

Communication Networks, Algorithms & Probability Theory

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Definition: Stopping Time

(Booklet page 99)

T random variable in $\mathbb{N} \cup \{+\infty\}$ is a
stopping time if,

$$\{T = n\} \in \mathcal{F}_n, \quad n \geq 1,$$

\mathcal{F}_n : past at time n ,
depends on X_0, X_1, \dots, X_n .

Translation:

An observer who looks at successive states
of (M_n) , knows when to stop at time T .

Stopping times: Examples

- ▶ $p \geq 0, T \equiv p$;
- ▶ Hitting time of a subset:
 $F \subset \mathcal{S}$,

$$T_F = \inf\{n : M_n = F\}$$

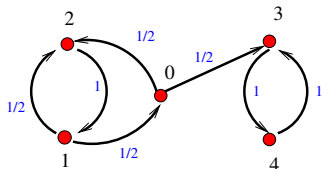
if $x \in \mathcal{S}$,

$$T_x = \inf\{n : M_n = x\}$$

- ▶ If \mathcal{S} finite: Cover time

$$\tau = \inf\{n : \{M_0, M_1, \dots, M_n\} = \mathcal{S}\}$$

Counter-example



$$M_0 = 0, \quad L = \sup\{n : M_n = 0\}$$

is not a stopping time.

The past before a stopping time T

$$\mathcal{F}_T = \bigvee_{t=0}^{+\infty} \bigvee_{A_0, \dots, A_t \subset \mathcal{S}}$$

$$\{M_0 \in A_0, M_1 \in A_1, \dots, M_t \in A_t, T \geq t\}$$

\mathcal{F}_T : all events before time T

Markov Property

A sequence (M_n) satisfies the **Markov property**: If for any m and $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{P}(M_{m+n} = y \mid M_n = x_n, \dots, M_0 = x_0) \\ = \mathbb{P}(M_{m+n} = y \mid M_n = x_n). \end{aligned}$$

Strong Markov

A Markov chain (M_n) (Booklet page 100)

satisfies the **strong Markov property**:

If for any **stopping time T** :

$$\begin{aligned} \mathbb{P}(M_{T+n} = y \mid T < +\infty, M_T = x_T, \dots, M_0 = x_0) \\ = \mathbb{P}(M_{T+n} = y \mid T < +\infty, M_T = x_T). \end{aligned}$$

Strong Markov \equiv **Markov** at non deterministic instants: **stopping times**.

Strong Markov (II)

Translation:

$$\begin{aligned} P(M_{T+n} \in \cdot \mid \mathcal{F}_T, T < +\infty) \\ = \mathbb{P}(M_{T+n} \in \cdot \mid M_T, T < +\infty) \end{aligned}$$

Example

If $x \in \mathcal{S}$, $T_x = \inf\{n : M_n = x\}$

Assuming that $T_x < +\infty$ \mathbb{P} -a.s.

Strong Markov:

$$\begin{aligned} \mathbb{P}(M_{T_x+n} \in A \mid M_{T_x}, \dots, M_0) \\ = \mathbb{P}(M_{T_x+n} \in A \mid M_{T_x} = x) \\ = \mathbb{P}(M_n \in A \mid M_0 = x). \end{aligned}$$

The sequence (M_{n+T_x}) is independent of the variables M_{T_x-1}, \dots, M_0 .

Decomposition into independent cycles.

Poisson and exponential distributions

Poisson distribution

Definition

(Booklet page 146)

Definition. A random variable ν has a Poisson distribution with parameter $\lambda > 0$, if for any $n \in \mathbb{N}$,

$$\mathbb{P}(\nu = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

Average:

$$\mathbb{E}(\nu) = \sum_{n \in \mathbb{N}} n \mathbb{P}(\nu = n) = \sum_{n \in \mathbb{N}} n \frac{\lambda^n}{n!} e^{-\lambda} = \lambda$$

Variance:

$$\mathbb{E}(\nu^2) - \mathbb{E}(\nu)^2 = \sum_{n \in \mathbb{N}} n^2 \mathbb{P}(\nu = n) - \lambda^2 = \lambda$$

Poisson distribution

Generating function: $|u| \leq 1$,

$$\begin{aligned} \mathbb{E}(u^\nu) &= \sum_{n \in \mathbb{N}} \mathbb{P}(\nu = n) u^n \\ &= \sum_{n \in \mathbb{N}} \frac{\lambda^n}{n!} e^{-\lambda} u^n = e^{-\lambda(1-u)}. \end{aligned}$$

Infinite radius of convergence.

Model to represent network accesses

Framework:

- ▶ N stations.
- ▶ Stations are independent.

p_N : Proba. of transmission per time unit for a station.

Assumption: N large and p_N small.

Model to represent network accesses

ν : Total nb. of transmissions per time unit.

$$\nu = \sum_{i=1}^N 1_{E_i}$$

E_i : station i transmits at time 0.

Assumption $p_N = \mathbb{P}(E_1) \sim \lambda/N$

$$\begin{aligned} \mathbb{P}(\nu = k) &= C_N^k \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k} \\ &= \frac{\lambda^k}{k!} \times \frac{N!}{(N-k)! N^k} \times e^{(N-k) \log(1-\lambda/N)} \sim \frac{\lambda^k}{k!} e^{-\lambda} \end{aligned}$$

ν has an asymptotic Poisson distribution with parameter λ .

General Properties

Addition (Booklet page 147)

If for $i = 1, 2$,

- ▶ X_i : Poisson with parameter λ_i ;
- ▶ X_1 and X_2 are independent;

then $X_1 + X_2$ is Poisson with parameter $\lambda_1 + \lambda_2$.

$$\begin{aligned}\mathbb{E}(u^{X_1+X_2}) &= \mathbb{E}(u^{X_1}) \mathbb{E}(u^{X_2}) \\ &= \exp(-(\lambda_1 + \lambda_2)(1 - u)),\end{aligned}$$

General Properties (II)

Thinning: If

- ▶ X Poisson with parameter λ ;
 - ▶ (B_i) Bernoulli i.i.d. parameter p
 $B_i \in \{0, 1\}$ and $\mathbb{P}(B_1 = 1) = p$,
- then $S = B_1 + B_2 + \dots + B_X$ has a Poisson distribution with parameter λp .

General Properties (III)

Thinning (II): If

- ▶ X Poisson with parameter λ ;
- ▶ (B_i) Bernoulli i.i.d. parameter p
 $B_i \in \{0, 1\}$ and $\mathbb{P}(B_1 = 1) = p$,

then

$B_1 + B_2 + \dots + B_X$ et $X - (B_1 + B_2 + \dots + B_X)$

are independent, and Poisson with parameter λp and $\lambda(1 - p)$.

Law of small numbers

Independent case (Booklet page 147)

If B_i^n , $1 \leq i \leq N_n$ independent Bernoulli r.v.

- $\lim_{n \rightarrow +\infty} \sup_i \mathbb{P}(B_i^n = 1) = 0$;
- $\lim_{n \rightarrow +\infty} \sum_i \mathbb{P}(B_i^n = 1) = \lambda < +\infty$,

then, as n goes to infinity,

$$S_n = B_1^n + B_2^n + \dots + B_{N_n}^n \xrightarrow{\text{Loi}} \text{Pois}(\lambda)$$

The exponential distribution

Advantages of Poisson distribution

Simplify complex combinatorial expressions.

Definition

(Booklet page 149)

For $\lambda > 0$, X is an exponential random variable with parameter λ if

$$\mathbb{E}(f(X)) = \int_0^{+\infty} f(x)\lambda e^{-\lambda x} dx,$$

$$\mathbb{P}(X \geq x) = \exp(-\lambda x), \quad x \geq 0.$$

Average:

$$\mathbb{E}(X) = \int_0^{+\infty} x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

Variance:

$$\mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{\lambda^2}$$

Laplace Transform:

$$\mathbb{E}(e^{-\xi X}) = \frac{\lambda}{\lambda + \xi}, \quad \operatorname{Re}(\xi) \geq 0.$$

Model to represent network accesses (II)

Framework:

- ▶ Small probability of transmission per time unit: $p_N = \lambda/N$.

τ_N first time when a given station transmits.

$$\mathbb{P}(\tau_N \geq k) = (1 - \lambda/N)^k,$$

$$\mathbb{P}\left(\frac{\tau_N}{N} \geq x\right) = (1 - \lambda/N)^{\lceil Nx \rceil} \sim e^{-\lambda x}$$

$$\frac{\tau_N}{N} \rightarrow X, \text{ where } X \text{ exp. param. } \lambda.$$

A crucial Property

If X exp r.v. with parameter λ , then

$$\mathbb{P}(X - x \geq y \mid X \geq x) = e^{-\lambda y} = \mathbb{P}(X \geq y).$$

Proof:

$$\begin{aligned}\mathbb{P}(X - x \geq y \mid X \geq x) &= \frac{\mathbb{P}(X \geq x + y)}{\mathbb{P}(X \geq x)} \\ &= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} = e^{-\lambda y}.\end{aligned}$$

Elementary property but **fundamental**.
Only continuous dist. with this property.

Exponential clocks

(Booklet page 150)

(E_1, \dots, E_N) , ind. exponential r.v. param. λ_i .

E_i : first time the i th clock rings.

The variable

$$\min(E_1, E_2, \dots, E_N)$$

has an exp. distribution with parameter

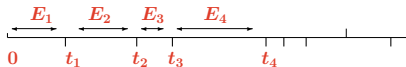
$$\lambda_1 + \dots + \lambda_N.$$

The first time one of the N clocks rings is equivalent to an exponential clock.

Poisson process

Definition

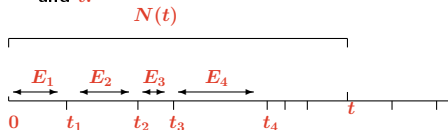
- ▶ (E_i) i.i.d. exponential r.v. param. λ ;
- ▶ $t_n = E_1 + \dots + E_n$;



$\{t_n, n \geq 1\}$: Poisson Process with rate/parameter λ .

Definition

- Equivalently: non-decreasing fn $t \rightarrow N(t)$,
- ▶ $N(t)$: number of points t_n between 0 and t .



$N(0) = 0$ and $N(t) - N(t-) \in \{0, 1\}$;
 $(N(t))$ is the counting measure of the Poisson process.

Example 1: Model for network accesses

Framework: (Booklet page 162)

- ▶ Proba. of transmission per time unit for a station: $p_N = \lambda/N$;

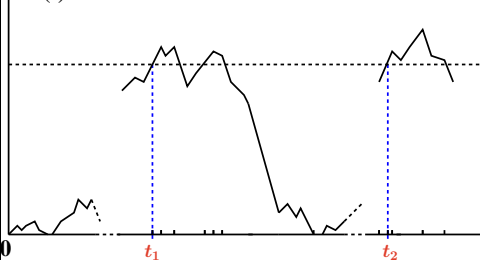
t_k^N : instant of transmission of the k th message of a station.

The sequence $(t_k^N/N, k \geq 0)$ converges in distribution to a Poisson process (t_k) with rate λ .

$$(t_k^N, k \geq 0) \stackrel{\text{dist}}{\sim} (Nt_k).$$

Example 2: Overflow phenomena

$X(t)$



$(X(t))$: nb of requests in a buffer.

Example 3: Throwing points at random on the real half-line

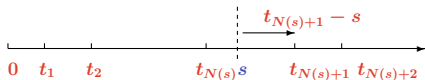
$[\lambda N]$ points are thrown uniformly on $[0, N]$.
 t_i^N : location of the i th point.

The sequence $(t_i^N) \xrightarrow{\text{dist.}}$ a Poisson process with rate λ on \mathbb{R}_+ when $N \rightarrow +\infty$.

Poisson Process:
"Uniform" points on \mathbb{R}_+

Properties of Poisson processes

Translation of a Poisson process



The Poisson process after time s :

La variable $t_{N(s)+1} - s$ is an exp. r.v. with param. λ

$t_{N(s)+1} - s$ is independent of $t_1, \dots, t_{N(s)}$

Translation of a Poisson process

Points seen from s : $(t_{N(s)+k} - s, k \geq 0)$

- ▶ a Poisson process with rate λ
 - ▶ are independent of $t_1, \dots, t_{N(s)}$.
- \Rightarrow Poisson proc. invariant by translation

Consequences

- ▶ nb of points in $[0, s]$ independent of nb of points in $[s, s + t]$
- ▶ $N(t + s) - N(s)$ same dist. as $N(t)$.

(Booklet page 153)

Independent increments

If $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$, the variables

$$N(a_1), N(a_2) - N(a_1), \dots, N(a_n) - N(a_{n-1})$$

are independent.

Consequences

The Poisson process is the **unique** integer valued process

- ▶ non-decreasing process with jumps of size $+1$;
- ▶ invariant by translation;
- ▶ independent increments.

Stochastic Modeling

Two fundamental processes:

- ▶ **Brownian motion.**
used in stochastic models whose evolution is continuous.

$$dB(t)$$

- ▶ **Poisson process.**
used in stochastic models whose evolution is discontinuous (jumps).

$$\Delta N(t) = N(t) - N(t-).$$

Operations on Poisson processes

Superposition

$M = (s_n)$, $N = (t_n)$ Independent Poisson proc. with rates μ and λ , then

$$P = M + N = (\{s_n\} \cup \{t_n\})$$

is Poisson with rate $\lambda + \mu$.

Operations on Poisson processes

Thinning

If (B_i) Bernoulli i.i.d. parameter p , indep. of $N = \{t_n\}$, the set of points

$$M_1 = \{t_n : B_n = 1\}$$

is a Poisson process with rate λp .

Operations on Poisson processes

Thinning (II)

Let (B_i) Bernoulli i.i.d. parameter p , indep. of $N = \{t_n\}$ with rate λ , then

$$M_1 = \{t_n : B_n = 1\} \text{ et } M_2 = \{t_n : B_n = 0\}$$

are independent Poisson processes, with rates λp and $\lambda(1 - p)$.

Some Calculations

Sums of exponentials

The variable $t_n = E_1 + \dots + E_n$ has the density

$$g_n(x) = \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x}.$$

The nb. $N(t)$ of points in $[0, t]$ is Poisson with parameter λt

$$\begin{aligned} \mathbb{P}(N(t) = n) &= \mathbb{P}(t_n \leq t < t_{n+1}) \\ &= \mathbb{P}(t_n \leq t < t_n + E_{n+1}) \\ &= \int_{\mathbb{R}_+^2} \mathbf{1}_{\{x \leq t \leq x+y\}} g_n(x) \lambda e^{-\lambda y} dx dy \\ &= \int_{\mathbb{R}_+^2} \mathbf{1}_{\{x \leq t \leq x+y\}} \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \lambda e^{-\lambda y} dx dy \\ &= e^{-\lambda t} \int_0^t \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} dx = \frac{(\lambda t)^n}{n!} e^{-\lambda t}. \end{aligned}$$

Poisson process as a stochastic process

Some definitions

\mathcal{F}_t : σ -field of events before time t

σ -field of events which can be expressed in terms of $N(s)$, $s \leq t$.

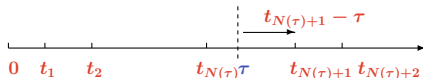
Stopping Time:

A r.v. τ with values in $\mathbb{R}_+ \cup \{+\infty\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$.

Examples

Translation of a Poisson process: II

- ▶ T constant equal to t .
- ▶ Pour $n \geq 1$, t_n is a stopping time:
 $\{t_n \leq t\} = \{N(t) \geq n\}$.
- ▶ $T = \inf\{t \geq 0 : N([0, t/2]) = N([t/2, t])\}$ is also a stopping time.



τ stopping time

The translated Poisson process at τ is

- ▶ a Poisson process with the same rate
- ▶ independent of locations of points before τ .

Translation of a Poisson process: II

Equivalently

$$(N(t + \tau) - N(\tau), t \geq 0) \perp \mathcal{F}_\tau,$$

$$(N(t + \tau) - N(\tau), t \geq 0) \stackrel{\text{dist.}}{=} (N(t), t \geq 0).$$

A **crucial** property in practice.

Infinitesimal properties

If $p_n(t) = \mathbb{P}(N(t) = n)$,

$$p_n(t + h) = \mathbb{P}(N([0, t + h]) = n)$$

$$\begin{aligned} &= \mathbb{P}(N([0, t]) = n, N([t, t + h]) = 0) + \mathbb{P}(N([0, t]) = n - 1, N([t, t + h]) = 1) \\ &\quad + \mathbb{P}(N([0, t + h]) = n, N([t, t + h]) \geq 2) \end{aligned}$$

$$p_n(t + h) = p_n(t) \times (1 - \lambda h) + p_{n-1}(t) \times \lambda h + o(h)$$

Chapman-Kolmogorov Equations

$t \rightarrow p_n(t) = \mathbb{P}(N(t) = n)$ satisfies

$$\frac{d}{dt} p_n(t) = \lambda p_{n-1}(t) - \lambda p_n(t), \quad n \geq 1.$$

$N(t)$ has a Poisson distribution with parameter λt

$$p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

C-K equations hold for **General Markov processes**.