Relaxation and discretization of control problems in the coefficients with a nonlinear functional in the gradient

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Model problem:

$\alpha, \beta, \mu > 0$, $\Omega \subset \mathbb{R}^N$ open, bounded, $f \in H^{-1}(\Omega)$

$F_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ Carathédory functions, $i = 1,2,$

$$|F_i(x,s,\xi)| \le C(1 + |s|^2 + |\xi|^2)$$

$$\text{(CP)} \quad \inf\left(\int_\omega F_1(x,u,\nabla u)dx + \int_{\Omega\setminus\omega} F_2(x,u,\nabla u)dx\right)$$

$$\begin{cases} -\mathrm{div}\Big(\big(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\big)\nabla u\Big) = f \text{ in } \Omega \\ u = 0 \; on \; \partial\,\Omega \end{cases}$$

$$\omega \subset \Omega \quad \text{measurable,} \quad |\omega| \le \mu$$

F. Murat. This problem has not a solution in general.

It is interesting to work with a relaxed formulation.

F. Murat, L.Tartar. If the functional to minimize is

$$\int_{\omega} G_1(x,u)dx + \int_{\Omega\setminus\omega} G_2(x,u)dx + \int_{\Omega} h(x,u)\big(\alpha\chi_{\omega} + \beta\chi_{\Omega\setminus\omega}\big)|\nabla u|^2 dx,$$

A relaxation of (CP) is given by

$$\text{(RCP)} \quad \inf \int_{\Omega} (\theta G_1(x,u) + (1-\theta)G_2(x,u) + h(x,u)M\nabla u\nabla u)dx$$

$$\begin{cases} -\mathrm{div}(M\nabla u) = f \text{ in } \Omega \\ u = 0 \; on \; \partial\,\Omega \end{cases} \qquad M \in \mathcal{K}(\theta), \qquad \int_{\Omega} \theta \le \mu$$

$\mathcal{K}(\theta)$ set of matrices constructed via homogenization using α with proportion θ and β with proportion $1-\theta$.

L. Tartar.   K. Lurie, A. Cherkaev characterize $\mathcal{K}(p),\ 0 \le p \le 1$

Define $\lambda(p) = \left(\frac{p}{\alpha} + \frac{1-p}{\beta}\right)^{-1}, \qquad \Lambda(p) = p\alpha + (1-p)\beta,$

If $N \ge 2, \mathcal{K}(p)$ is the set of symmetric matrices with eigenvalues satisfying

$$\lambda(p) \le \lambda_1 \le \cdots \le \lambda_N \le \Lambda(p)$$

$$\sum_{i=1}^{N} \frac{1}{\lambda_i - \alpha} \le \frac{N-1}{\Lambda(p) - \alpha} + \frac{1}{\lambda(p) - \alpha}$$

$$\sum_{i=1}^{N} \frac{1}{\beta - \lambda_i} \le \frac{N-1}{\beta - \Lambda(p)} + \frac{1}{\beta - \lambda(p)}$$

For our purpose it is enough to know $\mathcal{K}(p)\xi, \xi \in \mathbb{R}^N$

$$\mathcal{K}(p)\xi = \begin{cases} B\left(\frac{\lambda(p)+\Lambda(p)}{2}\xi, \frac{\Lambda(p)-\lambda(p)}{2}|\xi|\right) & \text{if } N \ge 2 \\ \lambda(p)\xi & \text{if } N = 1 \end{cases}$$

In general (JCD, J. Couce-Calvo, J.D. Martín-Gómez) the relaxed control problem has the form

$$\inf \int_{\Omega} H(x, u, \nabla u, M\nabla u, \theta) dx$$

$$\begin{cases} -\mathrm{div}(M\nabla u) = f \text{ in } \Omega \\ u = 0 \; on \; \partial\, \Omega \end{cases} \qquad M \in \mathcal{K}(\theta), \qquad \int_{\Omega} \theta dx \leq \mu,$$

or equivalently $$\inf \int_{\Omega} H(x, u, \nabla u, \sigma, \theta) dx$$

$$-\mathrm{div}\sigma = f \text{ in } \Omega, \; u \in H_0^1(\Omega), \; \sigma \in K(\theta)\nabla u, \; \int_{\Omega} \theta dx \; \leq \mu.$$

Related results: Allaire, Bellido, Grabovski, Gutiérrez, Maestre, Munch, Pedregal, Tartar,…

Remark: If (u_n, ω_n) are solution of

$$-\text{div}\Big(\big(\alpha\chi_{\omega_n} + \beta\chi_{\Omega\backslash\omega_n}\big)\nabla u_n\Big) = f \text{ in } \Omega,\ u_n = 0 \ on\ \partial\,\Omega,$$

then for a subsequence we have

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega), \quad \theta_n = \chi_{\omega_n} \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\Omega)$$

$$\sigma_n = \big(\alpha\chi_{\omega_n} + \beta\chi_{\Omega\backslash\omega_n}\big)\nabla u_n \rightharpoonup \sigma \ \ in\ \ L^2(\Omega)^N$$

$$\text{div}\sigma_n \to \text{div}\sigma \ \ in\ \ H^{-1}(\Omega).$$

We write $(u_n, \sigma_n, \theta_n) \overset{\tau}{\to} (u, \sigma, \theta)$.

$$\bar{\mathcal{F}}(u,\sigma,\theta) = \int_{\Omega} H(x,u,\nabla u,\sigma,\theta)dx$$

is the lower semicontinous envelope for the τ-convergence of $\mathcal{F}$ given by

$$\mathcal{F}(u,\sigma,\theta) = \int_{\omega} F_1(x,u,\nabla u)dx + \int_{\Omega\backslash\omega} F_2(x,u,\nabla u)dx$$

if $\theta = \chi_\omega$, $\sigma = \left(\alpha\chi_\omega + \beta\chi_{\Omega\backslash\omega}\right)\nabla u$,

$$\mathcal{F}(u,\sigma,\theta) = +\infty \text{ otherwise.}$$

If $N = 1$,

$$H(x,s,\xi,\eta,p) = \begin{cases} pF_1\left(x,s,\frac{\eta}{\alpha}\right) + (1-p)F_2\left(x,s,\frac{\eta}{\alpha}\right) & if\ \eta = \lambda(p)\xi \\ +\infty & \text{otherwise.} \end{cases}$$

If $N > 1$, we have

- $\mathrm{Dom}(H) = \{(x,s,\xi,\eta,p) : \eta \in \mathcal{K}(p)\xi\}$.
- $|H(x,s,\xi,\eta,p)| \leq C(1 + |s|^2 + |\xi|^2 + |\eta|^2)$
- H satisfies the following convexity property

$$H(x,s,\gamma\xi_1 + (1-\gamma)\xi_2, \gamma\eta_1 + (1-\gamma)\eta_2, \gamma p_1 + (1-\gamma)p_2)$$
$$\leq \gamma\, H(x,s,\xi_1,\eta_1,p_1) + (1-\gamma)H(x,\xi_2,\eta_2,p_2)$$
$$\text{if}\ \gamma \in [0,1],\ (\xi_2 - \xi_1)\cdot(\eta_2 - \eta_1) = 0.$$

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$$H(x,s,\xi,\eta,p) = pF_1\left(x,s,\frac{\beta\xi - \eta}{(\beta-\alpha)p}\right) + (1-p)F_1\left(x,s,\frac{\eta - \alpha\xi}{(\beta-\alpha)(1-p)}\right)$$
$$\text{if}\ \eta \in \partial\mathcal{K}(p)\xi$$

• If $F_i(x,s,\xi),\ i=1,2,$ are convex in ξ, we have

$$H(x,s,\xi,\eta,p)\geq pF_1\left(x,s,\frac{\beta\xi-\eta}{(\beta-\alpha)p}\right)+(1-p)F_1\left(x,s,\frac{\eta-\alpha\xi}{(\beta-\alpha)(1-p)}\right).$$

Cases where H is known

$$F_2(x,s,\xi)=r(x,s)|\xi|^2$$

$$Q(x,s,\xi)=F_1(x,s,\xi)-\frac{\alpha}{\beta}r(x,s)|\xi|^2 \text{ convex in } \xi .$$

$$\Longrightarrow \quad H(x,s,\xi,\eta,p)=\frac{h(x,s)}{\beta}\xi\cdot\eta+pQ\left(x,s,\frac{\eta-\beta\xi}{p(\beta-\alpha)}\right)$$

It contains some cases proved by Bellido, Pedregal, Grabovsky, ,…

$\forall(x,s,\xi),\ \exists\zeta\in\mathbb{R}^N$ such that the applications $t\to F_i(x,s,\xi+t\zeta)$ are linear.

$$\Rightarrow H(x,s,\xi,\eta,p)=pF_1\left(x,s,\frac{\beta\xi-\eta}{(\beta-\alpha)p}\right)+(1-p)F_1\left(x,s,\frac{\eta-\alpha\xi}{(\beta-\alpha)(1-p)}\right)$$

Numerical Aproximation

JCD, C. Couce-Calvo, M. Luna-Laynez, J.D. Martín-Gómez.

A discretization using an upper approximation of *H*

Take $\overline{H}(x,s,\xi,\eta,p)$, with

$$\overline{H}(x,s,\xi,\eta,p) \geq H(x,s,\xi,\eta,p)$$

$$\overline{H}(x,s,\xi,\alpha\xi,1) = F_1(x,s,\xi), \qquad \overline{H}(x,s,\xi,\beta\xi,0) = F_2(x,s,\xi).$$

For *h>0*, we consider a triangulation $\mathcal{T}_h = \{T_{i,h}\}_{i=1}^{n_h}$ of Ω

$$\Omega = \bigcup_{i=1}^{n_h} T_{i,h},\ T_{i,h} \text{ measurable},\ |T_{i,h}| > 0,\ \operatorname{diam}(T_{i,h}) \leq h$$

$$|T_{i,h} \cap T_{j,h}| = 0, \text{ if } i \neq j,$$

and a sequence of closed subspaces $V_h \subset H_0^1(\Omega)$

Discretized problem

$$\min \int_{\Omega} \overline{H}(x, u_h, \nabla u_h, M_h \nabla u_h, \theta_h) dx$$

$$0 \leq \theta_h \leq 1, \int_{\Omega} \theta_h dx \leq \mu^i, \quad M_h \in \mathcal{K}(\theta_h) \text{ a.e. in } \Omega$$

$$u_h \in V_h, \int_{\Omega} M_h \nabla u_h \cdot \nabla v_h dx = \int_{\Omega} f v_h \, dx, \; \forall v_h \in V_h$$

M_h, θ_h constants in the elements $T_{i,h}$

Assumptions on V_h

i)

$$\lim_{h\to 0} \min_{v_h\in V_h} \|v - v_h\|_{H_0^1(\Omega)} = 0, \qquad \forall v \in H_0^1(\Omega),$$

ii)

$$\lim_{h\to 0} \min_{v_h\in V_h} \|w_h\varphi - v_h\|_{H_0^1(\Omega)} = 0,$$

$$\forall w_h \in V_h \text{ bounded in } H_0^1(\Omega), \forall \varphi \in C_c^\infty(\Omega)$$

iii)

$$\varliminf_{h\to 0} \int_\Omega H(x, u_h, \nabla u_h, \sigma_h, \theta_h)dx \geq \int_\Omega H(x, u, \nabla u, \sigma, \theta)dx$$

$$\forall u_h \rightharpoonup u \text{ in } H_0^1(\Omega),\ u_h \in V_h, \qquad \forall \sigma_h \rightharpoonup \sigma \text{ in } L^2(\Omega)^N$$

$$\forall \theta_h \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\Omega), \qquad 0 \leq \theta_h \leq 1 \text{ a.e. in } \Omega$$

$$\lim_{h\to 0} \max_{v_h\in V_h} \frac{1}{\|v_h\|_{H_0^1(\Omega)}} \int_\Omega (\sigma_h - \sigma)\cdot \nabla v_h dx = 0.$$

Properties i), ii), iii) are satisfied for $V_h = H_0^1(\Omega)$.

If V_h is a usual space of finite elements, it satisfies i), ii).

In the examples where we know H, every sequence V_h satisfies iii).

Theorem: The discrete problem has a solution (u_h, M_h, θ_h)

Up to a subsequence $$\begin{cases} u_h \rightharpoonup u \text{ in } H_0^1(\Omega) \\ M_h \nabla u_h \rightharpoonup M\nabla u \text{ in } L^2(\Omega)^N \\ \theta_h \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\Omega) \end{cases}$$

(u, M, θ) solution of $$\begin{cases} \inf \int_\Omega H(x, u, \nabla u, M\nabla u, \theta)\,dx \\ -\operatorname{div}(M\nabla u) = f \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega \\ 0 \le \theta \le 1,\ \ M \in K(\theta) \text{ a.e.},\ \ \int_\Omega \theta\,dx \le \mu \end{cases}$$

$$\lim_{h\to 0} \int_\Omega \overline{H}(x, u_h, \nabla u_h, M_h \nabla u_h, \theta_h)\,dx = \int_\Omega H(x, u, \nabla u, M\nabla u, \theta)\,dx.$$

Example 1:

$$\overline{H}(x,s,\xi,A\xi,1)=F_1(x,s,\xi), \qquad \overline{H}(x,s,\xi,B\xi,0)=F_2(x,s,\xi)$$

$$\overline{H}(x,s,\xi,\eta,p)=+\infty \text{ otherwise.}$$

In this case, we are solving a discrete version of the original (unrelaxed) problem, i.e.

$$\inf\left(\int_{\omega} F_1(x,u_h,\nabla u_h)dx+\int_{\Omega\setminus\omega} F_2(x,u_h,\nabla u_h)dx\right)$$

$$\begin{cases} u_h \in V_h \\ \int_{\Omega}(\alpha\chi_{\omega_h}+\beta\chi_{\Omega\setminus\omega_h})\nabla u_h\cdot\nabla v_h dx=\int_{\Omega} f v_h\, dx,\ \forall v_h\in V_h \\ \omega_h \text{ a union of elements of } \mathcal{T}_h, \qquad |\omega_h|\leq\mu. \end{cases}$$

Example 2: $(N \geq 2)$ Since we know the values of $\overline{H}$ in the boundary of its domain, we can take

$$\overline{H}(x,s,\xi,\eta,p) \\ = pF_1\left(x,s,\frac{\beta\xi-\eta}{p(\beta-\alpha)}\right)+(1-p)F_2\left(x,s,\frac{\eta-\alpha\xi}{(1-p)(\beta-\alpha)}\right)$$

if $\eta \in \partial\mathcal{K}(p)\xi$

$\overline{H}(x,s,\xi,\eta,p) = +\infty$ elsewhere.

Clearly, when we know the function H we can just take $\overline{H} = H$.

A lower approximation of H

We consider $\underline{H}(x,s,\xi,\eta,p)$ with

$$\underline{H}(x,s,\xi,\eta,p) \leq H(x,s,\xi,\eta,p)$$

For $h>0$, consider $\mathcal{T}_h = \{T_{i,h}\}_{i=1}^{n_h}$ as above and closed subspaces $V_h \subset H_0^1(\Omega)$ satisfying properties i) and ii) as above and

$$\varliminf_{h\to 0} \int_\Omega \underline{H}(x,u_h,\nabla u_h,\sigma_h,\theta_h)dx \geq \int_\Omega \underline{H}(x,u,\nabla u,\sigma,\theta)dx$$

$$\forall u_h \rightharpoonup u \ \text{ in } H_0^1(\Omega),\ u_h \in V_h, \qquad \forall \sigma_h \rightharpoonup \sigma \ \text{ in } L^2(\Omega)^N$$

$$\forall \theta_h \overset{*}{\rightharpoonup} \theta \ \text{ in } L^\infty(\Omega), \qquad 0 \leq \theta_h \leq 1 \text{ a.e. in } \Omega$$

$$\lim_{h\to 0} \max_{v_h \in V_h} \frac{1}{\|v_h\|_{H_0^1(\Omega)}} \int_\Omega (\sigma_h - \sigma)\cdot \nabla v_h \, dx = 0.$$

Discretized problem $\mathfrak{C} = \text{co}\{(M,p)\colon\ M \in \mathcal{K}(p)\}$

$$\min \int_\Omega \underline{H}(x, u_h, \nabla u_h, M_h \nabla u_h, \theta_h)dx$$

$$0 \le \theta_h \le 1, \qquad \int_\Omega \theta_h dx \le \mu^i, \quad (\theta_h, M_h) \in \mathfrak{C} \text{ a.e. in } \Omega$$

$$u_h \in V^h, \ \int_\Omega M_h \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h \, dx, \quad \forall\, v_h \in V^h$$

M_h, θ_h constants in the elements $T_{i,h}$

Theorem: The discrete problem has a solution $\left(u_h, M_h, \theta_h\right)$

Up to a subsequence $\begin{cases} u_h \rightharpoonup u \ \ in\ H_0^1(\Omega) \\ M_h \nabla u_h \rightharpoonup M\nabla u \ \ in\ L^2(\Omega)^N \\ \theta_h \overset{*}{\rightharpoonup} \theta \ \ in\ L^\infty(\Omega) \end{cases}$

$$(u, M, \theta) \text{ solution of } \begin{cases} \inf \int_\Omega \underline{\underline{H}}\left(x, u, \nabla u, M\nabla u, \theta\right) dx \\ -\operatorname{div}\left(M\nabla u\right) = f \ \text{ in } \Omega, \ u = 0 \ \text{ on } \partial\Omega \\ 0 \le \theta \le 1, \ \ M \in K(\theta) \ \text{a.e.}, \ \ \int_\Omega \theta\, dx \le \mu \end{cases}$$

$$\lim_{h\to 0} \int_\Omega \underline{\underline{H}}\left(x, u_h, \nabla u_h, M_h \nabla u_h, \theta_h\right) dx = \int_\Omega \underline{\underline{H}}\left(x, u, \nabla u, M\nabla u, \theta\right) dx$$

Remark: Looking for the optimality conditions, we hope that the solution $(\hat{u}, \widehat{M}, \hat{\theta})$ satisfies

$$\widehat{M}\nabla\hat{u} \in \partial\mathcal{K}(\hat{\theta})\nabla\hat{u} \text{ a.e. in } \Omega.$$

Thus, we need to take $\underline{H}$ satisfying

$$\underline{H}(x,s,\xi,\eta,p) = H(x,s,\xi,\eta,p) \text{ if } \eta \in \partial\mathcal{K}(p)\xi.$$

Example: If $F_1(x,s,\xi), F_2(x,s,\xi)$ are convex in ξ take

$$\underline{H}\,(x,s,\xi,\eta,p) = H(x,s,\xi,\eta,p)$$

$$= pF\left(x,s,\frac{\beta\xi-\eta}{p(\beta-\alpha)}\right) + (1-p)F\left(x,s,\frac{\eta-\alpha\xi}{(1-p)(\beta-\alpha)}\right)$$

We have shown that we can solve numerically the control problem discretizing the unrelaxed or the relaxed problem.

What is better?

J. CD, C. Castro, M. Luna-Laynez, E. Zuazua consider the case $N = 1$.

Control problem (CP) $F_1, F_2 \in W^{1,\infty}$

$$\inf\left(\int_{\omega} F_1\left(x,u,\frac{du}{dx}\right)dx + \int_{(0,1)\setminus\omega} F_2\left(x,u,\frac{du}{dx}\right)dx\right)$$

$$\begin{cases} -\dfrac{d}{dx}\left(\left(\alpha\chi_{\omega} + \beta\chi_{(0,1)\setminus\omega}\right)\dfrac{du}{dx}\right) = f \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \qquad |\omega| \le \mu$$

Relaxed formulation (RCP)

$$\min \int_0^1 \left(\theta F_1\left(x,u,\frac{\lambda(\theta)}{\alpha}\frac{du}{dx}\right) + (1-\theta)F_2\left(x,u,\frac{\lambda(\theta)}{\beta}\frac{du}{dx}\right)\right)dx$$

$$\begin{cases} -\dfrac{d}{dx}\left(\lambda(\theta)\dfrac{du}{dx}\right) = f \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \qquad \int_0^1 \theta\, dx \le \mu.$$

Discretization

We take a partition $\mathcal{P}_r$ and refinement $\mathcal{Q}_h$ of respective diameters r and h

$$V_h = \{u \in C_0^0([0,1]) : u \text{ is affine in each interval of } \mathcal{Q}^h\}$$

We consider the discretized problems

$$\min\left(\int_\omega F_1\left(x,u,\frac{du}{dx}\right)dx + \int_{(0,1)\setminus\omega} F_1\left(x,u,\frac{du}{dx}\right)dx\right)$$

$$(DCP)\quad \begin{cases} u \in V_h \\ \displaystyle\int_0^1 \left(\alpha\chi_\omega + \beta\chi_{(0,1)\setminus\omega}\right)\frac{du}{dx}\frac{dv}{dx}dx = \int_0^1 f\, v dx,\ \forall v \in V_h \\ \omega \text{ is a union of intervals of } \mathcal{P}_r, \qquad |\omega| \le \mu. \end{cases}$$

$$\min \int_\omega \left(\theta F_1\left(x, u, \frac{\lambda(\theta)}{\alpha}\frac{du}{dx}\right) + (1-\theta)F_2\left(x, u, \frac{\lambda(\theta)}{\beta}\frac{du}{dx}\right)\right) dx$$

$$(DRP) \qquad \begin{cases} u \in V_h \\ \int_0^1 \lambda(\theta)\frac{du}{dx}\frac{dv}{dx}dx = \int_0^1 f\, v dx,\ \forall v \in V_h \end{cases}$$

$$\theta \text{ is constant in the intervals of } \mathcal{P}_r, \int_0^1 \theta dx \le \mu$$

Theorem. Taking $r = h$ and $f \in W^{1,l}(0,1)$ we have

$$|\min(RCP) - \min(DCP)| \le ch^{\frac{l+1}{l+2}}.$$

Taking $f \in L^\infty(0,1), r = \sqrt{h}$ and assuming that there exists an optimal control in $BV(0,1)$, we have

$$|\min(RCP) - \min(DRP)| \le ch.$$