An Intensity-Based Approach for Valuation of Mortgage Contracts Subject to Prepayment Risk

Yevgeny Goncharov*
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
Chicago, IL 60607
yevgeniy@math.uic.edu

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Abstract

The models in the literature for valuation of mortgage contracts subject to prepayment can generally be classified into one of two categories: the option-based models and the empirical mortgage-to-mortgage-refinancing models. Using risk-neutral martingale methods together with the intensity-based approach borrowed from credit risk modeling, this paper develops a framework that not only generalizes but considerably extends both approaches. In the case of option-based specifications based on non-optimal liability, our prepayment model is the first which is not tied to some particular numerical procedure. In the case where the state process is a diffusion, the mortgage equation is a semi-linear parabolic PDE. To illustrate this point we consider the continuous time limit of Stanton’s model [40] and show that the model is the first order version of splitting-up numerical method applied to our PDE. Throughout the paper we point out various new ways to develop mortgage modeling.

Keywords. Mortgage, prepayment, intensity, option-based approach.

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1 Introduction

Mortgage markets, both primary and secondary, evolved considerably during the last two decades. A large body of literature, both academic and institutional, is devoted to the rational pricing of mortgage-related assets. Considerable progress has been made with the understanding of relevant factors for mortgage pricing, but more work is required to get a satisfactory model.

The prepayment option is what makes mortgage related securities complicated assets to price. Mortgagors have the option to prepay fully or partially their loans prior to the maturity dates. This right has a dramatic effect on valuation by introducing cash flow uncertainty, which depends on the mortgagor’s view of possible future opportunities (e.g., the mortgagor’s expectation of the future behavior of the yield curve) to refinance the loan.

A very important feature of a mortgage model is how one models prepayment incentives of the borrower. This determines the borrower’s decision making implied by the model. Due to the different characteristics of approaches to model prepayment incentive measures (i.e., a borrower’s idea of profitability of prepayment), mortgage models in the literature can be classified into two groups. First is a type of model which is closely related to pricing of American options and therefore is called the option-based or option-theoretic approach. The most advanced in the sense of model completeness so far, this measure of prepayment incentive is completely endogenous. The central object of the approach is a borrower’s liability, which can be defined as the present value of cash flow that the borrower will pay off to repay his/her loan plus various transaction costs which are incurred in case of prepayment or default.\footnote{Some researchers, e.g., Stanton\cite{40}, understand transaction costs in a wide sense: it includes monetary as well as psychological costs, e.g., inconvenience associated with finding a bank, filling out forms, spending time, bad credit in the case of default, etc.} Option-based models measure the mortgagor’s incentive to prepay as the difference between this liability and outstanding principal.\footnote{If the possibility of default is also taken under consideration, then the option-based models measure the mortgagor’s incentive to default as the difference between this liability and the price of underlying real estate.}

\begin{quote}
As when, O lady mine!
With chiselled touch
The stone unhewn and cold
Becomes a living mould.
The more the marble wastes,
The more the statue grows.
\end{quote}

Michelangelo Buonarroti
(1474-1564)
mortgagor) and borrowers do not have preferences for a particular refinancing choice (see section 3.1 in the present paper for details).

The option-based approach has evolved from frictionless models with optimal prepayment behavior (e.g., Dunn and McConnell [12], Kau, Keenan, Muller and Epperson [31]), where the mortgage liability to the borrower and mortgage asset to the investor are not distinguished (or they differ due to fees only) and the borrowers terminate their mortgages if and only if it is financially optimal, to models which recognize the importance of taking into account substantial presence of transaction costs and non-optimal behavior (the borrower can fail to prepay when it is financially profitable and he/she can prepay when it is not profitable to do so) as well as heterogeneity of borrower characteristics. Transaction costs were first incorporated in the model as a part of refinancing threshold for optimal liability (e.g., Dunn and Spatt [13]). Then Johnson and Van Drunen [25], among others, extended the model to allow refinancing costs that vary across borrowers to capture burnout effect. All these mentioned models assumed optimal prepayment plus “background” prepayment due to relocation, divorce, etc. Stanton [40] first acknowledged the fact that borrowers fail to prepay even if it is optimal to do so. McConnell and Singh [34] agreed with Stanton on this point but still based their model on optimal frictionless liability while Stanton based his model on a “correct” value of liability. The reason for calling Stanton’s liability “correct” is the following. If one thinks about prepaying his/her mortgage, then he/she should consider the loss of the prospect of future prepayment which, in turn, is not optimal. Put another way, when he/she thinks that it may be worthwhile to postpone prepayment, then he/she should also consider the possibility of failing to use that future opportunity too. Stanton [40] included this possibility in his definition of the borrower’s liability while McConnel and Singh [34] ignored it. The other already mentioned models did not consider non-optimality of financially profitable decision at all.

The second class of models are sometimes called “option-based” models too (e.g., Deng and Quigley [7] and Deng, Quigley and Van Order [8]), though the name does not precisely (if we understand “option approach” in the sense of Merton [35]) reflect the nature of prepayment decision making that is assumed in the models. To avoid confusion we will call them mortgage-to-mortgage refinancing models or MMR for short (the reason for that will be clear shortly). In this type of model the prepayment policy is based on a comparison of prevailing mortgage and contract rates.\(^3\) This comparison can be done with a “naïve” mortgage formula (see formula (1) below) such as a precise calculation of how much is saved on, say, monthly payments by refinancing (e.g., Deng [6], Deng and Quigley [7], Deng, Quigley and Van Order [8]), or it can be done through a direct comparison such as the difference or ratio of two mortgage rates (e.g., Schwartz and Torous [38, 39], Richard and Roll [37], Kariya and Kobayashi [27], Kariya, Pliska and Ushiyama [26]). This borrower’s decision making assumption relates to the case where borrowers can refinance only to the same type of

\(^3\)For example, the mortgage holder prepay as soon as the mortgage rate drops below a specified threshold.
mortgage\(^4\) (as opposed to the already mentioned implied freedom of choice and absence of borrower’s preferences with the option-based approach). To the best of this author’s knowledge, nobody in this context used or even defined endogenous mortgage rates (see section 2.3 of this paper and Goncharov [17] for work in this direction). Because of exogenously (empirically) defined mortgage rate, these reduced models are often called empirical models (the model of Schwartz and Torous [38], for example, is called “purely empirical” by Stanton [40]). Usually, the 10 year Treasury yield is used as a proxy for the mortgage rate. Early MMR models (e.g., Schwartz and Torous [38]) recognized non-optimality of the borrower’s behavior; later models incorporated borrower heterogeneity (e.g., Deng, Quigley and Van Order [8]) and transaction costs as (e.g., Kariya, Pliska and Ushiyama [26], where the authors assumed optimal prepayment however).

A common feature of the literature on the option-based approach with non-optimal prepayment incentive measures (i.e., the possibility of non-optimal refinancing is included in the liability definition) is the way the dependence of prepayment behavior on future interest rates is handled. This is in the spirit of traditional backward binomial pricing, the illustrative and intuitive method often used to price American options. The option-based researchers discretize the termination process in time and then, at each time step, they consider the probabilities of prepayment and/or default. There are no attempts to find a continuous underlying process and/or its corresponding differential equation in the case when all factors are modeled as diffusion processes\(^5\). Accurate knowledge of the true underlying process is important since it allows one to employ powerful high-order numerical methods to solve the problem. At the same time the only advantage of a binomial model is its simplicity. If other processes (such as an interest rate process) are simulated by a binomial tree too (see Kau, Hilliard and Slawson [28]), then one should be careful with the state space mash because of the conditional stability of this explicit numerical algorithm. It is a well known fact of numerical analysis that the time step must be quadratically smaller than step sizes of the other variables in explicit approximations of diffusion processes. Therefore, if one wants to represent the state space accurately, the researcher must consider a small time step size, which is often undesirable for such a long term contract as a mortgage.

Stanton [40] and Downing, Stanton and Wallace [10] expressed the interest rate and the real estate (in the latter paper) part of the mortgage process with continuous diffusions and left only the borrower’s decision making as discrete. This allowed them to employ a stable numerical procedure for the continuous part. While this procedure guarantees stability, at the same time, as will be shown in the present paper, the explicit method which they employed for the

\(^4\)Or to some finite set of mortgage types (for example, 15- and 30- year fixed rate mortgages), if the approach is generalized (see Goncharov [17]).

\(^5\)It is done for optimal prepayment liability only. For example, Dunn and McConnell [12] presented a free boundary value problem for liability. Not surprisingly, this is exactly the problem of American options (optimal prepayment) as opposed to the solution of quasi-linear PDE’s with fixed boundaries in the case of non-optimal prepayment liability (as we will show in the present paper).
borrower’s decision making makes the model a first order approximation of the “underlying” continuous-time model, thereby reducing the efficiency of higher order numerical schemes for the continuous part (Crank-Nicolson in Stanton [40], parallel hopscotch in Downing, Stanton and Wallace [10]).

Meanwhile, there is a large credit-risk literature for securities that are subject to default. Much of this literature is concerned with an intensity-based approach and continuous time models. General models are created which can account for all possible default specifications (including possible occurrence of default at discrete dates). Some research is done on default risk premia and the relationship between the real world default probability and the risk-neutral probability measure, which is known to be a corner stone of pricing theory. Still, in spite of some successful mathematical theory for credit risk modelling, nobody has tried to use related results for mortgage modelling so far. But after all, from a mathematical point of view nothing precludes one from interpreting prepayment as a “default” in the intensity-based approach to pricing credit risk.

In this paper we use the intensity-based approach and risk-neutral, martingale methods to define a mortgage model in a rigorous way and derive formulae that are convenient for calculations. The formulae are valid for different type of mortgages, including adjustable rate and all kinds of fixed rate mortgages (such as graduated payment, growing equity or the most popular level-payment mortgages). Both the option-based and MMR approaches are special cases of our intensity-based approach. Moreover, in the case where all underlying factors are diffusions, we show that the main equation of the option-based approach is a semi-linear reaction-diffusion PDE.

As we already noted above, all option-based models with non-optimal-based borrower’s liability in the literature are effectively discrete-time models. Therefore our model is the first fully continuous-time model to use an option-based specification of prepayment. The other discrete-time option-based and MMR mortgage models can be viewed as various numerical approximations of versions of our continuous model, thus making it possible to speak about the “effectiveness” of the methods with respect to their continuous counterparts. As an example we consider Stanton’s option-based model [40] in section 4 and find its continuous-time limit. As will be seen, the intuitive econometric numerical procedure employed by Stanton in [40] (and by Dawning, Stanton and Wallace in [10], which is a straightforward generalization of the procedure employed by Stanton in [40]) is a variant of the fractional step numerical method of the first order of accuracy in time applied to our semi-linear “mortgage” PDE. At the end of section 4 we will show how a mortgage model with a one factor interest rate process (because of computational complexity, it is a favorite researcher’s

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6 And which is ignored with respect to prepayment probabilities in the existing literature on mortgage modelling.

7 In a more general setting, which was not considered in the literature yet, the PDE is quasi-linear. See a note to section 3.3.

8 Or splitting-up method, see, e.g., Marchuk [32] for description of the method and some applications, and Descombes and Schatzman [9] for a high order convergence scheme for a reaction-diffusion equation based on application of the splitting method.
choice for models which consider defaultable mortgages) can be “painless” expanded to a model with a simple two factor interest rate process. For example, this approach can reduce the 3-factor discrete model by Kariya, Ushiyama and Pliska [26] to effectively a 2-factor model which can be solved faster with a PDE approach rather than the Monte-Carlo method that they used.

The model in this paper allows one to accommodate default of the borrower. We make several comments about this. When working with mortgage default, an assumption that the default hazard function is absolutely continuous (which is implicit with the intensity-based approach) may not be adequate if one works with payments at discrete times. But in this case the researcher can work directly with the hazard default function, which will have jumps at payment days. In other words, the probability of default will be concentrated there. We do not develop the topic here. One of the reason for that is the author’s desire to present a new approach to mortgage modelling in the clearest fashion possible.

But, in general, the topic is important because default consideration allows the model to describe observed termination behavior better (compare, e.g., [40] vs. [10] and [38] vs. [39]). One of the ways to elaborate on prepayment/default structure is to consider the termination time as the minimum of prepayment and default random times under an assumption of conditional independence (see Goncharov [17]). The notion of conditional independence and the mathematical apparatus for working with the minimum of random times can be found in Bielecki and Rutkowski [1].

In summary, our new formulation of the general mortgage model allows one to see a mortgage from a fresh point of view. It can contribute to a deeper understanding of underlying processes and help researchers to find and apply related known results in default valuation and numerical analysis to mortgage modelling.

2 The Model

2.1 Introduction to Mortgage Securities

In this section we postulate nontechnical assumptions and notations for our basic security — “a mortgage”. Then we briefly review the relationships between a basic mortgage, a pool of mortgages, and mortgage pass-through securities, considering why it can be necessary to price a “basic mortgage” in order to price

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9 Whereas the same assumption is quite reasonable for the prepayment hazard process even in this discrete setting. Indeed, payment days are special for the default event: a borrower, if he/she has decided to default, can enjoy his/her underlying possession until the next payment day for free. Therefore the payment days are preferable for default. That does not pertain to the prepayment event. One still can assume continuity of prepayment hazard functions.

10 In the literature on models which acknowledge the presence of (non-optimal) prepayment for both exogenous and endogenous reasons, the prepayment function is represented as the summation of two processes which represent the corresponding parts of prepayment. The conditional independence condition should be stated for this popular splitting of the prepayment process too, but it is never mentioned in the literature on mortgage models.
more complicated mortgage instruments in the case of endogenously modeled prepayment incentive measures.

For descriptions and details on various mortgage securities the reader can consult Fabozzi [14].

2.1.1 A Basic Mortgage

We consider the following contract. A borrower takes a loan of $P_0$ dollars at some initial time and assumes the obligation to pay scheduled coupons at rate $c_t \geq 0$ continuously for duration $T$ of the contract. The loan is secured by the collateral of some specified real estate property, which obliges the borrower to make the payments. For a mortgage originated at time $s$, interest on the principal is compounded according to a contract rate $m_s^t$, $t \geq s$, where time dependence as the subscript $t$ reflects a possible change of the rate as specified by the contract and the superscript $s$ shows the time when this mortgage rate schedule was contracted. When $s = 0$ we omit the superscript, so $m_t = m_0^t$. When we are dealing with fixed-rate mortgages (no dependence on the subscript) we simply write $m_t$ (the maturity of the mortgage is implicit).

A straightforward argument shows that, in the absence of prepayment, the outstanding principal $P(t)$ at time $t$ equals the future (time-t) value of the original principal $P_0$ less the future (time-t) value of the cumulative coupon payments, that is,

$$P(t) = P_0 e^{\int_s^t m_s d\theta} - \int_0^t c_s e^{\int_s^\tau m_{\tau} d\theta} ds.$$  \hspace{1cm} (1)

The borrower has the right to settle his/her obligation during an interval specified by contract\textsuperscript{11} and prepay the outstanding principal $P(t)$ in a lump sum. If the borrower cannot fulfill his/her obligations then he defaults, i.e., turns over the underlying real estate. For a general payment upon mortgage termination we use $P_t$,\textsuperscript{12} which equals the outstanding principal $P(t)$ or the current price of the real estate depending on the prepayment/default outcome. Because we do not elaborate on the default topic, the reader can view our model as default-free\textsuperscript{13} with possibility of generalization to defaultable case. Thus, further in the paper termination time is called prepayment time and the difference between $P(t)$ and $P_t$ can be ignored to avoid confusion if desired.

\textsuperscript{11}Commercial mortgages, for example, often have a prepayment lockout period.

\textsuperscript{12}We do not introduce a notation for the real estate and deliberately run into a slight risk of confusion between $P(t)$ and $P_t$ because we do not want to overload the reader with different notations and we do not elaborate on the default topic.

\textsuperscript{13}A popular belief that if a mortgage is insured then it can be treated as a default-free mortgage (i.e., default can be considered as a form of prepayment) is not quite correct. Though for an investor it may look like this, nevertheless the default option has a specific effect on prepayment behavior. Compare termination models [40] and [10] in the context of the option-based approach. In the MMR case the default option (ideally) should be incorporated in the definition of the current (available for refinancing) mortgage rate.
The coupon payment rate $c_t$ and mortgage rate $m_t$ can depend on time and the current state of the economy. They can be deterministic functions and even stochastic processes. But they are not independent. In the case of a fully amortized mortgage, i.e., $P(T) = 0$, equation (1) implies that rates $c_t$ and $m_t$ must satisfy

$$P_0 = \int_0^T c_s e^{-\int_s^T m_t \, \mathrm{d}t} \, \mathrm{d}s. \tag{2}$$

Although some commercial mortgages may have a balloon payment at the maturity, throughout the paper we assume that the mortgage is fully amortized. The extension is trivial and can be taken care of within the framework we present.

For example, in the case of the most popular level-payment, fixed rate mortgage, both $c_t$ and $m_t$ are constants. If one fixes either rate, $c$ or $m$, then the other rate can be found from this equation. Other examples of fixed-rate mortgages are graduated payment mortgage (GPM) and growing equity mortgage (GEM). For GPM mortgage one chooses a mortgage rate $m$ and then finds a (deterministic) function $c_t$ which has some ability to reflect the borrower’s “preferences” when payments should increase and how they should increase. The payment schedule $c_t$ nevertheless must satisfy (2) for a fixed maturity $T$. GEM mortgages with a fixed mortgage rate $m$ start with the same payment as a level-payment mortgage (thus avoiding negative amortization which is common for a GPM mortgage), then payments are gradually increased with time. In this case $c_t$ is an increasing positive function ($c_0 = c$) and equation (2) produce some new maturity time $T'$ that will be less than the maturity time of the corresponding level-payment mortgage.

An adjustable-rate mortgage is an example of stochastic coupon payment rate $c_t$ and mortgage rate $m_t$. The contract mortgage rate $m_t$ is reset periodically in accordance with some appropriately chosen reference rate; the coupon rate $c_t$ may be found the same way as it is done in different kinds of fixed-rate mortgages. The reference rate depends on the current state of economy and, therefore, is stochastic.

Another situation where we may want to consider a stochastic coupon rate $c_t$ is to facilitate curtailment, i.e., partial prepayment. In this case maturity time $T$ depends on the whole history of the process $c_t$ via (1). Full prepayment and curtailment have different impacts on mortgage securities. If full prepayment reduces coupon payments in a pool, but leaves the maturity time the same, then curtailment, on the contrary, shortens the maturity time while leaving coupon payments the same. Curtailment has no received the attention it deserves (we can mention Chinloy [4] on the topic). As reported in Hayre and Rajan [20], it amounts roughly to the same portion of total prepayment in a pool as default (though both are still relatively low).

As can be seen, the outstanding principal $P(t)$ depends on the choice of the contracted mortgage rate $m_t$ and the contracted coupon payments $c_t$. In what follows this dependence will be suppressed for notational transparency.
2.1.2 Related Contracts

The mortgage market has undergone significant structural changes since the
1980’s. Innovations have occurred in terms of the design of new mortgage in-
struments and the development of products that use pools of mortgages as
collateral for the issuance of a security. Such securities, called mortgage-backed
securities (MBS), remove prepayment uncertainty to some extent but introduce
a strong history dependence in the form of so-called burnout. One simple
(not model-based) way to model the burnout effect is purely empirical. For
example, Schwartz and Torous [38] assume that a pool consists of borrowers
with homogenous characteristics and therefore are forced to take burnout (re-
call that burnout is a product of a pool’s heterogeneity) into account as the
explanatory variable \(\ln[P(t)/P^*(t)]\), where \(P(t)\) represents the dollar amount of
the pool outstanding at time \(t\), while \(P^*(t)\) is the pool’s principal that would
prevail at \(t\) in the absence of prepayments but reflecting the amortization of the
underlying mortgages. The adequacy of this approach is questionable because
the burnout effect is “non-Markovian”; it should depend on the whole history
path of prepayments. The modelling approach (e.g., Stanton [40]) is based on
the observation that one particular mortgage does not have burnout by defini-
tion. Therefore, using conditional independence of individual mortgages, one
can incorporate burnout into a model by modelling the prepayment behavior
for individual mortgages and constructing a pool’s prepayment behavior as a
combination of individual prepayments. A more practical way, however, is the
mix of these two approaches — one can assume existence of several homogenous
groups of borrowers, each group consisting of borrowers with identical charac-
teristics.

In the first approach due to homogeneity the price of a pool is the number of
borrowers in the pool multiplied by the price of one mortgage. In the modelling
approach the price of a pool is the summation of the prices of all individual
mortgages in the pool, while in the mixed approach it is a weighted average of
them. In both cases the price of a pool, and therefore the price of a mortgage
pass-through security, is based on a “basic mortgage”.

But what about other securities, collateralized mortgage obligations, for ex-
ample? The cash flow of such securities can be very complicated and numerical
calculations can be demanding. In order to “clear up the road” for using the
forward-looking (quasi) Monte-Carlo method, which is suitable for handling any
possible tangled historical dependence of a security’s cash flow, it is desirable to
remove future dependence of prepayment behavior, i.e., to define prepayment
behavior as a function of current information (this way McConnell and Singh
[34] priced CMO’s). With the reduced MMR approach, i.e., with an exoge-

\footnote{If the pool experienced a wave of prepayment then it is likely that most of those who
used this refinancing opportunity are “fast,” financially astute borrowers, whereas most of
those who remain are “slower” borrowers. Therefore, in the presence of another refinancing
opportunity, the pool is expected to be less active — the pool is “burned out”.

\footnote{A holder of this security receives a fixed portion of cash flow from the pool.}
nously modeled mortgage rate process, there is no such problem. But in the case of endogenously modeled prepayment one still needs to work with a “basic mortgage” to do that. Namely, in the option-based case one should price individual liabilities to borrowers (which can be priced as a mortgage with a broad interpretation of the transaction cost) and in the MMR case one should solve the equation “the price a newly originated generic pool of mortgages (which is equivalent, as we just discussed, to pricing an individual mortgage) is zero” for the mortgage rate process. Therefore the valuation of one mortgage (or a basic mortgage, as we called it in the previous section) is of fundamental importance and, keeping in mind possible generalizations to MBS, in the present paper we will deal only with a model for one mortgage with an emphasis on the way the mortgage prepayment can be defined.

2.2 Mortgage Model

We formalize our setup by introducing a completed filtered probability space \((\Omega, G, \{G_t\}_{t \geq 0}, Q)\)\(^\text{17}\), where the \(\sigma\)-algebra \(G_t\) represents all observations available to an investor at time \(t\), \(\Omega\) is a set of all possible outcomes and \(Q\) is the probability on \(G (\supseteq \bigcup_{t \geq 0} G_t)\). The prepayment time, for which we use notation \(\tau\), is then a positive stopping time on this filtered probability space (i.e., at any arbitrary time \(t\) we can tell if prepayment “occurred” given information \(G_t\)).

Let us “remove” knowledge about timing of prepayment from \(G_t\). Intuitively, lender is not notified about prepayment, but he is aware of other information (filtration \(F_t\) in what follows), e.g., interest rates movements, unemployment rates, etc. Then the lender cannot tell for sure if prepayment occurred or not since he does not have complete information about the borrower (such as intention to move, to divorce, etc.) and/or the borrower do not prepay as soon as it is profitable to do so (see, e.g., Hayre and Rajan [20]). This observation stands behind the following consideration.

We introduce information concerning only the timing of the prepayment as the filtration \(D_t = \sigma(\{I_{\{\tau \leq u\}} | u \leq t\})\). Now given \(D_t\) and the original filtration \(G_t\), we are interested in decomposing \(G_t\) into \(D_t\) and an additional filtration \(F_t\). Formally, it can be defined as a solution of the equality \(G_t = D_t \lor F_t\).\(^\text{18}\) We accept this complementary filtration \(\{F_t\}_{t \geq 0}\) as given. Now, based on the observation in the above paragraph, we can assume that prepayment time \(\tau\) is not an \(F_t\)-stopping time. This situation is the same as for default time in reduced-form modeling of credit derivatives. The technique is standard and we refer reader to Jeanblanc and Rutkowski [24] for detailed exposure of the topic.

\(^{16}\)For example, the 10 year T-note yield (it is “known today”) is often used as a proxy for the mortgage rate process.

\(^{17}\)We make throughout the technical assumption that this filtered probability space satisfy the “usual hypothesis” and all filtrations we work with later are \((Q, G)\)-complete.

\(^{18}\)It is clear that the equality does not define \(F_t\) uniquely. For example, if \(G_t = D_t\), we may take the trivial filtration, but also \(F_t = G_t\). We interested in having \(F_t\) be the “minimal” such filtration. In most applications, \(F_t\) is the natural filtration of a process which define the interest rate and real estate process.
Throughout this paper we assume that all given processes are positive and \( \mathcal{F}_t \)-progressively measurable. The latter condition is purely technical and is needed to have \( \mathcal{F}_t \)-adapted time-integrals of the processes. One simple (and sufficient for most applications) condition for a process to be \( \mathcal{F}_t \)-progressively measurable is that this process is adapted and right- or left- continuous (see, e.g., Métivier [36]).

A process \( r_t \) will represent a short-term interest rate, so that at any time \( t \) it is possible to invest one unit in a default-free deposit account and “roll-over” the proceeds until a later time \( s \) for a market value at that time of \( e^{\int_t^s r_u \, du} \).

The Fundamental Theorem of Arbitrage Pricing (see Harrison and Pliska [19]) states that absence of arbitrage implies that all securities are priced in terms of this short-rate process \( r_t \) and an equivalent martingale measure. Therefore, it is convenient to assume that the probability \( Q \) in the introduced probability space is this martingale measure and, thus, all expectations in the paper will be taken under this martingale measure \( Q \) without reminder. In a rough intuitive way, \( Q \) can be understood as the market’s insight about probabilities of possible outcomes. The connection between real-world and martingale probability measures will be briefly discussed at the end of this section.

Since our model is free of arbitrage opportunities, it is appropriate to assume (see, e.g., Bielecki and Rutkowski [1]) that the price of a security \( M_t \) that pays coupon payment continuously with the rate of \( c_t \) up to time \( \tau \wedge T \) and \( P_\tau \) in a lump sum at time \( \tau \), if \( \tau \leq T \) (e.g., as with a mortgage), equals the expected discounted value of the future cash flow, with expectation with respect to the martingale measure \( Q \). In other words, (formally) we have

\[
M_t = \mathbb{E} \left[ \int_0^{\tau \wedge T} c_s e^{-\int_s^T r_u \, du} \, ds + \mathbb{I}_{\{t < \tau \leq T\}} P_\tau e^{-\int_t^\tau r_u \, du} \bigg| \mathcal{F}_t \right].
\]  

(3)

It is easy to see from (1) that if coupon and contract mortgage rates \( c_t \) and \( m_t \) are positive \( \mathcal{F}_t \)-progressively measurable processes then the outstanding principal \( P(t) \) enjoys the same properties. The processes \( P_t \) and \( P(t) \) are assumed to be uniformly integrable. The following assumption is natural as we expect the cumulative payment to be bounded:

\[
\mathbb{E} \left[ \int_0^T c_s \, ds \right] < \infty.
\]  

(4)

Under these assumptions expectation (3) is well defined and finite. But (3) is not a convenient formula for calculation because of its explicit dependence on the stopping time \( \tau \). The purpose of the rest of this section is to remove this involvement.

The following definition is standard (see [24]).

**Definition 1** The process \( \Gamma_t = -\ln(1 - Q(\tau \leq t | \mathcal{F}_t)) \) is called the hazard process of the random time \( \tau \). Equivalently \( Q(\tau > t | \mathcal{F}_t) = e^{-\Gamma_t} \).
We assume that $\Gamma_t$ is an increasing process. Because there is no particular convenient or special day for prepayment, $\Gamma_t$ is assumed to be continuous. Indeed, we make the slightly more restrictive assumption that the process $\Gamma_t$ is an absolutely continuous process, i.e., $\Gamma_t = \int_0^t \gamma_\theta d\theta$ for some process $\gamma_t$, called the intensity of the random time $\tau$. In this case definition 1 has certain similarities with the definition of intensity of a Poisson process which can give us intuition behind the name “intensity” of $\gamma_t$.

Now we are in position to use the following result which is borrowed from credit-risk literature (e.g., we can reformulate proposition 8.2.1 from Bielecki and Rutkowski [1]). It removes explicit involvement of $\tau$ by employing its intensity process $\gamma_t$.

**Theorem 1** The value of the security $M_t$ admits the following representation

$$M_t = \mathbb{1}_{\{\tau > T\}} \mathbb{E} \left[ \int_t^T (c_s + P_s \gamma_s) e^{-\int_s^T (r_\theta + \gamma_\theta) d\theta} ds \bigg| \mathcal{F}_t \right].$$

**Note on discrete coupon payments.** If one assumes discrete coupon payments, then the analogous result can follow from the same proposition 8.2.1 in Bielecki and Rutkowski [1]) with a dividend process specified as a step-function. This would be the framework, for example, with (non-optimal prepayment version of) the set-up of Kau, Keenan, Muller and Epperson [31]. It can be analyzed along the same lines in what follows.

**Note on applications of Theorem 1.** If $P_t$ is the outstanding principal $P(t)$ and $\tau$ is called the prepayment time, then the security in Theorem 1 should be called a mortgage. But Theorem 1 is much more general. For example:
1) Later in section 3.2 we will use Theorem 1 to define the mortgagor’s liability.
2) Instead of representing the whole coupon payment, $c_t$ can be used for a specified part of the coupon payment, as in pricing interest only (IO) and principal only (PO) strips for example. Specifically, the interest only part, i.e., the part of the payment which covers interest on the principal $P(t)$, is $m_t P(t)$. The rest of the coupon payment $c_t - m_t P(t)$ goes to pay off the principal itself. Therefore the values of IO strip, for instance, is given by formula 5 with substitution of its “$c_t$” and “$P(t)$” processes with appropriate payments, namely, with $m_t P(t)$ and 0 respectively. Hence the value of an IO strip is given by:

$$IO(t) = \mathbb{1}_{\{\tau > T\}} \mathbb{E} \left[ \int_t^T m_s P(s) e^{-\int_s^T (r_\theta + \gamma_\theta) d\theta} ds \bigg| \mathcal{F}_t \right].$$

---

19For example, for defaultable mortgages with discrete payments it could be reasonable to assume that default times are distributed at due dates, in which case the due dates would be “special” for default. We work with continuous coupon cash flow.

20If $\tau$ is the first jump of a Poisson process with an intensity function $\gamma(t)$ then $Q(\tau > t) = e^{-\int_0^t \gamma(\theta) d\theta}$. Etymology of the term “intensity” (or “hazard rate”) for $\gamma(t)$ came from the fact that for small $\Delta t$ we have $Q(t < \tau \leq t + \Delta t | \tau > t) = \gamma(t) \Delta t + o(\Delta t)$ for almost all $t \in \mathbb{R}_+$. 

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3) It can be used to price complicated mortgage-backed securities such as CMO’s if we rewrite its another representation (7) stated in the corollary 1 below in the following way

\[
M_t = P(t) + \mathbb{E} \left[ \int_t^T (m_s - r_s) P(s) Q(\tau > s | \mathcal{F}_s) e^{-\int_t^s r_\theta d\theta} ds \bigg| \mathcal{F}_t \right],
\]

where the probability \(Q(\tau > t | \mathcal{F}_t)\) can be calculated via probabilities of individual borrowers.

For an insured mortgage, i.e., the contract with \(P_t = P(t)\), \(c_t\) and \(m_t\) satisfying equation (1), we have the following special version of equation (5):

**Corollary 1** The value of an insured mortgage is

\[
M_t = P(t) + \mathbb{E} \left[ \int_t^T (m_s - r_s) P(s) e^{-\int_t^s (r_\theta + \gamma_\theta) d\theta} ds \bigg| \mathcal{F}_t \right].
\]

**Proof.** Assume first that \(c_t\) and \(m_t\) are continuous and therefore, from (1), we get that \(P(t)\) is continuously differentiable with respect to \(t\). We integrate the second term in the expectation of (7), i.e., \(I_1\) in the terms of Theorem 1, by parts as follows:

\[
\int_t^T P(s) \gamma_s e^{-\int_t^s (r_\theta + \gamma_\theta) d\theta} ds = -\int_t^T P(s) e^{-\int_t^s r_\theta d\theta} ds \cdot \int_t^s \gamma_\theta d\theta = P(t) - \int_t^T (P'(s) - P(s)r_s) e^{-\int_t^s (r_\theta + \gamma_\theta) d\theta} ds,
\]

where we used the amortization condition \(P(T) = 0\). From (1) we get \(P'_t = m_t P(t) - c_t\). Combining and substituting the above results back into (5) we prove the corollary for continuous processes \(c_t\) and \(m_t\).

Now assume that \(c_t\) and \(m_t\) are arbitrary processes. Then there exist sequences of continuous processes \(c^n_t\) and \(m^n_t\) so that they converge to \(c_t\) and \(m_t\) in \(L^1(\Omega)\) \(\mathbb{Q}\)-a.s. The processes \(c^n_t\) and \(m^n_t\) can be chosen in such a way that they satisfy equation (1) and are bounded by \(c_t\) and \(m_t\). The corollary is already proven for continuous processes and, thus, we have the equality of the expectations in (5) and (7). Now all that we have to do is to prove the corollary for the original \(c_t\) and \(m_t\) is to take limit of both sides of that equality for \(n \to \infty\) and change the limits and the expectations. The last manipulation is justified by the dominated convergence theorem. \(\square\)
Note on implicit dependence. The above result does not mean that the price of a mortgage does not depend on a distribution of scheduled payments $c_t$, because the coupon rate implicitly enters the formula via (1) through outstanding principal $P(t)$ (the same way the mortgage rate $m_t$ implicitly enters formula (5)). For example, the price of a level-payment fixed-rate mortgage generally will not be the same as the price of other types of mortgages, such as graduated payment, growing equity or tiered-payment mortgages, even though they share the same fixed contracted mortgage rate $m_t$.

The corollary shows that a mortgage can be viewed as an interest-rate swap on notional amount $P(t)$. The parties exchange interest payments calculated according to the scheduled (usually fixed) rate $m_t$ and interest rate $r_t$ up to possible default, the timing of which is driven by the intensity $\gamma_t$.

Note on the real world and martingale measures. In the literature on mortgage valuation, authors often capture the stochastic nature of prepayments with an empirically estimated prepayment function of some state process such as the borrower’s prepayment incentive, loan-to-value ratio, interest rates, etc. The common feature of this literature is that the prepayment function is assumed to be the same under the real world and martingale measures. Jarrow, Lando and Yu, in their work [22] on default risk, argue that this equivalence is an example of an implicitly applied assumption of conditional diversification. Briefly (and rephrasing the authors so our wording is in terms of mortgage prepayment rather than default), the notion of conditional diversifiability requires that conditioning on the evolution of the state processes, the prepayment processes of borrowers are independent of each other. This captures the idea that once the systematic parts of prepayment risk have been isolated, the residual parts represent idiosyncratic, or borrower-specific, shocks that are uncorrelated across borrowers. Examples of such shocks include divorce, acquisition of a new job or its loss, advent of a new family member, etc. As the nonsystematic risk is not priced, it justifies the practice of using an empirically estimated prepayment function for valuation purposes.

For ourselves we can add that the creation of mortgage-backed securities, namely, the practice of pooling individual mortgages, can be seen as the “physical” implementation of the above argument. As such, the prepayment intensity $\gamma_t$ can be viewed as a model of a (deterministic) prepayment function. Of course, questions about whether the prepayment intensity is invariant under a change of the probability measure deserves deep empirical research.

2.3 Mortgage Rate

It is of a vital practical interest for Wall Street to define the mortgage rate endogenously, i.e., to find what mortgage rate is implied by the current (riskless) yield curve and the prepayment behavior of a representative mortgagor. This can be done through the postulate that at origination the value of a mortgage should be equal to the initial principal. For example, from (7), using the ar-
bitrage argument that \( M_0 = P_0 \), we can get that the mortgage rate \( m^t \) of a fixed-rate default-free mortgage at time \( t \) is a weighted average of the risk-free interest rate over the life-time \( T \) of the contract, that is,

\[
m^t = \mathbb{E} \left[ \int_t^{T+t} r_s P(s) e^{-\int_t^s (r_{\theta} + \gamma_\theta) d\theta} ds \mid \mathcal{F}_t \right].
\]

Unfortunately, this is an equation rather than a formula because \( P(s) \) and thus the right-hand side of (8) depends on the mortgage rate too. This dependence is quite different for option-based and MMR approaches. In the former case \( P(s) \) and \( \gamma_s \) depend just on the contract mortgage rate \( m^t \) for all \( s \in [t, T + t] \), but in the latter case \( \gamma_s \) depends on future mortgage rates \( m^s \) as well. This greatly complicates the problem with the MMR approach because equation (8) is a functional equation rather than just a nonlinear equation in one variable in the option-based approach.

Possibly due to these complications, equation (8) has not been studied in the current literature. Needless to say, its properties are worth empirical as well as analytical research (see the last paragraph in section 3.1 for one particular reason of importance and see Goncharov [17] for research in this direction).

3 Prepayment Intensity \( \gamma_t \)

3.1 Fixed-Rate Mortgages

Specification of the prepayment intensity process \( \gamma_t \) is a cornerstone of mortgage pricing. A large body of empirical literature studies mortgage prepayment. Wall street firms have sophisticated prepayment models. One of the features of many of such models, such as the Salomon Brothers prepayment model [20], is that incentive to prepay a mortgage is measured in terms of current \( m^t \) and contract \( m^0 \) mortgage rates (i.e., they endorse the MMR approach). This measurement can be the difference \( m^0 - m^t \) (as was argued in Hayre and Rajan [20] and Richard and Roll [37], it is able to capture prepayment incentive better than the difference), or through direct computation of how much the borrower will save on coupon payments with refinancing (Deng [?], Deng and Quigley [7], Deng, Quigley and Van Order [6]). Depending on the type (e.g., a 30-year or 15-year mortgage rate) of the mortgage rate \( m^t \) considered, the expressions above approximate/evaluate potential savings from refinancing a

\[21\]Mortgages are mostly traded in pools. Therefore, if one assumes heterogeneity of borrowers, the denominator and the numerator of equation (8) should be “averaged” over a possible “risk-neutral” distribution of borrowers prepayment processes \( \gamma_t \).
$m^0$-fixed-rate mortgage with a (30-year or 15-year mortgage rate) $m^t$-fixed-rate mortgage. Hayre and Rajan [20] report that the fixed-rate 30-year mortgage has retained its popularity as the refinancing vehicle of choice for mortgagors with an existing 30-year loan, although the possibility of being able to refinance from a 30-year loan into a 15-year loan or an adjustable-rate mortgage means that the mortgagor can consider a complex mix of rates and monthly payments. It is worth noting here that refinancing choice is nevertheless limited. There is no 23.5-year mortgage to refinance 6.5-year seasoned 30-year loans. To refinance the loan with a 15-year mortgage may be too expensive for the mortgagor, and an adjustable-rate mortgage can be too risky (and it might be more expensive because servicing fees are higher), although this type of loan can be attractive to mortgagors who plan to sell their property soon.

An option-based measure of prepayment incentive (which is implied in an option-based approach to mortgage pricing) is based on a comparison of outstanding principal $P(t)$ and the liability of the mortgagor. It is possible for an increasing yield curve to find a seasoned, say, 30-year loan such that the borrower’s liability will be higher than the outstanding principal (therefore making it profitable to refinance from the option-based approach point of view), but the contract rate $m^0$ is still lower than the current 30-year mortgage rate $m^t$ (therefore making refinancing to another 30-year mortgage unprofitable). This shows that if the choice of refinancing vehicles is limited then the option-based measure of prepayment incentive is not adequate. This measure implies total availability of funds to a borrower and neglects the borrower’s preferences and ability to pay a certain cash flow (it is not likely that a borrower with a 30-year mortgage will refinance it with a short, say 5-years loan). The outstanding principal $P(t)$ is compared with the liability, which is based on the behavior of economic variables over the period of time $[t, T]$, thus making this measure comparable with refinancing the current mortgage into a $(T - t)$-year fixed-rate mortgage (which is valid over the same period of time), the choice of which is not readily available to a residential mortgagor.

Another reason why the MMR choice of incentive measure may be more appropriate then the option-based measure is that the typical residential borrower is not financially astute, as sophisticated financial models are not freely accessible for the general public. A mortgagor calculates his/her amount of savings (true and ultimate measure of incentive to prepay his/her mortgage!) with refinancing, using currently available alternative mortgage rates and a straightforward ‘simple’ financial model (1) (like in Richard and Roll [37], from which the authors get $\frac{m^0}{m^t}$ as an appropriate incentive measure).

Nevertheless, the option-based approach has one vary important advantage over currently used MMR models: the prepayment behavior is defined endogenously with a low numerical price. For this reason the option-based approach is dominating the academic literature nowadays. But a natural question is: how much better (if at all) is the “fully featured” MMR approach? This question deserves further academic research, as potentially cheap and accurate approximations can be developed for the mortgage rate equation.
3.2 Definition of prepayment intensity for Option-based and MMR Approaches

The MMR approach assumes that the prepayment intensity $\gamma_t$ explicitly depends on the current (available for refinancing) mortgage rate $m_t$, i.e., $\gamma_t = \gamma(t, F_t, m_t)$. The mortgage rate can be defined exogenously (empirically) or endogenously via equation (8). In the former case a common choice is to model the mortgage rate as the 10-year long rate plus a constant (as reported in Boudoukh, Richardson, Stanton and Whitelaw [2], the 10-year long rate has correlation of 0.98 with the mortgage rate), though the robustness of the approach is questionable as it is known that the average mortgage lifetime varies under different economic circumstances. The case of endogenous mortgage rates can be numerically very demanding if not used with simplified assumptions and possible approximations (see Goncharov [17] for work in this direction), but it is capable of answering to what extent the option-based approach is applicable to the mortgage problem when there is a limited choice of refinancing vehicles.

In the option-based approach the prepayment intensity explicitly depends on a borrower’s liability along with the current information, i.e., $\gamma_t = \gamma(t, F_t, L_t)$, where we use $L_t$ for the value of the liability. We can define the liability $L_t$ with the help of Theorem 1. We note that the distinction between the liability to the mortgagor and the price of asset to the investor is that at time of prepayment a holder of the security receives $P_t$, but the mortgagor pays $P_t$ plus a transaction cost, which we denote by $F_t$. Therefore the liability $L_t$ has the interpretation of the price of a security paying coupons $c_t$ continuously up to time $\tau \lor T$, and $P_t + F_t$ upon prepayment of the mortgage. So we just use the formula (5) with $P_t + F_t$ instead of $P_t$:  

$$L_t = \mathbb{E} \left[ \int_t^T (c_s + [P_s + F_s] \gamma_s(L_s)) e^{-\int_t^s [r_u + \gamma_u(L_u)] du} ds \mid F_t \right].$$

(9)

After one solves the equation (it is not a formula anymore!) the intensity is defined as an $F_t$-adapted process, i.e., the time $t$ prepayment behavior is defined as a functional of the information available at that time. Now the road is open to use straightforward (quasi) Monte-Carlo simulation to price various types of mortgage-backed securities.

The existence of a solution to equation (9) is easy to prove under the current assumptions (section 2.2) if we additionally assume that $\gamma_t(x)$ is bounded and uniformly Lipschitz continuous.  

Recall that $P_t$ equals the outstanding principal $P(t)$ in the case of prepayment and the price of the underlying real estate in the case of default. Therefore the nature of the transaction cost $F_t$ can be quite different in these cases, e.g., the “moral” cost of default can be quite high. Of course, the transaction cost process is a defined part of the model.  

We could also change the coupon rate in the mortgage pricing formula to account for servicing fee. This is a relatively small amount and is usually neglected.  

Lipschitz continuity of $\gamma_t(L)$ is a quite natural assumption; we discuss this topic later in section 3.4.
Lemma 1 Let the processes $A_t(x)$ be Lipschitz continuous in $x$ uniformly in $t$ and $B_t(x)$ be uniformly Lipschitz continuous in $x$, i.e., for any $\omega \in \Omega$, $t \in [0, T]$

$$|A_t(x) - A_t(y)| \leq N(\omega)|x - y|, \quad |B_t(x) - B_t(y)| \leq L_1|x - y|,$$

where $E[N] \leq L_2$, and $L_1$ and $L_2$ are constants. Suppose $A_t(x)$ satisfies the integrability condition $E \left[ \int_t^T A_s(x_s) \, ds \right] \leq L_3$ for any bounded $\mathcal{F}_t$-measurable process $x_t$ and some finite constant $L_3$. Then there exists a solution to equation

$$x_t = E \left[ \int_t^T A_s(x_s) e^{-\int_t^s B_s(x_\theta) \, d\theta} \, ds \bigg| \mathcal{F}_t \right].$$

(10)

This solution is unique in the set of bounded $\mathcal{F}_t$-adapted processes.

Proof. Let $D_0^\alpha$ be a space of a bounded $\mathcal{F}_t$-adapted process with the norm

$$||x_t||_\alpha = \sup_{t \in [0,T], \omega \in \Omega} e^{-\alpha(T-t)}|x_t(\omega)|.$$

The theorem is equivalent to the problem of existence and uniqueness of the fixed point $x_t \in D_0^\alpha$ of the operator $J[x_t]$, which is defined as the right-hand side of equation (10). Clearly, $J$ maps $D_0^\alpha$ into itself. Moreover, the operator is a contraction in that space for $\alpha$ large enough. Indeed, for bounded $\mathcal{F}_t$-adapted processes $x_t$ and $y_t$ we have

$$|J[x_t] - J[y_t]| \leq E \left[ \int_t^T |A_s(x_s) - A_s(y_s)| e^{-\int_t^s B_s(x_\theta) \, d\theta} \, ds \bigg| \mathcal{F}_t \right] +

+ E \left[ \int_t^T A_s(y_s) \left| e^{-\int_t^s B_s(x_\theta) \, d\theta} - e^{-\int_t^s B_s(y_\theta) \, d\theta} \right| \, ds \bigg| \mathcal{F}_t \right] \leq

\leq E \left[ \int_t^T \left( N||x_t - y_t||_\alpha e^{\alpha(T-s)} + A_s(y_s) \int_t^s L_1||x_\theta - y_\theta||_\alpha e^{-\alpha(\theta-t)} \, d\theta \right) \, ds \right] \leq

\leq \frac{L_2 + L_1 L_3}{\alpha} e^{\alpha(T-t)}||x_t - y_t||_\alpha.$$

Taking the norm $|| \cdot ||_\alpha$ of this inequality we see that operator $J$ is a contraction if $\alpha > L_2 + L_1 L_3$ :

$$|J[x_t] - J[y_t]| \leq \frac{L(1 + M)}{\alpha} ||x_t - y_t||_\alpha.$$

We thus conclude that the equation $x_t = J[x_t]$ has a unique solution in the set of bounded $\mathcal{F}_t$-adapted processes. \(\square\)
Theorem 2 Suppose the process $\gamma_t(x)$ is uniformly Lipschitz continuous in $x$. Suppose the processes $P_t$ and $F_t$ are uniformly integrable, i.e., $E[P_t] \leq L$ and $E[F_t] \leq L$ for some constant $L$ and any $t \in [t, T]$. Then there exists a unique solution to equation (9) in the set of bounded $\mathcal{F}_t$-adapted processes.

Proof. The result is a straightforward consequence of Lemma 1. The conditions of Lemma 1 are easily verified. □

3.3 Diffusion State Process, Option-Based Approach

To formulate the mortgage model in a continuous-time diffusion state process setting, which is popular in financial applications, we assume that all relevant factors in the model (i.e., $r_t$, $c_t$, $P_t$ and $\gamma_t$) are deterministic locally Hölder continuous functions of time $t$ and a state process $X_t = (X_1^t, ..., X_n^t)$, for some $n$, where $\gamma_t$, additionally, depends on the borrower’s liability $L$ (this is the option-based approach after all!). Here $X_t$ is a diffusion process following the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad (11)$$

where the functions $\mu(t, x)$ and $\sigma(t, x)$ are locally Lipschitz continuous in $x$, uniformly in $t$. We assume that $X_t$ neither explodes nor leaves $D$ before $T$, i.e., $Q[\sup_{s \in [t, T]} |X_s| < \infty \text{ and } X_s \in D \text{ for all } s \in [t, T] \mid X_t \in D] = 1$, where $D \subset \mathbb{R}^n$ is a domain (i.e., open and connected set) which is called a support of the diffusion $X_t$. This guarantees that the strong solution $X_t$ exists on the interval $[0, T]$ and that this solution is continuous as a function of $t$ and initial conditions.

Additionally, we assume $\gamma_t(L) < C_1 L + C_2$ for some positive constants $C_1$ and $C_2$, and we need the matrix $a(t, x) = \sigma(t, x)\sigma^T(t, x)$ is elliptic for $(t, x) \in [0, T] \times D$, i.e., $\xi^T a(t, x) \xi > 0$ for all $\xi \in \mathbb{R}^n$.

Let operator $A$ be the generator of the diffusion state process $X_t$, i.e.,

$$A := \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \mu_i \frac{\partial}{\partial x_i}.$$

Then the Feynman-Kac representation states that to evaluate the borrower’s liability, i.e., to solve equation (9) which in the present setting is written as

$$L_t(x) = E \left[ \int_0^T \left( c_s + [P_s + F_s] \gamma_s(L_s) \right) e^{-\frac{1}{4} \int [r_s + \gamma_s(L_s)]^2 ds} ds \mid X_t = x \right], \quad (12)$$

one can solve the following backward semi-linear parabolic partial differential equation

$$\frac{\partial L}{\partial t} + AL - [r + \gamma(L)]L + c + (P + F)\gamma(L) = 0, \quad 0 < t < T \quad (13)$$

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\[ L_T(x) = 0, \quad x \in D, \]

where we left out the dependence of \( L, r, c, \gamma, P, F \) on time \( t \) and the value of the state variables \( x \) for notational simplicity and to underline the nonlinear dependence of \( \gamma \) on \( L \) (we will keep this practice in what follows). Under the stated assumptions a classical solution \( L \) to equation (13) exists and is unique. The proof of this result can be found in Theorem 1 of Hearth and Schweizer [21] which, as is easy to check, is valid for semi-linear PDE’s. Condition (A3) there can be verified by Theorem 7.4.9 of Friedman [16].

**Note on a stochastic intensity.** We can assume that the prepayment intensity depends on the borrower’s liability \( L \) through some of the components of \( X_t \) (only or along with the straightforward dependence), i.e., some coefficients in (11) can depend on \( L \). The existence of such components can be justified by the presence of randomness in the prepayment rate even if conditioned on “standard” information (interest rates, unemployment, loan-to-value ratio, etc.) for instance. The source of the randomness can come from the fact that human perception of the world changes with the influence of a lot of factors such as weather, health, social situation in the country, etc. This assumption leads to a quasi-linear PDE (some of the coefficients in operator \( A \) depend on \( L \)) as opposed to semi-linear PDE (13). This demands more attention from the researcher as the problem of existence of a solution to quasi-linear PDEs is more complex than in the case of linear equations and semi-linear equations.

**Note on technical conditions.** There are many sets of sufficient conditions for equivalence of martingale and PDE approaches (Feynman-Kac type results). The most popular conditions (see, e.g., appendix E of Duffie [11] for an overview of the conditions and references) are quite restrictive and are not satisfied for some popular set-ups (e.g., CIR interest rate model). However, the equivalence is almost universally assumed. For example, Friedlin [15] defined a generalized solution for linear (mortgage equation, MMR approach) and quasi-linear (the borrower’s liability, option-based approach) parabolic equations and studied their properties with the help of a probabilistic approach.

Our technical assumptions are general enough to work for many popular and reasonable set-ups.

### 3.4 Continuity of prepayment intensity \( \gamma_t \)

If the prepayment intensity function \( \gamma_t \) is not continuous (it is discontinuous in Stanton’s model [40], see section 4.2 and footnote 29 in particular), then a classical solution to (13) does not exist, but the Feynman-Kac representation still holds if we understand a solution to differential equation (13) in a generalized sense. While a discontinuity is a surmountable problem from a theoretical point of view, it is known to give troubles (if “neglected”) for numerical cal-
calculations such as a reduced convergence rate and spurious oscillations\textsuperscript{25} of the numerical solution. One should apply extra effort to overcome the problem. Thus it is worthwhile to observe here that from an intuitive interpretation of the prepayment intensity $\gamma_t$ it is natural to expect the intensity to be a continuous, even Lipschitz continuous, (deterministic) function of the state process $X_t$. Indeed, $\gamma_t$ corresponds to the rate of prepayment, and we use it to account for non-optimality of prepayment behavior, which is mainly based on “human” factors. Continuity of $\gamma_t$ would model a real “hesitating” human being; it states that a “small” change in economic variables leads to a small change in prepayment activity. A local Lipschitz condition would mean that there is no special economics states for the borrower, as this fuzzy boundary (between profitable and nonprofitable states) assumption is a manifestation of a person’s inherent behavioral uncertainty.

This observation is backed up by empirical work (see, e.g., Hayre and Rajan [20]), where traditionally the prepayment curve looks like a smoothed step function. A person hesitates when an object he/she wants costs around the price he/she believes is fair. As the price changes favorably, the person is becoming more decisive (cuspy part of the prepayment curve), and when the price is far beyond the point of profitability, it really does not matter a lot.\textsuperscript{26}

4 Intensity-Based Version of Stanton’s Model

4.1 Review of Stanton’s model

In Stanton’s model [40] a borrower is assumed to find refinancing profitable when his/her liability is higher than the outstanding principal plus the refinancing cost. The transaction cost should be understood in a wide sense, meaning it includes monetary as well as psychological costs (convenience to go to a bank, fill out forms, time spent on it, etc.). The transaction cost is assumed to be proportional to the principal, i.e., it is $P(t) F$, and the total borrower’s expenditure in the case of prepayment at time $t$ is $P(t)(1 + F)$, where $F$ is an exogenous constant.\textsuperscript{27} In addition, the borrower is assumed to check “financial news” (i.e. check if it is profitable to refinance) not continuously but at discrete stochastic time intervals, which are modeled as jumps of a Poisson process with the intensity $\rho$. At the same time the borrower is exposed to risks of terminating the mortgage prematurely due to exogenous reasons such as divorce, relocation, etc. This is modeled as a jump of another independent Poisson process with intensity $\lambda$. Finally, Stanton assumes that the risk-neutral dynamics of the interest rate process $r_t$ can be described by a one-factor, Cox, Ingersoll and Ross model [5],

\textsuperscript{25}They usually are negligible for valuation purposes but can have significant impact on hedging performance.

\textsuperscript{26}In old times people counted objects like this: 1,2,3,...,10, many, very many. It reflects human psychology very well.

\textsuperscript{27}Pricing mortgage pass-through securities, Stanton assumes that the transaction cost varies across the borrowers in a pool to model the burnout effect.
\[
\begin{align*}
\int_{a}^{b} 
\end{align*}
\]

Using econometric considerations, Stanton [40] comes up with the following two-step procedure to price a mortgage. The first step is to determine the borrower’s liability \( L_t(r) \). In order to find it, Stanton divides the life time of the mortgage into 360 intervals (the number of months in 30 years), each of length \( \Delta t = 1/12 \). He implements the following algorithm to find \( L_n \) given \( L_{n+1} \):

First, the coupon bond backward PDE is solved (using the Crank-Nicolson method) over the current time step \([n \Delta t, (n+1) \Delta t]\):

\[
\begin{align*}
\frac{\partial \hat{L}}{\partial t} + \sigma^2 \frac{\partial^2 \hat{L}}{\partial r^2} + a(b-r) \frac{\partial \hat{L}}{\partial r} - r \hat{L} + c &= 0 \quad (15)
\end{align*}
\]

with the terminal condition \( \hat{L}_{n+1}(r) = L_{n+1}(r), r \in \mathbb{R}^+ \). The solution \( \hat{L}_n \) (or \( \hat{L} \) for short) of this equation is the borrower’s liability conditional on the prepayment option remaining unexercised over the interval. Then the “true” liability \( L_n \) is found as an expectation of prepay/continue outcomes, i.e.,

\[
L_n = (1 - Pr(\hat{L}_n)) \hat{L}_n + Pr(\hat{L}_n) P(n \Delta t)(1 + F), \quad (16)
\]

where \( Pr(\hat{L}) \) is a probability of prepayment over the time interval \( \Delta t \) and is defined as

\[
Pr(\hat{L}) = \begin{cases} 
1 - e^{-\lambda \Delta t}, & \text{if } \hat{L} \leq P(t)(1 + F) \\
1 - e^{(-\lambda + \rho) \Delta t}, & \text{otherwise.}
\end{cases}
\]

The second main step is to evaluate the mortgage itself. Now prepayment probabilities are known and the mortgage price is found along the same lines as the liability in the first step \(28 \) (we have to change \( L_t \) to the value of the mortgage \( M_t \) with the only difference being that in formula (16) there will not be \( F \) (indeed, the investor receives just the outstanding principal \( P(t) \) upon prepayment, not \( P(t)(1 + F) \)).

4.2 Option-based continuous time model

In this section we “translate” Stanton’s assumptions to our framework. There are two sources of information assumed in the model: the interest rate and the timing of prepayment. Therefore the filtration \( \{F_t\}_{t \geq 0} \) can be defined as a natural filtration of the process (14).

Recall that Stanton assumes that borrowers prepay in an optimal manner given they know that it is profitable to do so (taking into account transaction costs). Therefore the intensity of the random time, which models the borrower’s refinancing decision, is \( \rho \mathbb{1}_{L_t(r) \leq P(t)(1 + F)} \), where \( L_t(r) \) is the borrower’s liability as a function of the interest rate \( r \) and time \( t \). The second relevant random time, which models prepayment for exogenous reasons, has constant intensity \( \lambda \).

\(28\)The second step, actually, is done each time interval right after the first step.
The prepayment time is constructed as a minimum of the two independent random times. As is well known, the intensity of the minimum of two independent random times, which coincide with probability zero, is just the summation of their intensities (see, e.g., Bielecki and Rutkowski [1]). Therefore we can define the intensity of the prepayment time \( \gamma_t \) as follows:

\[
\gamma_t = \begin{cases} 
\lambda , & L_t(r_t) \leq P(t)(1 + F) \\
\lambda + \rho , & \text{otherwise.}
\end{cases}
\]  

(17)

To find \( L_t(r) \) (and, consequently, to determine the prepayment behavior) we need to solve equation (13),\(^{29}\) which in the present setup is

\[
\frac{\partial L}{\partial t} + \frac{\sigma^2}{2} r \frac{\partial^2 L}{\partial r^2} + a(b - r) \frac{\partial L}{\partial r} - [r + \gamma(L)]L + c + P(1 + F)\gamma(L) = 0.
\]  

(18)

\[L_T(r) = 0, \quad 0 < r < \infty.\]

After we have found the liability function \( L_t(r) \), the prepayment intensity (as a function of interest rate) is known and defined by (17). The Feynman-Kac representation for the pricing formula (5) gives us the PDE for the mortgage price \( M_t(r) \) as a function of the given interest rate \( r \) and the time \( t \):

\[
\frac{\partial M}{\partial t} + \frac{\sigma^2}{2} r \frac{\partial^2 M}{\partial r^2} + a(b - r) \frac{\partial M}{\partial r} - [r + \gamma(L)]M + c + P\gamma(L) = 0,
\]  

(19)

\[M_T(r) = 0, \quad 0 < r < \infty.\]

4.3 Relation between Stanton’s Model and Its Continuous Counterpart

Using the idea of the splitting-up numerical method (see Marchuk [32] or [33] for details), we can apply the following procedure to solve equation (18) numerically. Let \( \Delta t \) be a time step and \( L_{n+1} \) be an approximation of \( L_{n}\Delta t \). We go backward in time and determine \( L_n \) (the approximation of \( L_n\Delta t \)) in two steps.

The first step: we consider the equation

\[
\frac{\partial \hat{L}}{\partial t} + \frac{\sigma^2}{2} r \frac{\partial^2 \hat{L}}{\partial r^2} + a(b - r) \frac{\partial \hat{L}}{\partial r} - r \hat{L} + c = 0
\]  

(20)

with the terminal condition \( \hat{L}_{n+1}\Delta t = L(n + 1) \). With its help we evaluate the “fractional” step \( L_{n+1/2} = \hat{L}_{n\Delta t} \).

The second step: we consider the remaining part of the original equation (18), which is the ODE

\[
L_t + \gamma(L)L + P(1 + F)\gamma(L) = 0
\]  

(21)

\(^{29}\)In fact, the discontinuity in the specification of \( \gamma_t \) implied by (17) does not allow for the existence of a classical solution, as we already discussed. We refer the reader to §5.3 of Friedlin [15] on this topic, where one can find a proof of existence of such a solution and some results on its smoothness.
with the terminal condition \( L_{(n+1)\Delta t} = L_{n+1}/2 \). Finally, we take \( L_n = L_{n\Delta t} \).

**Note on the accuracy of the splitting method.** When solving equations (20) and (21) it is natural to consider only some approximations of the equations since even after exact integration we get just an approximate solution of the original equation. The variant of the splitting-up method we used is of the second order (see section 4.3.1 of Marchuk [32] for a discussion of this “stabilization method.”)

The first step, as we can see, is really the same as the first step in Stanton’s procedure. To see how the second steps are related we explicitly solve the following approximation of (21) over the interval \([n\Delta t, (n+1)\Delta t]\). We freeze the argument of \( \gamma(\cdot) \) at time \( t = (n+1)\Delta t \), so that it is known over the interval \([n\Delta t, (n+1)\Delta t]\), i.e., \( \gamma \) is a constant \( \gamma(L_{(n+1)\Delta t}) \). The outstanding principal \( P(t) \) is frozen at time \( t = n\Delta t \). Thus we get the ODE with constant coefficients\(^{30}\)

\[
\frac{dL}{dt} + \gamma(\hat{M}_{n\Delta t})L + P(n\Delta t)(1 + F)\gamma(\hat{M}_{n\Delta t}) = 0,
\]

which is easy to integrate analytically. Its solution at time \( t = n\Delta t \) is

\[
L_n = e^{-\gamma(L_n)\Delta t}L_n + [1 - e^{-\gamma(L_n)\Delta t}]P(n\Delta t)(1 + F).
\]

As we can see it is exactly the same as the equation of the second step (16) in Stanton’s procedure. Therefore, Stanton’s procedure is a variant of the splitting (fractional step) method for numerical solutions of semi-linear PDE (13). It is important to note here that by freezing arguments as above we make (22) to be the first order approximation of the solution to the equation (21).

**Note on the advantage of the splitting method.** The splitting method for equation (18) that we have considered here is popular, in particular, for numeric integration of reaction-diffusion equations (a non-linear term depends just on the unknown function, not on its derivatives). It is motivated by the fact that the numerical integration of a linear PDE and a scalar ODE are easy, but a numerical integration of the original semi-linear equation involving the two operators together is troublesome. If we opt for an explicit scheme, the time step \( \Delta t \) is limited by \( O(\Delta x^2) \), but if we choose an implicit scheme, we have to solve a large system of nonlinear equations at each time step, a computationally expensive operation.

Now we can make several observations. Because the prepayment function has a jump, numerical methods, as seen above, cannot be of the second order approximation unless the discontinuity is specially treated. Even with a smooth \( \gamma(\cdot) \) Stanton’s computational method is of the first order in time because the second step is of the first order approximation of the original ODE

\(^{30}\)Here we use that for the frozen argument of \( \gamma(L_{(n+1)\Delta t}) \) we have \( L_{(n+1)\Delta t} = L_{n+1}/2 = \hat{M}_{n\Delta t} \).
(21), therefore, the method as a whole is of the first order in time. Plus, instead of working with (16) we can work with some approximation of equation (21) of an appropriate (second) order (or an asymptotic expansion ($\Delta t \to 0$) of (16)) with the benefit of faster calculations since algebraic operations are performed faster on a computer than evaluations of the exponential functions in (16) (recall that splitting-up method is itself of the second order). In summary, knowing the real PDE standing behind an option-based model, we can be more efficient by using the rich literature on numerical analysis. For example, for Stanton’s mortgage model with smooth prepayment rate we can employ two-step splitting method (e.g., [33]) to get the second order in time at little cost of additional computations. For construction of numerical schemes for differential equations with discontinuous coefficient the reader can consult Marchuk [32].

5 A Simple Model

A one factor model for interest rates is probably not adequate for long-lived securities such as mortgages. Two factor interest rate models fit data significantly better and, as Chen and Yang [3] found by testing different interest rate models for mortgage pricing, the ability to fit the initial term structure appears to be the most important characteristic of a sound interest rate model. It is common to consider two factor interest rate model for default-free models, but as soon as default is taken into consideration (e.g., a house price process is included, which is usually one factor diffusion process), researchers revert back to a one factor interest rate model (e.g., compare Schwartz and Torous [38] vs. [39]) because of the numerical burden multi-dimensional problems present. Clearly this should degrade performance of the mortgage models.

Here we propose a MMR model with a simple two factor interest rate model which can be implemented while essentially “paying the computational price” of just a one factor model. The possibilities of better fitting to the data and simple calculation in this setting promise better performance than option-based approaches with the same numerical burden (e.g., a one factor model for interest rate and a one factor model for house price).

We assume that the mortgage we want to price is a standard 30-year fixed rate mortgage. Let us write the instantaneous interest rate $r_t$ as a summation of the 10-year Treasury rate $l_t$ and a slope $\delta_t$, i.e. $r_t = l_t + \delta_t$. The same two factors are used by Boudoukh, Richardson and Stanton [2] in their non-parametric approach, where they use the yield on 10-year Treasury notes and the spread between the 10-year yield and the 3-month Treasury bill yield as factors that determine MBS prices. As the authors report, the 10-year Treasury yield has correlation of 0.98 with the mortgage rate (see Table 9.1 and Figure 9.1, [2]), therefore it is a good candidate to be used as a proxy for the mortgage rate. Thus we put in our model $\gamma = \gamma(l_t, t)$. The prepayment function $\gamma$ can depend on other process too if they are independent of $\delta_t$.

We assume that the processes $l_t$ and $\delta_t$ are independent. Let filtration $S_t$ be a natural filtration of the process $\delta_t$ and $L_t$ be a natural filtration of the process...
\(l_t\) and the other process which are supposed to be independent of \(\delta_t\). We put \(\mathcal{F}_t = S_t \vee L_t\). Then from (5) the price of the mortgage is

\[ M_t = \mathbb{E} \left[ \int_t^T \left[ c + P s \gamma(l_s) \right] e^{-\int_t^s [r_s + \gamma(l_s)] \, ds} \, ds \, | \, \mathcal{F}_t \right] = \]

\[ = \mathbb{E} \left[ \int_t^T \left[ c + P s \gamma(l_s) \right] e^{-\int_t^s [l_s + \delta_s + \gamma(l_s)] \, ds} \, ds \, | \, S_t \right] \left| L_t \right] = \]

\[ = \mathbb{E} \left[ \int_t^T \left[ c + P s \gamma(l_s) \right] \mathbb{E} \left[ e^{-\int_t^s \delta_s \, ds} \, | \, S_t \right] e^{-\int_t^s [l_s + \gamma(l_s)] \, ds} \, ds \, | \, L_t \right]. \]

The expectation \(\mathbb{E} \left[ e^{-\int_t^s \delta_s \, ds} \, | \, S_t \right]\) is the price \(\tilde{B}_s\) of a “bond” in a hypothetical world with “interest rate” \(\delta_t\). For many models (e.g., Vasicek) a closed form expression for \(\tilde{B}_s\) is known, say of the form \(\tilde{B}_s = q(t, s, \delta_t)\). Therefore we can implement a two factor model of interest rates in the present framework by “paying the price” of just a one factor model:

\[ M_t = \mathbb{E} \left[ \int_t^T \left[ c + P s \gamma(l_s) \right] \tilde{B}_s e^{-\int_t^s [l_s + \gamma(l_s)] \, ds} \, ds \, | \, \mathcal{F}_t \right] = \]

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\[ \] \[ \] \[ \] \[ \]

Conjecture. These calculations cannot be done in the case of the option-based approach since \(\gamma_t\) depends on \(M_t\) which by definition depends on both processes \(l_t\) and \(\delta_t\). However, knowing that the mortgage price depends on the 10-year Treasury yield \(l_t\) on a bigger scale than on the instantaneous interest rate \(r_t\), it is plausible to presume a “weak” dependence between processes \(\gamma_t\) and \(\delta_t\). That leads to the analogous one factor approximate equation for a borrower’s liability

\[ L_t = \mathbb{E} \left[ \int_t^T \left[ c + (P_s + F_s) \gamma(L_s) \right] \tilde{B}_s e^{-\int_t^s \tau_s \, ds} \, ds \, | \, M_t \right]. \]

This is a one factor interest rate equation with the term \(\tilde{B}_s\) that corrects “the cash flow” with attention to the way the yield curve is humped or sagged. This equation may give a good approximation to the original two factor equation and promises to give better results than those in the literature for option-based models with a one factor interest rate model.

\section{Conclusion.}

In this paper we developed a general model of default-free mortgages subject to prepayment risk by using an intensity-based approach. The model is flexible
and can be applied to all types of mortgages available on the market. Moreover, we showed that our approach generalizes all of the models in the literature. To do this, we classified (parametric) mortgage models into MMR (Mortgage-to-Mortgage Refinancing) and option-based groups due to the way the borrower’s prepayment incentive is measured. This division is natural and is validated by the fact that implementations of MMR and option-based models use different analytical and numerical approaches. We considered different specifications of our model in the view of this new classification.

Our general model is not tied to a particular numerical procedure as some existing models (e.g., Kau, Hilliard and Slawson [28]). As an example we showed that Stanton’s model [40] is in fact just a variant of a splitting-up, numerical method applied to our model. Knowledge of the real underlying process can give an edge to a researcher since he/she can be numerically more efficient. For example, Stanton’s approach,\(^{31}\) which is of the first order of convergence, can be easily "upgraded" to the second order. This is very important for mortgage securities, because modelling is a computationally heavy problem.

Throughout the text we pointed out new possible ways to develop mortgage modelling and to raise its efficacy. In forthcoming work we shall develop a mortgage model subject to default as well as prepayment risk. Another topic will be the study of the mortgage rate equation (8) and the relation between the mortgage rates implied by different model specifications (MMR and option-based approach) and long term Treasury yields. In the case of MMR specifications we shall generalize the mortgage equation to include the possibility of refinancing to different types of mortgages, say 30-year mortgage can be refinanced to 15- or 30-year mortgages. We will also consider numerical approximations of this equation.

References


\(^{31}\)It is the most advanced approach in the academic literature in certain sense.


