Communication Networks, Algorithms & Probability Theory

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Introduction

Access Protocols

(Booklet page 3)

- N stations scattered.
  The number N is unknown, variable.
  Topology unknown too.
- One communication channel.
- A station with a message
  must transmit it on the channel.
- Two attempts of transmission on the channel at the same time ⇒ failure.

The framework of distributed systems

- Set of independent nodes “connected” as a network.
- Centralized control not possible.

⇒ put the messages in a “queue” for transmission is impossible.

Information of a station

Each station can listen to the channel and can detect:

0 —nothing
No attempt of transmission on the channel

1 —success
One attempt

2 —collision
at least two attempts

The channel delivers a ternary information

Examples

- Wireless networks (Wifi, Bluetooth. . .)
- Local networks
- Cable network
- GSM network
  Access to control channel

Transmission policy

- Based only on the information delivered by the channel.
- Decision for a station with a message:
  try or not to transmit
- All stations use the same policy/algorithm.

Simplifications

- Discrete time.
- Decision to transmit or not:
  at the begining of every time unit.
- Duration of transmission of a message
  = one time unit.

Algorithms to transmit

A station with still a message to transmit for n time units,
let $O_1, \ldots, O_n$ the state of the channel

$$1 \leq i \leq n, \qquad O_i \in \{0, 1, 2\}.$$

Decision $t = n + 1$: $f_n : \{0, 1, 2\}^n \to \{0, 1\}$

Problem

Find a policy $\mathcal{P}$ such that :

- Admissible
  N messages to transmit:
  each message is transmitted with success
- Fairness: Stations use the same algorithm

Statistical Assumptions

- Number of stations N large.
- Each time unit: λ/N probability that a station has a new message to transmit.

At the level of the network: in average $\sim \lambda$ new messages per time unit.

Stability Property

- Stability of $\mathcal{P}$ under a flow of messages:

$L(t)$: nb of stations with a message at t

Existence of $\lambda_c(\mathcal{P}) > 0$ such that if $\lambda < \lambda_c(\mathcal{P})$, then

$$\sup_{t \geq 0} \mathbb{E}(L(t)) < +\infty.$$

Stability Property

Maximal throughput of the channel
$\lambda_{\text{max}} = \max_{\mathcal{P}} \lambda_c(\mathcal{P})$

Remark: $\lambda_{\text{max}} \leq 1$.

Aloha

Aloha

Some History (~ 1967) (Booklet page 7)

- Terminals on islands connected to the central server of the University of Hawaï.
- Connection with one radio frequency.

The algorithm

Abramson (1967)
At the begining of each time unit:
Each station flips a coin with bias p

- Result head
  Attempt to transmit.
- If Tail
  No attempt

Caracteristics

- Admissible.
- Randomized Algorithm
  Randomness reduces repeated collisions.

Randomized algorithms

1. Optimization: Simulated Annealing
2. Search in complex state spaces
3. Routing
4. computational geometry, mesh generation, Numerous applications.

Stochastic model

The framework

- nth time unit:
  A_n new messages
- (A_n) i.i.d. with average λ.
- L_n : number of messages to transmit at the end of the nth time unit.

Évolution

Relation

$$L_{n+1} = L_n + A_n - 1_{\{B_1^n + B_2^n + \cdots + B_{L_n}^n = 1\}}$$

B_i^n Bernoulli r.v. “coin” of the ith message at time n,

$$\mathbb{P}(B_i^n = 1) = p.$$

$(B_i^n, i \geq 0)$ i.i.d. and independent of L_n.

Properties of (L_n)

If $\mathbb{P}(A_1 > 0) > 0$ et $\mathbb{P}(A_1 = 0) > 0$
$\Rightarrow$ (L_n) irreducible Markov chain

Markovian view of stability

The Markov chain (L_n) is

- Ergodic:
  Irreducible with one invariant distribution

$$\lim_{n\to+\infty} \mathbb{P}(L_n = x) = \pi(x) > 0, \quad x \in \mathbb{N}$$

- Transient: $\mathbb{P}$-a.s. $L_n \to +\infty$.

Stability of algorithm $=$ Ergodicity of (L_n).
Instability " $=$ Transience of "

Stability Analysis

Control parameter: $\lambda = \mathbb{E}(A_1)$.

Existence of $\lambda_c > 0$ such that $\lambda < \lambda_c$
$\Rightarrow$ (L_n) ergodic ?

Stability Analysis

One time unit:

$$L_{n+1} = L_n + A_n - 1_{\{B_1^n + B_2^n + \cdots + B_{L_n}^n = 1\}}$$

Drift:

$$\mathbb{E}(L_{n+1} - L_n | L_n = x) = \lambda - xp(1-p)^{x-1},$$
$$\lim_{x\to+\infty} \mathbb{E}(L_{n+1} - L_n | L_n = x) = \lambda > 0.$$

If L_n is large enough
$\Rightarrow$ L_{n+1} grows in average

Conjecture: (L_n) is transient $\forall\lambda > 0$.

Transience criteria

General Criterion

(M_n) Markov chain

- transition matrix $(p(x, y))$
- irreducible
- aperiodic

F a finite subset of state space $\mathcal{S}$,

$$T_F = \inf\{n \geq 0 : M_n \in F\}.$$

Transience

The assertions are equivalent

- (M_n) is transient
- $\mathbb{P}_x(T_F = +\infty) > 0$ for $x \in \mathcal{S}$
- “the sequence (M_n) conv. to $+\infty$”: (M_n) visits any finite subset only a finite nb of steps

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Divergence at infinity

Energy function $f : \mathcal{S} \to \mathbb{R}_+$

1. $\sup(f(x) : x \in F) < +\infty$, $\forall F$ finite $\subset \mathcal{S}$
2. $\sup(f(x) : x \in \mathcal{S}) = +\infty$

- $\mathcal{S} = \mathbb{N}^d$, $f(x) = \sup(x_i : 1 \leq i \leq d)$

Transience

If f energy function and $(x_n) \in \mathcal{S}$, then if

$$\lim_{n \to +\infty} f(x_n) = +\infty$$

$\Rightarrow$ (x_n) visits a any finite subset only a finite number of time steps

Conclusion: (M_n) transient if there exists a function f such that $\mathbb{P}$-a.s.

$$\lim_{n \to +\infty} f(M_n) = +\infty.$$

Criteria for transience

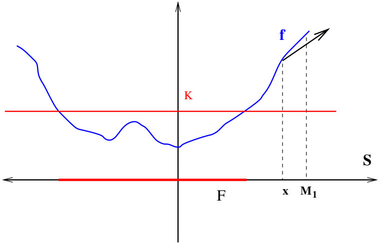
Theorem (Lamperti) (Booklet page 128)
If there exists f, K and $\gamma > 0$ such that

a) $\mathbb{E}_x(f(M_1) - f(x)) \geq \gamma$ if $f(x) \geq K$,

b) $\sup_{x \in \mathcal{S}} \mathbb{E}_x\left(|f(M_1) - f(x)|^2\right) < +\infty$,

then $\mathbb{P}$-a.s $f(M_n) \to +\infty$.

The Markov chain (M_n) is transient
f is Lyapounov function for (M_n)



Application to Aloha

Aloha

$\exists x_0$ such that if $x \geq x_0$,

$$\mathbb{E}(L_{n+1} - L_n | L_n = x) = \lambda - xp(1-p)^{x-1} \geq \frac{\lambda}{2}$$

if $\mathbb{E}(A_1^2) < +\infty$

$$\mathbb{E}\left(L_{n+1} - L_n\right)^2 \leq 2(A_n^2 + 1) < +\infty$$

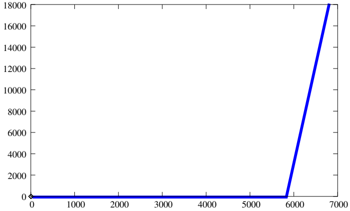
$\Rightarrow$ The Markov chain (L_n) is transient

L'algorithme ALOHA is always unstable

Aloha unstability

- Condition $\mathbb{E}(A_1^2) < +\infty$ not necessary
- $\mathbb{P}$-a.s. $\exists n_0$: no message is transmitted after time n_0.

Aloha unstability



Aloha λ=0.2

Stability Criteria

Notations

(M_n) Markov chain

- transition matrix $(p(x,y))$
- irreducible
- aperiodic

For $x \in \mathcal{S}$,

$$T_x^+ = \inf\{n > 0 : M_n = x\}$$

Ergodicity of a Markov chain

Proposition (Booklet page 114)

- For a $x_0 \in \mathcal{S}$, $\mathbb{E}_{x_0}\left(T_{x_0}^+\right) < +\infty$

$$\Updownarrow$$

- $\exists$ invariant proba $\pi = (\pi(x))$,

$$\pi(x) = \sum_y \pi(y) p(y,x), \qquad \forall x \in \mathcal{S}$$

(M_n) is ergodic (or positive recurrent)

In this case $(M_n) \xrightarrow{\text{dist.}} \pi$

Ergodicity criteria

Theorem (Foster) (Booklet page 125)

If there exist a fn f and K, $\gamma > 0$ such that

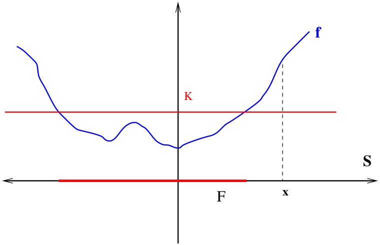
a) $\mathbb{E}_x(f(M_1) - f(x)) \leq -\gamma$ if $f(x) \geq K$,

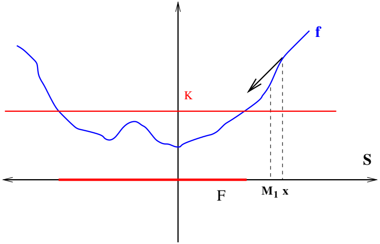
b) $\mathbb{E}_x\left(f(M_1)\right) < +\infty$ si $f(x) \leq K$,

c) $\{x : f(x) \leq K\}$ is finite

then the Markov chain (M_n) is ergodic

f is a Lyapounov function for (M_n)





Example: controlled Aloha

If x messages: proba for an attempt to transmit α/x, $\alpha > 0$

$$\widetilde{L}_{n+1} = \widetilde{L}_n + A_n - 1_{\{B_1^n + B_2^n + \cdots + B_{\widetilde{L}_n}^n = 1\}}$$

with $\mathbb{P}(B_i^n = 1 \mid \widetilde{L}_n = x) = p_x = \alpha/x$

Controlled Aloha

$$\mathbb{E}(\widetilde{L}_{n+1} - \widetilde{L}_n | \widetilde{L}_n = x) = \lambda - x p_x (1 - p_x)^{x-1}$$
$$= \lambda - \alpha \left(1 - \frac{\alpha}{x}\right)^{x-1} \sim \lambda - \alpha e^{-\alpha}$$

If $\lambda < \alpha e^{-\alpha}$, $\exists x_0 \geq 0$ et $\gamma > 0$, t.q. si $x \geq x_0$

$$\mathbb{E}(\widetilde{L}_{n+1} - \widetilde{L}_n \mid L_n = x) \leq -\gamma$$

$$\mathbb{E}(\widetilde{L}_1 \mid \widetilde{L}_0 \leq x_0) \leq x_0 + \mathbb{E}(A_0) < +\infty$$

Controlled Aloha

If $\alpha = 1$ and $\lambda < e^{-1} \sim 0.367$,
then the Markov chain $(\widetilde{L}_n)$ is ergodic

An algorithm with a throughput of 0.367 ?

Controlled Aloha :
x messages, proba to try: α/x, $\alpha > 0$

Controlled Aloha
not a distributed algorithm
nb x a priori UNKNOWN

Aloha algorithm unstable.

Aloha in practice

- Access to control channel of GSM network
- Satellite Networks
- A marginal use
- Generalization: Ethernet

Ergodicity/Transience: Example

The $M/G/1$ queue

(Booklet page 134)

- Arrivals $(\mathcal{N}_\lambda(t)) = (t_n)$ Poisson (λ)
- General Service dist. (σ_n)
  $\sigma(dx)$ law of σ_1
- FIFO policy

The $M/G/1$ queue

T_n : instant of departure of the nth job
L_n : nb of customers at $t = T_n$
If $L_{n-1} > 0$

$$T_n = T_{n-1} + \sigma_n,$$
$$L_n = L_{n-1} + \mathcal{N}_\lambda([T_n, T_n + \sigma_n[) - 1,$$

If $L_{n-1} = 0$

$$T_n = t_n$$
$$L_n = \mathcal{N}_\lambda([t_n, t_n + \sigma_n[),$$

Markov property of (L_n)

Strong Markov prop. of Poisson
+ independence of (σ_n)

- For $n \geq 1$, T_n stopping time
- if $\sigma_1 \not\equiv 0$
  (L_n) irreducible Markov chain on $\mathbb{N}$

The $M/G/1$ queue: Ergodicity

If $L_0 = x > 0$

$$L_1 = L_0 + \mathcal{N}_\lambda([T_0, T_0 + \sigma_1[) - 1,$$

$$\mathbb{E}(L_1 - L_0 \mid L_0 = x) = \lambda \mathbb{E}(\sigma_1) - 1.$$

If $\lambda \mathbb{E}(\sigma_1) < 1$ and $x > 0$

$$\mathbb{E}(L_1 - L_0 | L_n = x) = \lambda \mathbb{E}(\sigma_1) - 1 < 0,$$

$$\mathbb{E}(L_1 | L_0 = 0) \leq \lambda \mathbb{E}(\sigma_1) < +\infty.$$

The $M/G/1$ queue: Ergodicity

Foster:
$\lambda \mathbb{E}(\sigma_1) < 1 \Rightarrow$ the chain (L_n) is ergodic

The $M/G/1$ queue: Transience

If $L_0 = x > 0$,

$$L_1 = L_0 + \mathcal{N}_\lambda([T_0, T_0 + \sigma_1[) - 1,$$

$$\mathbb{E}(L_1 - L_0 \mid L_0 = x) = \lambda \mathbb{E}(\sigma_1) - 1.$$

If $\lambda \mathbb{E}(\sigma_1) > 1$, $\mathbb{E}(\sigma_1^2) < +\infty$ and $x > 0$,

$$\mathbb{E}(L_1 - L_0 | L_n = x) = \lambda \mathbb{E}(\sigma_1) - 1 > 0,$$

$$\mathbb{E}((L_1 - L_0)^2 | L_0 = x) \leq \lambda \mathbb{E}(\sigma_1) + \lambda^2 \mathbb{E}(\sigma_1^2) < +\infty$$

The $M/G/1$ queue: Transience

Lamperti:

$\lambda \mathbb{E}(\sigma_1) > 1$ and $\mathbb{E}(\sigma_1^2) < +\infty$

$\Rightarrow$ the chain (L_n) is transient

Condition $\mathbb{E}(\sigma_1^2) < +\infty$ not necessary