$ is defined by (21). Moreover, $\tau_3(F')$ depends only on the metric $C' = F'^T F'$.

Proof. Let F' be an element of $\mathcal{M}'$ of maximal rank. By definition, we have

$$n_0(F') = \arg \min_{F_3 \in T_0^*((-1,1);\mathbb{R}^3)} W_2(F', F_3).$$

For every rotation $R \in \mathrm{SO}(TS')$, the isotropy property yields

$$n_0(F'R) = \arg \min_{F_3 \in T_0^*((-1,1);\mathbb{R}^3)} W_2(F'R, F_3) = \arg \min_{F_3 \in T_0^*((-1,1);\mathbb{R}^3)} W_2(F', F_3) = n_0(F').$$

In particular, this entails that $n(-F') = n(F')$. What is more, due to frame-indifference, we have from Lemma 3

$$n(F') = F'\tau'(F') + n_{F'}\tau_3(F'),$$

where τ' is a bundle map from $\mathcal{M}'$ into TS' and τ_3 a map from $\mathcal{M}'$ into $\mathbb{R}$, both of them depending only on the metric $C' = F'^T F'$. Thus,

$$\begin{aligned} F'\tau'(F') + n_{F'}\tau_3(F') &= n(F') = n(-F') = -F'\tau'(-F') + n_{-F'}\tau_3(-F') \\ &= -F'\tau'(F') + n_{F'}\tau_3(F'). \end{aligned}$$

Consequently, $F'\tau'(F') = 0$ and $n(F') = n_{F'}\tau_3(F')$ as claimed. □

Proposition 4. *Assume that the shell is isotropic along its midsection, then the flexural energy $I_{flex}(\psi_0)$ depends only on the metric and the second fundamental form of the deformed surface. Namely, we have*

$$I_{flex}(\psi_0) = \frac{1}{3}\int_{S'} P_{C'}(|n(D'\psi_0)| b_{D'\psi_0})\, dx',$$

where $b_{D'\psi_0}$ is the second fundamental form of ψ_0, i.e.,

$$b_{F'} = D'N^T F' + F'^T D'N, \tag{27}$$

with $N = n_{F'}/|n_{F'}|$ and $P_{C'}$ is defined by Proposition 3.

Proof. Since $n(D'\psi_0)$ (denoted n for short) is normal colinear to $n_{D'\psi_0}$ and thus to the normal N to the deformed surface, we get $D'n^T D'\psi_0 + D'\psi_0{}^T D'n = (n \cdot N)(D'N^T D'\psi_0 + D'\psi_0{}^T D'N)$. □

4.4. Right invariance under the special linear group

We denote by $\mathrm{SL}(TS')$ the special linear group over TS', that is the fiber bundle over S' whose fiber at x' is the linear diffeomorphisms of $T_{x'}S'$ of determinant equal to one. In this section, we consider the case where the energy W_2 is right invariant under the special linear group, that is $W_2(F', F_3) = W_2(F'U, F_3)$, for every $x' \in S'$, $U \in \mathrm{SL}(T_{x'}S')$ and $(F', F_3) \in T^*_{x'}(S'; \mathbb{R}^3) \times T^*((-1,1); \mathbb{R}^3)$.

Proposition 5. *Assume that W_2 is right invariant under $\mathrm{SL}(TS')$, then the flexural energy $I_{flex}(\psi_0)$ depends only on the metric, and on the mean curvature $H = \mathrm{Tr}(C'^{-1/2} b_{D'\psi} C'^{-1/2})$ of the deformation. More precisely, we have*

$$I_{flex}(\psi_0) = \frac{1}{3}\int_{S'} K_{x',\det(C')}(H)\, dx', \tag{28}$$

where $K : S' \times \mathbb{R} \to \Pi_2$; Π_2 is the set of polynomials of degree lower or equal to 2. Moreover, if the shell is homogeneous along its fibers, then

$$I_{flex}(\psi_0) = \frac{1}{3}\int_{S'} \kappa_{x',\det(C')} |H|^2\, dx', \tag{29}$$

where κ is a map from $S' \times \mathbb{R}^+$ into $\mathbb{R}^+$.

Proof. Let $\mathcal{O}$ be the fiber bundle over S' whose fibers are the maps from $T_{x'}S'$ into itself of zero trace. For every $O \in \mathcal{O}$ and $x' = \pi_{S'}(O)$, there exists a regular map $U : (0,1) \to \mathrm{SL}(T_{x'}S')$ such that $\dot{U}(0) = O$ and $U(0) = \mathrm{Id}$. Let F be a section of $\mathcal{M}$, $(G', s, v) \in T^*_{x'}(S'; \mathbb{R}^3) \times \mathbb{R} \times (\mathbb{R}')^3$, and let $\gamma(t) = (\gamma'(t), \gamma_3(t))$ be a curve in $T^*(S; \mathbb{R}^3)$ such that $\dot{\gamma}(0) = (G'_{F'}, s_0, v_{n(F')})$ as in (11). From the right invariance under the special group, for every $U_0 \in \mathrm{SL}(T_{x'}S')$, we have $W_2(\gamma(t)) = W_2(\gamma'(t)U_0U(t), \gamma_3(t))$. As a consequence,

$$D^2W_2\,[\dot{\gamma}(0)]^2 = D^2W_2\left[\frac{d}{dt}(\gamma' U_0 U, \gamma_3)_{|t=0}\right]^2. \tag{30}$$

Then, a simple computation yields

$$\frac{d}{dt}(\gamma' U_0 U, \gamma_3)_{|t=0} = \left((G'U_0 + F'U_0O)_{F'U_0}, s_0, v_{n(F')}\right),$$

which, owing to (30) and (10), leads to $Q^0_{F'}(G', s) = Q^0_{F'U_0}(G'U_0 + F'U_0O, s)$. From (23), recalling that $C' = F'^T F'$, we get

$$\begin{aligned}
P_{C'}(F'^T G' + G'^T F') &= Q^0_{F'}(G', 1) = Q^0_{F'U_0}(G'U_0 + F'U_0O, 1)\\
&= P_{U_0^T F'^T F' U_0}((F'U_0)^T(G'U_0 + F'U_0O) + (G'U_0 + F'U_0O)^T(F'U_0))\\
&= P_{U_0^T C' U_0}(U_0^T(F'^T G' + G'^T F')U_0 + U_0^T C' U_0 O + O^T U_0^T C' U_0)\\
&= P_{C'_0}\Big((C'_0)^{1/2}[C_0'^{-1/2} U_0^T (F'^T G' + G'^T F')U_0 C_0'^{-1/2}\\
&\qquad + C_0'^{1/2} O C_0'^{-1/2} + C_0'^{-1/2} O^T C_0'^{1/2}] C_0'^{1/2}\Big)
\end{aligned}$$

where $C' = F'^T F'$ and $C'_0 = U_0^T C' U_0$. Since the map $O \mapsto C_0'^{1/2} O C_0'^{-1/2} + C_0'^{-1/2} O^T C_0'^{1/2}$ is a diffeomorphism over the set of symmetric trace-free matrices, the above expression leads to

$$P_{C'}(F'^T G' + G'^T F') = P_{C'_0}\left(\mathrm{Tr}\left(C_0'^{-1/2} U_0^T (F'^T G' + G'^T F') U_0 C_0'^{-1/2}\right) C'_0/2\right).$$

In addition, $\mathrm{Tr}\left(C_0'^{-1/2} U_0^T (F'^T G' + G'^T F') U_0 C_0'^{-1/2}\right) = \mathrm{Tr}\left(C'^{-1/2}(F'^T G' + G'^T F')C'^{-1/2}\right)$, so that we may write

$$P_{C'}(F'^T G' + G'^T F') = P_{C'_0}\left(\mathrm{Tr}\left(C'^{-1/2}(F'^T G' + G'^T F')C'^{-1/2}\right) C'_0/2\right).$$

Since C' is symmetric and nonnegative, there exists a rotation $R \in \mathrm{SO}(T_{x'}S')$ and λ_1, λ_2 nonnegative reals such that

$$C' = R^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R.$$

Let us choose $U_0 \in \mathrm{SL}(T_{x'}S')$ in this fashion

$$U_0 = (\det C')^{1/4} R^T \begin{pmatrix} \lambda_1^{-1/2} & 0 \\ 0 & \lambda_2^{-1/2} \end{pmatrix}.$$

Hence, $C_0' = U_0^T C' U_0 = (\det C')^{1/2}\,\mathrm{Id}$, so that

$$P_{C'}(F'^T G' + G'^T F') = P_{(\det C')^{1/2}\,\mathrm{Id}}\left(\mathrm{Tr}\left(C'^{-1/2}(F'^T G' + G'^T F')C'^{-1/2}\right)(\det C')^{1/2}\,\mathrm{Id}\,/2\right). \tag{31}$$

Using the definition of I_{flex}, and the fact that $D'\psi_0{}^T D'n + D'n^T D'\psi_0 = |n(D'\psi_0)| b_{D'\psi_0}$, we infer that

$$\begin{aligned} I_{flex}(\psi_0) &= \frac{1}{2}\int_{S'} P_{(\det C')^{1/2}\,\mathrm{Id}}\left(|n(D'\psi_0)|\mathrm{Tr}\left(C'^{-1/2} b_{D'\psi} C^{-1/2}\right)(\det C')^{1/2}\,\mathrm{Id}\,/2\right)\,dx' \\ &= \frac{1}{2}\int_{S'} P_{(\det C')^{1/2}\,\mathrm{Id}}\left(|n(D'\psi_0)| H (\det C')^{1/2}\,\mathrm{Id}\,/2\right)\,dx'. \end{aligned}$$

Finally, the right invariance of W_2 with respect to $\mathrm{SL}(TS')$ implies that $|n(F')|$ depends only on $\det C'$. Setting $K_{x',\det C'}(H) = P_{(\det C')^{1/2}\,\mathrm{Id}}(|n(F')| H (\det C')^{1/2}\,\mathrm{Id}\,/2)$, we get (28). Moreover, if the shell is homogeneous along its fibers, then $P_{(\det C')^{1/2}\,\mathrm{Id}}$ is homogeneous of degree two and, accordingly, $K_{x',\det C'}(H)$ is a monomial. $\square$

5. Geometric Configuration

Classically, the energy of an elastic body is not written in terms of the deformation ψ^ε of S^ε, but in terms of the deformation $\widetilde{\psi}^\varepsilon$ of the *geometric* configuration $\widetilde{S}^\varepsilon := g_\varepsilon(S^\varepsilon)$, where

$$\begin{array}{rcl} g_\varepsilon : S' \times (-\varepsilon, \varepsilon) & \to & \mathbb{R}^3 \\ (x', x_3) & \mapsto & x' + x_3 n'(x'), \end{array}$$

$n' : S' \to \mathbb{R}^3$ being the normal to S'. We set $\widetilde{S} := \widetilde{S}^{\varepsilon_0}$, for a small enough ε_0 such that g_{ε_0} is one to one. In the following, we will always assume that $\varepsilon \leq \varepsilon_0$. We intend to recast our results in this geometric configuration. This is easily achieved by a mere change of variables. To begin with, we have to recast our initial three-dimensional problem in the geometric configuration.

5.1. Recast of the problem

We denote by $\widetilde{J}_\varepsilon(\widetilde{\psi}^\varepsilon)$ the elastic energy of a deformation $\widetilde{\psi}^\varepsilon$ of $\widetilde{S}^\varepsilon$ and assume that it has the following form

$$\widetilde{J}_\varepsilon(\widetilde{\psi}^\varepsilon) := \int_{\widetilde{S}^\varepsilon} \widetilde{W}^\varepsilon\left(\widetilde{x}, \nabla\widetilde{\psi}^\varepsilon\right) d\widetilde{x}, \tag{32}$$

where $\widetilde{W}^\varepsilon$ stands for the stored energy function of the solid. Furthermore, we assume the shell to be submitted to dead body loads $\widetilde{f}_\varepsilon$, so that the total energy of the system is given by

$$\widetilde{I}_\varepsilon(\widetilde{\psi}^\varepsilon) := \widetilde{J}_\varepsilon(\widetilde{\psi}^\varepsilon) - \int_{\widetilde{S}^\varepsilon} \widetilde{f}_\varepsilon \cdot \widetilde{\psi}^\varepsilon \, d\widetilde{x}. \tag{33}$$

Finally, clamping boundary conditions are added on a part of the boundary $g_\varepsilon(\Gamma^\varepsilon) = g_\varepsilon(\gamma \times (-\varepsilon, \varepsilon))$, where $\gamma \subset \partial S'$.

Our aim is to determine the behavior of the minimizers $\widetilde{\varphi}^\varepsilon$ of $\widetilde{I}_\varepsilon$ over

$$\widetilde{\Psi}^\varepsilon := \left\{\widetilde{\psi}^\varepsilon \in W^{1,\infty}(\widetilde{S}^\varepsilon)^3 : \widetilde{\psi}^\varepsilon \circ g_\varepsilon(x^\varepsilon) = \phi^\varepsilon(x^\varepsilon),\ x^\varepsilon \in \Gamma^\varepsilon \text{ a.e. } \right\},$$

as ε goes to zero under the assumptions, on the stored energy and the applied loads, made hereunder.

In order to apply our results, several assumptions, similar to the ones we made on W^ε and φ^ε, have to be imposed on $\widetilde{W}^\varepsilon$ and on the minimization sequences $\widetilde{\varphi}^\varepsilon$.

Dependence of the stored energy with respect to the thickness. We assume that there exists continuous nonnegative maps $\widetilde{W}_2$ and $\widetilde{W}_0$ such that for every $(\widetilde{x}, F) \in \widetilde{S} \times \mathbb{R}^{3\times 3}$, we have

$$\widetilde{W}^\varepsilon(\widetilde{x}, F) = \varepsilon^{-1}\left(\varepsilon^{-2}\widetilde{W}_2(\widetilde{x}, F) + \widetilde{W}_0(\widetilde{x}, F)\right). \tag{34}$$

Behavior of strongly extended fibers. We assume that the stored energy $\widetilde{W}_2$ is bounded from below by a positive constant for strongly extended fibers, namely, there exists $\delta, c > 0$ such that

$$\forall \widetilde{x} \in \widetilde{S},\ \forall F' \in T^*_{x'}(S'; \mathbb{R}^3),\ \forall F_3 \in \mathbb{R}^3 \text{ such that } |F_3| \geq \delta, \text{ we have } \widetilde{W}_2(\widetilde{x}, F' \circ \pi'_{x'} + F_3 \otimes n'(x')) \geq c, \tag{35}$$

where x' is the projection of $\widetilde{x}$ onto S' and $\pi'_{x'}$ is the projection of $\mathbb{R}^3$ onto $T_{x'}(S'; \mathbb{R}^3)$.

Zero set of $\widetilde{W}_2$. We assume that $\widetilde{W}_2$ is a C^2 nonnegative function and denote by $\widetilde{\mathcal{M}}$ the restriction of its zero set to $S' \times \mathbb{R}^{3\times 3}$, that is

$$\widetilde{\mathcal{M}} := \left\{(x', F) \in S' \times \mathbb{R}^{3\times 3} \text{ such that } \widetilde{W}_2(x', F) = 0)\right\}. \tag{36}$$

Let $\mathcal{M}'$ be the projection of $\widetilde{\mathcal{M}}$ onto $T^*(S'; \mathbb{R}^3)$, that is,

$$\mathcal{M}' := \bigcup_{x' \in S'} \left\{F' \in T^*_{x'}(S'; \mathbb{R}^3) \ :\ \text{there exists } n \in \mathbb{R}^3 \text{ such that } (x', F' \circ \pi'_{x'} + n \otimes n'(x')) \in \widetilde{\mathcal{M}}\right\}. \tag{37}$$

Once again, we assume that the projection of $\widetilde{\mathcal{M}}$ onto $\mathcal{M}'$ to be one to one, that is there exists map $n : \mathcal{M}' \to \mathbb{R}^3$ such that

$$\widetilde{\mathcal{M}} = \left\{(x', F' \circ \pi'_{x'} + n(F') \otimes n'(x')) \in S' \times \mathbb{R}^{3\times 3} \ :\ F' \in \mathcal{M}'\right\}. \tag{38}$$

Interpenetration. To avoid interpenetration of matter, it is geometric to expect $D\widetilde{\psi}^{\varepsilon}$ to be invertible. To this end, we require that $\widetilde{W}_0(x', F) = \infty$ for every $(x', F) \in \widetilde{\mathcal{M}}$ such that $\det F < 0$, and that

$$\widetilde{W}_0(x', F) \to \infty \quad \text{if} \quad (x', F) \in \widetilde{\mathcal{M}} \quad \text{and} \quad \det F \to 0. \tag{39}$$

Applied loads. The volumic loads are assumed to scale as the inverse of the thickness of the shell and more precisely that there exists $\widetilde{f} : \widetilde{S} \to \mathbb{R}^3$ such that

$$\widetilde{f}_{\varepsilon}(\widetilde{x}) = \varepsilon^{-1}\widetilde{f}(\widetilde{x}) \qquad \text{for every } \widetilde{x} \in \widetilde{S}. \tag{40}$$

Ansatz. We assume that the minimizers of the energy admit an asymptotic expansion in this fashion

$$\widetilde{\varphi}^{\varepsilon}(x' + \varepsilon x_3 n') = \sum_{k \geq 0} \varepsilon^k \varphi_k(x', x_3). \tag{41}$$

5.2. Change of variable

In order to apply our result, we first have to rewrite the energy in terms of the associated deformation $\psi^{\varepsilon} = \widetilde{\psi}^{\varepsilon} \circ g_{\varepsilon}$ of S^{ε}. We have

$$\widetilde{J}(\widetilde{\psi}^{\varepsilon}) = J(\psi^{\varepsilon}) = \int_{S^{\varepsilon}} W^{\varepsilon}(D\psi^{\varepsilon})\, dx, \tag{42}$$

with $W^{\varepsilon} = \varepsilon^{-1}\left(\varepsilon^{-2}W_2 + W_0\right)$ and for every $F \in T_x^*(S;\mathbb{R}^3)$,

$$W_k(F) = \widetilde{W}_k(F \circ (Dg_{\varepsilon}(x))^{-1}) \det(Dg_{\varepsilon}(x)), \qquad k = 0, 2. \tag{43}$$

Note that W_2 and W_0 are independent of ε since $Dg_{\varepsilon} = (\mathrm{Id}', n') + x_3(D'n', 0)$ (which is denoted by Dg hereafter). In addition, these energies satisfy the assumptions made in section 2.1. Finally, the minimizers $\varphi^{\varepsilon} = \widetilde{\varphi}^{\varepsilon} \circ g_{\varepsilon}$ admit the same asymptotic expansion than $\widetilde{\varphi}^{\varepsilon}$. Thus, all of the results of section 1 and 4 apply and may be expressed in terms of $\widetilde{W}_0$ and $\widetilde{W}_2$ up to a change of variable. Moreover, the definitions of $\mathcal{M}'$ and of the map $n : \mathcal{M}' \to \mathbb{R}^3$ are independent of the chosen approach.

Lemma 5. *If function W_2 is defined by (43), and F' is a section of $\mathcal{M}'$, then for every $G' \in T_{x'}^*(S';\mathbb{R}^3)$ and $s \in \mathbb{R}$, we have*

$$Q^0_{F'}(G', s) = \widetilde{Q}^0_{F'}(G' - sF'D'n', s),$$

where

$$\widetilde{Q}^0_{F'}(G', s) := \inf_{v \in \mathbb{R}^3} D^2\widetilde{W}_2(x', F)\,[sn', G'\pi'_{x'} + v \otimes n']^2,$$

where $\pi'_{x'}$ is the projection of $\mathbb{R}^3$ onto $T_{x'}S'$ and $F = F'(x') \circ \pi'_{x'} + n(F'(x')) \otimes n'(x')$.

Proof. Let F' be a section of $\mathcal{M}'$, x' be an element of S' and $G' \in T_{x'}^*(S';\mathbb{R}^3)$. From the definition of $Q^0_{F'}$, we have

$$Q^0_{F'}(G', s) = \inf_{v \in \mathbb{R}^3} D^2W_2[G'_{F'}, s_0, v_{n(F')}]^2,$$

where $(G'_{F'}, s_0, v_{n(F')}) = \dot{\gamma}(0)$ and $\gamma(t) = (F'(x') + tG', ts, n(F'(x')) + vt) \in T^*(S';\mathbb{R}^3) \times (-1,1) \times (\mathbb{R}')^3 = T^*(S';\mathbb{R}^3) \times T^*((-1,1);\mathbb{R}^3) = T^*(S;\mathbb{R}^3)$. From (43), we deduce

$$Q^0_{F'}(G', s) = \det(Dg(x', 0)) \inf_{v \in \mathbb{R}^3} D^2\widetilde{W}_2[\dot{\tilde{\gamma}}(0)]^2 = \inf_{v \in \mathbb{R}^3} D^2\widetilde{W}_2[\dot{\tilde{\gamma}}(0)]^2, \tag{44}$$

where $\tilde{\gamma}(t) = \gamma(t) \circ Dg(x', ts)^{-1}$. On the other hand, $Dg = (\mathrm{Id}', n') + x_3(D'n', 0)$, and

$$\begin{aligned}
(Dg)^{-1} &= ((\mathrm{Id}', n') + x_3(D'n', 0))^{-1} \\
&= \left((\mathrm{Id}', n')(\mathrm{Id} + x_3(\mathrm{Id}', n')^{-1}(D'n', 0))\right)^{-1} \\
&= \left(\mathrm{Id} - x_3(\mathrm{Id}', n')^{-1}(D'n', 0)\right)(\mathrm{Id}', n')^{-1} + o(x_3).
\end{aligned}$$

Since $(\mathrm{Id}' + n' \otimes e_3)^{-1} = \begin{pmatrix} \pi' \\ n'^T \end{pmatrix}$, the above identity reads

$$\begin{aligned}(Dg)^{-1} &= \begin{pmatrix} \pi' \\ n'^T \end{pmatrix} - x_3 \begin{pmatrix} \pi' \\ n'^T \end{pmatrix} (D'n', 0) \begin{pmatrix} \pi' \\ n'^T \end{pmatrix} + o(x_3) \\ &= \begin{pmatrix} \pi' \\ n'^T \end{pmatrix} - x_3 \begin{pmatrix} \pi' D'n'\pi' \\ 0 \end{pmatrix} + o(x_3).\end{aligned}$$

It follows that (using the notation F' for $F'(x')$ for short)

$$\begin{aligned}\tilde{\gamma}(t) &= \left(x' + tsn', ((F', n(F')) + t(G', v))\left(\begin{pmatrix} \pi' \\ n'^T \end{pmatrix} - ts \begin{pmatrix} \pi' D'n'\pi' \\ 0 \end{pmatrix}\right)\right) + o(t) \\ &= \left(x' + tsn', (F', n(F'))\begin{pmatrix} \pi' \\ n'^T \end{pmatrix} + t(G'\pi' + v \otimes n') - tsF'D'n'\pi'\right) + o(t).\end{aligned}$$

Consequently,

$$\begin{aligned}\dot{\tilde{\gamma}}(0) &= \left[\left(x', (F', n(F'))\begin{pmatrix} \pi' \\ n'^T \end{pmatrix}\right), (sn', G'\pi' + v \otimes n' - s(F'D'n'\pi'))\right] \\ &= [(x', F'\pi' + n(F') \otimes n'), (sn', (G' - sF'D'n')\pi' + v \otimes n')] \\ &= [(x', F), (sn', (G' - sF'D'n')\pi' + v \otimes n')]\end{aligned}$$

The conclusion follows from (44). □

From now on, we limit our analysis to the case where standard boundary conditions (18) are applied. From Proposition 2, we immediately infer the next result.

Proposition 6. *Assume that the standard boundary conditions (18) are applied to the shell. Let $\widetilde{\varphi}^\varepsilon$ be the minimizer of the total energy $\widetilde{I}_\varepsilon(\widetilde{\varphi}^\varepsilon)$ over the space of admissible deformations. If $\widetilde{\varphi}^\varepsilon$ admits an asymptotic expansion as in (41), and if the total energy $\widetilde{I}_\varepsilon(\widetilde{\varphi}^\varepsilon)$ remains bounded, then*

$$\varphi_0 = \arg \min_{\psi_0 \in \Psi_0} \widetilde{I}_0(\psi_0),$$

where

$$\widetilde{I}_0(\psi_0) = \frac{1}{3}\int_{S'} \widetilde{Q}^0_{D'\psi_0}(D'n - D'\psi_0 D'n', 1)dx' + 2\int_{S'} \widetilde{W}_0(x', (D'\psi_0, n))dx' - 2\int_{S'} \widetilde{f}_0 \cdot \psi_0 \, dx', \tag{45}$$

$\widetilde{f}_0(x') = \widetilde{f}(x', 0)$ for every $x' \in S'$, $n = n(D'\psi_0)$, and

$$\Psi_0 := \left\{\psi_0 \in W^{1,\infty}(S')^3 \ : \ D'\psi_0 \in \mathcal{M}', \ n = n(D'\psi_0) \in L^\infty(S')^3, \ \psi_0(x') = \phi_0(x'), \text{ and } n(x') = n_\gamma(x') \text{ for a.e. } x' \in \gamma\right\}.$$

Note that a result equivalent to Lemma 2 may similarly be stated in the geometric configuration which applies to slightly more general Dirichlet conditions.

5.3. Homogeneity along the fibers

We say that the shell is homogeneous along its fibers in the geometric configuration if for every $x' \in S'$, $s \in (-1, 1)$ and $F \in \mathbb{R}^{3\times3}$, we have $\widetilde{W}_2(x' + sn', F) = \widetilde{W}_2(x', F)$.

Proposition 7. *If the shell is homogeneous along its fibers in the geometric configuration, then $\widetilde{Q}^0_{F'}(G', s)$ is independent of s, and is denoted by $\widetilde{Q}^0_{F'}(G')$.*

5.4. Frame-indifference

In the following, we assume the stored energy to be frame-indifferent, that is, $\widetilde{W}^\varepsilon(\widetilde{x}, RF) = \widetilde{W}^\varepsilon(\widetilde{x}, F)$, for every $(\widetilde{x}, F) \in \widetilde{S}^\varepsilon \times \mathbb{R}^{3\times 3}$ (with $\varepsilon > 0$ small enough), and every rotation $R \in \mathrm{SO}(3)$. Note that it is equivalent to the frame-indifference of W^ε.

Proposition 8. *If the stored energy function $\widetilde{W}_2$ is frame-indifferent, then there exists a bundle map $\widetilde{P} : C' \mapsto \widetilde{P}_{C'}$ over S' from $\mathcal{E}_{S'}$ into $\mathcal{P}$ such that for every deformation ψ_0 of finite energy $\widetilde{I}_0(\psi_0)$, we have for all $G' \in T^*(S'; \mathbb{R}^3)$*

$$\widetilde{Q}^0_{D'\psi_0}(G', 1) = \widetilde{P}_{C'}(D'\psi_0{}^T G' + G'^T D'\psi_0),$$

with $C' = D'\psi_0{}^T D'\psi_0$ and $n = n(D'\psi_0)$. Moreover, if the shell is homogeneous along its fibers in the geometric configuration, then $\widetilde{P}_{C'}$ is homogeneous of degree two.

Proof. The proof is similar to the one devised for the abstract configuration. Once again, there exists a map $\hat{\widetilde{W}}$ such that, at least in a neighborhood of $\widetilde{\mathcal{M}}^+ = \{F \in \widetilde{\mathcal{M}} \ : \ \det F > 0\}$, we may write $\widetilde{W}_2(x, F) = \hat{\widetilde{W}}(x, F^T F)$. After some computations, we derive the claimed result with

$$\widetilde{P}_{C'}(M) = \inf_{w' \in T_{x'}S',\, w_3 \in \mathbb{R}} D^2\hat{\widetilde{W}}\left(x', \widetilde{C}\right)\left[n', \pi'^T M\pi' + n'w'^T\pi' + \pi'^T w'n'^T + n'w_3n'^T\right]^2, \tag{46}$$

where $\widetilde{C} = (D'\psi_0\pi' + n(F)\otimes n')^T(D'\psi_0\pi' + n(F)\otimes n')$. Moreover, frame indifference implies also that $\widetilde{C}$ depends only on C'. Finally, if the shell is homogeneous along its fibers in the geometric configuration, we have

$$\widetilde{P}_{C'}(M) = \inf_{w' \in T_{x'}S',\, w_3 \in \mathbb{R}} \frac{\partial^2\hat{\widetilde{W}}}{\partial\widetilde{C}^2}\left(x', \widetilde{C}\right)\left[\pi'^T M\pi' + n'w'^T\pi' + \pi'^T w'n'^T + n'w_3n'^T\right]^2. \tag{47}$$

□

5.5. Planar isotropy

We say the shell is isotropic along its midsection, if for every $x' \in S'$, and $(F', F_3) \in T^*_{x'}(S'; \mathbb{R}^3) \times \mathbb{R}^3$, we have

$$\widetilde{W}^\varepsilon\left(x', F'R\pi'_{x'} + F_3 \otimes n'\right) = \widetilde{W}^\varepsilon\left(x', F'\pi'_{x'} + F_3 \otimes n'\right),$$

for every $R \in \mathrm{SO}(T_{x'}S')$. This is equivalent to the definition used in the abstract configuration. We investigate the consequences of planar isotropy on the flexural part of the energy

$$\widetilde{I}_{flex}(\psi_0) := \frac{1}{3}\int_{S'} \widetilde{Q}^0_{D'\psi_0}(D'n - D'\psi_0 D'n', 1)dx'.$$

Proposition 9. *If the shell is isotropic along its midsection, then*

$$\widetilde{I}_{flex}(\psi_0) = \frac{1}{3}\int_{S'} \widetilde{P}_{C'}(|n(D'\psi_0)|b_{D'\psi_0} - (C'D'n' + (D'n')^T C'))\,dx',$$

where $b_{D'\psi_0}$ is the second fundamental form of the deformed surface, given by (27).

Proof. The proof is similar to Proposition 4. □

5.6. Right-invariance under the special linear group

We say that the stored energy $\widetilde{W}_2$ is invariant under the special linear group, if for every $\widetilde{x} = x' + x_3n'$, and $(F', F_3) \in T^*_{x'}(S'; \mathbb{R}^3) \times \mathbb{R}^3$, we have

$$\widetilde{W}_2\left(\widetilde{x}, F'U\pi'_{x'} + F_3 \otimes n'\right) = \widetilde{W}_2\left(\widetilde{x}, F'\pi'_{x'} + F_3 \otimes n'\right),$$

for every $U \in \mathrm{SL}(T_{x'}S')$. Note that this definition is equivalent to the one given in the abtract configuration.

Proposition 10. *Assume that $\widetilde{W}_2$ is right-invariant under the special linear group, then*

$$\widetilde{I}_{flex}(\psi_0) = \frac{1}{3}\int_{S'} \widetilde{K}_{x',\det(C')}\left(|n(D'\psi_0)|H - H_0\right) dx', \tag{48}$$

where H and H_0 are respectively the mean curvatures of the deformed shell $\psi_0(S')$ and undeformed shell S'. Moreover, if the shell is homogeneous along its fibers, then

$$\widetilde{I}_{flex}(\psi) = \frac{1}{3}\int_{S'} \widetilde{\kappa}_{x',\det(C')}(|n(D'\psi_0)|H - H_0)^2 dx'.$$

Proof. For all section F' of $T^*(S';\mathbb{R}^3)$ and $G' \in T^*_{x'}(S';\mathbb{R}^3)$, be have

$$\widetilde{P}_{C'}(F'^T(G' - F'D'n') + (G' - F'D'n')^T F') = P_{C'}(F'^T G' + G'^T F').$$

From Proposition 5 and (31), we deduce that

$$\begin{aligned}\widetilde{P}_{C'}(F'^T(G' - F'D'n') + (G' - F'D'n')^T F') = \\ \widetilde{P}_{(\det C')^{1/2}\,\mathrm{Id}}\left(\mathrm{Tr}\left(C'^{-1/2}(F'^T(G' - F'D'n') + (G' - F'D'n')^T F')C'^{-1/2}\right)(\det C')^{1/2}\,\mathrm{Id}/2\right).\end{aligned}$$

We thus obtain (48) with

$$\widetilde{K}_{x',\det(C')}(H) = \widetilde{P}_{\det(C')\,\mathrm{Id}}(H(\det C')^{1/2}\,\mathrm{Id}/2). \tag{49}$$

If the shell is homogeneous along its fibers, then the $\widetilde{P}_{(\det C')\,\mathrm{Id}}$ is homogeneous of degree two, wherefrom the conclusion in this case. □

6. Examples

We are now in position to apply our formal convergence result to derive different modelings for isometric bending shells, vesicles and RBCs. Note that, in our setting, we do not derive the nonlinear membrane shell model (see [26]) since W_2 cannot be chosen to be equal to zero. In all this section, we assume that W^ε or $\widetilde{W}^\varepsilon$ satisfy assumptions (2,3) and (35,38), that the Dirichlet boundary conditions on Γ^ε are given by (7,18) and that the minimizers φ^ε or $\widetilde{\varphi}^\varepsilon$ of the total energy do admit an asymptotic expansion as in (6,41), while their total energies $I_\varepsilon(\varphi^\varepsilon)$ and $\widetilde{I}_\varepsilon(\widetilde{\varphi}^\varepsilon)$ remain bounded. Moreover, the stored energies are assumed to be frame indifferent.

6.1. *Isometric bending shells*

In this section, we recover the isometric bending shell model by choosing $\widetilde{W}_0 = 0$ and the set $\widetilde{\mathcal{M}}$ of the zeros of $\widetilde{W}_2$ restricted to the midsection to be equal to

$$\widetilde{\mathcal{M}}_{iso} := S' \times \mathrm{SO}(3). \tag{50}$$

The sequence of minimizers of the energy converges toward the minimizer of an energy whose elastic part depends only on the difference between the second fundamental form of the deformed shell and that of its reference configuration.

Proposition 11. *If $\widetilde{W}^\varepsilon = \varepsilon^{-3}\widetilde{W}_2$;if $\widetilde{W}_2$ is such that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_{iso}$ given by (50) then*

$$\varphi_0 = \arg\min_{\psi_0 \in \Psi_0} \widetilde{I}_0(\psi_0),$$

where

$$\widetilde{I}_0(\psi_0) = \frac{1}{3}\int_{S'} \widetilde{P}_{x',\mathrm{Id}}\left((D'\psi_0)^T D'n + D'nD'\psi_0 - (D'n' + D'n'^T)\right)dx' - 2\int_{S'} f_0 \cdot \psi_0\, dx', \tag{51}$$

$f_0(x') = f(x', 0)$ for every $x' \in S'$, $n = n(D'\psi_0)$, $\widetilde{P}_{x',\mathrm{Id}}$ is a polynomial of degree at most equal to 2 given by (46) and

$$\Psi_0 = \{\psi_0 \in W^{1,\infty}(S')^3 : (D'\psi_0)^T D'\psi_0 = \mathrm{Id}, n = n(D'\psi_0) \in W^{1,\infty}(S')^3,\ \psi_0(x') = \phi_0(x')$$
$$\text{and } n(x') = n_\gamma(x') \text{ for a.e. } x' \in \gamma\}.$$

Moreover, if the shell is homogeneous along its fibers, then $\widetilde{P}_{x',\mathrm{Id}}$ is homogeneous of degree 2.

Proof. Its is a straightforward application of Proposition 6 and Proposition 8. □

Example. Let us give a practical example. For instance, one can choose the Saint Venant-Kirchhoff nonlinearly elastic stored energy function

$$\widetilde{W}_2(F) = \mu\,\mathrm{Tr}((F^TF - \mathrm{Id})^2) + \frac{\lambda}{2}\,\mathrm{Tr}(F^TF - \mathrm{Id})^2.$$

A simple computation leads to the energy

$$\widetilde{I}_0(\psi_0) = \frac{1}{3}\int_{S'} 2\mu\,\mathrm{Tr}((b - b_{ref})^2) + \frac{\lambda\mu}{2\mu + \lambda}\,\mathrm{Tr}(b - b_{ref})^2 - f_0 \cdot \psi_0 dx'$$

where $b = (D'\psi_0)^T D'n + (D'n)^T D'\psi_0$ is the second fundamental form of the deformed shell and $b_{ref} = D'n'^T + D'n'$ is the second fundamental form of the undeformed shell.

6.2. *Vesicles*

In this section, we derive Helfrich functionals, with or without spontaneous curvature, from three-dimensional elasticity. The main difference with the isometric case lies in the fact that we assume the energy to be right-invariant under the special linear group $\mathrm{SL}(TS')$. Note that this readily implies that it may not be chosen to be isotropic. The Helfrich functional without spontaneous curvature is derived using the abstract configuration S, while the one with spontaneous curvature is obtained using the geometric configuration $\widetilde{S}$ of the shell.

6.2.1. Without spontaneous curvature

In this section, we consider the case, where the zero set of W_2 restricted to the midsection is given by

$$\mathcal{M}_H := \left\{(F', F_3) \in T^*(S'; \mathbb{R}^3) \times T_0^*((-1,1); \mathbb{R}^3) : \det(F) = 1,\ F_3 \cdot v = \det(F', v) \text{ for all } v \in T_0^*((-1,1); \mathbb{R}^3)\right\}. \quad (52)$$

From Propositions 2 and 5, we obtain that the minimizers of the energy formally converge toward the Helfrich functional with no spontaneous curvature.

Proposition 12. *If* $W^\varepsilon = \varepsilon^{-3} W_2$ *;if* W_2 *is right-invariant under* $\mathrm{SL}(TS')$*, and such that* $\mathcal{M} = \mathcal{M}_H$ *given by (52); If the shell is homogeneous along its fibers in the abstract configuration then*

$$\varphi_0 = \arg\min_{\psi_0 \in \Psi_0} I_0(\psi_0),$$

where

$$I_0(\psi_0) = \frac{1}{3}\int_{S'} \kappa |H|^2 dx' - 2\int_{S'} f_0 \cdot \psi_0 dx', \quad (53)$$

$f_0(x') = f(x', 0)$ *for every* $x' \in S'$*,* H *is the mean curvature of the deformed shell* $\psi_0(S')$*,* $\kappa(x') = P_{\mathrm{Id}}(\mathrm{Id}/2)$*, where* P_{Id} *is given by (25), and*

$$\Psi_0 = \{\psi_0 \in W^{1,\infty}(S')^3 \ :\ n \in W^{1,\infty}(S')^3 \textit{ while } \det((D'\psi_0)^T D'\psi_0) = 1,\ \psi_0(x') = \phi_0(x') \text{ and } n(x') = n_\gamma(x') \textit{ for a.e. } x' \in \gamma\},$$

and n *is the normal to the deformed surface* $\psi_0(S')$*.*

Example. Proposition 12 can be applied with

$$W_2(F) = \hat{W}(C) = \alpha(\det(C) - 1)^2 + \beta|Ce_3 - e_3|^2, \quad (54)$$

where $C = F^T F$ and α and β are positive real constants. A simple computation leads to

$$D^2\hat{W}\left[0, \begin{pmatrix} \mathrm{Id}/2 & w' \\ w' & w_3 \end{pmatrix}_{\mathrm{Id}}\right]^2 = 2\alpha(1 + w_3)^2 + 2\beta|(w', w_3)|^2.$$

Then, from the expression (25) of P_{Id}, we get

$$\kappa = P_{\mathrm{Id}}(\mathrm{Id}/2) = \inf_w 2\alpha(1 + w_3)^2 + 2\beta|(w', w_3)|^2 = 2(\alpha^{-1} + \beta^{-1})^{-1}.$$

Hence, the limit energy in this case is

$$I_0(\psi_0) = \frac{2}{3}\int_{S'} (\alpha^{-1} + \beta^{-1})^{-1} |H|^2 dx' - 2\int_{S'} f_0 \cdot \psi_0 dx'.$$

6.2.2. With spontaneous curvature

In this section, we derive from three-dimensional elasticity a model of shells whose limit energy is the Helfrich functional with nonzero spontaneous curvature. Basically, such a model is obtained by using the same assumptions as in the previous case but cast in the geometric configuration, with a set of zeros restricted to the midsection for $\widetilde{W}_2$ given by

$$\widetilde{\mathcal{M}}_H := \left\{(x', F) \in S' \times (\mathbb{R}^{3\times 3}) : \det(F) = 1, \text{ and } (\mathrm{Cof}\, F - F)n' = 0\right\}. \quad (55)$$

The following Proposition is a direct application of Proposition 6 and Proposition 10.

Proposition 13. *If $\widetilde{W}^{\varepsilon} = \varepsilon^{-3}\widetilde{W}_2$;if $\widetilde{W}_2$ is right-invariant under* $\mathrm{SL}(TS')$*, and such that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_H$ given by (55); if the shell is homogeneous along its fibers in the geometric configuration then*

$$\varphi_0 = \arg\min_{\psi_0\in\Psi_0} \widetilde{I}_0(\psi_0),$$

where

$$\widetilde{I}_0(\psi_0) = \frac{1}{3}\int_{S'} \widetilde{\kappa}|H - H_0|^2 dx' - 2\int_{S'} f_0 \cdot \psi_0 dx', \tag{56}$$

$f_0(x') = \widetilde{f}(x', 0)$ for every $x' \in S'$, where H is the mean curvature of the deformed shell $\psi_0(S')$ and H_0 is the mean curvature of S', with

$$\widetilde{\kappa}(x') = \widetilde{P}_{\mathrm{Id}}(\mathrm{Id}/2),$$

where $\widetilde{P}_{\mathrm{Id}}$ is given by (47) and

$$\Psi_0 = \{\psi_0 \in W^{1,\infty}(S')^3 \ : \ n \in W^{1,\infty}(S')^3 \text{ while } \det(D'\psi_0{}^T D'\psi_0) = 1,\ \psi_0(x') = \phi_0(x') \text{ and } n(x') = n_\gamma(x') \text{ for a.e. } x' \in \gamma\}, \tag{57}$$

where n is the normal to the deformed surface $\psi_0(S')$.

Example. The stored energy function

$$\widetilde{W}_2(x', F) = \hat{\widetilde{W}}(x', \widetilde{C}) = \alpha(\det(F^T F) - 1)^2 + \beta|F^T F n' - n'|^2.$$

satisfies the assumptions of Proposition 13, and we have

$$\widetilde{\kappa} = \inf_{w'\in T_{\cdot}S'\, w_3\in\mathbb{R}} \frac{\partial\hat{\widetilde{W}}}{\partial\widetilde{C}^2}(x', \mathrm{Id})\left[\pi'^T\pi'/2 + n'w'^T\pi' + \pi'^T w' n'^T + n' w_3 n'^T\right]^2.$$

Furthermore, we have

$$\frac{\partial\hat{\widetilde{W}}}{\partial\widetilde{C}^2}(x', \mathrm{Id})\,[\delta C]^2 = 2(\alpha\,\mathrm{Tr}(\delta C)^2 + \beta|\delta C n'|^2),$$

so that,

$$\widetilde{\kappa} = \inf_{w'\in T_{\cdot}S'\, w_3\in\mathbb{R}} 2\left(\alpha(1 + w_3)^2 + \beta(|w'|^2 + w_3^2)\right) = 2(\alpha^{-1} + \beta^{-1})^{-1}.$$

For such a choice of $\widetilde{W}_2$, and under the assumptions made in Proposition 13 on the sequence of minimizers $\widetilde{\varphi}^{\varepsilon}$, these converge formally toward a minimizer of

$$\widetilde{I}_0(\psi_0) = \frac{2}{3}\int_{S'} (\alpha^{-1} + \beta^{-1})^{-1}|H - H_0|^2 dx' - 2\int_{S'} \widetilde{f}_0 \cdot \psi_0 dx',$$

over Ψ_0 given by (57). Note that this is the set of deformations that preserve the local area of the shell supplemented with boundary conditions on a subset of the boundary.

6.3. *Red Blood Cells*

The mechanical behavior of a Red Blood Cell (RBC) is driven by the nature of its membrane which is mainly made of a lipid bilayer. Note that in addition to the lipid bilayer, RBCs are also composed of a protein skeleton. This skeleton ensures a small resistance of the RBCs to shear stress. Such a model may be obtained as the limit of the three-dimensional elasticity. In this section, we derive a model of the mechanical behavior of RBCs as the limit of genuine three-dimensional elasticity. To this end, we consider a stored energy W^{ε} whose asymptotic assumption reads as

$$W^{\varepsilon} = \varepsilon^{-1}(\varepsilon^{-2} W_2 + W_0),$$

where W_2 satisfies the same assumption as in the study of vesicles without spontaneous curvature (see section 6.2.1), namely, its zero set restricted to the midsection is given by (52). We get that the sequence φ^ε of minimizers formally converges toward

$$\varphi_0 = \arg\min_{\psi_0\in\Psi_0} I_0(\psi_0),$$

where

$$I_0(\psi_0) = \frac{1}{3}\int_{S'} k|H|^2 dx' + 2\int_{S'} W_0(D'\psi_0, n)dx' - 2\int_{S'} f_0\cdot\psi_0 dx',$$

$f_0(x') = f(x', 0)$, n is the normal to the deformed shell $\psi_0(S')$ and Ψ_0 is the set of deformations that preserve the local area of the shell and satisfy the boundary conditions

$$\psi_0(x') = \phi_0(x') \quad \text{and} \quad n(x') = n_\gamma(x') \quad \text{for every } x' \in \gamma.$$

Example. As an example, we can choose the nonlinearly elastic Saint Venant-Kirchhoff stored energy function

$$W_0(F) = \mu\,\mathrm{Tr}\left((C-\mathrm{Id})^2\right) + \frac{\lambda}{2}\,\mathrm{Tr}\,(C-\mathrm{Id})^2\,,\ \text{with } C = F^T F,$$

and $W_2(F)$ as in (54). This leads to a limit energy

$$I_0(\psi_0) = \frac{2}{3}\int_{S'}(\alpha^{-1}+\beta^{-1})^{-1}|H|^2 dx' + 2\int_{S'}\mu\,\mathrm{Tr}\left((D'\psi_0{}^T D'\psi_0 - \mathrm{Id}')^2\right) + \frac{\lambda}{2}\,\mathrm{Tr}\left(D'\psi_0{}^T D'\psi_0 - \mathrm{Id}'\right)^2 dx'$$
$$- 2\int_{S'} f_0\cdot\psi_0 dx'.$$

7. Conclusion

In this article, we prove, using a formal approach, that new nonlinearly elastic shell models may be derived assuming the shell to be highly anisotropic. Notably, it enables us to derive some models used in the study of vesicles and RBCs. Part of the results presented in this article have since been proved by a Γ-convergence approach in an Eulerian setting for the justification of the modeling of vesicles by Merlet [32, 33]. Finally, let us recall and emphasize the fact that the computation of the limit energy should include a relaxation step that is not taken into account in our formal framework. The only interesting case being the one where the flexural term $Q^0_{F'}(G, s)$ is not fully degenerate, that is not independent of G. In such a case, a relaxation of the membrane term of the limit energy is expected to take place. The correct limit energy in Proposition 2 should read

$$I_0{}'(\psi_0) = \frac{1}{3}\int_{S'} Q^0_{D'\psi_0}(D'n, 1)dx' + 2\int_{S'} Q' W_0(D'\psi_0, n)dx' - 2\int_{S'} f_0 \cdot \psi_0 dx',$$

where $Q'W_0$ is the in-plane quasiconvexification of W_0, defined for every $F' \in T^*(S'; \mathbb{R}^3)$ by

$$Q'W_0(F', n) = \inf_{\varphi \in C_0^\infty(\omega; T_{x'}S')} |\omega|^{-1} \int_\omega W_0(F'(\mathrm{Id}' + D'\varphi), n)\, dy',$$

where $x' = \pi_{S'}(F')$ and ω is a bounded regular open set of $T_{x'}S'$.

Notations

- $\mathbb{R}$, set of reals
- $\mathbb{R}'$, dual set of reals
- $\mathbb{N}$, set of non-negative integers
- S', midsurface of the shell in the reference configuration
- ε, thickness of the shell
- $S^\varepsilon := S' \times (-\varepsilon, \varepsilon)$, abstract reference configuration of the shell
- $S := S^1$, rescaled abstract reference configuration of the shell
- $\widetilde{S}^\varepsilon$, geometric reference configuration
- $\widetilde{S}$, geometric reference configuration of maximum thickness
- x^ε, element of S^ε
- $\widetilde{x}$, element of $\widetilde{S}$
- x', element of S'
- TM, tangent space to M
- T_xM, tangent fiber to M at x
- T^*M, cotangent space to M
- T^*_xM, cotangent space to M at x
- $T(M;\mathbb{R}^3) := TM \oplus TM \oplus TM$, Whitney triple sum of TM
- $T_x(M;\mathbb{R}^3) := T_xM \oplus T_xM \oplus T_xM$, Whitney triple sum of T_xM
- $T^*(M;\mathbb{R}^3) := T^*M \oplus T^*M \oplus T^*M$, Whitney triple sum of T^*M
- $T^*_x(M;\mathbb{R}^3) := T^*_xM \oplus T^*_xM \oplus T^*_xM$, Whitney triple sum of T^*_xM
- $\pi_B : \mathcal{F} \to B$, projection of a fiber bundle $\mathcal{F}$ onto its base B
- $\pi'_{x'} : \mathbb{R}^3 \to T_{x'}S'$, projection onto $T_{x'}S'$
- ψ^ε, deformation of S^ε
- $\psi(\varepsilon) : S \to \mathbb{R}^3$, rescaled map of the deformation ψ^ε
- $\boldsymbol{\psi} = (\psi_k)$, expansion of the deformation $\psi(\varepsilon) = \sum_k \varepsilon^k \psi_k$ with respect to the thickness
- $\boldsymbol{\varphi} = (\varphi_k)$, expansion of the minimizers $\varphi(\varepsilon) = \sum_k \varepsilon^k \varphi_k$ with respect to the thickness
- $D\psi^\varepsilon$, differential of $\psi^\varepsilon : S^\varepsilon \to \mathbb{R}^3$
- $D\psi^\varepsilon(x^\varepsilon)$, differential of $\psi^\varepsilon : S^\varepsilon \to \mathbb{R}^3$ at x^ε
- $(D'\psi^\varepsilon, D_3\psi^\varepsilon)$, decomposition of $D\psi^\varepsilon \in T^*(S^\varepsilon;\mathbb{R}^3)$ on $T^*(S';\mathbb{R}^3) \times T^*((-\varepsilon,\varepsilon);\mathbb{R}^3)$ where $\psi^\varepsilon : S^\varepsilon \to \mathbb{R}^3$
- $(D'\psi^\varepsilon(x), D_3\psi^\varepsilon(x))$, decomposition of $D\psi^\varepsilon(x) \in T^*_x(S^\varepsilon;\mathbb{R}^3)$ on $T^*_{x'}(S';\mathbb{R}^3) \times T^*_{x_3}((-\varepsilon,\varepsilon);\mathbb{R}^3)$ where $\psi^\varepsilon : S^\varepsilon \to \mathbb{R}^3$

- ∂_3, partial differentiation along the fibers
- $J_\varepsilon(\psi^\varepsilon)$, elastic energy of a deformation $\psi^\varepsilon : S^\varepsilon \to \mathbb{R}^3$
- $I_\varepsilon(\psi^\varepsilon)$, total energy of a deformation $\psi^\varepsilon : S^\varepsilon \to \mathbb{R}^3$
- $L_\varepsilon(\psi^\varepsilon)$, work of the external loads
- $J(\varepsilon)(\psi(\varepsilon))$, rescaled elastic energy of the deformation ψ^ε
- $I(\varepsilon)(\psi(\varepsilon))$, rescaled total energy of the deformation ψ^ε
- $\mathbf{J}(\varepsilon)(\boldsymbol{\psi})$, elastic energy of the deformation of asymptotic expansion $\boldsymbol{\psi}$
- $\mathbf{I}(\varepsilon)(\boldsymbol{\psi})$, total energy of the deformation of asymptotic expansion $\boldsymbol{\psi}$
- $I_0(\psi_0)$, limit of the total energy (for standard boundary conditions)
- f_ε, external loads
- $W^\varepsilon : T^*(S^\varepsilon; \mathbb{R}^3) \to \overline{\mathbb{R}}^+$, stored energy
- Ψ^ε, set of admissible deformations of S^ε
- $\boldsymbol{\Psi}$, set of admissible asymptotic expansions for the deformations of S^ε
- $W_k : T^*(S; \mathbb{R}^3) \to \mathbb{R}^+$, k-th term of the asymptotic expansion of the stored energy
- $\mathcal{M} \subset T^*(S; \mathbb{R}^3)$, the restriction of the zero set of W_2 to the midsection S'
- $\mathcal{M}' \subset T^*(S'; \mathbb{R}^3)$, projection of $\mathcal{M}$ on $T^*(S'; \mathbb{R}^3)$
- $n_0 : \mathcal{M}' \to T_0^*((-1, 1); \mathbb{R}^3)$, orientation of the normal fiber in the deformed configuration
- $n : \mathcal{M}' \to \mathbb{R}^3$, orientation of the normal fiber in the deformed configuration (only the vectorial part)
- n', normal to the midsection S' of the reference configuration
- $D^2 W_2$, second derivative of W_2 on $\mathcal{M}$
- $a[\cdot]^2 = a(\cdot, \cdot)$, where a is a bilinear form
- $Q_F : T^*(S'; \mathbb{R}^3) \times \mathbb{R}^3 \to \mathbb{R}$, quadratic form associated to the flexural limit energy, where F is a section of $\mathcal{M}'$
- $Q_F^0 := \inf_v Q_F(\cdot, v) : T^*(S'; \mathbb{R}^3) \to \mathbb{R}$
- $I_{flex}(\psi)$, flexural limit energy of a deformation ψ
- dx', 2-dimensional Hausdorff measure restricted to S'
- $\mathrm{Tr}(A)$, the trace of the matrix A
- $\widetilde{\psi}^\varepsilon$, $\widetilde{W}^\varepsilon$, $\widetilde{W}_k$, $\widetilde{\mathcal{M}}$, $\widetilde{L}_\varepsilon$, $\widetilde{f}_\varepsilon, \cdots$ overtilded variables are defined on the geometric configuration
- $\mathrm{SO}(TM)$, rotations of TM, that is the fiber bundle made of all rotations of $T_x M$ (with $x \in M$), where M is a Riemann manifold
- $\mathrm{SO}(n)$, rotations of $\mathbb{R}^n$
- $\mathrm{SL}(TM)$, special group of TM, that is the fiber bundle made of all linear diffeomorphism of $T_x M$ of determinant equal to one (with $x \in M$), where M is a Riemann manifold
- $\mathcal{E}_M$, set of symetric bilinear forms on the tangent space of M
- $\mathcal{O}$, fiber bundle over S' whose fibers are the maps from $T_{x'} S'$ into itself of zero trace
- $\mathcal{P}$, set of polynomials of degree lower than or equal to two on $\mathcal{E}_{S'}$

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