Communication Networks, Algorithms & Probability Theory

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A Reminder

Ergodicity criteria

Theorem (Foster) (Booklet page 125)
If there exist a fn f and $K, \gamma > 0$ such that

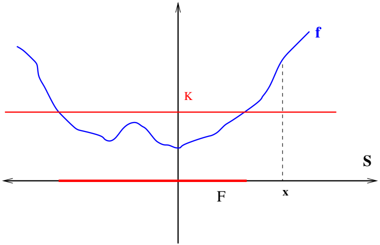
a) $\mathbb{E}_x(f(M_1) - f(x)) \leq -\gamma$ if $f(x) \geq K$,

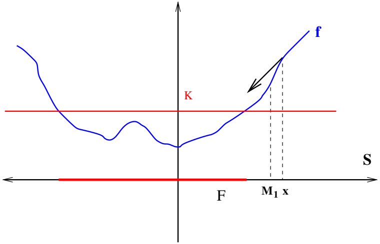
b) $\mathbb{E}_x\left(f(M_1)\right) < +\infty$ si $f(x) \leq K$,

c) $\{x : f(x) \leq K\}$ is finite

then the Markov chain (M_n) is ergodic

f is a Lyapounov function for (M_n)





Criteria for transience

Theorem (Lamperti) (Booklet page 128)

If there exist f, K and $\gamma > 0$ such that

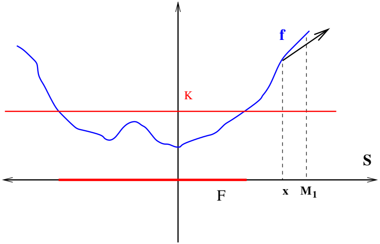
a) $\mathbb{E}_x(f(M_1) - f(x)) \geq \gamma$ if $f(x) \geq K$,

b) $\sup\limits_{x \in \mathcal{S}} \mathbb{E}_x\left(|f(M_1) - f(x)|^2\right) < +\infty$,

then $\mathbb{P}$-a.s $f(M_n) \to +\infty$.

The Markov chain (M_n) is transient

f is Lyapounov function for (M_n)



Ethernet

Access Protocols

(Booklet page 3)

- N stations scattered.
  The number N is unknown, variable.
  Topology unknown too.
- One communication channel.
- A station with a message
  has to transmit it on the channel.
- Two attempts of transmission on the channel at the same time $\Rightarrow$ failure.

Information of a station

Each station can listen to the channel and can detect:

0 —Nothing
No attempt of transmission on the channel

1 —Success
One attempt

2 —Collision
At least two attempts

The channel delivers a ternary information

The Ethernet Algorithm

Metcalf (Harvard) 1973 (Booklet page 12)

Each station/message has a "counter" C.

- When it arrives: $C = 0$;
- Each attempt and failure $C \to C + 1$.

The algorithm (II)

A Station with counter k:

Flip a coin with bias $1/2^k$

- Head (proba $1/2^k$): Attempt to transmit.
- Tail: No attempt.

Characteristics

A station with k failures

tries to transmit with proba $1/2^k$:

- With C, history is (slightly) taken into account.
- New messages have a higher priority.
- Reduces repeated collisions.

Stochastic Model

Framework

- nth time unit: A_n new messages.
- (A_n) i.i.d. Poisson with parameter λ.

State space for a Markovian model ?

Markovian model

State space:

$\mathcal{S}$: space of integer valued finite sequences.

$x \in \mathcal{S}$,
$x = (x_0, \dots, x_p, 0, 0, \dots) = (x_0, \dots, x_p)$,

x_k : nb of stations whose counter is k

For each of the x_k stations: proba $1/2^k$ to transmit.

$x_0 + \cdots + x_p$: total nb of stations waiting for transmission.

Markovian model: transitions

If $X(0) = x$, $x = (x_0, \dots, x_p)$,
$(B_i^k, i \geq 1)$ Bernoulli parameter $1/2^k$,

$$Y_k = \sum_{i=1}^{x_k} B_i^k,$$

Y_k:nb of stations with counter k which try to transmit

$$Z_k = x_k - \sum_{i=1}^{x_k} B_i^k,$$

Z_k: nb of stations with counter k : no attempt.

Markovian model: transitions

If $X(0) = x$ and if collision

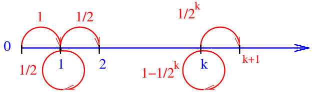
$$X(1) = (A_1, Y_0, Y_1 + Z_2, \\ Y_2 + Z_2, \dots, Y_{p-1} + Z_p, Y_p)$$

$(X(n))$ irreductible Markov chain on $\mathcal{S}$.

An associated Markov chain $(b(n))$ on $\mathbb{N}$

Transitions $b(0) = 0$ (Booklet page 13)

- $\mathbb{P}(b(n) = k + 1 \mid b(n-1) = k) = \dfrac{1}{2^k}$
- $\mathbb{P}(b(n) = k \mid b(n-1) = k) = 1 - \dfrac{1}{2^k}$



Rate of growth of $(b(n))$

Proposition. *For* $t \geq 0$,

$$\frac{\log t}{\log 2} + \frac{\log\log 2}{\log 2} \leq \mathbb{E}(b(t)) \leq \frac{\log(t+1)}{\log 2}.$$

Remark
$b(t) - \log t/\log 2$ does not converge in distribution.
“Oscillating” random variable.

Model of the saturated channel

- Assumption: no successful transmission.
- State:
  $Y(n) = (y_1(0), y_1(n), \dots, y_k(n), \dots)$
  with initial state $Y(0) = (0)$.

Model of the saturated channel

Proposition (Booklet page 13)

For $t \geq 1, y_k(t), k \geq 1$ are independent

$y_n(t)$: Poisson with parameter

$$\lambda \sum_{k=0}^{t} \mathbb{P}(b(k) = n).$$

Number $Z(t)$ of attempts at t is Poisson with parameter

$$\nu_t = \lambda \mathbb{E}(b(t+1)).$$

Unstability of Ethernet

Proposition (Booklet page 15)

If $\lambda > \log 2 \sim 0.693$ then the Markov chain $(X(n))$ is transient and $\mathbb{P}$-a.s. only a finite number of messages are transmitted.

Unstability of Ethernet

Proof: $\nu_n = \lambda \mathbb{E}(b(n+1)) \sim \lambda \log n / \log 2$

$$\sum_{n\geq 1} \mathbb{P}(Z(n){=}1){=}\sum_{n\geq 1} \nu_n e^{-\nu_n} \sim \frac{\lambda}{\log 2} \sum_{n\geq 1} \frac{\log n}{n^{\lambda/\log 2}}$$

Si $\lambda > \log 2,$ then

$$\mathbb{E}\left(\sum_{n\geq 1} 1_{\{Z(n)=1\}}\right) = \sum_{n\geq 1} \mathbb{P}(Z(n) = 1) < +\infty$$

$\mathbb{P}$-a.s. a finite nb of transmissions $\Rightarrow$ transience.

Unstability of Ethernet

Theorem, Aldous (1987)

For any $\lambda > 0 \Rightarrow$ the Markov chain $(X(n))$ is transient.

If $\lambda < \log 2$,
$\mathbb{P}$-a.s. an infinite nb of messages are transmitted.

Conclusions

- Unstable algorithm.
- Unstability theoretical
- Unstability in practice
- Open protocol (Xerox PARC)
- Industrial success. IEEE 802.3 norm.

The tree algorithm

1979
Capetanakis (MIT)
Tsybakov et Mikhailov (Moscow)

The Algorithm

(Booklet page 27)

Each station: a “counter” variable C.

- If $C = 0$: attempt to transmit

  1. Success: the end.

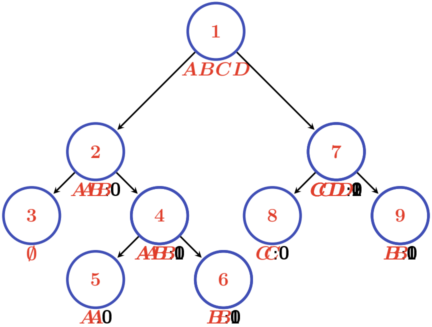
  2. Collision: coin tossing:
     if head, $C = 0$, otherwise $C = 1$.

The Algorithm

- If $C > 0$, no attempt.
  on the channel
  1. if success or silence $C \to C - 1$.
  2. if collision $C \to C + 1$.

Example

	1	2	3	4	5	6	7	8	9
C	No	No	∅	No	Ok	Ok	No	Ok	Ok
0	ABCD	AB		AB	A	B	CD	C	D
1		CD	AB	CD	B	CD		D	
2			CD		CD				



Characteristics

- Continuous probing of the channel.
- Ternary information is taken into account.

Stochastic Model

- Arrivals Poisson with parameter λ.
- $X(n) = (x_0(n), x_1(n), \ldots)$. $x_k(n)$: nb of stations with counter k at $t = n$.

Example

	1	2	3	4	5	6	7	8	9
C	No	No	$\emptyset$	No	Ok	Ok	No	Ok	Ok
0	ABCD	AB		AB	A	B	CD	C	D
1		CD	AB	CD	B	CD		D	
2			CD		CD				

1	2	3	4	5	6	7	8	9
(4)	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}0\\2\\2\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}1\\1\\2\end{pmatrix}$	$\begin{pmatrix}1\\2\end{pmatrix}$	(2)	$\begin{pmatrix}1\\1\end{pmatrix}$	(1)

Protocol with blocked arrivals

A simplified model

- Split into sessions.
- Messages arrived during the nth session $\Rightarrow$ transmitted in the $(n+1)$th.

R_n duration of a session starting with n messages.

Recurrence Relation

$R_0 = R_1 = 1$. For $n \geq 2$,

$$R_n \overset{\text{dist}}{=} 1 + R_{X_n} + \overline{R}_{n-X_n}$$

with

$$X_n = B_1 + B_2 + \cdots + B_n.$$

(B_i) i.i.d. Bernoulli parameter $1/2$

Sequence $(\overline{R}_n)$ same law as (R_n)

$(\overline{R}_n)$ independent of (R_n)

Poisson Transform

If $r_n = \mathbb{E}(R_n)$ (Booklet page 31)

$$r(x) = \sum_{n\geq 0} r_n \frac{x^n}{n!} e^{-x} = \mathbb{E}(r_{N[0,x]}) = \mathbb{E}(R_{N[0,x]})$$

$(N[0,t], t \geq 0)$ Proc. Poisson intensité 1 indépendant de (R_n)

Recurrence Relation

$R_0 = R_1 = 1$
If $n \geq 2n \geq 0$ and $Y_n = n - X_n$,

$$R_n \overset{\text{dist}}{=} 1 + R_{X_n} + \overline{R}_{Y_n} - 2_{\{n\leq 1\}}$$

$$R_{N[0,x]} \overset{\text{dist}}{=} 1 + R_{X_{N[0,x]}} + \overline{R}_{Y_{N[0,x]}} - 2_{\{N[0,x]\leq 1\}}$$

Poisson thinning
$X_{N[0,x]}$ and $Y_{N[0,x]}$
indep. Poisson variables $(x/2)$

$$r(x) = 2r(x/2) + 1 - 2(1+x)e^{-x}$$

Poisson Transform

Iterating

$$r(x) = \sum_{n\geq 0} \mathbb{E}(R_n) \frac{x^n}{n!} e^{-x}$$
$$= 1 + \sum_{k\geq 0} 2^{k+1} \left(1 - \left(1 + \frac{x}{2^k}\right) e^{-x/2^k}\right)$$

identification of coefficients $\Rightarrow \mathbb{E}(R_n)$

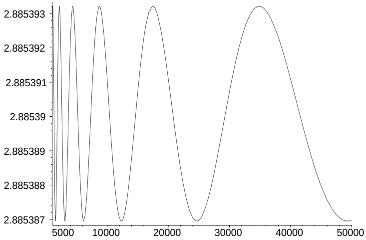
Asymptotic Throughput

$\mathbb{E}(R_n)/n$
Average transmission time for 1 message.
Throughput

$$\lambda_c = \neq \lim_{n\to+\infty} \frac{n}{\mathbb{E}(R_n)}$$

- Pb: limit does not exist !
- $(n/\mathbb{E}(R_n))$ oscillating around
  $\log 2/2 \sim 0.3465735903$
- But
  $\liminf_{n\to+\infty} n/\mathbb{E}(R_n) \sim 0.34657 > 0.$

Evolution of $n \to \mathbb{E}(R_n)/n$



Stability of tree algorithm

Theorem (Booklet page 36)
If $\lambda < 0.34657$ then the tree algorithm with blocked arrivals is stable

Theorem
If $\lambda < 0.36017$ then the tree algorithm is stable

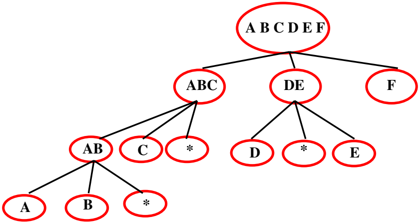
Improvements

d-ary tree

- Splitting into d groups.
- d large: Quick isolation
- d large: Many silences

Optimal value $d = 3$
If $\lambda < 0.40159$ then the tree algorithm with blocked arrivals is stable.

Ternary Tree Algorithm



Theoretical Bounds

Hybrid algorithm based on ternary tree

$\exists$ stable algorithm if $\lambda < 0.487$.

$\Rightarrow \lambda_{\text{max}} \geq 0.487$

Maximal throughput $\lambda_{\text{max}} \leq 0.56$.

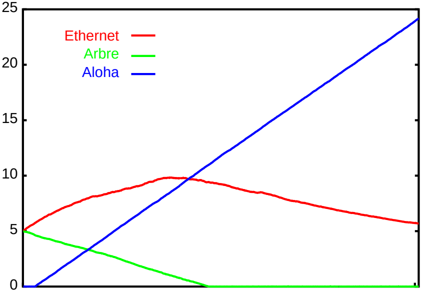
Conjecture: $\lambda_{\text{max}} \leq 0.52$

Theoretical Bounds

Tree algorithm

Quasi-optimal use of ternary information of the channel

Weak industrial impact



Algorithms & Data structures

Leader election

Context:

- N unreliable stations
- Common communication channel

Problem

- Determine a leader

Application: Sensor networks

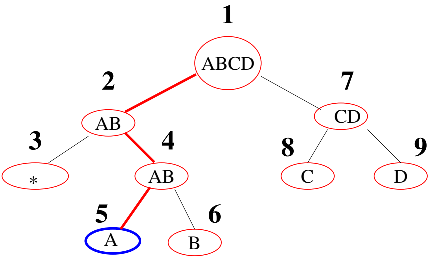
Leader election

Each station has a counter C
If $C = 0$: transmission of its Id;

- One Id on channel: Fin.
- Collision,

$$C \to \begin{cases} 0 \text{ avec proba } 1/2; \\ 1 \text{ avec proba } 1/2; \end{cases}$$

- Silence. Stations having transmitted just before set their counter to 0.

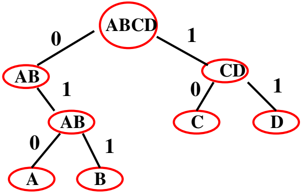


Related Problems

Binary search trees

- N elements x_1,. . . , $x_N \in \mathcal{S}$
- Pb: If $y \in \mathcal{S}$, determine if $y \in \{x_1, \ldots, x_n\}$
- Minimum nb of steps.

A=0101 B=0111
C=101 D=11



Algorithms “Divide and Conquer”

n elements

- Condition to stop
  If $n < D \longrightarrow$ Stop.
- Tree structure
  Si $n \geq D$, division at random into 2 groups with sizes n_1 and n_2, $n_1 + n_2 = n$

  $\longrightarrow$ Apply $\mathcal{A}(n_1)$ et $\mathcal{A}(n_2)$

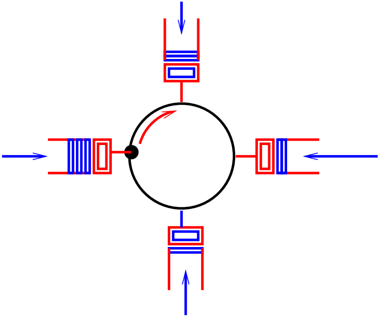
Search, sort algorithms,. . .

Token Ring Algorithm

Algorithm

Framework (Booklet page 20)

- Stations on a ring
- Token circulating on the ring clockwise
- Station grabs token $\Rightarrow$ transmission
- no transmission without token



History

- Cambridge ring (1974).
- IBM (1980). Token Ring.
- Industrial norms IEEE 802.4 and 802.5.

Algorithmic framework

Characteristics

- Assumption on topology
- Weakness of token (may be lost).

Mathematical models

- Polling systems.

Stability

Theorem (Booklet page 25)
Propagation time negligible:
$\lambda < 1 \Rightarrow$ stability of network.

Ring topology
$\Rightarrow$ Optimal maximal throughput.