A Post-treatment of the homogenization method in shape optimization

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Abstract
In most shape optimization problems, the optimal solution does not belong to the set of genuine shapes but is a composite structure. The homogenization method consists in relaxing the original problem thereby extending the set of admissible structures to composite shapes. From the numerical viewpoint, an important asset of the homogenization method with respect to traditional geometrical optimization is that the computed optimal shape is quite independent from the initial guess (even if only a partial relaxation is performed). Nevertheless, the optimal shape being a composite, a post-treatment is needed in order to produce an almost optimal non-composite (i.e. workable) shape. The classical approach consists in penalizing the intermediate densities of material, but the obtained result deeply depends on the underlying mesh used and the details level is not controllable. In a previous work, we proposed a new post-treatment method for the compliance minimization problem of an elastic structure. The main idea is to approximate the optimal composite shape with a locally periodic composite and to build a sequence of genuine shapes converging toward this composite structure. This method allows us to balance the level of details of the final shape and its optimality. Nevertheless, it was restricted to particular optimal shapes, depending on the topological structure of the lattice describing the arrangement of the holes of the composite. In this article, we lift this restriction in order to extend our method to any optimal composite structure for the compliance minimization problem.

Keywords: Shape optimization, homogenization, elasticity, compliance.

1. Introduction
A paradigm of structural optimization is the compliance minimization problem in elasticity. Namely, we look for the optimal shape of a structure that is of maximal rigidity for a given weight (or quantity of material).

Let $D$ be the optimization domain and let $\Omega \subset D$ be the reference configuration of an isotropic elastic body in $\mathbb{R}^2$. Let $\Gamma$, $\Gamma_N$ and $\Gamma_D$ be a partition of the boundary of $\Omega$. Function $u(\Omega)$ denotes the displacement of the structure $\Omega$ which is assumed to be clamped on $\Gamma_D$ and submitted to surface loads $g$ on $\Gamma_N$:

$$
\begin{align*}
\text{div } \sigma &= 0 \quad \text{in } \Omega, \text{ with } \sigma = Ae(u), \\
u &= 0 \quad \text{on } \Gamma_D, \\
\sigma \cdot n &= g \quad \text{on } \Gamma_N, \\
\sigma \cdot n &= 0 \quad \text{on } \Gamma = \partial \Omega \setminus (\Gamma_N \cup \Gamma_D),
\end{align*}
$$

(1)

where $e(u) = (\nabla u + \nabla u^T)/2$ is the linearized metric tensor, $A$ is Hooke’s law or the elasticity tensor defined by

$$
A\xi = 2\mu \xi + \lambda (\text{Tr } \xi) \text{Id},
$$

(2)

with Lamé moduli $\lambda$ and $\mu$, and $\sigma$ is the constraint tensor. We consider the following compliance minimization problem

$$
\min_{|\Omega|=V, \Gamma_D \cup \Gamma_N \subset \partial \Omega} \int_{\Gamma_N} g \cdot u(\Omega) \, ds,
$$

(3)

where $V$ is a given volume.

It turns out that the optimal solution to the problem above is a shape made of composite material (see the books [1], [6], [8] and [10]). This is obtained by the homogenization method which consists in relaxing the problem by enlarging the set of admissible shapes to include composite shapes; see for instance [1] and [12]. However, our wish is to produce a sequence of workable shapes that converge towards the optimal shape instead of the optimal shape itself.
A naive way of tackling this issue consists in rebuilding on each triangle of the mesh a composite (laminate or periodical). This procedure depends on the size of the mesh and on the mesh itself. Moreover, it leads to the appearance of boundary-layers that do not, depending on the cost function (for instance, if it depends on the gradient of the deformation) may not yield the correct result. Another issue with the homogenization method is the identification of the optimal composite materials and their respective Hooke laws which may be an impediment to the construction of the optimal sequence. In the case of compliance minimization, optimal composites have been identified and their Hooke laws computed.

A means to circumvent the obstacles underlined above is to explicitly devise a set of convergent shape sequences for which the associated Hooke laws are well-known. Then, it suffices to minimize the cost function in this partially relaxed problem where admissible shapes are taken in the former set.

2. Shape sequences
The first step of the scheme presented in the introduction consists in defining sequences of shapes for which the behavior of the limit composite shape is computable. The set of composite shapes thus obtained has to be rich enough for the minimization of the cost function over this set to be the closest to the optimal cost.

The most simple composites for which an explicit converging shape sequence is known are periodic composites (presented in section 4.1). However, this set is too restrictive and has to be enriched. In a second step, we introduce locally periodic shapes on a regular lattice, obtained by the transformation of a periodic composite by a diffeomorphism. For particular shape optimization problems, this set is large enough to capture an almost optimal solution. However, it remains limited in the general case. In particular, it does not allow the presencesence of singularities in the lattice which often precludes the obtention of a satisfying solution. This problem is lifted in the last part of this section 2.4, where locally periodic shapes on a lattice containing a finite number of singularities is introduced. We send the reader to [14] for a detailed description of the non singular case. We also remark that

2.1. Periodic composites
Let $Y = [0, 1]^2$ and let

$$U_{\ell} = \{\omega \subset \mathbb{R}^2 : x \in \omega \Leftrightarrow x + f_1 \in \omega \Leftrightarrow x + f_2 \in \omega\},$$

where $(f_1, f_2)$ is the canonical basis of $\mathbb{R}^2$, be the set of $Y$-periodic open subset of $\mathbb{R}^2$. The construction of homogenous periodic solids is obtained as limit of the open sets:

$$\Omega_{\varepsilon} = D \cap \omega_{\varepsilon},$$

where $D$ is the optimization domain,

$$\omega_{\varepsilon} = \{x \in \mathbb{R}^2 : \varepsilon^{-1} x \in \omega\}, \text{ and } \omega \in U_{\ell}.$$

The sequence

$$\Omega_{\varepsilon} = \{x \in D : \varepsilon^{-1} x \in \omega\}$$

is a shape sequence that converges to a periodic composite shape. Now, if we call $H^1_\#(Y)$ the space of $Y$-periodic $H^1$ functions, the Hooke law of the composite material reads as:

$$A^* \sigma \cdot \sigma = \inf_{\tau \in H^1_\#(Y)} \int_{Y \cap \omega} A \tau \cdot \tau \, dx.$$  \hfill (4)

2.2. Locally Periodic composites on structured lattices
This time around $\omega$ depends on $x$ so that $\omega : \mathbb{R}^2 \to U_{\ell}$ is a function. We define

$$\omega_{\varepsilon} = \{x \in \mathbb{R}^2 : x \in \varepsilon \omega(x)\},$$

in the same fashion as above. The Hooke law $A^*$, in this case, depends on $x$, and is given by the same formula (4) above.

2.3. Locally periodic composites on regular lattices

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Let \( \varphi : D \to \mathbb{R}^2 \) be a regular local diffeomorphism. In this paragraph, we consider the case where \( D \) is simply connected. The sequence of open sets
\[
\Omega_{\varphi,c} = \{ x \in D : x \in \varphi^{-1}(c \omega(x)) \}
\]
converges to a composite shape whose Hooke law \( A^* \) is, at every point \( x \), that of a periodic homogenous shape of period \( D_x \varphi^{-1}Y \) and is given by the following
\[
A^* \xi : \xi = \inf_{\tau \in L^2_2((D_x \varphi^{-1}Y \cap \omega(x)))} \int_{D_x \varphi^{-1}Y \cap \omega(x)} A \tau \cdot \tau \, dx.
\]
If we set \((u_1, u_2) = D_x \varphi^{-1}\), and define vectors \(v_1\) and \(v_2\) as follows
\[
v_1 = e^r u_1, \quad v_2 = e^r u_2 \quad \text{and} \quad 1 = v_1 \wedge v_2 = e^{2r} u_1 \wedge u_2,
\]
so that
\[
r = -\frac{1}{2} \ln(\det(u_1 \wedge u_2)),
\]
then the Hooke law \( A^* \) depends only on \(v_1, v_2\) and \(\omega\), and, consequently, the partially relaxed optimization problem only depends on \(v_1, v_2\) and the function \(\omega\).

Now, if we define \((u_1^*, u_2^*)\) conjugate vectors of \((u_1, u_2)\) (that is \((u_1^*, u_2^*) = D \varphi(x)\)) and \((v_1^*, v_2^*)\) conjugate vectors of \((v_1, v_2)\), then
\[
u_1^* = e^r v_1^* \quad \text{and} \quad u_2^* = e^r v_2^*
\]
are gradients. Therefore,
\[
0 = \nabla \wedge u_1^* = e^r (\nabla r \wedge v_1^* + \nabla \wedge v_1^*) \quad \text{and} \quad 0 = \nabla \wedge u_2^* = e^r (\nabla r \wedge v_2^* + \nabla \wedge v_2^*)
\]
Thus,
\[
0 = \nabla r \wedge v_1^* + \nabla \wedge v_1^* \quad \text{and} \quad 0 = \nabla r \wedge v_2^* + \nabla \wedge v_2^*,
\]
and
\[
\nabla r = (\nabla \wedge v_1^*) v_2^* - (\nabla \wedge v_2^*) v_1^*.
\]
In particular, for any test function \(p : D \to \mathbb{R}\), such that \(p = 0\) on \(\partial D\), we have
\[
\int_D ((\nabla \wedge v_1^*) v_2^* - (\nabla \wedge v_2^*) v_1^*) \wedge \nabla p \, dx = 0.
\]
Note that \(r\) depends only on \(v_1^*\) and \(v_2^*\) up to a constant.

The relaxed problem consists in minimizing the cost function for the composite shape, the Hooke law of which is defined by
\[
A^* \xi : \xi = \inf_{\tau \in L^2_2((v_1^*, v_2^*) \cap \omega))} \int_{(v_1^*, v_2^*) \cap \omega)} A \tau \cdot \tau \, dx,
\]
over the set of elements \(((v_1^*, v_2^*), \omega) \in V_{ad} \times S_{per}\) where
\[
V_{ad} = \{(v_1^*, v_2^*) : D \to \mathbb{R}^2 \times \mathbb{R}^2 \text{ satisfying (10) and } v_1^* \wedge v_2^* = 1\}
\]
and \(S_{per}\) is the set of regular maps from \(D\) into a subset of the \(Y\)-periodic open set \(U_0\).

For a given solution \((v_1^*, v_2^*, \omega)\) of the optimization domain, we can reconstruct the corresponding sequence of shapes. As \((v_1^*, v_2^*)\) fulfills (10) for all test functions \(p\), there exists a map \(r\) such that the equation (1) is satisfied. Moreover, the map \(r\) is defined up to a constant \(c_r \in \mathbb{R}\) by this relation
\[
r = r_0 + c_r.
\]
By equations (7), we have
\[
u_1^* = e^{r_0 + c_r} v_1^* \quad \text{and} \quad u_2^* = e^{r_0 + c_r} v_2^*
\]
Figure 1: Construction of $\Omega^e_{\varphi}$

From equation (8), and thanks to the simple connectedness of $D$, there exists $\varphi$ satisfying $D\varphi = (u_1^*, u_2^*)$, and

$$\varphi = e^{c e} (\varphi_0 + c_{\varphi}).$$

By (5), the shape sequences defined by $\varphi$ are

$$\Omega_{\varphi, \epsilon} = \{ x \in D : \varphi_0(x) + c_{\varphi} \in (e^{-c e} \epsilon \omega(x)) \},$$

or, equivalently,

$$\Omega_{\varphi, \epsilon} = \Omega_{\varphi_0 + c_{\varphi}, \epsilon e^{-c e} \epsilon}.$$  \hspace{1cm} (14)

It follows that the shape sequence is uniquely determined by $(u_1^*, u_2^*)$ up to a rescaling $e^{-c e}$ of the periodicity cells and a phase difference $c_{\varphi}$. See Figure 1 for an illustration of the procedure.

2.4. Locally periodic composites on non regular lattices

Locally periodic composites on a regular lattice have a major limitation for shape optimization. Indeed, the arrangement of periodicity cells has to fulfill a global topological constraint. For instance, if one considers non simply connected optimization domains, the arrangements presented in Figure 3 cannot be produced by the previous construction. This global topological constraint strongly limits the applicability of the method presented in the previous paragraph. In this section, we extend the previous construction to lattices that can present defects as in Figures 2-3. Nonetheless, we restrict our analysis to symmetric periodic cells

$$U^*_\omega := \{ \omega \in U_\epsilon \text{ such that } x \in \omega \text{ iif } -x \in \omega \}.$$  

We assume that the optimization domain is of the form

$$D = D_0 \setminus \bigcup_{i=1, \cdots, N_\epsilon} B_i,$$

where the $B_i$ are disjoint simply connected open subsets of a domain $D_0$ of $\mathbb{R}^2$. We want to be able to consider composite shapes whose lattices have topological defects of the kind presented in Figures 2-3. For such lattices, the periodicity cells cannot be oriented, so that $(u_1, u_2)$ has to be defined modulo the equivalence relation

$$(u_1, u_2) \simeq (\tilde{u}_1, \tilde{u}_2) \text{ iif } (u_1, u_2) = -(\tilde{u}_1, \tilde{u}_2).$$
Figure 2: Nonorientable eigenvector field. The cell at the singular point has five sides.

Following the same steps as in the previous section, the composite shapes are described by \((v_1^*, v_2^*) \in V_{ad}\) and \(\omega \in S_{per}\), where

\[
V_{ad} = \{(v_1^*, v_2^*) : D \to \mathbb{R}^2 \times \mathbb{R}^2 / \simeq \text{ satisfying (10) } \forall p \in \mathcal{T}_D, \text{ and } v_1^* \wedge v_2^* = 1\},
\]

where

\[
\mathcal{T}_D = \{p \in C^\infty(D) : p = 0 \text{ on } \partial D_0, \text{ and } \forall i \in \{1, \cdots, N_s\}, p \text{ is constant on } \partial B_i\},
\]

and \(S_{per}\) is the set of regular maps from \(D\) into a subset of the \(Y\)-periodic open set \(U_s\).

The Hooke law associated to such a composite is given by (11) (Note that, as we have assumed that \(\omega\) belongs to \(U_s\), \(A^\ast\) is independent of the choice of the element of the class of \((v_1^*, v_2^*)\) made).

It remains to describe, how for a given element \((v_1^*, v_2^*, \omega) \in V_{ad} \times S_{per}\) we construct a shape sequence converging to a corresponding composite.

First, since

\[
\int_D ((\nabla \wedge v_1^*)v_2^* - (\nabla \wedge v_2^*)v_1^*) \wedge \nabla p \, dx = 0, \forall p \in \mathcal{T}_D,
\]

there exists \(r : D \to \mathbb{R}\) such that

\[
\nabla r = (\nabla \wedge v_1^*)v_2^* - (\nabla \wedge v_2^*)v_1^*.
\]

Moreover, \(r\) is defined up to a constant, i.e. \(r = r_0 + c_r\).

Next, setting

\[
u_1^* = e^r v_1^* \quad \text{and} \quad u_2^* = e^r v_2^*,
\]

we have

\[
\nabla \wedge u_1^* = 0 \quad \text{and} \quad \nabla \wedge u_2^* = 0.
\]

As this stage, the study departs from the previous case, because (16) does not imply that \(u_1^*\), \(u_2^*\) are gradients. This is due to the presence of the holes \(B_i\) in the optimization domain \(D\) and to the fact that \((u_1^*, u_2^*)\) is only defined modulo the equivalence relation \(\simeq\).

Let \(P_\Sigma\) be the projection of \(\Omega \times \mathbb{R}^2 \times \mathbb{R}^2\) onto \(\Omega \times \mathbb{R}^2 / \simeq\) and set

\[
W = P_\Sigma^{-1}(\{(x, w_1, w_2) \in \Omega \times (\mathbb{R}^2 \times \mathbb{R}^2) / \simeq \text{ such that } (w_1, w_2) = (v_1(x), v_2(x))\})
\]

and

\[
H = \{\psi : W \to \mathbb{R} : \psi(x, w_1, w_2) = -\psi(x, -w_1, -w_2)\}.
\]

To each hole \((B_i)_{i=1,\cdots,N_s}\), we associate a map (called a singular function) \(\psi_i : W \to \mathbb{R}/\mathbb{Z}\) such that for all \((x, v_1, v_2) \in W\),

\[
\psi_i(x, v_1, v_2) = -\psi_i(x, -v_1, -v_2),
\]
as follows. Let $T_i$ be a small tubular neighborhood in $D$ of a path connecting $\partial B_i$ to $\partial D \setminus \bigcup_{j \neq i} \partial B_j$.

Let $\phi_i$ be a diffeomorphism from $[0, 1]^2$ into the tubular neighborhood $T_i$ such that

$$\phi_i(-1 \times [0, 1]) = \partial T_i \cap \partial B_i \quad \text{and} \quad \phi_i(1 \times [0, 1]) = \partial T_i \cap \partial D \setminus \partial B_i.$$  

Let $\psi$ be the map defined on $[0, 1]^2$ by

$$\psi(a, b) = b.$$  

We define $\psi_i$ as the regular map satisfying

$$\psi_i(x, v_1, v_2) = 0 \quad \text{if} \quad x \not\in T_i,$$

$$\psi_i(x, v_1, v_2) = \pm \psi(\phi_i(x)) \quad \text{if} \quad x \in T_i.$$  

Note that since $P_\sigma^{-1}(T_i)$ has two disjoint connected components, $\psi_i$ is correctly and uniquely defined (up to a sign) by (19), (20) and (21).

One can prove that for all $(v_1^*, v_2^*) \in \mathcal{V}^s_{ad}$, there exists $\varphi \in H$ and $(c_i) \in (\mathbb{R}^2)^{N_s}$ such that

$$\nabla \varphi + \sum_{i=1}^{N_s} c_i \nabla \psi_i = (u_1^*, u_2^*).$$  

For a given element $(v_1^*, v_2^*) \in \mathcal{V}^s_{ad}$, and a given function $r$ satisfying (), functions $c_i$ are uniquely determined. The same remark applies to $\varphi$ as long as $N_s \neq 0.$
The shape sequence is then defined by

\[ \Omega_{\varphi,(c),\varepsilon} = \{ x \in D : \varepsilon^{-1}\varphi(x) + \sum_{i=1}^{N_s} \varepsilon^{-1}c_i \psi_i \in \omega(x) \}. \] (23)

Note that the shape sequence is uniquely defined by \((\varepsilon_1^*, \varepsilon_2^*)\) up to a rescaling.

3. Optimization Procedure

In this section, we present a simplified version of the optimization procedure introduced in section 2.4. It relies on the introduction of an approximated Hooke law in a relatively rough fashion which is nonetheless quite accurate in practise as least for the compliance optimization problem studied in the next paragraph. The said approximation allows, in addition, to set free from the condition imposed on the \(e^*_i\).

3.1. Approximation of the Hooke laws

The main difficulty here is the lack of analytical formulae for the computation of Hooke laws in terms of the parameters \(u_1, u_2, \text{and} \omega\). We may replace, in certain instances, the Hooke law by an approximation (this is done in the sequel) using rank 2 laminates (which are optimal for compliance optimization - see next paragraph). Another alternative consists in carrying out a pre-computation for a certain pack of parameters then use some interpolation. However, this procedure seems quite heavy and has not been attempted yet.

At any rate, our method considers rectangular cells and their associated Hooke laws.

3.2. Optimization Process

We proceed in two steps. First, we solve the compliance minimization problem in the set of rank 2 laminates. This provides us the orientation \(\vartheta_c\) of the laminates and the proportions of lamination \((p_1, p_2)\) in each direction as well as the density \(\vartheta\) of the optimal shape. Next, we remove from the optimization domain \(D_0\) small neighborhoods of the points where the field \(\vartheta_c\) is not regular (that is not locally orientable) to obtain a new optimization domain \(D\). We denote by \(N_s\) the number of holes thus created and by \((B_i)_{i=1,\ldots,N_s}\) the removed holes. Then, we build the singular functions \((\psi_i)_{i=1,\ldots,N_s}\) associated to the holes and compute a function \(\varphi\) and the reals \((c_i)_{i=1,\ldots,N_s}\) satisfying (22). Finally, the shape sequence is computed using (23). Note that in [7], a partial relaxation of the problem similar to ours is performed albeit for a more restrictive set of functions \(\varphi\). Also, in [9], a different construction based on the deformation of a periodic lattice is proposed.

4. Numerical Examples: Compliance Optimization

We consider the minimization problem (3) of the compliance of an elastic structure. We restrict our analysis to the two-dimensional case. The compliance of the structure \(\Omega\) is defined as

\[ J(\Omega) = \int_{\Gamma_N} g \cdot u(\Omega) \, ds, \]

where \(u(\Omega)\) is the displacement of the structure, unique solution of the problem of elasticity (1). The problem we focus on consists in minimizing the compliance \(J\) over the shapes \(\Omega\) belonging to the admissible set

\[ U_{ad} = \{ \Omega \text{ open set in} \, \mathbb{R}^2 \text{ such that} \, \Gamma_N \cup \Gamma_D \subset \partial \Omega \text{ and} \, |\Omega| \leq V \}. \]

In Figures 4-6, we display results obtained for different shapes namely the cantilever, a double bridge (with two loads) and a triple bridge (with three loads). Note that the black rectangles denote clamping conditions and the red bars and arrows show the loads and their directions. The cantilever exhibits no singularity whereas the double bridge has a singular point at the center (between the loadings, see Figure 2 where the optimal composite bridge is displayed as well as a zoom of the region where the singularity lies) and the triple bridge has essentially two singularities (each one between two loadings).

The holes that were nucleated to get rid of the singularities are also displayed. The effect of the parameter \(\varepsilon\) which regulates the level of detail is clearly exposed: the level of details increases as \(\varepsilon\) tends zero. Similarly and quite understandably the compliance decreases proportionally with respect to \(\varepsilon\).
5. Conclusion
In this work, we have set free from the obstruction caused by non-regular lattices. Indeed, we have improved our initial method presented in [12]. It is clear that one may carry on with the optimization procedure using a geometrical optimization method in order to sharpen the contours of the shapes we have obtained. This may be done using a level set approach for instance as it uses a fixed mesh; see [2], [3], [4], [5], [13] and [14]).

Our next move is to generalize the procedure to the compliance optimization in $\mathbb{R}^3$, and to other objective functions that can be tackled by the homogenization method.

References


Figure 5: Elements of the shape sequence indexed by the parameter $\varepsilon$ converging to the optimal double bridge


Figure 6: Elements of the shape sequence indexed by the parameter $\varepsilon$ converging to the optimal triple bridge