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Simultaneous shape, topology and homogenized properties optimization

O. Pantz, K. Trabelsi

Abstract In this brief note, we present an approach that combines the three classical techniques in structural optimization, i.e. the boundary variation, the topological and the homogenization methods. As a first test of this method, we apply it to the compliance optimization in $\mathbb{R}^2$.

1 Introduction

The standard problem in structural optimization consists in finding the optimal shape of a structure which is at once of minimal weight and of maximal rigidity. One way of tackling the problem is to perform variations of the boundary that is to say perturbations of some set Ω representing the initial shape. After defining the shape derivative, one associates to the latter a direction of slope which is then determined by the gradient method. This technique is also known as shape sensitivity, see (Pironneau 1984). However this method bears an important drawback which is that the resulting shape depends heavily on the initial one. Namely, the topology of the shape remains unchanged. Actually, this method yields local minima which may be remote from global ones. An alternative approach is the homogenization method which optimizes the topology and generates global minima. It is grounded on the introduction of composite materials which can be shown to produce optimal shapes in a relaxed or generalized sense. The technique consists first in computing an optimal composite shape whose areas of intermediate densities are then removed through some penalization procedure, see (Allaire 2001). Yet another way of undertaking optimal shape design is by topological gradient methods. Namely, the optimal shape is sought in a larger class of domains $\mathcal{F} = \{\Omega \subset D\}$ where the domain D is determined by the desired bulk of the final structure. Typically, the notion of topological gradient is required to assess where material shall be removed, see (Sokołowski and Żochowski 1999) and (Céa *et al.* 2000).

A classical approach to structural optimization is the sequential implementation of the topological and geometric optimization albeit for a non-composite material, see the pioneering paper of (Olhoff *et al.* 1992); see also (Eschenauer and Schumacher 1994) and (Beuzit and Habbal 1998). Note that topological and shape sensitivity have already been coupled via a level set method, see for instance (Allaire *et al.* 2004) and (Burger *et al.* 2004).

In the work described hereafter, we combine these three methods in one procedure. In short, we apply shape sensitivity to a homogenized composite structure which we perforate by a criterion similar to the topological gradient. In doing so, we aim at melding the advantages of the three methods just outlined thus raising a highly efficient shape design routine. Numerical simulations are made with `FreeFem++`, see (Hecht *et al.* 2005).

2 Setting of the initial problem

Let Ω be the reference configuration of an isotropic elastic body in $\mathbb{R}^N$. Let Γ, Γ_N and Γ_D be a partition of the boundary of Ω. Function $u(\Omega)$ denotes the displacement of the structure $\Omega \subset D$ which is assumed to be clamped on Γ_D and submitted to surface loads g on Γ_N

$$\begin{cases} \operatorname{div}\sigma = 0 \text{ in } \Omega, \text{ with } \sigma = Ae(u), \\ u = 0 \quad \text{ on } \Gamma_D, \quad \sigma\cdot n = g \text{ on } \Gamma_N, \\ \sigma\cdot n = 0 \text{ on } \Gamma = \partial\Omega \setminus (\Gamma_N \cup \Gamma_D), \end{cases}$$

where $e(u) = (\nabla u + \nabla u^T)/2$ and A is Hooke's law or the elasticity tensor defined by

$$A\xi = 2\mu\xi + \lambda(\operatorname{Tr}\xi)\,\mathrm{Id}, \tag{1}$$

with Lamé moduli λ and μ. We consider the following compliance minimization problem

$$\min_{|\Omega|=V,\ \Gamma_D\cup\Gamma_N\subset\partial\Omega} \int_{\Gamma_N} g\cdot u(\Omega)\,ds, \tag{2}$$

where V is a given volume.

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O. Pantz, K. Trabelsi

CMAP, Ecole Polytechnique, 91128 Palaiseau, France
e-mail: trabelsi@cmap.polytechnique.fr

3
A brief reminder of structural optimization methods

3.1
Shape sensitivity

Geometric optimization is an application of the gradient method to shapes. We define perturbations of the set Ω by

$$\Omega(d) = (\mathrm{Id} + d)(\Omega) \equiv \{x + d(x) \text{ such that } x \in \Omega\},$$

where $d : \Omega \to \mathbb{R}^N \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ is a given *small* vector field. Now, if the map $F_\Omega : d \to J(\Omega(d))$ is differentiable at $d = 0$, we define the shape derivative

$$\langle J'(\Omega), d\rangle = \langle F'_\Omega(0), d\rangle.$$

Note that by the Hadamard structure theorem, it is known that the above derivative is carried only on the boundary of the shape, see Theorem 1. Next, we apply the gradient method by defining a direction of slope h in the following fashion. We endow the space of vector fields from Ω into $\mathbb{R}^N$ with the $H^1(\Omega)^N$ Hilbert structure, thence the descent direction is the unique element $h \in H(\Omega) = \{d \in H^1(\Omega)^N : d_{|\Gamma_D \cup \Gamma_N} = 0\}$ that satisfies

$$\int_\Omega (\nabla h \cdot \nabla d + h \cdot d)\, dx = \langle J'(\Omega), d\rangle + \alpha \int_\Gamma (d \cdot n)\, ds, \quad (3)$$

for every $d \in H(\Omega)$, where α is the Lagrange multiplier corresponding to the volume constraint $|\Omega| = V$.

Remark 1 The choice of the space of variations is ruled by technical implementation considerations. For further remarks we refer to (Allaire and Pantz 2006). For a thorough introduction to geometric optimization and its implementation, we send the reader to (Allaire 2005) and to (Sokołowski and Zolesio 1992).

3.2
The homogenization method

Minimizing sequences of (2) converge to a composite shape and the principle of the homogenization method is to extend the minimization problem to such generalized or composite shapes. The composite shape is described by two variables, the material density $\theta(x) : \Omega \to (0, 1)$ and the homogenized Hooke tensor $A^*(x)$, which represents the underlying micro-structure (the shape of the holes). The displacement of the composite structure is the solution of

$$\begin{cases} \operatorname{div} \sigma = 0 \text{ in } \Omega, \text{ with } \sigma = A^* e(u), \\ \sigma \cdot n = g \text{ on } \Gamma_N, \text{ and } \sigma \cdot n = 0 \text{ on } \Gamma \\ u = 0 \quad \text{ on } \Gamma_D. \end{cases} \quad (4)$$

The homogenized optimization problem is defined as

$$\begin{aligned} J_m(\Omega) &= \min_{\theta, A^*} J(\Omega, \theta, A^*), \\ J(\Omega, \theta, A^*) &= \int_{\Gamma_N} g \cdot u\, ds = \int_\Omega {A^*}^{-1} \sigma \cdot \sigma\, dx, \end{aligned} \quad (5)$$

where the minimization takes place over all couples (θ, A^*) such that A^* is a homogenized Hooke law corresponding to a material density θ such that $\int_\Omega \theta\, dx = m$, and m is a given quantity of matter.

The minimum is reached by special composites called sequential laminates, obtained as successive layerings of void and material in N orthogonal directions and with adequate proportions. The directions of lamination are given by the eigenvectors of the stress tensor σ. Moreover, it is shown that the energy satisfies a Hashin and Shtrikman bound that is

$$HS(\sigma) = \min_{A^* \in G_\theta} {A^*}^{-1} \sigma \cdot \sigma = \min_{A^* \in L_\theta} {A^*}^{-1} \sigma \cdot \sigma, \quad (6)$$

where G_θ is the set of homogenized elasticity tensors and L_θ is the subset of G_θ consisting of the composite laminated materials. For instance, in dimension $N = 2$, a tedious computation leads to (see (Allaire 2001))

$$HS(\sigma) = A^{-1} \sigma \cdot \sigma + \frac{1 - \theta}{\theta} \frac{\kappa + \mu}{4\mu\kappa} (|\sigma_1| + |\sigma_2|)^2, \quad (7)$$

where σ_1 and σ_2 are the eigenvalues of σ and $\kappa = \lambda + \mu$. Finally, the optimal density θ is

$$\theta_m^{opt}(\Omega) = \min\left(1, \sqrt{\frac{\kappa + \mu}{4\mu\kappa\ell}} (|\sigma_1| + |\sigma_2|)\right), \quad (8)$$

where ℓ is the Lagrange multiplier corresponding to the volume constraint $\int_\Omega \theta\, dx = m$.

For an in depth survey of the homogenization method, we send the reader to (Allaire 2001).

3.3
Topological gradient

The effect of the nucleation of holes on the optimality of the shape is rigorously informed by the notion of topological gradient (cf. (Sokołowski and Żochowski 1999), see also (Céa *et al.* 2000) and the references therein) which measures the perturbation of the objective function, say $J(\Omega)$, with respect to a (small) perforation of the domain through a so-called topological asymptotic expansion with the size of the hole as the small parameter. More precisely, for the compliance in $\mathbb{R}^2$, f is the topological derivative of J if

$$J(\Omega_\rho) = J(\Omega) + \rho^2 f(x_0, \omega) + o(\rho^2), \quad (9)$$

where $\Omega_\rho = \Omega \setminus (x_0 + \rho\omega)$, $x_0 \in \Omega$, $\rho > 0$ and ω is a bounded open set containing the origin.

In our implementation, we did not use the topological gradient as a criterion for nucleating holes. Actually, it seems to us that removing matter in those areas where the density (or stress) is close to zero after convergence of the shape sensitivity loop and the homogenization one is quite a natural criterion which is as good as using the topological gradient.

4 Combining the three methods

We present now our method which consists in combining the above three techniques in one.

4.1 Numerical algorithm

Let V be the volume of the optimization domain Ω, m the quantity of material available such that $m \leq V$, θ the density of the material and A^* the corresponding microstructure, and call $\mathcal{P}_{V,m}$ the problem that consists in minimizing $J_m(\Omega)$ under the constraints $|\Omega| = V$. Our algorithm consists in solving a sequence of problems $\mathcal{P}_{V_n,m}$ while $V_n \to m$. When $V = m$, the density θ is constant equal to 1 and a real (non-composite) optimal shape is thereby obtained.

The algorithm consists of two subroutines and a main loop. The first subroutine determines $\Omega_m^{geo}(\Omega, V)$ the solution of the minimization of J_m by the boundary variation (geometric) method, initialized by Ω:

(i) Initialization $\Omega_0 = \Omega$.
(ii) Iteration until convergence:
  (a) update the Lagrange multiplier α_n,
  (b) compute h_n solution of problem (3) with $\Omega = \Omega_n$ and $J = J_m$,
  (c) set $\Omega_{n+1} = (\mathrm{Id} - \varepsilon_n h_n)\Omega_n$, where $\varepsilon_n > 0$ is a small real.

The second subroutine computes $\Omega_m^{topo}(\Omega, V)$ the solution of the minimization of J_m by the geometric and topological methods, initialized by $\Omega_m^{geo}(\Omega, V)$:

(i) Initialization $\Omega_0 = \Omega_m^{geo}(\Omega, V)$.
(ii) Iteration until $\min\limits_{\Omega_n} \theta_m^{opt}(\Omega_n) \leq \beta \min\limits_{\Gamma_n} \theta_m^{opt}(\Omega_n)$ with $0 < \beta < 1$, set $\Omega_{n+1} = \Omega_m^{geo}(\Omega_n \backslash B(x_n, \rho_n), V)$ where x_n is the element of Ω_n of minimum density and $\rho_n > 0$ is a small parameter.

The main loop runs as follows:

(i) Initialization Ω_0 such that $|\Omega_0| = V_0 > m$.
(ii) While $V_n > m$,
  (a) compute $\Omega_n = \Omega_m^{topo}(\Omega_{n-1}, V_n)$,
  (b) set $V_{n+1} = \max(V_n - \delta V, m)$, where $\delta > 0$ is a small parameter.

Note that to compute $\Omega_m^{geo}(\Omega, V)$, we need the shape derivative which follows.

Theorem 1 *The map $J_m(\Omega)$ is differentible at zero and for every d such that $d = 0$ on $\Gamma - D \cup \Gamma_N$, we have*

$$\langle J_m'(\Omega), d\rangle = \int_\Gamma \Big(-\frac{\kappa+\mu}{4\mu\kappa}|\sigma_2|^2 + \Big[\ell - \frac{\kappa+\mu}{4\mu\kappa}|\sigma_2|^2\Big]_-\Big)(d\cdot n)\, ds, \tag{10}$$

where σ_2 is the greatest eigenvalue of the stress tensor σ corresponding to the optimal composite given by (5).

In the above, $[f]_- = f$, if $f \leq 0$, and 0 otherwise.

4.2 Numerical example

As an illustration of the above algorithm, we consider the compliance minimization of the following cantilever Fig. 1. All figures below display the density of the material. The final shapes obtained (after 1000 iterations) are displayed on Fig. 2-3. Let us remark that from a

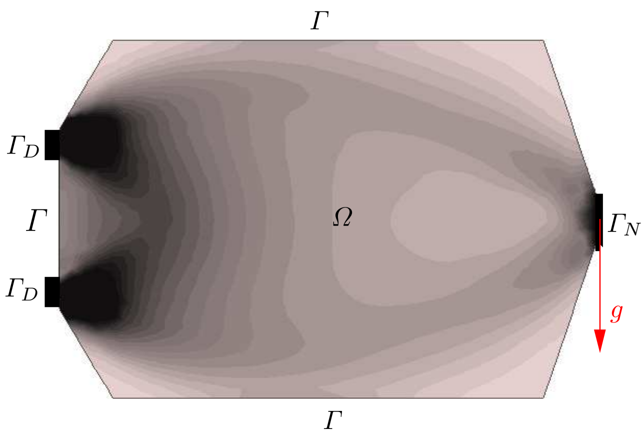

Fig. 1 Initial shape

practical viewpoint we had to regularize the mesh using a fine mesh for computations and a coarser one for moving the nodes. Moreover, the shape gradient had also to be regularized at the corners where boundary conditions change from Dirichlet to Neumann. These numerical simulations have been carried out using `FreeFem++`, see (Hecht *et al.* 2005). For more details regarding implementation issues under `FreeFem++`, we send the reader to (Allaire and Pantz 2006). Practically, the nucleation of holes was carried out according to two criteria. Once the $H^1(\Omega)^2$ norm of slope's direction h_n is small enough, that is when the boundary variation and homogenization methods have converged, we either locate the minimum density (respectively, stress) in the domain Ω and make a hole if it is small compared to the density (resp. stress) on the boundary, see Fig. 2 (resp. Fig. 3).

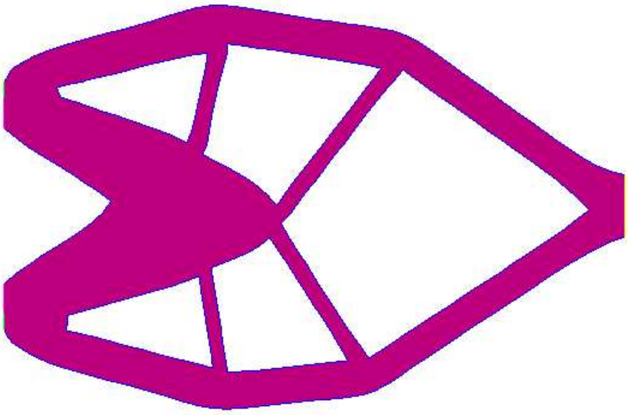

Fig. 2 Final shape

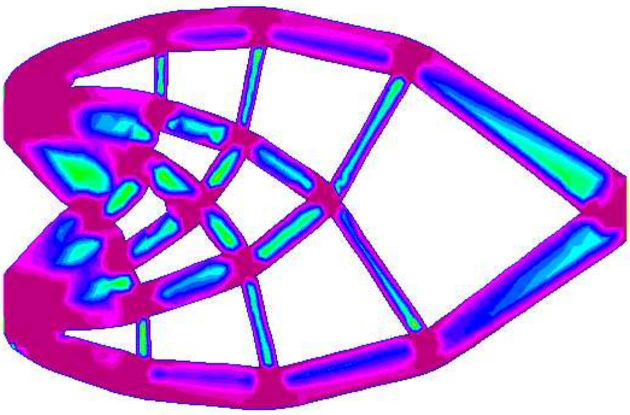

Fig. 3 Final shape (alternate topology optimization)

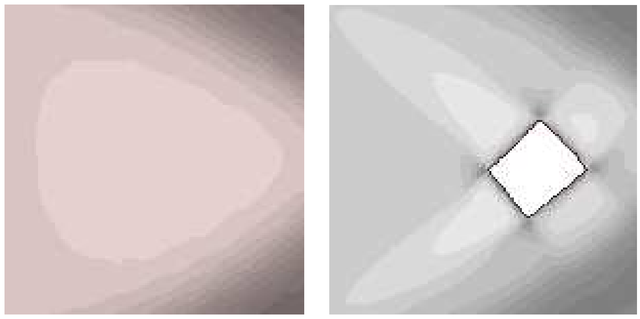

Fig. 4 Zoom before (left) and after the nucleation of a hole (iteration 60)

4.3 Remarks on the numerical results

Fig. 4 represents the optimal density of the cantilever before and after the hole was made. Indeed, lower density areas appear around the hole just after its nucleation. It turns out that the algorithm tends to reproduce the laminated homogenized pattern at a macroscopic level and thus were it not for the imposed geometrical constraints that are the minimal size of the hole and the minimal distance between holes, an infinity of holes would be birthed. This is a confirmation of the fact that problem $\mathcal{P}_{V,m}$ is ill-posed in general, see (Allaire 2001).

Let us also mention that a drawback of a geometric formulation where the shape is defined by a mesh is that holes are hard to manipulate. For instance, it is difficult to merge two holes which we do not do..

5 Conclusion

To conclude, we first mention that our results are comparable to those obtained solely by one of the methods i.e. either the boundary variation or the homogenization, see (Allaire and Pantz 2006). Moreover, the optimal shape obtained does not depend on the geometry nor the topology of the initial shape.

Our next move is to implement this method in a level set framework which is a priori more adapted to the needs of this method. For instance, a level set formulation allows for the merging of holes thereby making possible the nucleation of a large quantity of holes at once which seems to be the natural will of this algorithm as was remarked in the previous section.

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References

Allaire G.: *Shape optimization by the homogenization method*, Springer Verlag, New York, 2001.

Allaire G., de Gournay F., Jouve F., Toader A.: Structural optimization using toplogical and shape sensitivity via a level set method, *Control and Cyb.*, **34** (2004), 59-80.

Allaire G.: *Conception optimale de structures*, Editions de l'Ecole Polytechnique, 2005.

Allaire G., Pantz O.: Structural Optimization with `FreeFem++`, to appear in *Struct. Multidisc. Optim.* (2006).

Bendsoe M. P., Sigmund O.: *Topology optimization. Theory, methods and applications*, Springer Verlag, Berlin (2001).

Beuzit S., Habbal A.: Design of an automatic topology/geometry optimization software, *Proceedings of the 2nd International Conference on Integrated Design and Manufacturing in Mechanical Engineering*, Jean-Louis Batoz *et al.* Editor, 133-140, Kluwer, 1998.

Burger M., Hackl B., Ring W.: Incorporating topological derivatives into level set methods, *J. Comput. Phys.* **194** (2004), no. 1, 344-362.

Céa J., Garreau S., Guillaume P., Masmoudi M.: The shape and topological optimizations connection. IV WCCM, Part II (Buenos Aires, 1998), *Comput. Methods Appl. Mech. Engrg.* **188** (2000), no. 4, 713-726.

Eschenauer H., Schumacher A.: A bubble method for topology and shape optimization of structures, *Struct. Multidisc. Optim.*, **8**, (1994), 52-51.

Hecht F., Pironneau O., Ohtsuka K., *FreeFem++ Manual*, downloadable at `http://www.freefem.org`.

Olhoff N., Bendsoe M. P., Rasmussen J.: CAD-integrated structural topology and design optimization, *Shape and layout optimization of structural systems and optimality criteria methods*, 171-197, CISM Courses and Lectures **325**, Springer, Vienna, 1992.

Pironneau O.: *Optimal shape design for elliptic systems*, Springer-Verlag, New York, 1984.

Sokołowski J., Żochowski A. : On the topological derivative in shape optimization, *SIAM J. Control Optim.* **37** (1999), no. 4, 1251-1272.

Sokołowski J., Zolesio J.P., *Introduction to shape optimization: shape sensitivity analysis*, Springer Series in Computational Mathematics, Vol. 10, Springer, Berlin, 1992.