Communication Networks, Algorithms & Probability Theory

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2011

Poisson process as a stochastic process

Some definitions

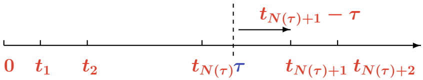
$\mathcal{F}_t$: σ-field of events before time t
σ-field of events which can be expressed in terms of $N(s)$, $s \leq t$.

Stopping Time:
A r.v. τ with values in $\mathbb{R}_+ \cup \{+\infty\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$.

Examples

- T constant equal to t.
- Pour $n \geq 1$, t_n is a stopping time: $\{t_n \leq t\} = \{N(t) \geq n\}$.
- $T = \inf\{t \geq 0 : N([0, t/2]) = N([t/2, t])\}$ is also a stopping time.

Translation of a Poisson process: II



τ stopping time

The translated Poisson process at τ is

- a Poisson process with the same rate
- independent of locations of points before τ.

Translation of a Poisson process: II

Equivalently

$$(N(t+\tau) - N(\tau), t \geq 0) \perp \mathcal{F}_\tau,$$

$$(N(t+\tau) - N(\tau), t \geq 0) \overset{\text{dist.}}{=} (N(t), t \geq 0).$$

A crucial property in practice.

Infinitesimal properties

If $p_n(t) = \mathbb{P}(N(t) = n)$,

$$p_n(t+h) = \mathbb{P}(N(]0, t+h]) = n)$$
$$= \mathbb{P}(N(]0,t]) = n, N(]t, t+h]) = 0) = \mathbb{P}(N(]0,$$
$$+ \mathbb{P}(N(]0,t]) = n-1, N(]t, t+h]) = 1) \quad + \mathbb{P}$$
$$+ \mathbb{P}(N(]0, t+h]) = n, N(]t, t+h]) \geq 2) \quad + \mathbb{P}$$

$$p_n(t+h) = p_n(t) \times (1 - \lambda h) + p_{n-1}(t) \times \lambda h + o(h)$$

Chapman-Kolmogorov Equations

$t \to p_n(t) = \mathbb{P}(N(t) = n)$ satisfies

$$\frac{d}{dt} p_n(t) = \lambda p_{n-1}(t) - \lambda p_n(t), \quad n \geq 1.$$

$N(t)$ has a Poisson distribution with parameter λt

$$p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

C-K equations hold for General Markov processes.

Jump processes

The basic parameters

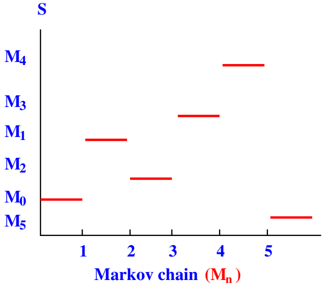
A jump process: (Booklet page 175)

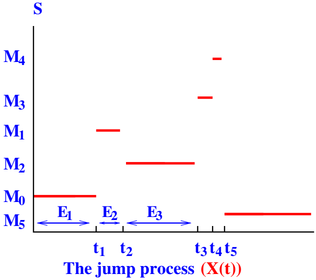
- (M_n) a Markov on $\mathcal{S}$
  with transition matrix $(p(x, y))$.
- $(q_x, x \in \mathcal{S})$ real numbers > 0.

A jump process $(X(t))$:

- $X(0) = M_0 = x \in \mathcal{S}$
  $X(t) = x$ pour $t < t_1 = E_1$
  E_1: exponential r.v. with parameter q_x.
- $X(E_1^x) = M_1$
  $X(E_1^x + t) = M_1$ for $t < E_2$
  E_2 exp. v.a. with parameter q_{M_1}.
- $t_2 = E_1 + E_2$, $X(t_2) = M_3$, ...

$(X(t))$ visits the same states and in the same order as (M_n).





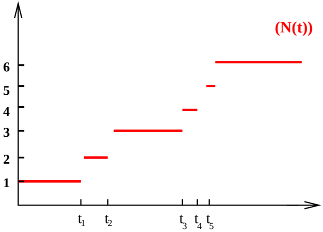
Sequence of jump instants : (t_n) :

- $(X(t_n)) = (M_n)$;
- $t_{n+1} - t_n$ conditionally on $X(t_n) = z$: exponential r.v. with parameter q_z.

Jump process :
A Markov chain
with a random time scale.
Jump processes: fundamental models for discrete modeling

Some examples

Poisson Process



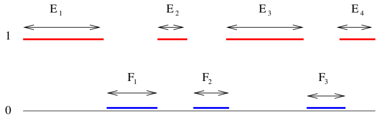
Alternating process

A machine

- Working. Duration: exp. dist. with parameter λ
- Failure. Repair time: exp. dist. with parameter μ

State of the system:

$$X(t) = \begin{cases} 1 & \text{working at time } t \\ 0 & \text{out} \qquad\qquad\quad t \end{cases}$$



(E_i) i.i.d. exp. par. λ:
$\mathbb{P}(E_1 \geq x) = \exp(-\lambda x)$
(F_i) i.i.d. exp. par. μ:
$\mathbb{P}(F_1 \geq x) = \exp(-\mu x)$

The $M/M/1$ queue

Requests to a node: (Booklet page 178)

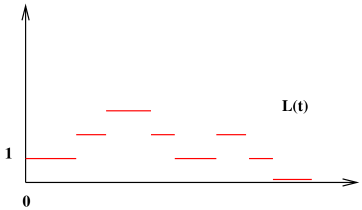
- Arrivals: Poisson with rate λ.
- Duration of a communication: exponential dist. with parameter μ.
- Service in the order of arrivals. FIFO (First In First Out).

The $M/M/1$ queue (II)

$L(t)$: nb of communications waiting at time t.

If $L(t) = n \geq 1$, next event after t occurs at $t + T(t)$, it is either :

- An arrival. $L(t + T(t)) = n + 1$.
- A departure. $L(t + T(t)) = n - 1$.



Busy period of the M/M/1 queue

The $M/M/1$ queue (III)

At time t:

- The exp. distribution “forgets”: The dist. of residual service time σ : exp. r.v. (μ).
- Poisson property: Next arrival at τ : exp. r.v. (λ) independent of arrivals and services before time t.

The $M/M/1$ queue (IV)

At t, two exponential clocks: σ and τ,

$$T(t) = \min(\sigma, \tau).$$

$T(t)$ is exp. dist. with parameter $\lambda + \mu$.

$$\mathbb{P}(T(t) \geq x, \tau < \sigma) = \mathbb{P}(\tau \geq x, \tau < \sigma)$$
$$= e^{-(\lambda+\mu)x}\frac{\lambda}{\lambda+\mu} = \mathbb{P}(T(t) \geq x)\mathbb{P}(\tau < \sigma)$$

$L(t + T(t)) = n - 1$ si $\sigma < \tau$
$L(t + T(t)) = n + 1$ si $\sigma > \tau$.

The $M/M/1$ queue (V)

$L(t))$ is a jump process:
If $L(0) = x > 0$ and E exp. r.v. with par. $\lambda + \mu$: $L(t) = L(0)$ for $t < E$ and

$$L(E) = \begin{cases} L(0) + 1 \text{ with proba } \lambda/(\lambda+\mu) \\ L(0) - 1 \text{ with proba } \mu/(\lambda+\mu) \end{cases}$$

If $L(0) = 0$ and E exp. var. par. λ :
$L(t) = L(0)$ for $t < E$ et $L(E) = 1$.

The $M/M/1$ queue (End)

$L(t))$ is a jump process:

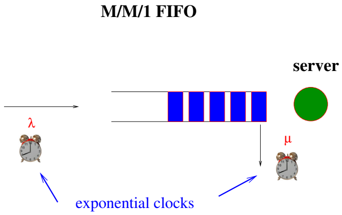
- **Markov chain**:

$$p(x, x+1) = \frac{\lambda}{\lambda+\mu}$$
$$p(x, x-1) = \frac{\mu}{\lambda+\mu}, x > 0$$

et $p(0,1) = 1$.

- **The sequence** (q_x)

$q_x = \lambda + \mu$ if $x > 0$ and $q_0 = \lambda$.



Starting from $x \in \mathbb{N}$, it is easier to phrase the model as follows:

An exponential clock for each possible jump with rates λ and $\mu 1_{\{x>0\}}$

rather than

A global exponential clock with rate $\lambda + \mu 1_{\{x>0\}}$.

General case

Equivalent formulation

If $X(0) = x$ (Booklet page 176):

- for $z \in \mathcal{S}$, E^z exp. r.v. with par.

$$q(x, z) \stackrel{\text{def.}}{=} q_x p(x, z);$$

- The r.v. E^z, $z \in \mathcal{S}$ are independent.

  exponential clocks starting from x.

$t_1 = \inf\{E^z : z \in \mathcal{S}\}$ an exp. r.v. with par.

$$\sum_{y \in \mathcal{S}} q(x, y) = q_x \sum_{y \in \mathcal{S}} p(x, y) = q_x;$$

$$\mathbb{P}(t_1 = E^y) = \mathbb{P}(E^y \le \min(E^z, z \in \mathcal{S})) = \frac{q_x p(x, y)}{q_x} = p(x, y).$$

Exponential clocks

If $X(0) = x$:

$X(t)$ stays at x until time t_1 when the first clock “rings”

$X(t_1) = y$ si $t_1 = E^y$.

and so on, . . .

An alternative construction of jump processes

Jump matrix

A key quantity:

$$Q = (q(x, y), x, y \in \mathcal{S})$$

$$q(x, y) = q_x p(x, y) \qquad x \ne y.$$
$$q(x, x) = -q_x.$$

contains all parameters of jump process:

$$q_x = \sum_{y \in \mathcal{S}} q(x, y),$$
$$p(x, y) = q(x, y)/q_x, \qquad x \ne y.$$

Examples

Poisson Process

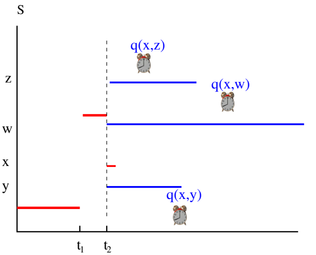
$$q(x, x + 1) = \lambda, x \ge 0.$$

Alternate process

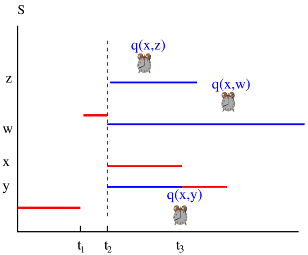
$$q(0, 1) = \lambda, \qquad q(1, 0) = \mu.$$

The $M/M/1$ queue

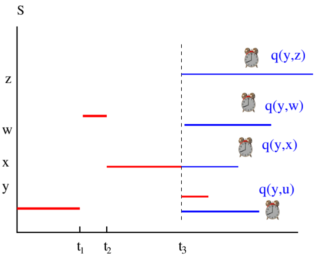
$$q(x, x + 1) = \lambda;$$
$$q(x, x - 1) = \mu 1_{x>0};$$



Jump process (X(t))



Jump process (X(t))



Jump process (X(t))

Non-explosive jump processes

Definition If (t_n) sequence of jump instants of $(X(t))$ and if $\mathbb{P}$ a.s.

$$\lim_{n \to +\infty} t_n = +\infty,$$

The jump processus is non-explosive.

Explosive processes delicate to investigate.

The Markov property

$$[X(s+t)\,|\,X(u), u \le t, X(t) = x] \overset{\text{dist.}}{=} [X(s+t)\,|\,X(t) = x]$$

Homogeneous Markov property:

$$[X(s+t)\,|\,X(u), u \le t, X(t) = x] \overset{\text{dist.}}{=} [X(s)\,|\,X(0) = x]$$

Theorem
A non-explosive jump process has the Markov property

The Markov property

$\mathcal{F}_t$: set of the events before time t.

$$\mathcal{F}_t = \langle X(s), s \le t \rangle$$

on the event $\{X(t) = x\}$.

$$\mathbb{E}\,[f(X(s+t))\,|\mathcal{F}_t] = \mathbb{E}\,[f(X(s))\,|X(0) = x]$$

$$\overset{\text{def.}}{=} \mathbb{E}_x\,[f(X(s))]\,.$$

If $p^t(x,y) = \mathbb{P}(X(t) = y \mid X(0) = x)$

$$p^{t+s}(x,y) = \mathbb{P}(X(t+s) = y \mid X(0) = x) = \sum_{z \in \mathcal{S}} p^t(x,z)p^s(z,y).$$

If $P(t) = (p^t(x,y), x, y \in \mathcal{S})$,

$$P^{s+t} = P^s \circ P^t,$$

Semi-group formula $\Leftrightarrow$ Markov property
$\Rightarrow$ exponential "formula"
$P^t = \exp(tA)P(0)$.

The $M/M/1$ process

If $p^t(x,n) = \mathbb{P}(L(t) = n \mid L(0) = x)$

$$p^{t+h}(x,n) = \lambda h p^t(x,n-1) + \mu h p^t(x,n+1) + (1-(\lambda+\mu)h)p^t(x,n) + o(h)$$

$$\frac{d}{dt}p^t(x,n) = \lambda p^t(x,n-1) - (\lambda+\mu)p^t(x,n) + \mu p^t(x,n+1$$

$$\frac{d}{dt}p^t(x,0) = -\lambda p^t(x,0) + \mu p^t(x,1)$$

Kolmogorov equations for the $M/M/1$ process

$$\frac{d}{dt}p^t(x,\cdot) = p^t(x,\cdot) \times Q.$$

$Q = (q(x,y))$ jump matrix $(L(t))$,

$$q(x,x+1) = \lambda;$$
$$q(x,x-1) = \mu 1_{x>0};$$

and $q(x,x) = -\lambda - \mu 1_{x>0}$

Kolmogorov Equations

Markovian notation

If $(X(t))$ is a jump process

$$\mathbb{P}_x(A) \stackrel{\text{def.}}{=} \mathbb{P}(A \mid X(0) = x),$$

and A an event and if F is an integrable r.v.

$$\mathbb{E}_x(F) \stackrel{\text{def.}}{=} \mathbb{E}(F \mid X(0) = x).$$

Kolmogorov equations: general case

Pour $x,\ y \in \mathcal{S},\ p^t(x,y) = \mathbb{P}_x(X(t) = y)$,

$$\begin{aligned}\frac{d}{dt}p^t(x,y) &= \sum_{z\neq y} p^t(x,z)q(z,y) \\ &\quad - \left(\sum_{z\neq y} q(y,z)\right) p^t(x,y) \\ &= \sum_{z} p^t(x,z)q(z,y) \\ &= \sum_{z} q(x,z)p^t(z,y)\end{aligned}$$

Kolmogorov equations: forward and backward

Si $P(t) = (p^t(x,y), x, y \in \mathcal{S})$,

$$\frac{d}{dt}P(t) = P(t)\cdot Q = Q\cdot P(t).$$

Intuitively (not rigorous !!):

$$P(t) = e^{tQ}P(0);$$

Consistent with semi-group relation:

$$P^{s+t} = P^s \circ P^t.$$

Markov chain: analogue $Q = P - I$;
Brownian Motion $\Rightarrow Q = \Delta$.

Characterization of Markov processes

If $\mathbb{P}$-a.s. (t_n) goes to infinity,
exists a unique solution to K equations.
A jump process $(X(t))$ is uniquely characterized by

- initial value $X(0)$;
- jump matrix Q.

General case (including explosive proc.) delicate.

Functional Aspects

Functional formulation of K equations

(Booklet page 181)

Definition If $f : \mathcal{S} \to \mathbb{R}$ is bounded,
$P(t) : f \to P(t, f)$

$$P(t,f)(x) = \mathbb{E}_x(f(X(t)) = \sum_{z \in \mathcal{S}} f(z)\mathbb{P}_x(X(t) = z)$$

If $Q(f)$ is bounded, equations K :

$$\frac{d}{dt}P(t)(f) = P(t) \circ Q(f) = Q \circ P(t,f),$$

$$P(t)(f) = P(0)(f) + \int_0^t P(s) \circ Q(f)\, ds$$

$$\mathbb{E}_x[f(X(t))] = \mathbb{E}_x[f(X(0))] + \int_0^t \mathbb{E}_x[Q(f)(X(s))]\, ds$$

integral representation:

$$\mathbb{E}_x[f(X(t))] = f(x) + \mathbb{E}_x\left(\int_0^t (Q \cdot f)(X(s))ds\right).$$

For all $t \geq 0$ the variable

$$M(t) = f(X(t)) - f(x) - \int_0^t (Q \cdot f)(X(s))\, ds$$

has an average 0.

$(M(t))$ is a martingale,

$$f(X(t)) = f(x) + M(t) + \int_0^t (Q \cdot f)(X(s))\, ds$$

"Itô formula " for jump processes.