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The Modeling of Deformable Bodies with Frictionless (Self-)Contacts

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Abstract

We propose a mathematical model for m-dimensional deformable bodies moving in $\mathbb{R}^n$, that allows for frictionless contacts or self-contacts while forbidding transversal (self-)intersection. To this end, a topological constraint is imposed to the set of admissible deformations. We restrict our analysis to the static case (although the dynamic case is briefly addressed at the end of the article). In this case, no transversal self-intersection can occur as long as $2m < n$, so our modeling is mainly designed to handle the case $2m \geqq n$. For nonlinear hyperelastic bodies, we prove the existence of at least one minimizer of the energy on the set of admissible deformations, under suitable assumptions on the stored energy function. Moreover, for certain choices of m and n, under regularity assumptions on the minimizers, the solutions of the minimization problem satisfy Euler–Lagrange equations.

1. Introduction

Contacts and self-contacts arise in many practical situations. In this paper, we focus our attention on the case of hyperelastic bodies. Nevertheless, part of our work could be applied to other types of materials. In particular, bending effects can be added, which are often suitable for thin structures (that is $m < n$). Moreover, we mainly focus on the static case, although the dynamic case is briefly discussed at the end of this article. One way of describing a system of elastic bodies is to use a variational approach. Minimizers of the energy over the set of admissible deformations are stable equilibrium states. For a hyperelastic body M, submitted to dead loads, the total energy of a deformation $\varphi : M \to \mathbb{R}^n$ has the following form.

$$I(\varphi) = \int_M W(D\varphi)\,\mathrm{d}x - L(\varphi), \tag{1}$$

where $W(\cdot)$ is the stored energy function (depending on the material of which the body is made) and $L(\cdot)$ is the work of the external loads. The first theoretical question that arises in such a framework is whether such an energy admits at least one minimizer. The answer relies on the given stored energy function $W(\cdot)$ and the admissible set. Assume that the minimizing sequences are compact for a given topology. Existence of minimizers then follows from the sequential lower semicontinuity of I and the closure of the admissible set for this topology.

For finite valued stored energy function and under growth assumptions on $W(\cdot)$, MORREY [15, 16] proved the sequential lower semicontinuity of I to be equivalent to the quasiconvexity of the stored energy function W. When $W(\cdot)$ satisfies further additional coercivity conditions, existence of minimizers of the energy I on Sobolev spaces $W^{1,p}(M;\mathbb{R}^n)$ is obtained. Nevertheless, his modeling allows for existence of non-physical solutions. For instance, in the case $\dim(M) = n$, minimizers of I can "reverse" the body. In other words, deformations such that $\det(D\varphi) < 0$ are permitted. In order to overcome this problem, BALL [3] introduced the notion of polyconvexity of the stored energy function W. It allows him to handle the case of non-finite-valued stored energy functions, and more precisely to prove the existence of minimizers for stored energy functions such that

$$W(F) \to +\infty \quad \text{when } \det(F) \to 0,$$

and $W(F) = +\infty$ for all F such that $\det(F) \leqq 0$. If such an additional condition ensures local injectivity almost everywhere of the minimizers, it does not forbid non-injective solutions except if Dirichlet conditions are applied on the whole boundary of M (see [4]). Existence of almost everywhere global injective minimizers was obtained for mixed displacement–traction boundary conditions by CIARLET & NEČAS [6, 7]. Their approach rests on the additional constraint

$$\int_M \det(D\varphi)\,\mathrm{d}x \leqq \mathrm{Vol}(\varphi(M)) \tag{2}$$

on the admissible deformations. BAIOCCHI et al. prove the existence of solutions of pure traction boundary conditions where the body is constrained in a possibly unbounded region. In another spirit, again in the context of hyperelasticity and $\dim(M) = n$, GIAQUINTA et al. [8, 9] prove the existence of minimizers of the energy over a set of weak diffeomorphisms.

Another important challenge consists in deriving the Euler–Lagrange equations fulfilled by the minimizers of the energy. For finite-valued stored energy functions without contact such necessary conditions of optimality are easy to obtain, under appropriate growth conditions on W. Unfortunately, such conditions are incompatible with energies that blow up as $\det(D\varphi)$ goes to zero. In this case, Euler–Lagrange equations can only be obtained under strong smoothness assumptions on the minimizers as it is done by Ciarlet and Nečas. They proved, for smooth minimizers of the energy, that self-contact forces are normal to the boundary, that is frictionless. The determination of the nature of the contact forces against a rigid obstacle has been obtained by SCHURICHT [22] without any smoothness assumption (with finite-valued stored energy function). In the case $\dim(M) = 1$, SCHURICHT [20] studied contact problems of nonlinear rods in $\mathbb{R}^2$ against rigid bodies (see also the

Ball thesis [2]). He established existence of solutions and derived Euler–Lagrange equations for a large class of materials. In [21], Schuricht applied the global injectivity condition of Ciarlet and Nečas to rods moving in $\mathbb{R}^3$ and verified existence of solutions without self-penetration. Another approach, based on the global curvature of a curve, has been introduced by Gonzalez et al. [11] and Schuricht & Mosel [23] to study self-contacts for rods.

In this paper, we propose a new condition on the admissible deformations in order to prevent transversal self-intersections of m-dimensional deformable bodies moving in $\mathbb{R}^n$ for $2m \geqq n \geqq m$. Let M be a submanifold of $\mathbb{R}^n$. The reference injection of M into $\mathbb{R}^n$ is denoted j_M. We define the set of admissible deformations simply as the closure of the embeddings isotopic to the reference injection j_M for an appropriate topology. With such a definition, it is straightforward to prove the existence of minimizers of the energy (under suitable assumptions on the stored energy function). Nevertheless, the implicit definition of the admissible set of deformations has at least one drawback: it is not clear how to recover the Euler–Lagrange equations. In order to achieve such a goal, the usual method is to perform small variations around the minimizer in the admissible set. Then, the differentiability of the energy functional leads to the Euler–Lagrange equations. Unfortunately, the definition of the admissible set does not provide an explicit description of the neighborhood of a minimizer, or of the allowed variations. To overcome this problem, we prove that admissible deformations fulfill an explicit topological constraint. More precisely, we show that the self-intersections of a deformation φ are, at least partially, described by a topological invariant $\phi(\varphi)$. A deformation such that $\phi(\varphi)$ is equal to $\phi(j_M)$ is called ϕ-admissible. We prove that any admissible deformation is ϕ-admissible. Conversely, in the cases $\dim(M) = n$ or $\dim(M) = 1$ and $n = 2$, every immersion that is ϕ-admissible belongs to the admissible set. This allows us to prove that any immersion that is a minimizer of the energy satisfies the expected Euler–Lagrange equations.

The plan of the paper is as follows. We first recall some basic definitions of differential geometry (Section 2). Then, we give a description of the self-intersections of a deformation (Section 3), in particular we define the ϕ-admissible set and study its main properties. In Section 4, we set up the minimization problem for nonlinear hyperelastic bodies and prove the existence of at least one solution using classical arguments. Section 5 is devoted to determining whether or not solutions of the minimization problem satisfy the expected Euler–Lagrange equations. First, we consider the case of n-dimensional bodies moving in $\mathbb{R}^n$ (Section 5.1). In this case, in order to obtain a reasonable modeling for frictionless contact, we do not need to assume that the stored energy goes to infinity as the determinant of the gradient goes to zero. Moreover, we compare our model with the one introduced by Ciarlet & Nečas [7] and show that every ϕ-admissible deformation satisfies condition (2), whereas the converse is not true. The Euler–Lagrange equations fulfilled by regular minimizers of the energy over the ϕ-admissible set (or over the admissible set) can be derived as it is done in [7]. A similar result is obtained for one-dimensional structures moving in a two-dimensional Euclidean space in Section 5.2. (Proofs for this part have been skipped and can be found in [17]). The case of shells or thin films, that is, surfaces moving in $\mathbb{R}^3$, is discussed in Section 5.3. In such cases, the constraint added does not prevent the formation of some nontransversal self-intersections. This precludes the recovery of the Euler–Lagrange equations. As stated before, our modeling is essentially relevant in the cases $2m \geqq n$. In particular, for one dimensional structures moving in $\mathbb{R}^3$, ϕ-admissibility is usually not a restrictive enough condition as illustrated in Section 6. The dynamic case is mentioned at the end of the article (Section 7).

Let us specify the various notations that we shall use:

j_A^B : injection of A into B.
$\triangle(A) = \{(x, y) \in A \times A \ : \ x = y\}$: diagonal of $A \times A$.
A^c : complement of the set A.
$A \setminus B = A \cap B^c$.
$B^n(x, r)$: open ball centered at $x \in \mathbb{R}^n$ and of radius r.
$S^{n-1}(x, r)$: sphere centered at $x \in \mathbb{R}^n$ and of radius r.
$\dot{\varphi}$: derivative of φ (If φ is a regular one variable function).
$D_x\varphi$: differential of φ at x (If φ is a multi-variable function).
TM : tangent bundle of the manifold M.
T_xM : fiber at x of the tangent bundle TM.
$\Omega^k(M)$: set of differential forms of degree k on the manifold M.
$H^k(M)$: real cohomology group of M of degree k.
$f^*(\alpha)$: pull back by f of the differential form α.

2. Preliminaries

We recall in this section some basic definitions and notions of differential geometry and topology. For a comprehensive treatment of the topic, we refer, for instance, to GODBILLON [10], ARNOLD [1], and BOTT & TU [5].

2.1. Differential geometry

Let M and N be differentiable manifolds of dimension m and n, respectively. Let f be a regular map from M into N. The map f is an immersion if and only if $D_x f$ is of rank m for every x. An injective immersion is an embedding. Two embeddings f and g are said to be isotopic if there exists a regular map F from $M \times [0, 1]$ into N, such that $F(0) = f$, $F(1) = g$, and $F(t)$ is an embedding. For every t in $[0, 1]$ ($F(t)$ is the map from M into N defined by $F(t)(x) = F(x, t)$).

2.2. Differential forms

Let us recall the definition of differential forms for open subsets of $\mathbb{R}^n$. Let Λ^k be the set of k-linear antisymmetric forms on $\mathbb{R}^n$. The exterior product between a k form α and an l form ω is the $k + l$ form $\alpha \wedge \omega$ defined by

$$\alpha \wedge \omega(X_1, \ldots, X_{k+l}) = \sum_{\sigma} (-1)^{|\sigma|} \alpha(X_{\sigma_1}, \ldots, X_{\sigma_k}) \omega(X_{\sigma_{k+1}}, \ldots, X_{\sigma_{k+l}}),$$

where the sum is taken over by the permutations σ of $\{1,\dots,k+l\}$. Let $(e_1,\dots,e_n)$ be the canonical basis of $\mathbb{R}^n$ and let $(\mathrm{d}x_1,\dots,\mathrm{d}x_n)$ be the canonical basis of $(\mathbb{R}^n)^*$, that is $\mathrm{d}x_k(e_i)=\delta_{i,k}$. Let $I=(i_1,\dots,i_k)$, with $1\leqq i_1<\cdots<i_k\leqq n$, we denote $\mathrm{d}x_I$ as the k-linear alternate form defined by

$$\mathrm{d}x_I=\mathrm{d}x_{i_1}\wedge\cdots\wedge dx_{i_k}.$$

Let U be an open subset of $\mathbb{R}^n$. A differential form of degree k on U is a $\mathcal{C}^\infty$-mapping from U to Λ^k. The set of k-differential forms on U is denoted $\Omega^k(U)$.

Let $f:V\to U$ be a $\mathcal{C}^\infty$-mapping from an open subset V of $\mathbb{R}^p$ into an open subset U of $\mathbb{R}^n$. Let $\alpha\in\Omega^k(U)$ be a differential form. We define the pullback $f^*\alpha$ of α by f as the k-differential form on V defined by

$$f^*(\alpha)(x)(X_1,\dots,X_k)=\alpha(f(x))(D_xf(X_1),\dots,D_xf(X_k)).$$

The operator of differentiation d defined for any $\mathbb{R}$-valued function g by

$$\mathrm{d}g=\sum_k\frac{\partial g}{\partial x_k}\mathrm{d}x_k,$$

can be extended to an operator $d:\Omega^k(V)\to\Omega^{k+1}(V)$ defined by

$$\mathrm{d}\alpha=\sum_I\mathrm{d}f_I\wedge\mathrm{d}x_I,$$

where $\alpha=\sum_I f_I\mathrm{d}x_I$, and $\mathrm{d}f_I$ is the differential of the real valued function f_I. By the Schwarz equality, we have $d\circ d=0$. The cohomology group $H^k(V)$ of degree k of V is the quotient space of the kernel of d [as a mapping from $\Omega^k(V)$ into $\Omega^{k+1}(V)$] by the image of d [as a mapping from $\Omega^{k-1}(V)$ into $\Omega^k(V)$]. If f is a $\mathcal{C}^\infty$-mapping from $V\to U$, the mapping f^* from $\Omega^k(U)\to\Omega^k(V)$ induced a mapping from $H^k(U)\to H^k(V)$. Moreover, if two mappings f and g are homotopic, then f^* and g^* define the same mapping from $H^*(U)$ into $H^*(V)$.

All those notions can be extended to differentiable manifolds.

3. Description of the self-intersections of a deformation

Let M be an m-dimensional submanifold (with or without boundary) of $\mathbb{R}^n$ and let j_M be the injection of M in $\mathbb{R}^n$. We say that a deformation $\psi:M\to\mathbb{R}^n$ is admissible if it belongs to the $\mathcal{C}^0$-closure of the embeddings isotopic to the reference injection j_M. The set of admissible deformations is denoted by $\mathcal{A}(j_M)$. In this section we address the problem of describing the self-intersections of a deformation. This leads us to associate to any deformation φ a topological invariant $\phi(\varphi)$ which describes, at least partially, the self-intersections of the deformation φ. The set $\mathcal{A}_\phi(j_M)$ of deformations that has the same topological invariant as the reference injection j_M is called the ϕ-admissible set. Every admissible deformation is ϕ-admissible. It follows, for instance, that no deformation with transversal self-intersection belongs to the set $\mathcal{A}(j_M)$. Before broaching the general case, we focus our attention on the case of a thin structures moving in $\mathbb{R}^2$.

3.1. *Case of thin structures in* $\mathbb{R}^2$

The aim of this section is to give a straightforward description of the self-intersections of a continuous deformation. The approach in this section is heuristic and is not intended to provide us with complete proofs of the statements made. We begin with the simplest case, that is the study of the intersection between two deformations from one-dimensional manifolds M_1 and M_2 into $\mathbb{R}^2$. If the intersection is transversal (see below), the intersection is completely described by a set of oriented points in $M_1 \times M_2$. The definition of this oriented set could be extended to the nontransversal case. However, this generalization failed to detect some self-intersections of deformations of the circle S^1 into $\mathbb{R}^2$. A later definition is introduced in order to solve this problem.

3.1.1. The transversal case. Let us consider two one-dimensional bodies M_1 and M_2 moving in $\mathbb{R}^2$ (M_1 and M_2 are assumed to be diffeomorphic either to [0, 1] or S^1). For two given deformations φ and ψ of M_1 and M_2, respectively, that is, mappings from M_1 or M_2 into $\mathbb{R}^2$, we want to define the intersection between φ and ψ. We define the set of common points between φ and ψ as

$$K(\varphi, \psi) := \{(x, y) \in M_1 \times M_2 \ : \ \varphi(x) = \psi(y)\}. \tag{3}$$

The intersection is said to be transverse if, for every $(x, y) \in K(\varphi, \psi)$, the family $(\dot{\varphi}(x), \dot{\psi}(y))$ is free. In this case, the set $K(\varphi, \psi)$ is a finite set of points and is stable under small $\mathcal{C}^1$-perturbations of φ and ψ. Furthermore, each of them could be endowed with a sign $s_{\varphi,\psi}(x, y)$, depending on the orientation of the basis $(\dot{\varphi}(x), \dot{\psi}(y))$.

$$s_{\varphi,\psi}(x, y) := \operatorname{sign}(\det(\dot{\varphi}(x), \dot{\psi}(y))).$$

As transversal intersections are stable under small perturbations, a deformation without self-intersection could only have nontransversal intersections.

3.1.2. The nontransversal case. It remains for us to consider the nontransverse case. Figure 1 represents two cases of nontransverse intersections. The images of M_1 under φ and of M_2 under ψ are represented with a dashed line and a continuous line, respectively. In the first configuration the beams intersect with each other whereas they just make contact in the second one. Let us first note that it is clear, from this example, that only a global criterion will enable us to distinguish deformations with

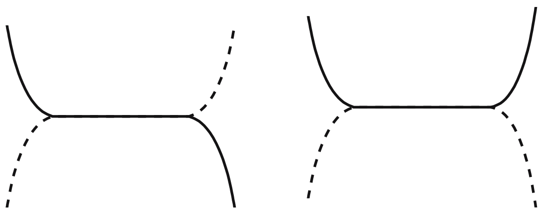

Fig. 1. Nontransverse intersections

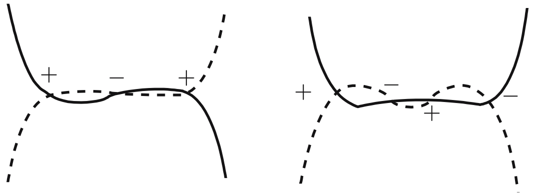

Fig. 2. Perturbations of non-transverse configurations

intersections from deformations without intersections. Moreover, the set $K(\varphi, \psi)$ is the same in both cases, and thus does not fully describe the intersection between φ and ψ.

Let V be a small neighborhood of $K(\varphi, \psi)$. There exist two deformations, $\widetilde{\varphi}$ and $\widetilde{\psi}$, close to φ and ψ, respectively, such that $K(\widetilde{\varphi}, \widetilde{\psi}) \subset V$, and such that the intersection between $\widetilde{\varphi}$ and $\widetilde{\psi}$ is transverse. To each connected component U of V, one can associate an integer $s_{\varphi,\psi}(U)$, equal to the sum of the sign of the points $(x, y) \in K(\widetilde{\varphi}, \widetilde{\psi}) \cap U$.

$$s_{\varphi,\psi}(U) := \sum_{(x,y)\in K(\widetilde{\varphi},\widetilde{\psi})\cap U} s_{\widetilde{\varphi},\widetilde{\psi}}(x, y).$$

This integer does not depend on the choice of $\widetilde{\varphi}$ and $\widetilde{\psi}$ made as long as $\widetilde{\varphi}$ and $\widetilde{\psi}$ are close enough to φ and ψ. If φ and ψ belong to the admissible set, one can choose $\widetilde{\varphi}$ and $\widetilde{\psi}$ such that $K(\widetilde{\varphi}, \widetilde{\psi}) = \emptyset$, thus,

$$\begin{aligned} s_{\varphi,\psi}(U) = 0 \quad & \text{for every connected component } U \text{ of } V \\ & \text{and for any neighborhood } V \text{ of } K(\varphi, \psi). \end{aligned} \tag{4}$$

Let us compute $s_{\varphi,\psi}(U)$ in the two configurations represented in Fig. 1. We obtain $s_{\varphi,\psi}(U) = +1$ in the first case. Thus, this configuration is not admissible. In the second case, $s_{\varphi,\psi}(U) = 0$ (see Fig. 2).

3.1.3. The case of self-intersections. We will now investigate the case of self-intersections. If φ belongs to the admissible set $\mathcal{A}(j_M)$, the condition (4) is satisfied with $M_2 = M_1$ and $\psi = \varphi$. Nevertheless, the converse is not true. Assume that M_1 is homeomorphic to S^1. For any integer k, the deformation φ_k defined by

$$\begin{aligned} \varphi_k : S^1 &\to \mathbb{R}^2 \\ \theta &\mapsto (\cos(k\theta), \sin(k\theta)), \end{aligned} \tag{5}$$

where the circle S^1 is parameterized by the angle θ fulfills the criterion (4). Indeed, let $\widetilde{\varphi}_k = (1 + \varepsilon)\varphi_k$, where ε is a small positive real, then $K(\widetilde{\varphi}_k, \varphi_k) = \emptyset$ and for any connected component U of any neighborhood V of $K(\varphi_k, \varphi_k)$,

$$s_{\varphi_k,\varphi_k}(U) = \sum_{(x,y)\in K(\widetilde{\varphi}_k,\varphi_k)\cap U} s_{\widetilde{\varphi}_k,\varphi_k}(x, y) = 0.$$

Even so the deformations φ_k are not always intersection-free: there is no embedding close to φ_k as long as $k \neq \pm 1$. Thus, the mapping $s_{\varphi,\varphi}$ does not give us a complete description of the self-intersections of φ.

In order to solve this particular problem, let us go back to the study of the intersections between two deformations. As before, φ and ψ denote deformations from one-dimensional manifolds M_1 and M_2 into $\mathbb{R}^2$. We define the mapping $d_{\varphi,\psi}$ from $M_1 \times M_2 \setminus K(\varphi, \psi)$ into $\mathbb{R}^2_*$ by

$$d_{\varphi,\psi}(x, y) := \varphi(x) - \psi(y).$$

Let $\phi_{\mathbb{R}^2_*}$ be the closed 1-form defined on $\mathbb{R}^2_*$ by

$$\phi_{\mathbb{R}^2_*} := \frac{1}{2\pi}\left(\frac{x}{x^2+y^2}\mathrm{d}y - \frac{y}{x^2+y^2}\mathrm{d}x\right).$$

We denoted $\phi(\varphi, \psi)$ the pullback of $\phi_{\mathbb{R}^2_*}$ by $d_{\varphi,\psi}$

$$\phi(\varphi, \psi) := d^*_{\varphi,\psi}(\phi_{\mathbb{R}^2_*}).$$

Let U be an oriented open set of $M_1 \times M_2$, such that $\partial U \subset M_1 \times M_2 \setminus K(\varphi, \psi)$. We assert that (see Proposition 5)

$$\int_{\partial U} \phi(\varphi, \psi) = s_{\varphi,\psi}(U). \tag{6}$$

Hence, the mapping $s_{\varphi,\psi}$ is completely described by the integration of the 1-form $\phi(\varphi, \psi)$ on loops in $M \times M \setminus K(\varphi, \psi)$. It remains to apply this analysis to the study of self-intersections.

Let φ be a deformation; we define $\phi(\varphi)$ as the 1-form on $M \times M \setminus K(\varphi, \varphi)$

$$\phi(\varphi) = \phi(\varphi, \varphi).$$

If φ belongs to the set of admissible deformations, and if U is an open set such that

$$\inf_{x \in U} \left|d_{\varphi,\varphi}(x)\right| > 0,$$

then the restriction of $d_{\varphi,\varphi}$ to U is homotopic to d_{j_M, j_M}. Thus, there exists a mapping $u : U \to \mathbb{R}$ such that

$$\phi_U(\varphi) - \phi_U(j_M) = \mathrm{d}u, \tag{7}$$

where $\phi_U(\varphi)$ and $\phi_U(j_M)$ are the restriction of $\phi(\varphi)$ and $\phi(j_M)$ to the open set U. In other words, $\phi_U(\varphi)$ and $\phi_U(j_M)$ are equal up to an exact form. The set of ϕ-admissible deformations will be defined in 3.2 as the deformations that fulfill this criteria. The condition (7) implies that the integration of the 1-form $\phi(\varphi)$ and $\phi(j_M)$ on any loop of $M \times M \setminus K(\varphi, \varphi)$ are equal. Thus, by (6), this condition is at least as strong as the condition (4). Actually, the 1-form $\phi(\varphi)$ contains more information about the intersection than $s_{\varphi,\varphi}$, and the condition (7) is strictly stronger than (4). Indeed, let γ be the loop in $S^1 \times S^1$ defined by

$$\begin{aligned}\gamma : S^1 &\to S^1 \times S^1\\ \theta &\mapsto (\theta, \theta + h),\end{aligned}$$

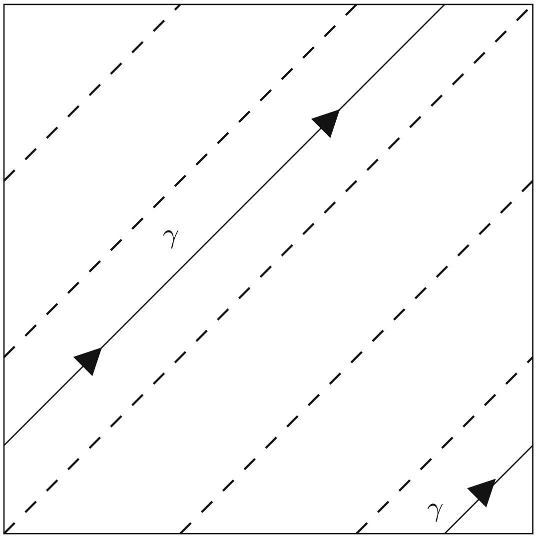

Fig. 3. The test loop γ in the torus $S^1 \times S^1$, with $\varphi = \varphi_3$

where h is a small positive real. A simple computation shows that

$$\int_\gamma \phi(\varphi_k) = k.$$

The integer k is called the turning number of the deformation. The Fig. 3 represents the loop γ in the torus $S^1 \times S^1$ (a square whose opposite edges are identified) for $\varphi = \varphi_3$. The set $K(\varphi_3, \varphi_3)$ is drawn with dashed lines.

If j_{S^1} is the canonical injection of S^1 into $\mathbb{R}^2$ (that is $j_{S^1} = \varphi_1$), we have

$$\int_\gamma \phi(j_{S^1}) = 1.$$

As long as $k \neq 1$,

$$\int_\gamma \phi(\varphi_k) \neq \int_\gamma \phi(j_{S^1}),$$

$\phi(\varphi_k) - \phi(j_{S^1})$ is not an exact form and the condition (7) is not fulfilled. Hence, the deformation φ_k is not an admissible deformation as long as $k \neq 1$.

Remark 1. Using the de Rahm duality theorem, one can prove that the condition (7) is equivalent to

$$\langle \mathcal{I}(\varphi), \omega \rangle = \langle \mathcal{I}(j_M), \omega \rangle \tag{8}$$

for any closed form ω on $M \times M$ with compact support included in the set $(K(\varphi, \varphi) \cup \partial(M \times M))^c$, where

$$\langle \mathcal{I}(\varphi), \omega \rangle := \int_{M \times M} \phi(\varphi) \wedge \omega.$$

3.2. Definition of the ϕ-admissible deformations

Let n be an integer and M be a submanifold of $\mathbb{R}^n$. Let j_M be the injection of M into $\mathbb{R}^n$.

For all mappings φ from M into $\mathbb{R}^n$, we denote by d_φ the mapping

$$\begin{aligned} d_\varphi : M \times M &\to \mathbb{R}^n \\ (x, y) &\mapsto \varphi(x) - \varphi(y) \end{aligned}$$

and by $K(\varphi)$ the non-injective set, that is,

$$K(\varphi) := \{(x, y) \in M \times M \ : \ \varphi(x) = \varphi(y)\}.$$

For any open subset U of $M \times M$ such that

$$\inf_{x \in U} |d_\varphi(x)| > 0, \tag{9}$$

we denote $\phi_U(\varphi)$ the element of $H^{n-1}(U)$ defined as the pullback of $\phi_{\mathbb{R}^n_*}$ by $d_{\varphi,U}$, the restriction of d_φ to the open set U,

$$\phi_U(\varphi) := d^*_{\varphi,U}(\phi_{\mathbb{R}^n_*}) \in H^{n-1}(U),$$

where $\phi_{\mathbb{R}^n_*}$ is the canonical $n-1$ non-exact closed form on $\mathbb{R}^n_*$ defined by

$$\phi_{\mathbb{R}^n_*}(x)(X_1, \ldots, X_{n-1}) := \det(x/|x|, X_1, \ldots, X_{n-1})/|S^{n-1}|$$

and $|S^{n-1}|$ is the $n-1$ Hausdorff measure of the unit sphere S^{n-1} of $\mathbb{R}^n$.

Remark 2. We recall that $H^{n-1}(U)$ is the quotient space of $n-1$-closed forms by the $n-1$ exact forms on U.

The mapping that maps any open subset U of $M \times M$ for which (9) holds to $\phi_U(\varphi)$ is denoted $\phi(\varphi)$.

We say that a deformation is ϕ-admissible if and only if for any open subset U

$$\phi_U(\varphi) = \phi_U(j_M), \tag{10}$$

as an element of $H^{n-1}(U)$, that is, if $\phi_U(\varphi) - \phi_U(j_M)$ is an exact form on U.

We denote by $\mathcal{A}_\phi(j_M)$ the set of ϕ-admissible deformations, that is

$$\begin{aligned} \mathcal{A}_\phi(j_M) := \Big\{ \varphi \in \mathcal{C}^0(M; \mathbb{R}^n) \ : \ \phi_U(\varphi) = \phi_U(j_M) \text{ in } H^{n-1}(U), \\ \text{for any open set } U \text{ such that } \inf_{x \in U} |d_\varphi(x)| > 0 \Big\}. \end{aligned} \tag{11}$$

Remark 3. The element $\phi_U(\varphi)$ of $H^{n-1}(U)$ is well defined, even if φ is only continuous. Indeed, if $\widetilde{\varphi}$ is a regular approximation of φ, then $\phi_U(\widetilde{\varphi}) \in H^{n-1}(U)$ is independent of $\widetilde{\varphi}$ as long as $\|\varphi - \widetilde{\varphi}\|_{\mathcal{C}^0}$ is small enough.

Remark 4. One can define $\phi(\varphi)$ as an element of the inverse limit of the groups $H^1(U)$, where U is any open subset of $M \times M$ which fulfills the condition (9). Then $\phi(\varphi)$ is a mapping, which maps every open subset U which fulfills (9) to an element $\phi_U(\varphi)$ of $H^1(U)$. Moreover, if $U \subset V$, then

$$\phi_U(\varphi) = j_U^{V*}(\phi_V(\varphi)),$$

where j_U^V is the injection of U in V.

3.3. Elementary properties of the ϕ-admissible set

3.3.1. $\mathcal{C}^0$-Closure The set of ϕ-admissible deformations is closed for the $\mathcal{C}^0$ topology. Furthermore, the set of admissible deformations is included in the set of ϕ-admissible deformations.

Proposition 1. *$\mathcal{A}_\phi(j_M)$ is closed for the $\mathcal{C}^0$ topology.*

Proof. Let φ_n be a sequence of ϕ-admissible deformations and φ be a deformation of M such that φ_n converges toward φ for the $\mathcal{C}^0(M;\mathbb{R}^n)$ topology. Let U be an open subset of $M \times M$ and δ be a positive real such that

$$\inf_{x\in U} |d_\varphi(x)| > \delta > 0.$$

Let $\widetilde{\varphi}$ be a $\mathcal{C}^\infty$ regularization of φ such that

$$\|\varphi - \widetilde{\varphi}\|_{\mathcal{C}^0} < \delta/3.$$

Let $\widetilde{\varphi}_n$ be $\mathcal{C}^\infty$ regularization of φ_n such that

$$\|\varphi_n - \widetilde{\varphi}_n\|_{\mathcal{C}^0} < \delta/3.$$

There exists N such that $\|\varphi - \varphi_N\|_{\mathcal{C}^0} < \delta/3$, so that if $\widetilde{\varphi}_t = t\widetilde{\varphi} + (1-t)\widetilde{\varphi}_N$, then for all $(x, y) \in U$,

$$|\widetilde{\varphi}_t(x) - \widetilde{\varphi}_t(y)| > 0.$$

Thus, the restriction $d_{\widetilde{\varphi}_t,U}$ of $d_{\widetilde{\varphi}_t}$ to U defines a homotopy from $d_{\widetilde{\varphi},U} : U \to \mathbb{R}^n_*$ to $d_{\widetilde{\varphi}_N,U} : U \to \mathbb{R}^n_*$. As $\widetilde{\varphi}_N$ belongs to the set of ϕ-admissible deformations,

$$\begin{aligned}\phi_U(\varphi) = \phi_U(\widetilde{\varphi}) &= d^*_{\widetilde{\varphi},U}\left(\phi_{\mathbb{R}^n_*}\right) = d^*_{\widetilde{\varphi}_N,K}\left(\phi_{\mathbb{R}^n_*}\right)\\ &= \phi_U(\widetilde{\varphi}_N) = \phi_U(\varphi_N) = \phi_U(j_M).\end{aligned}$$

Hence, we have proved that $\phi_U(\varphi) = \phi_U(j_M)$ for every open subset U of $M \times M$, such that (9) holds. In other words that φ belongs to the ϕ-admissible set $\mathcal{A}_\phi(j_M)$.

Remark 5. The proof of the Proposition 1 shows that the element $\phi_U(\varphi)$ is, as stated in the previous section, correctly defined for any continuous deformation and any open subset U for which (9) holds.

Proposition 2. *The admissible set $\mathcal{A}(j_M)$ is included in the ϕ-admissible set $\mathcal{A}_\phi(j_M)$.*

Remark 6. Under some conditions on the dimension of M and the dimension n of the space, we proved that $\mathcal{A}(j_M) \cap \mathrm{Imm}(M;\mathbb{R}^n) = \mathcal{A}_\phi(j_M) \cap \mathrm{Imm}(M;\mathbb{R}^n)$ (see Sections 5.1, 5.2, and 5.3).

Proof. Let φ be an embedding isotopic to j_M. There exists an isotopy φ_t such that $\varphi_0 = \varphi$ and $\varphi_1 = j_M$. Let U be an open subset of $\triangle(M)^c$. Then $d_{\varphi_t,U} : [0,1]\times U \to \mathbb{R}^n_*$ is a regular homotopy from $d_{\varphi,U} : U \to \mathbb{R}^n_*$ to $d_{j_M,U} : U \to \mathbb{R}^n_*$, and

$$d^*_{\varphi,U}\left(\phi_{\mathbb{R}^n_*}\right) = d^*_{j_M,U}\left(\phi_{\mathbb{R}^n_*}\right),$$

as an element of $H^1(U)$. Hence, φ belongs to the set of ϕ-admissible deformations. The conclusion follows from the previous proposition.

Remark 7. If $n \geqq 3$ and $\dim(M) \geqq 2$, there exist deformations $\varphi : M \to \mathbb{R}^n$ that belong to the ϕ-admissible but not to $\mathcal{A}(j_M)$ (see Section 5.3). Moreover, we do not know whether or not $\mathcal{A}_\phi(j_M) = \mathcal{A}(j_M)$ when $n = 2$.

3.3.2. Right and left invariance.

Proposition 3. *Let $g : M \to M$ be a homeomorphism isotopic to the identity, then*

$$(\varphi \in \mathcal{A}_\phi(j_M)) \Rightarrow (\varphi \circ g \in \mathcal{A}_\phi(j_M)).$$

Proof. Let U be an open set of $M \times M$ such that

$$\inf_{x \in U} |d_\varphi(x)| > 0.$$

Let $V = (g, g)^{-1}(U)$. There exists regularizations $\widetilde{g}$ and $\widetilde{\varphi}$ of g and φ such that

$$\phi_U(\varphi) = d^*_{\widetilde{\varphi},U}\left(\phi_{\mathbb{R}^n_*}\right)$$

$$\phi_V(\varphi \circ g) = d^*_{\widetilde{\varphi}\circ\widetilde{g},V}\left(\phi_{\mathbb{R}^n_*}\right).$$

Moreover, $\widetilde{g}$ can be chosen such that it is diffeomorphic to the identity. In the following, $(\widetilde{g}, \widetilde{g})$ will be understood as its restriction to V with values in U.

$$\begin{aligned} d_{\widetilde{\varphi},U} \circ (\widetilde{g}, \widetilde{g}) &= d_{\widetilde{\varphi}\circ\widetilde{g},V} \\ d^*_{\widetilde{\varphi}\circ\widetilde{g},V}\left(\phi_{\mathbb{R}^n_*}\right) &= (\widetilde{g}, \widetilde{g})^* \circ d^*_{\widetilde{\varphi},U}\left(\phi_{\mathbb{R}^n_*}\right) = (\widetilde{g}, \widetilde{g})^* \circ d^*_{j_M,U}\left(\phi_{\mathbb{R}^n_*}\right) \\ &= d^*_{j_M\circ\widetilde{g},V}\left(\phi_{\mathbb{R}^n_*}\right) = d^*_{j_M,V}\left(\phi_{\mathbb{R}^n_*}\right). \end{aligned}$$

Proposition 4. *Let g be a diffeomorphism from $\mathbb{R}^n$ into itself that preserves the orientation, then*

$$(\varphi \in \mathcal{A}_\phi(j_M)) \Rightarrow (g \circ \varphi \in \mathcal{A}_\phi(j_M)).$$

Proof. As g is a diffeomorphism that preserves the orientation, (g, g) defines a diffeomorphism from $\mathbb{R}^n \times \mathbb{R}^n \setminus \triangle(\mathbb{R}^n)$ into itself and $(g, g)^*$ from $H^{n-1}(\mathbb{R}^n \times \mathbb{R}^n \setminus \triangle(\mathbb{R}^n))$ into itself is nothing else but the identity. Let U be a subset of $M \times M$ such that

$$\inf_{x \in U} |d_\varphi(x)| > 0,$$

then

$$\inf_{x \in U} |d_{g\circ\varphi}(x)| > 0.$$

Let p be the mapping

$$\begin{aligned} p : \mathbb{R}^n \times \mathbb{R}^n \setminus \triangle(\mathbb{R}^n) &\to \mathbb{R}^n_* \\ (x, y) &\mapsto x - y. \end{aligned}$$

Without loss of generality, we can assume that φ is regular and that

$$\phi_U(\varphi) = d^*_{\varphi,U}\left(\phi_{\mathbb{R}^n_*}\right) \quad \text{and} \quad \phi_U(g \circ \varphi) = d^*_{g\circ\varphi,U}\left(\phi_{\mathbb{R}^n_*}\right).$$

We have $d_{g\circ\varphi,U} = p \circ (g, g) \circ (\varphi, \varphi)_{|U}$ and

$$\begin{aligned} d^*_{g\circ\varphi,U}\left(\phi_{\mathbb{R}^n_*}\right) &= \left(p \circ (g, g) \circ (\varphi, \varphi)_{|U}\right)^*\left(\phi_{\mathbb{R}^n_*}\right) = (\varphi, \varphi)^*_{|U} \circ (g, g)^* \circ p^*\left(\phi_{\mathbb{R}^n_*}\right) \\ &= (\varphi, \varphi)^*_{|U} \circ p^*\left(\phi_{\mathbb{R}^n_*}\right) = d^*_{\varphi,U}\left(\phi_{\mathbb{R}^n_*}\right) = d^*_{j_M,U}\left(\phi_{\mathbb{R}^n_*}\right). \end{aligned}$$

3.3.3. Transversal self-intersections. In this section, we prove that any deformation with transverse self-intersection does not belong to the set of ϕ-admissible deformations $\mathcal{A}_\phi(j_M)$ and thus to the admissible set $\mathcal{A}(j_M)$. More precisely,

Proposition 5. *Let $\varphi : M \to \mathbb{R}^n$ be a continuous mapping. Assume that φ has a transverse self-intersection at points $(x, y) \in M \times M$, such that $x \neq y$, that is*

- $\varphi(x) = \varphi(y)$
- *The mapping φ is of class $\mathcal{C}^1$ in the neighborhoods of x and y*
- $D_x\varphi(T_x M) + D_y\varphi(T_y M) = \mathbb{R}^n$.

Then φ does not belong to the set of ϕ-admissible deformations $\mathcal{A}_\phi(j_M)$.

Proof. The proof is based on the construction of two functions γ_φ and γ_{j_M} from S^{n-1} into S^{n-1} whose degree depends, respectively, on $\phi(\varphi)$ and $\phi(j_M)$. As shown thereafter, $\deg(\gamma_\varphi) = 1$ whereas $\deg(\gamma_{j_M}) = 0$, and thus, $\phi(\varphi) \neq \phi(j_M)$. It follows that φ does not belong to the set $\mathcal{A}_\phi(j_M)$ of ϕ-admissible deformations.

As $D_x\varphi(T_x M) + D_y\varphi(T_y M) = \mathbb{R}^n$, the mapping

$$\begin{aligned} d_\varphi : M \times M &\to \mathbb{R}^n \\ (a, b) &\mapsto \varphi(a) - \varphi(b) \end{aligned}$$

is a submersion at (x, y). From the implicit function theorem, we deduce that there exists a neighborhood V of 0 in $\mathbb{R}^{2m}$, a neighborhood W of (x, y) in $M \times M$ and a local diffeomorphism $g : V \to W$ such that $g(0) = (x, y)$ and

$$d_\varphi \circ g(x_1, \ldots, x_{2m}) = (x_1, \ldots, x_n).$$

Furthermore, one can choose g such that the unit disk D^n is included in V, and such that $W \cap \triangle(M) = \emptyset$, where

$$\triangle(M) = \{(a, b) \in M \times M : a = b\}.$$

Let $\delta = 1/2$ and

$$V_\delta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{2m-n} \cap V \ : \ |x| > \delta\}.$$

Let $h : V_\delta \to U = g(V_\delta) \subset W \setminus \{(x, y)\}$ be the restriction of g to V_δ. There exists a real δ' such that

$$\inf_{x \in U} |\varphi(x)| > \delta' > 0.$$

We set

$$\gamma_\varphi = \theta \circ d_{\varphi,U} \circ h \circ j^{V_\delta}_{S^{n-1}} \quad \text{and} \quad \gamma_{j_M} = \theta \circ d_{j_M,U} \circ h \circ j^{V_\delta}_{S^{n-1}},$$

where $\theta(z) = z/|z|$. The degree of the mapping γ_φ and γ_{j_M} depends only on $\phi_U(\varphi)$ and $\phi_U(j_M)$. Indeed, denoting by $\phi_{S^{n-1}}$ the generator of $H^{n-1}(S^{n-1})$, that is the $n - 1$ differential form defined by

$$\phi_{S^{n-1}}(x)(X_1, \ldots, X_n) = \det(n, X_1, \ldots, X_n)/|S^{n-1}|,$$

we have

$$\begin{aligned}\deg(\gamma_\varphi) &= \int_{S^{n-1}} \gamma_\varphi^*(\phi_{S^{n-1}})\\ &= \int_{S^{n-1}} \left(h \circ j_{S^{n-1}}^{V_\delta}\right)^* \circ d_{\varphi,U}^*(\theta^*(\phi_{S^{n-1}}))\\ &= \int_{S^{n-1}} \left(h \circ j_{S^{n-1}}^{V_\delta}\right)^* \circ d_{\varphi,U}^*(\phi_{\mathbb{R}_*^n}).\end{aligned}$$

Thus, the degree of γ_φ is

$$\deg(\gamma_\varphi) = \int_{S^{n-1}} \left(h \circ j_{S^{n-1}}^{V_\delta}\right)^* (\phi_U(\varphi)), \tag{12}$$

whereas the degree of γ_{j_M} is

$$\deg(\gamma_{j_M}) = \int_{S^{n-1}} \left(h \circ j_{S^{n-1}}^{V_\delta}\right)^* (\phi_U(j_M)). \tag{13}$$

As $\gamma_\varphi : S^{n-1} \to S^{n-1}$ is nothing else but the identity,

$$\deg(\gamma_\varphi) = 1.$$

On the other hand,

$$\begin{aligned}\deg(\gamma_{j_M}) &= \int_{S^{n-1}} \left(h \circ j_{S^{n-1}}^{V_\delta}\right)^* (\phi_U(j_M))\\ &= \int_{S^{n-1}} \left(h \circ j_{D^n}^{V}\right)^* (\phi_W(j_M))\\ &= \int_{\partial D^n} \left(h \circ j_{D^n}^{V}\right)^* (\phi_W(j_M))\\ &= \int_{D^n} d\left(\left(h \circ j_{D^n}^{V}\right)^* (\phi_W(j_M))\right) = 0.\end{aligned}$$

As claimed, $\deg(\gamma_{j_M}) \neq \deg(\gamma_\varphi)$ and from (12) and (13) we deduce that $\phi_U(\varphi) \neq \phi_U(j_M)$ and that φ does not belong to the set of ϕ-admissible deformations.

4. The minimization problem

In this section, we consider nonlinear hyperelastic bodies. With suitable assumptions on the stored energy function W of the material, we show that there exists at least one minimizer of the energy over the set of admissible deformations. Moreover, this existence results remains true if one considers ϕ-admissible deformations instead. The proof of existence relies on standard arguments and on the fact that both $\mathcal{A}(j_M)$ and $\mathcal{A}_\phi(j_M)$ are closed for $\mathcal{C}^0$ topology. For the sake of completeness, we treat in detail the case of nonlinear hyperelastic bodies, but similar results can easily be extended to other types of materials (see Section 4.3).

4.1. Setting of the problem

Let M be a differentiable submanifold of $\mathbb{R}^n$ (with or without boundary), m the dimension of M, and j_M the injection of M into $\mathbb{R}^n$. The manifold M is implicitly endowed with the differential structure and the Riemann metric induces by j_M. Furthermore, the m-dimensional Hausdorff measure in $\mathbb{R}^n$ induced a measure on M noted $\mathrm{d}x$. We define $\mathcal{F}(M;\mathbb{R}^n)$ as the vector bundle of base M whose fiber at $x \in M$ is the set of linear mappings from $(T_x^*M)^m$ into $\mathbb{R}^n$. Let π be the projection of this vector bundle on its base. The stored energy function W is a mapping from $\mathcal{F}(M;\mathbb{R}^n)$ into $\mathbb{R}^+$. We assume that W is a Carathéodory function: the restriction of W to a fiber $\pi^{-1}(x)$ is $\mathcal{C}^0$ for almost every $x \in M$, and the restriction of W to any section is measurable (for any regular mapping $G : M \to \mathcal{F}(M;\mathbb{R}^n)$ such that $\pi \circ G = \mathrm{Id}_M$, the mapping $W \circ G$ is measurable). We assume W to be quasiconvex, that is, for every $F \in \mathcal{F}(M;\mathbb{R}^n)$,

$$\int_U W(F)\,\mathrm{d}x \leqq \int_U W(F + D\varphi)\,\mathrm{d}x,$$

where U is the unit ball of $T_{\pi(F)}M$, $\varphi \in \mathcal{C}_0^\infty(U;\mathbb{R})$. Moreover, we assume that there exists $p > m$ such that the following growth and coercivity conditions are fulfilled

$$\begin{aligned} \forall F \in \mathcal{F}(M;\mathbb{R}^n), &\quad |W(F)| \leqq C(1 + |F|^p) \\ \forall F \in \mathcal{F}(M;\mathbb{R}^n), &\quad W(F) \geqq \alpha |F|^p + \beta, \end{aligned}$$

where C, α, and β are constants, and $\alpha > 0$. We consider the case where M is submitted to dead body forces $f \in L^2(M;\mathbb{R}^n)$ and fixed on a subset N of M such that every connected component of M intersects N. Let $I : W^{1,p}(M;\mathbb{R}^n) \to \mathbb{R}$ be the energy functional

$$I(\psi) = \int_M W(D\psi)\,\mathrm{d}x - \int_M f(x)\cdot\psi(x)\,\mathrm{d}x. \tag{14}$$

The set of admissible deformations of finite energy is

$$\mathcal{A}^p(j_M) = \left\{\varphi \in W^{1,p}(M;\mathbb{R}^n) \cap \mathcal{A}(j_M) \ : \ \varphi(x) = j_M(x) \text{ for all } x \in N\right\}, \tag{15}$$

whereas the set of ϕ-admissible deformations with finite energy is

$$\mathcal{A}_\phi^p(j_M) = \left\{\varphi \in W^{1,p}(M;\mathbb{R}^n) \cap \mathcal{A}_\phi(j_M) \ : \ \varphi(x) = j_M(x) \text{ for all } x \in N\right\}. \tag{16}$$

We consider the two following minimization problems

$$\text{Find } \varphi \in \mathcal{A}^p(j_M) \quad \text{such that } I(\varphi) = \inf_{\psi \in \mathcal{A}^p(j_M)} I(\psi) \tag{$\mathcal{P}$}$$

and

$$\text{Find } \varphi \in \mathcal{A}_\phi^p(j_M) \quad \text{such that } I(\varphi) = \inf_{\psi \in \mathcal{A}_\phi^p(j_M)} I(\psi). \tag{$\mathcal{P}_\phi$}$$

Remark 8. We would like to emphasize that our formulation is not limited to the study of a single body. If one considers two bodies whose reference configurations are the submanifolds A and B, respectively, we set $M = A \cup B$. Note that we assume for simplicity that all the connected components of M have the same dimension m. This assumption could be removed at no extra cost.

Remark 9. The reference configuration used in our formulation is a differentiable manifold M. Usually, an open subset of $\mathbb{R}^m$ is used instead. Our choice allows us to treat more kinds of topology.

Remark 10. As we will see in Section 5, not only does the minimization problem $(\mathcal{P})$ have a physical meaning but so does the minimization problem $(\mathcal{P}_\phi)$, at least if $n = 2$ or $\dim(M) = n$. Indeed, in such cases, minimizers of the total energy fulfill the Euler–Lagrange equations, describing an elastic body with frictionless contacts.

4.2. Existence

Proposition 6. *Both minimization problems* $(\mathcal{P})$ *and* $(\mathcal{P}_\phi)$ *have at least one solution.*

Proof. The quasiconvexity of W, coercivity and growing conditions imply that the functional I is sequentially lower semicontinuous (see [3, 15, 16]) for the weak topology of $W^{1,p}(M;\mathbb{R}^n)$. Let φ_n be a minimization sequence of I over $\mathcal{A}^p(j_M)$. The Dirichlet conditions combined with the coercivity ensures that the sequence φ_n is bounded in $W^{1,p}(M;\mathbb{R}^n)$. One can extract a subsequence φ_{n_k} weakly converging toward an element $\varphi \in W^{1,p}(M;\mathbb{R}^n)$. As I is sequentially lower semicontinuous, we have

$$I(\varphi) \leqq \inf_{\psi \in \mathcal{A}^p(M)} I(\psi).$$

Since $p > m$, the injection of $W^{1,p}(M;\mathbb{R}^n)$ into $\mathcal{C}^0(M;\mathbb{R}^n)$ is compact. Hence, φ_{n_k} converges in $\mathcal{C}^0(M;\mathbb{R}^n)$ and, as $\mathcal{A}^p(j_M)$ is closed for the $\mathcal{C}^0$ topology, φ belongs to $\mathcal{A}^p(j_M)$. Thus, the minimization problem $(\mathcal{P})$ has at least one solution. The existence of a solution to the problem $(\mathcal{P}_\phi)$ ensues from the closure of the set $\mathcal{A}^p_\phi(j_M)$ for the $\mathcal{C}^0$ topology given by Proposition 1.

Remark 11. Existence results could also be obtained if N does not intersect every connected component of M. Yet, the loads f have to fulfill compatibility conditions to ensure the existence of a minimizer. For instance, if M has only one connected component and if $\int_M f(x)\mathrm{d}x = 0$, then there exists at least one solution to the minimization problem.

Remark 12. The existence could also be obtained under other assumptions on the stored energy function W. In particular, it remains true under the hypothesis made by BALL [3], which allows for the case $W(F) \to +\infty$, if $\det(F) \to 0$ in the case $m = n$. Furthermore, the definition of the admissible set given here only holds for continuous functions and thus requires $p > m$. This condition could probably be weakened using methods similar to the one used by TANG [25] and ŠVERÁK [24].

4.3. Non-hyperelastic bodies

In the case $m < n$, if W satisfies the classical assumption of frame indifference [$W(RF) = W(F)$ for all $R \in SO(n)$] and if the natural injection vanishes the energy $W(Dj_M) = 0$, then the quasiconvexity of W implies that M does not sustain any compression. The elastic energy vanishes for any deformation whose pull-back metric is less than or equal to the identity and in particular for constant maps (see [12, 13, 19]). This degenerated behavior is directly linked to the fact that bending effects are neglected. Such an approximation is in many cases too coarse. In consequence, existence results of minimizers for models that include such effects are likely. For curvature-dependent energies, if an existence result is available without taking into account any non-self-intersection constraints, it will usually remain true if one compels the deformations to belong to either the admissible or ϕ-admissible set. Indeed, the existence of minimizers (regardless of the type of energy considered) usually relies upon the same arguments as those which we used in the hyperelastic case: relative compactness of the minimization sequences, sequential lower semicontinuity of the energy and closeness of the function space X of optimization. In such a case, if the injection of X into the set of continuous maps is continuous, then there exists also at least one minimizer of the energy over $X \cap \mathcal{A}(j_M)$ or $X \cap \mathcal{A}_\phi(j_M)$ as both of those sets are closed for the topology of X. If bending terms are considered, the space X is expected to be included in $W^{2,1}(M; \mathbb{R}^n)$ and the injection of X into $W^{2,1}(M; \mathbb{R}^n)$ endowed with the weak topology is expected to be continuous. In consequence, the injection of X into $\mathcal{C}^0$ is also continuous and the energy admits at least one minimizer over each of the sets $X \cap \mathcal{A}(j_M)$ and $X \cap \mathcal{A}_\phi(j_M)$.

5. Interpretation of the minimizers of the energy

In this section, we examine whether or not solutions of the minimization problems $(\mathcal{P})$ or $(\mathcal{P}_\phi)$ are solutions of the Euler–Lagrange equations describing the behavior of elastic bodies with frictionless contacts. All results and proofs are expressed with respect to the problem $(\mathcal{P}_\phi)$. They can easily be adapted to the study of the problem $(\mathcal{P})$. To this end, it suffices to use the fact that the admissible set $\mathcal{A}(j_M)$ is included in the ϕ-admissible set $\mathcal{A}_\phi(j_M)$ (Proposition 2). Throughout this section, we assume the stored energy function W to be $\mathcal{C}^1$.

5.1. The case of n-dimensional bodies in $\mathbb{R}^n$

As we recalled in Section 1, the case of n-dimensional bodies moving in $\mathbb{R}^n$ has been studied by Ciarlet & Nečas in [7]. Our modeling differs from theirs in the definition of the admissible set of deformations. Furthermore, their assumptions on the stored energy W are stronger than ours. They choose W in a way that forces local injectivity almost everywhere for deformations with finite elastic energy. To this end, they assume that W takes its values in $\overline{\mathbb{R}}^+$ and verifies

$$\begin{cases} W(F) = +\infty & \text{if } \det(F) \leqq 0 \\ W(F) \to +\infty & \text{as } \det(F) \to 0. \end{cases} \tag{17}$$

Their admissible deformations are defined as those that satisfy

$$\int_M \det(D\varphi)\,\mathrm{d}x \leqq \mathrm{Vol}(\mathrm{Im}(\varphi)). \tag{18}$$

It is easy to show that, if a deformation fulfills this constraint and has a finite energy, it is almost everywhere injective not only locally but globally. This follows from the equation (18) and the identity

$$\int_{\varphi(M)} \mathrm{Card}(\varphi^{-1}(y))\,\mathrm{d}y = \int_M \det(D\varphi(x))\,\mathrm{d}x$$

fulfilled by every $W^{1,p}$ function such that $\det(D\varphi) > 0$ almost everywhere. Using the notation introduced in Section 4.1, CIARLET & NEČAS [7] define their set of admissible deformations as

$$\mathcal{A}^p_{CN}(j_M) = \Big\{\varphi \in W^{1,p}(M;\mathbb{R}^n) \;:\; \varphi(x) = j_M(x) \text{ for all } x \in N,$$
$$\det(D\varphi) > 0 \text{ almost everywhere}, \int_M \det(D\varphi)\,\mathrm{d}x \leqq \mathrm{Vol}(\mathrm{Im}(\varphi))\Big\}.$$

Under other assumptions on the stored energy, W, Ciarlet & Nečas prove that there exists a minimizer of the energy on their set of admissible deformations whenever the infimum is finite.

The condition $\phi(\varphi) = \phi(j_M)$ is at least as strong as the condition (18) introduced by Ciarlet & Nečas (see Proposition 8). Every ϕ-admissible deformation $\varphi \in W^{1,p}(M;\mathbb{R}^n)$ such that $\det(D_x\varphi) \neq 0$ almost everywhere satisfies (18) and thus is admissible according to the Ciarlet–Nečas definition. In particular, every element $\varphi \in \mathcal{A}_\phi(j_M)$ is injective almost everywhere: for almost all $z \in \mathbb{R}^n$, $\mathrm{Card}(\varphi^{-1}(z)) = 0$ or 1. Consequently, our model inherits several properties of theirs. In particular, this feature enables us to extend their formal interpretation of the minimizers of the energy. If a solution of the minimization problems $(\mathcal{P})$ or $(\mathcal{P}_\phi)$ is a regular immersion, it fulfills the Euler–Lagrange equations (see Proposition 7). In our case, the proof can be run word by word, as in the formulation of Ciarlet & Nečas . Actually, our criterion is stronger: some deformations that belong to the set $\mathcal{A}^p_{CN}(j_M)$ do not belong to $\mathcal{A}_\phi(j_M)$. For instance, the deformation represented in Fig. 4 fulfills criterion (18) but does not belong to $\mathcal{A}_\phi(j_M)$ as it has a transversal self-intersection. The existence of such deformations prevents us from hoping that an asymptotic analysis, performed on the Ciarlet–Nečas model, could lead to a reasonable model of a thin structures with frictionless self-contacts, without self-intersection.

Finally, let us remark that every deformation φ belonging to $\mathcal{A}^p_\phi(j_M)$ preserves the orientation, without any assumption on the stored energy W (see Proposition 9). This is another important difference between our model and the Ciarlet–Nečas formulation. A deformation $\varphi \in \mathcal{A}^p_\phi(j_M)$ only satisfies

$$\det(D\varphi) \geqq 0,$$

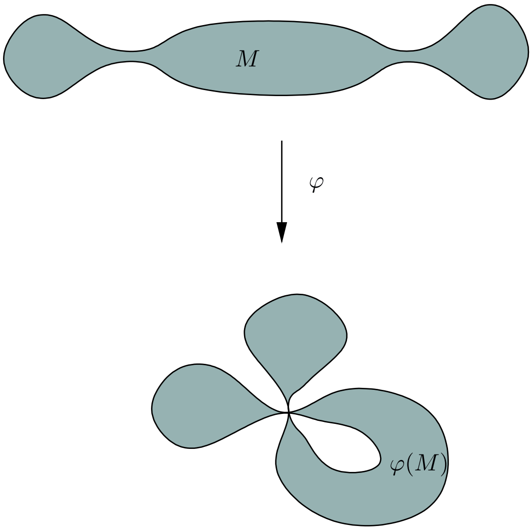

Fig. 4. Admissible deformation according to the Ciarlet–Nečas model with transversal self-intersection

and not

$$\det(D\varphi) > 0 \text{ almost everywhere.}$$

In the next proposition, we will assume that $N \subset \partial M$, that is, the body is fixed only at its boundary.

Proposition 7. *Let M be an n-dimensional submanifold of $\mathbb{R}^n$ such that ∂M is smooth. Let φ be a solution of the minimization problem*

$$\varphi \in \mathcal{A}^p_\phi(j_M), \quad I(\varphi) = \inf_{\psi \in \mathcal{A}^p_\phi(j_M)} I(\psi), \tag{$\mathcal{P}_\phi$}$$

where the set $\mathcal{A}^p_\phi(j_M)$ and the functional I are defined as in (14) *and* (15)*. Assume in addition that $\varphi \in \mathcal{C}^1(\overline{M})$ and $\det(D\varphi) > 0$ on $\overline{M}$ then the minimizer φ is a solution of the following boundary-value problem:*

$$-\operatorname{div} DW(D\varphi) = f \quad \text{in } M \setminus \partial M, \tag{19}$$

$$\varphi = j_M \quad \text{on } N, \tag{20}$$

$$DW(D\varphi(x)) \cdot n'(x) = \lambda(x)n(x) \quad \text{with } \lambda(x) \leqq 0, \text{ for all } x \in \partial M \setminus N; \tag{21}$$

the last equations correspond to one of the following situations (*for all $x \in \partial M \setminus N$*)*:*

$$\varphi^{-1}(\varphi(x)) = \{x\} \quad \text{whence } \lambda(x) = 0, \tag{22}$$

$$\varphi^{-1}(\varphi(x)) = \{x, y\}, \quad \text{with } y \in \partial M \text{ whence}$$
$$n(x) + n(y) = 0 \text{ and } \lambda(x)\mathrm{d}a(x) = \lambda(y)\mathrm{d}a(y), \tag{23}$$

where $n'(z)$ and $n(z)$ denote the unit's outer normal vectors along ∂M and $\varphi(\partial M)$ at the point z and $\varphi(z)$, respectively, and $\mathrm{d}a(z)$ denotes the differential area along ∂M at the point z.

Remark 13. Equation (19) has to be understood in a weak form.

Proof. Let φ be a smooth ϕ-admissible deformation such that $\det(D\varphi) > 0$. By Proposition 8, φ belongs to the Ciarlet–Nečas admissible set $\mathcal{A}^p_{CN}(j_M)$. In [7], CIARLET & NEČAS prove that there are enough variations $F : [0, 1] \times M \to \mathbb{R}^3$ such that, $F(0, x) = \varphi(x)$ and $F(t, \cdot)$ is an embedding for every $t > 0$ to recover the given Euler–Lagrange equations, using the differentiability of the energy $I(\cdot)$.

Proposition 8. *Let M be an n-dimensional differentiable submanifold of $\mathbb{R}^n$ (with boundary). Every deformation $\varphi \in \mathcal{A}_\phi(j_M) \cap W^{1,p}(M; \mathbb{R}^n)$ is such that*

$$\int_M |\det(D\varphi)|\, \mathrm{d}x = \mathrm{Vol}(\mathrm{Im}(\varphi)),$$

and φ is injective almost everywhere, that is for almost all $z \in \mathbb{R}^m$, $\varphi^{-1}(z)$ is either empty or contains only one element.

Proof. Let $\varphi \in \mathcal{A}_\phi(j_M) \cap W^{1,p}(M; \mathbb{R}^n)$. Let x and y be two distinct elements of the interior of M such that

$$\varphi(x) = \varphi(y).$$

We consider a chart $g : B^n(0, 1) \to M$ such that $y \notin \mathrm{Im}\, g$ and $g(0) = x$ (We recall that $B^n(0, 1)$ is the open ball of radius 1 centered at the origin). For all $0 < r < 1$, we set

$$S_r^{n-1} = g(S^{n-1}(0, r)),$$

and

$$K_r = \{x\} \times S_r^{n-1} \cup \{y\} \times S_r^{n-1}.$$

Assume that

$$\varphi \circ g(S_r^{n-1}) \cap \varphi(x) = \emptyset, \tag{24}$$

then K_r is a compact subset of ${d_\varphi}^{-1}(\mathbb{R}^n_*)$ and as φ belongs to the set of admissible deformations $\mathcal{A}_\phi(j_M)$, we have

$$\int_{\{x\}\times g\left(S_r^{n-1}\right)} d_\varphi^*\left(\phi_{\mathbb{R}^n_*}\right) = \int_{\{x\}\times g\left(S_r^{n-1}\right)} d_{j_M}^*\left(\phi_{\mathbb{R}^n_*}\right) = 1,$$

and

$$\int_{\{y\}\times g\left(S_r^{n-1}\right)} d_\varphi^*\left(\phi_{\mathbb{R}^n_*}\right) = \int_{\{y\}\times g\left(S_r^{n-1}\right)} d_{j_M}^*\left(\phi_{\mathbb{R}^n_*}\right) = 0,$$

On the other hand, as $\varphi(x) = \varphi(y)$, we have

$$\int_{\{x\}\times g\left(S_r^{n-1}\right)} d_\varphi^*\left(\phi_{\mathbb{R}_*^n}\right) = \int_{\{y\}\times g\left(S_r^{n-1}\right)} d_\varphi^*\left(\phi_{\mathbb{R}_*^n}\right),$$

which cannot hold when considered together with the previous equations. Hence, our assumption (24) is false. For all positive real $0 < r < 1$, we have

$$\varphi \circ g\left(S_r^{n-1}\right) \cap \varphi(x) \neq \emptyset.$$

We have just shown that for all $x \in M \setminus \partial M$,

$$\mathrm{Card}(\varphi^{-1}(x)) > 1 \Rightarrow \mathrm{Card}(\varphi^{-1}(x)) = +\infty. \tag{25}$$

(Here and in the following of the proof exposition, φ is understood as a mapping from $M \setminus \partial M$ into $\mathbb{R}^n$.) Let P be the set of non-injective points, that is

$$P = \{z \in \mathbb{R}^n : \ \mathrm{Card}(\varphi^{-1}(z)) > 1\}.$$

For all $z \in P$, from (25), we deduce that $\mathrm{Card}(\varphi^{-1}(z)) = +\infty$. Moreover, a theorem from Marcus & Mizel [14] shows that, as $\varphi \in W^{1,p}(M, \mathbb{R}^n)$ with $p > n$,

$$\int_M |\det(D\varphi)|\,\mathrm{d}x = \int_{\varphi(M)} \mathrm{Card}(\varphi^{-1}(z))\,\mathrm{d}z.$$

It brings up

$$\begin{aligned}\int_M |\det(D\varphi)|\,\mathrm{d}x &= \int_P \mathrm{Card}(\varphi^{-1}(z))\,\mathrm{d}z + \int_{\varphi(M)\setminus P} \mathrm{Card}(\varphi^{-1}(z))\,\mathrm{d}z \\ &= +\infty|P| + \mathrm{Vol}(\mathrm{Im}(\varphi)) - |P|.\end{aligned}$$

This implies that the measure $|P|$ of the set of non-injective points is zero and that

$$\int_M |\det(D\varphi)|\,\mathrm{d}x = \mathrm{Vol}(\mathrm{Im}(\varphi)).$$

Proposition 9. *Let M be an n-dimensional differentiable submanifold of $\mathbb{R}^n$ (with boundary). Every deformation $\varphi \in \mathcal{A}_\phi(j_M) \cap W^{1,p}(M; \mathbb{R}^n)$ is such that* $\det(D\varphi) \geqq 0$.

Proof. Let $z \in \mathbb{R}^n \setminus \varphi(\partial M)$; the degree $\deg(\varphi, M, z)$ is defined as

$$\deg(\varphi, M, z) = \int_{\partial M} d(\varphi, z, \partial M)^* \left(\phi_{\mathbb{R}_*^n}\right)$$

where

$$\begin{aligned} d(\varphi, z, \partial M) : \partial M &\to \mathbb{R}_*^n \\ x &\mapsto \varphi(x) - z. \end{aligned}$$

The following formula holds (see [24], Corollary 1)

$$\int_M \det(D\varphi)\,\mathrm{d}x = \int_{\mathbb{R}^n} \deg(\varphi, M, z)\,\mathrm{d}z. \tag{26}$$

Furthermore, if $z \in \operatorname{Im}(\varphi)$, there exists $y \in M$ such that $\varphi(y) = z$, and

$$\begin{aligned}\deg(\varphi, \partial M, z) &= \int_{\partial M} d(\varphi, z, \partial M)^* \left(\phi_{\mathbb{R}^n_*}\right) \\ &= \int_{\partial M} (p_y \circ d_{\varphi,U})^* \left(\phi_{\mathbb{R}^n_*}\right)\end{aligned}$$

where $p_y : \partial M \to M \times M$ is defined by $p_y(t) = (t, y)$ and U is a neighborhood of $\partial M \times \{y\}$. Thus,

$$\deg(\varphi, \partial M, z) = \int_{\partial M} d^*_{\varphi,U} \circ p^*_y \left(\phi_{\mathbb{R}^n_*}\right).$$

As φ belongs to $\mathcal{A}_\phi(j_M)$, we have

$$\begin{aligned}\deg(\varphi, \partial M, z) &= \int_{\partial M} d^*_{j_M,U} \circ p^*_y \left(\phi_{\mathbb{R}^n_*}\right) \\ &= \int_{\partial M} d(j_M, j_M(y), \partial M)^* \left(\phi_{\mathbb{R}^n_*}\right) \\ &= \deg(j_M, \partial M, j_M(y)) \\ &= 1.\end{aligned}$$

Moreover, if $z \notin \operatorname{Im}(\varphi)$, $\deg(\varphi, \partial M, z) = 0$. From (26), it follows that

$$\int_M \det(D\varphi)\,\mathrm{d}x = \int_{\varphi(M)} 1\,\mathrm{d}x = \operatorname{Vol}(\operatorname{Im}(\varphi)),$$

and from Proposition 8, we deduce that

$$\int_M \det(D\varphi)\,\mathrm{d}x = \int_M |\det(D\varphi)|\,\mathrm{d}x.$$

Hence, $\det(D\varphi) \geqq 0$ almost everywhere.

5.2. The case of thin structures in $\mathbb{R}^2$

In this section, we state a similar results for one-dimensional structures moving in $\mathbb{R}^2$ to those obtained in the case $n = m$ in Section 5.1, Proposition 7. If a minimizer of the energy over $(\mathcal{P})$ or $(\mathcal{P}_\phi)$ is a regular immersion, then it fulfills Euler–Lagrange equations. To avoid unnecessary technicalities, we will only consider the case of one unfixed circle ($M = S^1$ and $N = \emptyset$). The results given could be extended to other cases, that is $M = [-1, 1]$ or $N \neq \emptyset$ or even to a problem involving several bodies. Let us first set the main result of this section.

Proposition 10. *Let $M = S^1$ and $N = \emptyset$. Let $\varphi : S^1 \to \mathbb{R}^2$ be a solution of the minimization problem:*

$$\varphi \in \mathcal{A}^p_\phi(j_M), \quad I(\varphi) = \inf_{\psi \in \mathcal{A}^p_\phi(j_M)} I(\psi), \tag{$\mathcal{P}_\phi$}$$

where the functional I and the set $\mathcal{A}_\phi^p(j_M)$ are defined as in (14) *and* (15)*. Assume in addition that φ is a $\mathcal{C}^1$ immersion. Then for all $z \in \mathrm{Im}(\varphi)$, there exists a family $(x_0(z), \dots, x_N(z))$ of elements of S^1 such that*

$$\varphi^{-1}(z) = \{x_0(z), \dots, x_N(z)\},$$

and an unitary vector $n(z)$ normal to $\mathrm{Im}(\varphi)$ such that for all test function $\psi \in \mathcal{C}^\infty(S^1; \mathbb{R}^2)$ such that for all $z \in \mathrm{Im}(\varphi)$ and all $0 \leqq i < \mathrm{Card}(\varphi^{-1}(z))$,

$$(\psi(x_{i+1}(z)) - \psi(x_i)(z)) \cdot n(z) \geqq 0, \tag{27}$$

we have

$$\int_{S^1} DW(\dot{\varphi}) \cdot \dot{\psi} + f \cdot \psi \, \mathrm{d}x \leqq 0. \tag{28}$$

Moreover, if $DW(\dot{\varphi})$ belongs to $H^1(M; \mathbb{R}^2)$, there exists, for all $z \in \mathrm{Im}(\varphi)$, a family $(\lambda_{k+1/2}(z))_{k=-1,\dots,N(z)}$ of nonnegative reals such that

$$\lambda_{-1/2}(z) = \lambda_{N(z)+1/2}(z) = 0,$$

where $N(z) = \mathrm{Card}(\varphi^{-1}(z))$ such that for almost all $x \in S^1$ we have

$$-\frac{\mathrm{d}DW(\dot{\varphi})}{\mathrm{d}x}(x) = f(x) + (\lambda_+(x) - \lambda_-(x))\dot{\varphi}(x)^\perp,$$

where $\dot{\varphi}(x)^\perp$ is the image of $\dot{\varphi}(x)$ under a rotation of angle $\pi/2$ and λ_+ and λ_- are defined by

$$\lambda_+(x_k(z)) = \lambda_{k+n(z)\cdot n(x_k(z))/2} \quad \textit{and} \quad \lambda_-(x_k(z)) = \lambda_{k-n(z)\cdot n(x_k(z))/2} \tag{29}$$

for all $z \in \mathrm{Im}(\varphi)$, $n(x)$ being the normal to $\dot{\varphi}(x)$ such that $\dot{\varphi}(x) \wedge n(x) > 0$.

Remark 14. The first part of Proposition 10 remains true for any energy function $I \in C^1(W^{1,p}(M; \mathbb{R}^2); \mathbb{R})$. Let φ be a minimizer of I over $\mathcal{A}_\phi(M)$. Assume that φ is a $\mathcal{C}^1$ immersion and that the function I is derivable at φ, then for any test function $\psi \in \mathcal{C}^\infty(M; \mathbb{R}^2)$ verifying (27), we have

$$\langle DI(\varphi), \psi \rangle \leqq 0.$$

Moreover, if $DI(\varphi)$ belongs to $L^2(M; \mathbb{R}^2)$, the second part of the Proposition 10 still holds and there exists two functions λ_+ and $\lambda_- \in L^2(M; \mathbb{R}^2)$ of the form (29) such that

$$DI(\varphi) = (\lambda_+ - \lambda_-)\dot{\varphi}^\perp.$$

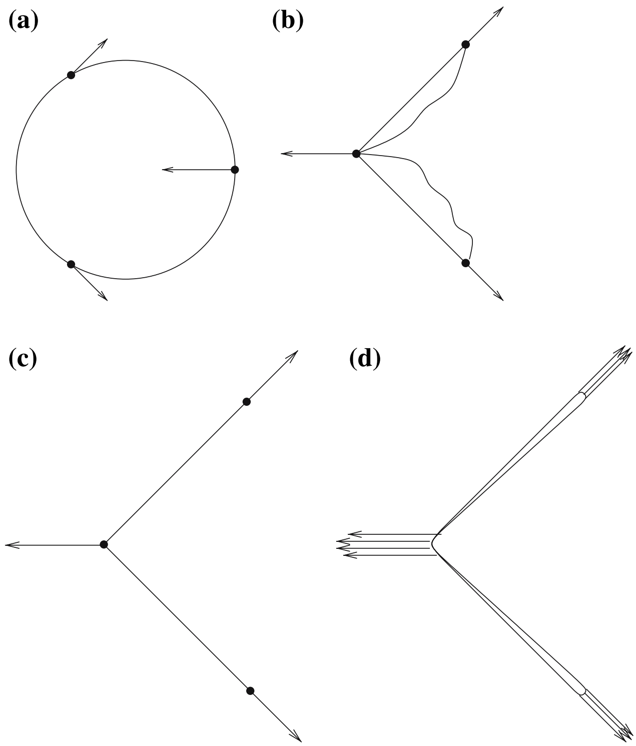

Fig. 5. Construction of a minimizer with self-contacts

Remark 15. Proposition 10 does not tell us anything about the regularity of the solutions φ of the minimization problem $(\mathcal{P}_\phi)$ or of the Lagrange multipliers $\lambda_\pm$. If the stored energy W is convex and if $W(F) = 0$ for any unitary vector $F \in \mathbb{R}^2$, the membrane does not sustain compression. Without applied load, any deformation φ such that $|\dot{\varphi}| \leqq 1$ is such that $I(\varphi) = 0$ and is a minimizer of I. Thus, we cannot expect to prove any regularity result in the general case. Regularity can be expected only if M is stretched far enough so that $|\dot{\varphi}| \geqq 1$. We conjecture that, if the minimizer is such that $|\dot{\varphi}| > 1$ almost everywhere and if the applied loads belongs to L^2, then φ belongs to H^2. One can question whether any self-contacts can occur in such a case and also whether Proposition 10 is not completely formal. Special loads can be chosen, in order to ensure that self-contacts occur. To this end, one can apply three punctual loads $f_0 = (-2, 0)$, $f_1 = (1, 1)$, and $f_2 = (1, -1)$ to the circle at $x_k = (\cos(2\pi k/3), \sin(2\pi k/3))$ $(k = 0, \dots, 2)$ (see Fig. 5a). If no self-contacts occur, then $\varphi_{|[x_0,x_1]}$, $\varphi_{|[x_1,x_2]}$, and $\varphi_{|[x_2,x_0]}$ are affine. It is easy to check that such a situation cannot occur without reversing the orientation of the circle (which is not allowed). Moreover, multiple self-contacts are not energy competitive. So, the minimizer of φ has one self-contact, for instance $\varphi(x_0)$ belongs to $\varphi(]x_1, x_2[)$ (see Fig. 5b). Even if it means increasing the applied load f_k, we can assume that $|\dot{\varphi}|$ will be greater than one almost everywhere (see Fig. 5c). Finally, it is reasonable to expect that a regularization of the applied loads will lead to a minimizer φ which will satisfy the assumptions of Proposition 10 (see Fig. 5d).

The proof of Proposition 10 is rather long and technical, albeit not difficult. The main idea is to give a geometrical definition of the ϕ-admissible set of deformations. Next, we prove that this definition is equivalent with the algebraic one (used earlier to prove the existence) for immersions. Then, using the geometric definition, we prove that the set of regular embeddings isotopic to j_{S^1} are dense in the set of ϕ-admissible immersions for the $\mathcal{C}^1$ topology. This allows us to build enough "variations" around the minimizer in the set of ϕ-admissible deformations $\mathcal{A}_\phi(j_{S^1})$ to obtain the Euler–Lagrange equations. Moreover, the set of ϕ-admissible immersions is nothing else but the set of the admissible immersions, that is

$$\mathcal{A}_\phi(j_M) \cap \mathrm{Imm}(S^1; \mathbb{R}^2) = \mathcal{A}(j_M) \cap \mathrm{Imm}(S^1; \mathbb{R}^2).$$

The complete proof can be found in [17].

Remark 16. Actually, we conjecture that $\mathcal{A}_\phi(j_M) = \mathcal{A}(j_M)$ in the case $n = 2$.

5.3. The case of shells or thin films

In the case of shells, that is, $n = 3$ and $\dim(M) = 2$, we cannot give a similar interpretation to the solutions of the minimization problem $(\mathcal{P}_\phi)$ as in the cases $n = m$ or $n = 2$ and $m = 1$. Indeed, even if a solution of the minimization problem $(\mathcal{P}_\phi)$ has no transversal self-intersection, it might have nontransversal self-intersections. In such a case, not enough test functions can be built to recover the Euler–Lagrange equations. Let us give an example of a deformation that belongs to the set of admissible deformations $\mathcal{A}_\phi(j_M)$, but which has self-intersections and does not belong to the admissible set $\mathcal{A}(j_M)$. Let $M = S^1 \times [0, 1]$. We define the reference deformation of M as the mapping

$$\begin{aligned} j_M : M &\to \mathbb{R}^3 \\ (\theta, h) &\mapsto (\cos(\theta), \sin(\theta), h). \end{aligned}$$

Let k be an integer and φ_k the deformation of M defined by

$$\varphi_k(\theta, h) = (\cos(k\theta), \sin(k\theta), h).$$

One can easily show that for every k the deformation φ_k belongs to $\mathcal{A}_\phi(j_M)$. Nevertheless, if $|k| > 1$, it does not belong to the $\mathcal{C}^0$-closure of the embeddings. This counterexample is very similar to the one that we have met (and solved) in the study of self-contacts of a body homeomorphic to S^1, moving in $\mathbb{R}^2$ (see Section 3.1).

6. One-dimensional structures moving in $\mathbb{R}^3$

As we pointed out in Section 1, our modeling is essentially relevant in the cases $2m \geqq n$. Thus, it fails to describe self-contacts occurring between rods or strings in $\mathbb{R}^3$. This is due to the fact that no transversal intersections can occur in such a case: if x_1 and x_2 are contact points, that is such that $\varphi(x_1) = \varphi(x_2)$, we obviously cannot expect $(\varphi(x_1), \varphi(x_2))$ to be a basis of $\mathbb{R}^3$. As our modeling is designed to prevent transversal self-intersections, the constraint $\varphi \in \mathcal{A}_\phi$ is almost empty in such cases. As showed hereafter, this is indeed the case for $M = S^1$.

Consider a reference injection $j_M : S^1 \to \mathbb{R}^3$ of nontrivial knot type. There exists a continuous isotopy F such that $F(0) = j_M$ and $F(1)$ is a standard injection of S^1 in $\mathbb{R}^3$. Indeed, the knot can vanish into a point, as illustrated by Fig. 6. Moreover, $\mathcal{A}_\phi$ contains any deformation $\mathcal{C}^0$-isotopic to j_M. Thus, the standard injection of S^1 in $\mathbb{R}^3$ always belongs to the set $\mathcal{A}_\phi$. Such a behavior is physically reasonable for strings (one can indeed tighten a knot in order to make it invisible) but not for rods, where flexure is taken into account. Consider now j_M to be the standard injection of S^1 into $\mathbb{R}^3$. Using the same isotopy (but with time reversed), any knot can emerge out of any point of the curve, as if it had already been present in the reference injection. Even for strings, this is not sensible. Thus for $M = S^1$, $\mathcal{A}_\phi$ is nothing but the set of all continuous mappings of S^1 into $\mathbb{R}^3$, and the criterion $\varphi \in \mathcal{A}_\phi$ is empty.

Nevertheless, if we consider several closed curves, that is where M is the union of circles, then $\mathcal{A}_\phi$ is not the set of all continuous maps from M into $\mathbb{R}^3$. Figure 7 represents two injections of a body M constituted by two circles M_1 and M_2. Let the reference j_M be the deformation drawn on the left hand side of the figure and φ be the deformation on the right. It is easy to check that

$$\int_{M_1 \times M_2} \phi(j_M) = 0$$

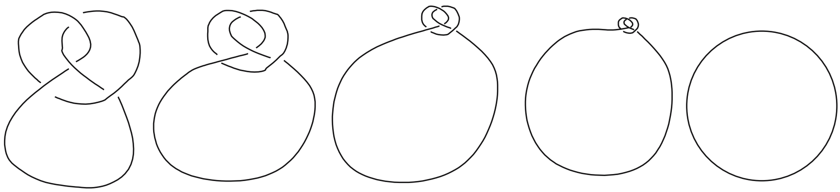

Fig. 6. Vanishing of a knot into a point

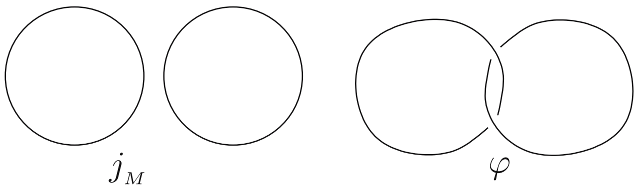

Fig. 7. Injection of two unknotted circles (*left*), and a non-ϕ-admissible deformation (*right*)

while

$$\int_{M_1 \times M_2} \phi(\varphi) = \pm 1.$$

It follows that φ does not belong to the ϕ-admissible set.

7. Remarks on the dynamic case

In the dynamic case, the deformation φ of the body M depends on the time. As we have done for the static case, we look for a necessary (or even sufficient) and explicit condition for a deformation $\varphi : M \times [0, T] \to \mathbb{R}^n$ to be physically admissible. Assume that $\varphi(t = 0)$ is the reference injection j_M, the simplest extension of the static case to the dynamic case will be to impose $\varphi(t)$ to be ϕ-admissible for each time t. Such a condition is clearly too weak. Indeed, the sequence of deformations represented on the Fig. 8 (with $M = [0, 1]$ and $n = 2$) is not physically admissible even if for each time t, $\varphi(t)$ is ϕ-admissible. At time $t = 0.5$, the moving part of the body crosses the fixed one, moving from the right to the left. We propose the following extension of the definition of the ϕ-admissible set to the dynamic case. For every deformation $\varphi : [0, T] \times M \to \mathbb{R}^n$, we define d_φ as the map

$$\begin{aligned} d_\varphi : [0, T] \times M \times M &\to \mathbb{R}^n \\ (t, x, y) &\mapsto \varphi(t, x) - \varphi(t, y). \end{aligned}$$

The admissible set of time-dependent deformations is then defined as the static case, once $M \times M$ is replaced by $[0, T] \times M \times M$. In other words, for any open set U of $[0, T] \times M \times M$ such that

$$\inf_{x \in U} |d_\varphi(x)| > 0,$$

we denote by $\phi_U(\varphi)$ the element of $H^{n-1}(U)$ defined as the pullback of $\phi_{\mathbb{R}^n_*}$ by $d_{\varphi,U}$, the restriction of d_φ to U,

$$\phi_U(\varphi) := d^*_{\varphi,U}(\phi_{\mathbb{R}^n_*}) \in H^{n-1}(U).$$

The set of time-dependent ϕ-admissible deformations is then defined by

$$\begin{aligned} \mathcal{A}_\phi(T, j_M) := \Big\{ \varphi \in \mathcal{C}^0([0, T] \times M; \mathbb{R}^n) \ : \ & \varphi(t = 0) = j_M \\ & \text{and } \phi_U(\varphi) = \phi_U(j_M) \text{ in } H^{n-1}(U), \\ & \text{for any open set } U \text{ such that } \inf_{x \in U} |d_\varphi(x)| > 0 \Big\}. \end{aligned} \tag{30}$$

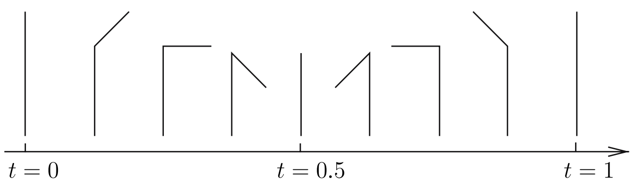

Fig. 8. Nonadmissible time-dependent deformation

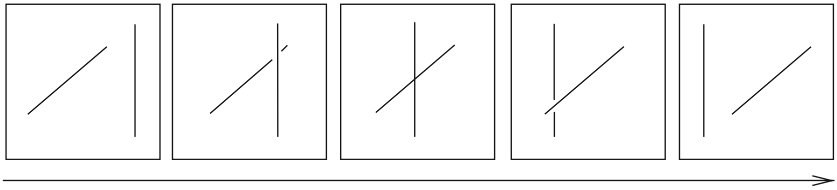

Fig. 9. Time-dependent deformation with transversal intersection

It is easy to extend some results obtained in the static case to the dynamic case. For instance, every deformation φ such that $\varphi(t)$ is an embedding at each time t and such that $\varphi(t = 0) = j_M$ belongs to $\mathcal{A}_\phi(T, j_M)$. The set $\mathcal{A}_\phi(T, j_M)$ is closed for the $\mathcal{C}^0$ topology. We say that a time-dependent deformation φ has a transversal self-intersection if there exists $(t, x, y) \in [0, T] \times M \times M$ such that

- $\varphi(t, x) = \varphi(t, y)$
- The mapping φ is $\mathcal{C}^1$ in neighborhoods of (t, x) and (t, y)
- $\left(\frac{\partial\varphi}{\partial t}(t, x) - \frac{\partial\varphi}{\partial t}(t, y)\right)(\mathbb{R}) + \frac{\partial\varphi}{\partial x}(t, x)(T_x M) + \frac{\partial\varphi}{\partial x}(t, y)(T_y M) = \mathbb{R}^n.$

We can prove that any deformation with a transversal self-intersection does not belong to $\mathcal{A}_\phi(T, j_M)$. Note that, as long as $2m + 1 \geqq n$, there exist time-dependent deformations with transversal self-intersections. In particular, Fig. 9 represents the evolution of two beams moving in $\mathbb{R}^3$ with a transversal intersection at time $t = 0.5$. Thus, the condition $\varphi \in \mathcal{A}_\phi(T, j_M)$ is not empty. Nevertheless, it is still too weak in certain particular cases. For elastic strings, when the energy only depends on $D\varphi$ and when no flexure term is taken into account, any knot can be born from any point of the string (see Section 6). However, flexure terms will forbid such deformations as they will require an infinite amount of energy.

8. Conclusion

The modeling presented in this article allows us to consider contacts between elastic bodies of dimension m moving in $\mathbb{R}^n$, as long as $2m \geqq n$ for the static case and $2m + 1 \geqq n$ for the dynamic case. Using this modeling, numerical simulations have been preformed, in the case $n = 2$ (see [18]). For higher dimensions ($n = 3$ and $\dim(M) = 2$ or 3), similar numerical methods could be applied. However, high-dimensional problems are encountered in dimension at least four. To solve these, one has to use some adaptive methods, which may be difficult to tackle. Finally, the ϕ-admissible set is too large in the case $n = 3$ and $m = 2$ and a more restrictive condition has to be found in order to forbid deformations with degenerate intersections.

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