Equilibrium Impact of Value-At-Risk *

Markus Leippold†
University of Zürich

Fabio Trojani ‡
University of Southern Switzerland

Paolo Vanini §
University of Southern Switzerland

This Version: January 31, 2003
First Version: December 14, 2001

*The authors thank seminar participants at the University of Zürich, the Swiss Federal Institute of Technology Zürich, the University of Southern Switzerland, the Bendheim Center of Finance at Princeton University, the Quantitative Methods in Finance 2002 Conference, and the 9th Symposium on Finance, Banking, and Insurance in Karlsruhe. In particular, we thank Giovanni Barone-Adesi, Damir Filipovic, Rajna Gibson, Michel Habib, Ronnie Sircar, and Liuren Wu for their valuable comments. Fabio Trojani and Paolo Vanini acknowledge the financial support of the Swiss National Science Foundation (NCCR FINRISK). We welcome comments, including references to related papers we inadvertently overlooked.

†Correspondence Information: Markus Leippold, Swiss Banking Institute, University of Zürich, Plattenstrasse 14, 8032 Zürich, Switzerland, tel: (01) 634 39 62, mailto:leippold@isb.unizh.ch

‡Correspondence Information: Fabio Trojani, Institute of Finance, University of Southern Switzerland, Via Buffi 13, 6900 Lugano, Switzerland, mailto: fabio.trojani@lu.unisi.ch.

§Correspondence Information: Paolo Vanini, Institute of Finance, University of Southern Switzerland, Via Buffi 13, 6900 Lugano, Switzerland, mailto: paolo.vanini@lu.unisi.ch.
Abstract

We analyze Value-at-Risk based regulation rules and their effects on financial markets. Our model is formulated in a continuous-time economy where investors maximize expected utility subject to some regulatory Value-at-Risk constraint when asset price dynamics exhibit stochastic volatility. We show that in partial equilibrium the effectiveness of VaR regulation is closely linked to the “leverage effect”, the tendency of volatility to increase when prices decline. We then extend our analysis to a pure exchange economy and explore the implications of VaR regulation on equilibrium quantities such as interest rates and volatilities. Analysis of the general equilibrium model with heterogeneous investors indicates ambiguous economic effects. Nevertheless, we identify a tendency for lower interest rates and higher risk premia. Moreover, regulation leads to changes in the cross-sectional wealth distributions, which are independent of the banks’ risk aversion. Summarizing, it is not possible to make a clear statement whether Value-at-Risk as an endogenous risk measure supports the aim of regulation. In order to maintain the power of analytical expressions, different approximations are necessary in the model. Particular care is given to control the approximation errors either analytically or numerically.

JEL Classification Codes: G11, G12, G28, D92, C60, C61.

Key Words: Value-at-Risk, Regulatory Policy, Dynamic Financial Equilibria, Perturbation Theory, Stochastic Volatility, Monte Carlo.
Regulation aims at maintaining and improving the safety of the financial industry. At the root of every regulation framework lies the basic idea of calculating capital reserves as a function of the firm’s total capital and its risk. It might be tedious, but straightforward to calculate the firm’s capital. However, a less obvious task is the measurement of risk. In the 1996 Amendment of the Basle 1988 Accord, the Bank for International Settlement (BIS) extended their regulatory framework from credit risk to market risk. Regulators advocate normative and simplified rules applicable to all financial institutions. However, the Amendment recognizes the complexity of correctly assessing a risk figure to market risk and allows banks, satisfying some predefined qualitative standards, to use their internal models for regulatory reporting. In 1996, Value-at-Risk (VaR) based risk management had already emerged as common market practice. Thus, the BIS chose VaR as standard tool for regulatory reporting. In this paper, we explore the impacts of VaR based regulation, both in partial and general equilibrium, in economies with a stochastic opportunity set where stock returns exhibit stochastic volatility.

To our knowledge, Basak and Shapiro (2001) were the first to analyze the equilibrium impact of VaR constraints on asset prices in a model with continuous trading. In their model, VaR constrained investors take on a larger risk exposure than unconstrained investors, implying a deepening and prolongation of market downturns. This unappealing effect can be attributed to the lack of subadditivity of VaR. Indeed, adopting a coherent risk measure, Basak and Shapiro (2001) show that the above deficiency can be removed. Finally, they extend their model to a general equilibrium populated by homogenous log-investors. Closely related to Basak and Shapiro (2001) is the work by Berkelaar, Cumberayot, and Kouwenberg (2002). In a similar setup, they find that the presence of risk managers tends to reduce market volatility, as intended by regulators. Only in very bad states of the economy, risk managers would adopt a gambling strategy such that volatility is increased. They further explore the effects of VaR regulation on volatility smiles.

Cuoco, He, and Issaenko (2001) attribute the results of Basak and Shapiro (2001) to their static definition of the VaR constraint. In Basak and Shapiro (2001), banks have to report their VaR to the regulatory authority only at some specific date, while they are allowed to
continuously trade before and after that reporting date. This is clearly not consistent with the current regulatory framework, where banks need to calculate and adjust their VaR at least daily. Cuoco, He, and Issaenko (2001) extend the Basak and Shapiro (2001) model by letting the VaR limit be dynamically updated. Assuming geometric Brownian motion for asset price dynamics Cuoco, He, and Issaenko (2001) show that neither VaR nor Expected Shortfall based regulation induce riskier portfolio policies in partial equilibrium. However, they do not provide a general equilibrium analysis, nor do they extend their model to non-normal asset returns.

Danielsson and Zigrand (2001) analyze the general equilibrium effects of VaR constraints in a static economy and find that VaR regulation adversely affects prices, liquidity and volatility relative to a benchmark unregulated economy. In Danielsson, Shin, and Zigrand (2001) this analysis is extended to a multiperiod but still myopic setting, where it is shown by simulation that VaR exacerbates market shocks.¹

Our model is formulated in continuous time: Investors derive utility both from intermediate consumption and terminal wealth. In partial equilibrium, we compute optimal policies under VaR constraints, when stock price drift and volatility depend on a stochastic state variable. Therefore, our model is a fully dynamic intertemporal choice model which exhibits stochastic volatility. The non-constancy of volatility is well known in practice especially when markets are under pressure. Since volatility essentially drives VaR, it is in such stress situations where risk regulation is likely to become effective by risk figures binding at the risk limits. However, the price to pay when analyzing a model that comes closer to reality regarding its underlying assumptions about price dynamics, is the impossibility of obtaining fully analytical solutions. Since we nevertheless insist on analyticity we rely on approximation procedures. Besides the approximations, which gives us exact comparative static results, we carefully analyze the errors and higher order corrections: In particular, we use an Itô-Taylor approximation of VaR and consider the error bounds analytically and for different input parameters. For the optimal policy we rely on perturbation theory (see Kogan and Uppal (2002), Trojani and Vanini (2001)). In this approach, solutions can be constructed to arbitrary precision around the closed-form optimal policies of the log-utility investor. In partial equilibrium higher order terms are
derived and convergence under a-priori bounds is proven. The accuracy of the policies up to second order are compared to numerical solutions obtained by Monte Carlo simulations (see Detemple, Garcia, and Rindisbacher (2003) and Cvitanic, Goukasian, and Zapatero (2003)). A basic finding is the link between the effectiveness of VaR regulation to the ”leverage effect”, the tendency of volatility to increase when prices decline. According to the sign of the correlation between the risky asset and its volatility, the VaR constraint reduces or raises the risk exposure. If the correlation is negative, investment is reduced before the risk limits are reached due to the VaR constraint. Therefore, in this case incentives of the bank are not distorted. Such a negative correlation was found for example by Anderson, Benzoni, and Lund (2002) for S&P daily returns. Under the normality assumptions of Cuoco, He, and Issaenko (2001) the interrelationship between asset prices and volatility remains hidden. By “distortion of incentives”, we mean that, taking the regulator’s perspective, the presence of regulation has a destabilizing effect on the financial system by increasing the default probability of the financial institutions. In our paper, we directly connect such an increase in default probability to the bank’s exposure in the risky asset. If, compared to the unregulated economy, regulation leads to an increase in the bank’s exposure and to a suboptimal policy for the bank, we say that regulation distorts incentives.

After having analyzed our model in partial equilibrium, we embed the model in a general equilibrium with heterogeneous agents and explore the effects of VaR based regulation on equilibrium interest, asset price dynamics, and portfolio policies. Hence, VaR becomes an endogenous measure of risk. From this perspective, we follow the general equilibrium analysis in Basak and Shapiro (2001), imposing more realistic assumptions on a) the VaR measure used, b) the price dynamics, and c) allowing the economy to be populated by heterogeneous agents with risk-aversion different from log-utility. A main conclusion is the ambiguity of the results with respect to any incentive distortions. In other words, no decisive answer can be given in the models under consideration whether incentives of the financial institution are distorted or not when they are exposed to VaR constraints. In general, only tendencies can be deduced from our analysis. This is certainly not a result the financial industry and its regulators await,
as the implementation of industry-wide regulation standards is very costly. One would rather
expect to have a clear-cut knowledge about the outcome of applying such standards to the
financial sector.

There is more than one reason for the ambiguous results. First, rational general equilibrium
problems with stochastic volatility are hard to solve per se. To this complexity VaR adds even
more. The simplicity of VaR when used with exogenous investment strategy turns into a
difficult to a hardly-tractable problem if the strategies are the outcome in equilibrium. The
implicit definition of VaR and its non-linearity under stochastic volatility is not well-suited to
test for the sensitivity of regulation on economic variables. More specifically, for different model
specifications under different parameter constellations we find that for a low rate of economic
growth, there is a tendency for VaR regulation to decrease interest rates and to increase
volatility as well as the banks’ exposure to the risky asset. Finally, the heterogenous results
follow with minimal heterogeneity assumptions on the banks. More precisely, the institutions
differ only with respect to their initial wealth and the risk aversion. This contrasts the often
heard, but never proved, argument that uniformity of the regulation can pronounce systemic
crisis. Furthermore, if the risk heterogeneity is removed and the population of institutions is
split into a regulated and not regulated one, the VaR regulation still has an impact on the
homogeneous economy: Wealth is shifted from the regulated to the unregulated institutions.
This is surprising, since the VaR limits are chosen such that the explicit risk constraints are
wealth level independent. This shows that VaR as an endogenous risk measure has an effect on
economic variables in equilibrium which is not apparent in the formulation of the risk measure.

We remark that the framework offered in this paper is not only suitable to analyze the
effect of VaR regulation, but also would allow to analyze the consequences of VaR limits for
traders within a large bank. If traders can move the markets, VaR becomes an endogenous
risk measure. VaR limited traders would try to move the markets in a way that might not
yield the most favorable outcome for the bank.

The paper is structured as follows. The next section introduces the basic model setup.
Section ?? derives the VaR constraint and its approximation. Section 2 elaborates on the
bank's optimization problem and presents its solution in the presence of VaR constraints. Section 3 investigates the extension to general equilibrium with VaR constrained investors and derives the implications on interest rates, volatilities, and portfolio policies. Section 4 concludes.

1 The Model

The financial market consists of a risky asset with price $P_t$ and an instantaneous risk-free money-market account with value $B_t$. The dynamics of $B_t$ and $P_t$ are

$$dB_t = r(X_t)B_t dt, \quad B_0 = 1,$$

with $r(X_t)$ the risk-free rate process, and

$$dP_t = \alpha(X_t)P_t dt + \sigma(X_t)P_t dZ_t, \quad P_0 = p.$$ (2)

Uncertainty is modelled by a two dimensional Brownian motion $(Z_t, Z^K_t)$ having correlation $\mathbb{E}[dZ_t dZ^K_t] = \rho dt$. Drift and diffusion parameters in the asset’s dynamics depend on a one-dimensional state variable $X_t$. The process $X_t$ also follows an Itô diffusion

$$dX_t = \mu_X(X_t) dt + \sigma_X(X_t) dZ^K_t, \quad X_0 = x.$$ (3)

In the sequel, we split the drift of the asset price process into the short rate component $r(X_t)$ and a risk premium component $\lambda(X_t)$, i.e. $\alpha(X_t) = r(X_t) + \lambda(X_t)$.

We consider a bank selecting a portfolio fraction $w_t (1 - w_t)$ of current wealth $W_t$ invested in the risky asset (riskless asset). Thus the bank’s wealth dynamics are

$$\frac{dW_t}{W_t} = (w_t \lambda(X_t) + r(X_t))dt + w_t \sigma(X_t) dZ_t.$$ (4)

The regulator is supposed to define the bank’s constraint on market risk by means of a
VaR risk measure\(^2\).

**Definition 1.** The time-\(t\) VaR of a portfolio \(w_t\) given \(\mathbb{P}\)-probability level \(\nu \in (0, 1)\) and for a fixed time-horizon \(\tau > 0\) is defined by

\[
\text{VaR}^{\nu,w}_t = \inf \{ L \geq 0 \mid \mathbb{P}(W_t - W_{t+\tau} \geq L \mid \mathcal{F}_t) < \nu \},
\]

(5)

where \(W_{t+\tau}\) is the portfolio value at time \(t + \tau\) of a fixed-weight strategy with initial weight \(w_t\) at time \(t\).

Definition 1 is consistent with the VaR concept adopted by regulators to measure market risk. For reporting purposes the time-horizon \(\tau\) is chosen to be 1 day or 10 days.

In our model, the bank’s VaR is bounded at time \(t\) by an exogenous limit \(\text{VaR}_t\) for the given time-horizon \(\tau\). In the sequel we work with a VaR limit \(\text{VaR}_t\) proportional to current wealth\(^3\), i.e.,

\[
\text{VaR}_t(W, t) = \beta W_t, \beta \in [0, 1].
\]

(6)

The wealth dynamics (4) depend on the stochastic opportunity set \(X_t\). Thus, we cannot expect to obtain generally closed-form solution for the bank’s intertemporal decision problem in the presence of VaR constraints. To retain analytical tractability, we therefore carefully approximate the VaR constraint implied by (5), (6) and we apply the Itô Taylor formula to define the first order approximation,

\[
\log W_{t+\tau} \approx \log W_{t+\tau}^{(1)} = \log W_t + \left( r(X_t) + w_t \lambda(X_t) - \frac{1}{2} w_t^2 \sigma(X_t)^2 \right) \tau.
\]

(7)

The accuracy of the approximation (7) is quantified by the next proposition.

**Proposition 1.** The approximation error of the first-order approximation \(W_{t+\tau}^{(1)}\) for the value
of a fixed-weight portfolio with initial weight $w_t$ is bounded by

$$
P \left( \left| \log W_{t+\tau}^{(1)} - \log W_{t+\tau} \right| \geq M \, |\mathcal{F}_t| \right) \leq \frac{1}{M} \mathbb{E} \left[ |\mathcal{R}| \right],$$

where

$$
\mathbb{E} \left[ |\mathcal{R}| \right] = \left| \int_t^{t+\tau} \int_s^t \mathbb{E} \left[ \mathcal{L}r(X_u) + w_t \mathcal{L} \chi(X_u) + \frac{1}{2} w_t^2 \mathcal{L} \sigma(X_u)^2 \right] \mathrm{d}u \mathrm{d}s \right|,
$$

and $\mathcal{L} = \mu_X \frac{\partial}{\partial X} + \frac{1}{2} \sigma_X^2 \frac{\partial^2}{\partial X^2}$ is the infinitesimal generator of $X$.

Proposition 1 can be interpreted as follows: $P(\cdot |\mathcal{F}_t)$ is the conditional probability that the logarithmic difference between the approximated wealth and the true wealth exceeds a prespecified value $M$ at time $t + \tau$. If we choose, say, $M = 0.10$, Proposition 1 offers a bound on the probability that $\log W_{t+\tau}^{(1)}$ and $\log W_{t+\tau}$ differ by more than 10%. The accuracy of the VaR approximation (8) is illustrated in Table 1, where we assumed a mean-reverting geometric Brownian motion for the volatility process. In this case, $\mathbb{E} \left[ |\mathcal{R}| \right]$ can be computed in closed form. The results are presented for two different conditioning values of the state variable, $X_t = 1$ and $X_t = 3$. For lower values of $X_t$ the approximation is even more accurate. Table 1 shows that the approximation bounds are generally very tight. For instance, the bound on the error probability for $M = 1$, $X = 1$ and an horizon $\tau = 10$ is always below 0.0001. The bounds for $X = 3$ are always below 0.01 and in many cases below 0.0005.

The quality of the above approximation results suggests that an approximate VaR calculation based on (7) could be reasonably used to investigate the theoretical properties of portfolio selection under VaR constraints. In excess of this, the common market practice, where VaR figures are reported based on a conditional normal distribution assumption, further motivates our approach.

The advantage of using (7) for computing an approximate VaR constraint is that it implies some direct portfolio bounds on the optimal policy of the VaR constrained bank. These are given in the next proposition.

**Proposition 2.** To first order, the constraint $\text{VaR} \leq \overline{\text{VaR}}$ is equivalent to the following upper
and lower bound on the fraction $w_t$ of wealth invested in the risky asset,

$$w_b^-(X_t) \leq w_t \leq w_b^+(X_t),$$

where

$$w_b^\pm(X) = \frac{\lambda(X)}{\sigma(X)^2} + \frac{v}{\sigma(X)\sqrt{\tau}}$$

$$\pm \frac{\sqrt{(\lambda(X)\tau + \sigma(X)\sqrt{\tau}v)^2 + 2\sigma(X)^2\tau(r(X)\tau - \log(1 - \beta))}}{\sigma(X)^2\tau},$$

with $v = N^{-1}(\nu)$ the $\nu$-quantile of the standard Normal distribution.

The functional form (6) for the VaR limit implies a bound on the optimal portfolio fraction that is wealth independent$^4$.

The basic structure of the bounds in (10) corresponds to a discrete time confidence interval for $w_t$. The interval is centered at the point

$$\frac{\lambda(X)\tau + \sigma(X)\sqrt{\tau}v}{\sigma(X)^2\tau}.$$

This entity can be interpreted as the optimal policy of a log-investor that computes expected excess returns based on a Gaussian VaR threshold return for the time horizon $\tau$. By inspection of equation (10), we have

$$w_b^+(X) \geq 0, \quad w_b^-(X) \leq 0,$$

for any $X$. This holds for all functional forms $\lambda(X)$ and $\sigma(X)$. Hence, when volatility is an increasing function of $X$ we can expect the portfolio bounds $w_b^\pm$ to tend to $\pm \infty$ as $X \to \infty$.

In the next sections we will make use of some polynomial functional forms

$$\lambda(X) = \lambda X_t^{n_1}, \quad \sigma(X) = \sigma X_t^{n_2}, \quad r(X) = r X_t^{n_0},$$

where $\lambda, \sigma, r > 0$ and $n_0, n_1, n_2 \in \mathbb{N}$ in order to obtain some more explicit characterizations of
the impact of VaR constraints on the optimal policies of a VaR constrained investor; cf. also Assumption 2 below. For these model settings, the parameters \( n_0, n_1, \) and \( n_2 \) will have to be selected with some care in order to avoid trivial settings where VaR constraints are either vacuous or binding a priori.

2 Partial Equilibrium

This section is devoted to the analysis of VaR constrained optimal policies in a partial equilibrium framework.

2.1 Optimal Policies

To define the bank’s partial equilibrium optimization problem under VaR constraints we start from the following standard assumption on preferences.

**Assumption 1.** The bank derives utility from final wealth \( W_T \) according to a CRRA-utility function

\[
    u(W, T) = \frac{W^\gamma - 1}{\gamma}, \gamma < 1,
\]

where \( \gamma > 0 \) is a time preference rate.

According to Assumption 1, the bank derives utility from terminal wealth \( W_T \) only. The model will be extended in Section 2.4 to allow also for the presence of intermediate consumption streams.

The major goal of this section is to investigate the partial equilibrium impact of VaR constraints in the presence of a stochastic opportunity set. Detailed characterizations of these issues are derived under the following parametric assumptions on \( X' \)’s dynamics, on \( \lambda_t, \sigma_t \) and \( r_t \).
**Assumption 2.** $X_t$ follows a mean reverting process given by

$$dX_t = (\theta - \kappa X_t)dt + \sigma_X X_t^m dZ_t^X,$$

where $\kappa, \theta, \sigma_X$ are positive constants. The risk premium, the risky asset volatility and the interest rate processes are of the form

$$\lambda(X_t) = \lambda X_t^{n_1}, \quad \sigma(X_t) = \sigma X_t^{n_2}, \quad r(X_t) = r,$$

where $\lambda, \sigma, r, m, n_1, n_2$ are positive constants.

Notice that Assumption 2 comprises model settings where volatility is stochastic or where variance-in-mean effects are present, for instance when $n_1 \neq 0$, $n_2 \neq 0$ and $m \neq 0$. On the other hand, when $n_1 = 0$ we can investigate the impact of VaR constraints under a pure stochastic volatility setting. In the case $n_1 = 2n_2$, the unconstrained Merton policy of a log-utility investor is a constant

$$\lambda(X_t)/\sigma(X_t)^2 = \lambda/\sigma^2.$$

Therefore, in this case VaR constraints are either vacuous or binding a priori. Further, when restricting the parameter $m$ to $m = 0$, $m = \frac{1}{2}$ or $m = 1$ we obtain a model class that encompasses the most popular stochastic volatility models, like the lognormal, the square-gaussian, or the square-root model of Heston (1993), and at the same time allows for closed forms of all moments and cross-moments of $X_t$. Existence of such closed form expressions are necessary to obtain analytical asymptotic characterizations of the relevant optimal policies and equilibrium quantities in the presence of VaR constraints. Generally, computation of the moments of $X_t$ can be achieved by means of the next technical result.

**Lemma 1.** Under Assumption 2 the $k$-th moment $\mathbb{E}[X_{t+\tau}^k | \mathcal{F}_t]$, $k \geq 1$, solves the differential
equation

\[
\frac{d\mathbb{E} \left[ X_{t+\tau}^k \mid \mathcal{F}_t \right]}{dt} = k \left( \theta \mathbb{E} \left[ X_{t+\tau}^{k-1} \mid \mathcal{F}_t \right] - \kappa \mathbb{E} \left[ X_{t+\tau}^k \mid \mathcal{F}_t \right] + \frac{k-1}{2} \sigma^2 \mathbb{E} \left[ X_{t+\tau}^{k+2} \mid \mathcal{F}_t \right] \right),
\]

subject to \( \mathbb{E} \left[ X_t^k \mid \mathcal{F}_t \right] = X_t^k \).

We now define the VaR-constrained portfolio optimization problem of our bank, based on the approximate VaR constraint implied by Proposition 1\textsuperscript{5}. Denoting by \( \text{BR}(W,X) \) the budget constraints (3), (4) and by \( \mathcal{C}(X) \) the approximate VaR constraint (10), the optimization problem of a VaR constrained investor can be written as

\[(P1) : \quad J(W,X,t) = \max_{w \in \mathcal{R}(W,X)} \mathbb{E} [u(W_T,T) \mid \mathcal{F}_t],\]

where \( \mathcal{R}(W,X) = \{ w \in \text{BR}(W,X) \} \cap \mathcal{C}(X) \). Intuitively, the solution of problem \((P1)\) has to provide optimal investment strategies characterized by a region where the VaR constraint binds and a region where it does not bind. This is the content of the next proposition.

**Proposition 3.** Consider the control problem \((P1)\) under the Assumptions 1 and 2. In a region where the \( J \)-function is increasing and jointly strictly concave in \( W \) and \( X \), the policy solving \((P1)\) is

\[
w^*(X,t) = \begin{cases} 
  w_b^+(X,t), & \text{if } w_f(X,t) \geq w_b^+(X,t), \\
  w_b^-(X,t), & \text{if } w_f(X,t) \leq w_b^-(X,t), \\
  w_f(X,t), & \text{else},
\end{cases}
\]

where

\[
w_f(X,t) = -X^{n_1-2n_2} \frac{\lambda J_W}{\sigma^2 W J_{WW}} - \rho X^{m-n_2} \frac{\sigma X J_W X}{\sigma W J_{WW}}, \tag{11}
\]

and \( w^\pm_b \) is given in (10).

Apparently, inserting the optimal policies back into the HJB equation for problem \((P1)\) leads to a non-explicitly solvable PDE. We therefore apply an asymptotic analysis that computes the value function and the implied optimal policies as a power series in \( \gamma \).
2.2 Perturbation with respect to Risk Aversion

From the homogeneity properties of problem (P1), the value function $J$ has to be of the form

$$J(W, X, t) = e^{\gamma g(X, t)} W^{\gamma - 1},$$

with $g(X, t)$ an unknown function. Expanding the function $g(X, t)$ in the parameter $\gamma$, i.e.

$$g(X, t) = g_0(X, t) + \gamma g_1(X, t) + \frac{1}{2} \gamma^2 g_2(X, t) + O(\gamma^3),$$

the optimal policy up to first order in $\gamma$ is easily obtained from (11), (12), and (13) as

$$w_f^{(1)}(X, t) = (1 + \gamma) X_t^{n_1 - 2n_2} \frac{\lambda}{\sigma^2} + \gamma X_t^{n_2 - n_2} \frac{\rho \sigma X}{\sigma} \frac{\partial g_0(X_t, t)}{\partial X}.$$ 

Therefore, if $g_0(X, t)$ can be determined, the first-order policy of an investor with risk aversion $1 - \gamma$ is also determined. Equation (14) gives the approximate optimal policy as a sum of two approximate subpolicies. A myopic portfolio

$$(1 + \gamma) X_t^{n_1 - 2n_2} \frac{\lambda}{\sigma^2}$$

and an intertemporal hedging position

$$\gamma X_t^{n_2 - n_2} \frac{\rho \sigma X}{\sigma} \frac{\partial g_0(X_t, t)}{\partial X}.$$ 

As usual, the difficulty in characterizing the optimal portfolio solution under a stochastic opportunity sets consists precisely in determining the optimal intertemporal hedging policy. Moreover, characterizing the hedging policy of a VaR constrained investor is a crucial issue in order to understand the equilibrium impact of VaR constraints, as we show below. Since we want to analyze the impact of VaR constraints (i) under quite general model settings and (ii) for several dynamic specifications of the economy, we make use of a perturbation approach that allows us to analyze the qualitative impact of VaR constraints for a quite broad range of
In the perturbation approach, the problem of characterizing the intertemporal hedging policy is easily circumvented as soon as the function $g_0$ can be computed. By construction, the unknown function $g_0$ is fully determined as soon as we can compute the value function of a log-utility investor. This is the basic idea behind the next proposition, which gives a way to compute $g_0$ as a conditional expectation of some random variables that involve only some known functions of the future states of the process $(X_t)$.

**Proposition 4.** Consider the optimization problem (P1). The function $g_0$ in (14) reads

$$g_0(X, t) = g_0^f(X, t) - \frac{\sigma^2}{2} \int_t^T \mathbb{E} \left[ \mathbb{1}_{|w_{\log}^f| > |w_{b}^{\pm}|} (\hat{w}_{b}^{\pm} (X, s) X_s^{n_2})^2 | \mathcal{F}_t \right] ds,$$

$$g_0^f(X, t) = r(T - t) + \frac{\lambda^2}{2 \sigma^2} \int_t^T \mathbb{E} \left[ X_s^{2(n_1 - n_2)} | \mathcal{F}_t \right] ds,$$

$$w_{\log}^f(X, t) = X_t^{n_1 - 2n_2} \frac{\lambda}{\sigma^2},$$

$$\hat{w}_{b}^{\pm} (X, t) = w_{b}^{\pm} (X, t) - X_t^{n_1 - 2n_2} \frac{\lambda}{\sigma^2},$$

where $g_0^f$ is the function $g_0$ prevailing in the absence of VaR constraint, and $w_{\log}^f(X, t)$ is the uncontrained optimal portfolio policy of a log utility investor.

With Proposition 15 we are now in a position where a computation of the expectations in $g_0$ and $g_0^f$ immediately provides a first order approximation to the optimal portfolio of a VaR constrained investor for the class of models defined in Assumption 2. The quality of these approximations can be improved by considering higher order terms in the perturbation series. Before studying in detail the partial equilibrium effects of VaR constrains, we show in the next section how such higher order approximations can be obtained and how convergence to the underlying solution can be achieved under appropriate assumptions.

### 2.3 Accuracy and Convergence of the Perturbation Approach

Before analyzing the impact of VaR constrains in more detail, we illustrate for an explicit model setting the accuracy of the above perturbative approach. We proceed in three steps.
First, we show how higher order corrections for the optimal policies of problem \( P_1 \) can be obtained. Second, we demonstrate that for some given “reasonable” a priori bounds on the terms in the perturbation series the whole series converges. Third, we compute different finite order approximations and investigate in a numerical example the speed of convergence of the perturbation series. Specifically, we apply Lemma 1 to calculate the function \( g_0 \). In order to calculate \( g_0 \), we have generally to resort to numerical methods.

To analyze the accuracy of the first order approximations in the last section we first consider some higher order policy approximations. From (11), (12), and (13), the \( n \)-th order policy approximation is easily obtained as

\[
\begin{align*}
  w_f^{(n)}(X,t) &= X_t^{n_1-2n_2} \frac{\lambda}{\sigma^2} \frac{1 - \gamma^{n+1}}{1 - \gamma} + X_t^{m-n_2} \frac{\rho \sigma X}{\sigma} \sum_{j=0}^{n-1} \frac{\gamma^{j+1}}{j!} \frac{\partial g_j(X,t)}{\partial X} \frac{1 - \gamma^{n-j}}{1 - \gamma}. 
\end{align*}
\]  

Some natural a-priori bound of on the unknown functions \( g_j(X,t), j \in \mathbb{N} \), which ensure convergence of the perturbation theory to the correct limit policy are given in the next result.

**Corollary 1.** Consider the control problem \( (P_1) \). The optimal policy \( w_f^{(n)} \) for a non-binding VaR constraint up to order \( n \) converges toward \( w_f \) if \( \frac{\partial g_j(X,t)}{\partial X} \leq K X_t^{2j(n_1-n_2)}, K > 0, j = 0, \ldots, n \). The limit policy is bounded by

\[
\begin{align*}
  w_f(X,t) &= \lim_{n \to \infty} w_f^{(n)}(X,t) \leq X_t^{n_1-2n_2} \frac{\lambda}{\sigma^2} \frac{1}{1 - \gamma} + X_t^{m-n_2} \frac{\rho \sigma X \gamma e X_t^{2(n_1-n_2)}}{\sigma(1 - \gamma)} K. 
\end{align*}
\]  

Having established convergence of the perturbation series, we need to compute the higher order functions \( g_j, j \geq 1 \), in order to improve the accuracy of a first order approximation based on \( g_0 \). This task is accomplished next. Specifically, we first define \( \log W_T = \log W_t + H_{t,T} \) for the wealth dynamics, where

\[
H_{t,T} := \int_t^T \left( r + \lambda X_s^{n_1} w_s - \frac{1}{2} w_s^2 \sigma^2 X_s^{2n_2} \right) ds + \int_t^T w_s \sigma X_s^{n_2} dZ_s.
\]
Then, we first expand $g(X, t)$ as a power series in $\gamma$, we insert it in the equality

$$
\mathbb{E}\left[\frac{W^\gamma - 1}{\gamma} \bigg| \mathcal{F}_t\right] = \frac{e^{\gamma g(X)}W^\gamma - 1}{\gamma},
$$

and finally expand also the LHS and the RHS of (18) as a power series in $\gamma$. By matching the resulting coefficient for each power of $\gamma$ in the resulting power series we obtain the following characterizations for some of the higher order functions $g_j(X, t)$.

**Corollary 2.** Consider problem (P1). If policies are not binding in the interval $[t, T]$, we have:

\[
\begin{align*}
g_0(X, t) &= \mathbb{E}[H_{t,T} | \mathcal{F}_t] \\
g_1(X, t) &= \frac{1}{2} \text{Var}[H_{t,T} | \mathcal{F}_t], \\
g_2(X, t) &= \frac{1}{6} \left( \mathbb{E}[H_{t,T}^3 | \mathcal{F}_t] + 2\mathbb{E}[H_{t,T}^2 | \mathcal{F}_t]^3 - 3\mathbb{E}[H_{t,T} | \mathcal{F}_t] \mathbb{E}[H_{t,T}^2 | \mathcal{F}_t] \right), \\
g_3(X, t) &= \frac{1}{8} \left( \mathbb{E}[H_{t,T}^4 | \mathcal{F}_t] - 6\mathbb{E}[H_{t,T}^3 | \mathcal{F}_t]^4 + 12\mathbb{E}[H_{t,T} | \mathcal{F}_t]^2 \mathbb{E}[H_{t,T}^2 | \mathcal{F}_t] \right. \\
&\quad \left. - 3\mathbb{E}[H_{t,T}^2 | \mathcal{F}_t]^2 - 4\mathbb{E}[H_{t,T} | \mathcal{F}_t] \mathbb{E}[H_{t,T}^3 | \mathcal{F}_t] \right).
\]

Corollary 2 shows the direct relation that exists between the order of the $g_j$-functions in the perturbation series and the higher moments of the return $H_{t,T}$ on optimally invested wealth. In particular, we see that when the $n$-order $g$-function enters is the approximation for the optimal portfolio policy, the approximation is taking into account the $(n + 1)$-th moment of $H_{t,T}$. Moreover, the influence of the $(n + 1)$-th moment on the $n$-th order approximation decreases rapidly, i.e. proportionally to $1/(n + 1)!$.

To illustrate more concretely the accuracy of the above approximation results we now compare for an explicit model setting the finite order portfolio policy approximations obtained with the above procedure and the corresponding exact portfolio policy. We do this for a model setting corresponding to $n_1 = 2$, $n_2 = 1$, $r = 0$ in Assumption 2. Exact solutions are obtained by means of Monte Carlo simulation methods; see for instance Cvitanic, Goukasian, and Zapatero (2003) and Detemple, Garcia, and Rindisbacher (2003).
Further, since we perform perturbations exclusively with respect to the risk aversion parameter $\gamma$ and since our form of VaR constraint is $\gamma$-independent, we focus for brevity on an unconstrained portfolio optimization.

Since for the given model setting we know the log-investor’s portfolio decision in closed-form, we use this solution as a control variate. The following parametric model specification is assumed in the sequel:

$$
\frac{dP_t}{P_t} = 0.05X_t^2 dt + 0.25X_t dZ_t,
$$

$$
dX_t = (0.8 - 0.8X_t) dt + 0.2X_t dZ_t.
$$

Figure 1 illustrates the findings of our Monte Carlo simulations. Point $A$ in Figure 1 gives the log-investor’s optimal portfolio policy. The bold straight line in the figure is the first-order approximation to the underlying portfolio policy. The second-order approximation is represented by the bold dashed line. Finally, the line with circle marks is the exact portfolio policy estimated by Monte Carlo simulation. already comes fairly close to the Monte Carlo simulation. As is apparent from Figure 1 the second order order approximations produces very accurate results over the given support for the parameter $\gamma$. The first order approximation does not fit the concavity of the portfolio profile as a function of $\gamma$, but still produces the correct sign about the effects obtained for higher/lower risk aversion parameters than in the log utility case.

### 2.4 Incorporating Intermediate Consumption

This section extends the previous analysis to incorporate intermediate consumption in the bank’s optimal wealth dynamics. In this way we can consider the impact of VaR constraints on the optimal expenditure policy of a bank. Moreover, introducing consumption will allow us to endogeneize in a later section the risky assets dynamics in the presence of VaR constrained investors. We interpret intermediate consumption as intermediate cash-outflows caused by the bank’s expenditure policy. This puts us back to the known portfolio problem where utility is
derived from both terminal wealth and intermediate consumption. Assumption 1 is extended as follows.

Assumption 3. Banks derive utility from both an intermediate consumption process \((C_t)_{0 \leq t \leq T}\) as well as from terminal wealth \(W_T\). They maximize the expected value of the random variable

\[
V(W_T, C, T) = \int_0^T e^{-\delta s} \frac{C_T^\gamma - 1}{\gamma} ds + e^{-\delta T} \frac{W_T^\gamma - 1}{\gamma}, \quad 0 \leq \delta < 1,
\]

where \(\delta \geq 0\) is the subjective discount rate.

In the presence of intermediate consumption, the wealth dynamics are

\[
dW_t = \left((r - c_t) + w_t \lambda(X_t)\right) W_t dt + w_t \sigma(X_t) W_t dZ_t,
\]

where \(c_t = C_t/W_t\) is the consumption rate. It follows from equation (20) that future current wealth in the presence of a consumption policy is given by

\[
\log W_T = \log W_t + H_{t,T},
\]

with

\[
H_{t,T} := \int_t^T \left(w_s \lambda(X_s) + r - c_s - \frac{1}{2} w_s^2 \sigma(X_s)^2 \right) ds + \int_t^T w_s \sigma(X_s) dZ_s.
\]

We define gain the relevant VaR limit as a fraction of current wealth, as in (6). Further, recall that there we assumed a constant portfolio \(w_t\) over the relevant time horizon \((t, t + \tau)\). In a similar vain, we define a new VaR constraint where consumption is locked-in at the consumption rate prevailing at time \(t\). Under Assumption 2 and using the results and the proof of Proposition 2, the Itô-Taylor approximated VaR constraint in the presence of intermediate consumption is given by

\[
Q(w, c) = r - c + \lambda X^{n_1} w - \frac{1}{2} w^2 \sigma^2 X^{2n_2} - \frac{1}{\tau} \log(1 - \beta) + \frac{1}{\sqrt{\tau}} w \sigma X^{n_2} \leq 0.
\]
For \((c, w)\), we thus obtain a constrained set \(C(X)\) given by

\[(c, w) \in C(X) = \{Q(w, c) \leq 0\} \cap \{c > 0\}.\]

Since \(Q(w, c)\) is a quadratic function of \(w\) and a linear function of \(c\), the set \(\{(c, w) : Q(w, c) = 0\}\) describes a parabola in \((w, c)\)-space. This is illustrated in Figure 2.

The portfolio fractions \(w^\pm_b\) are obtained as the solutions of the equation \(Q(c, w) = 0\), given any feasible consumption rate \(c\). Note that the convexity of the constrained set \(C(X)\) is ensured by the fact that \(\partial^2 Q(c, w)/\partial w^2 < 0\). Consequently, an increase in volatility due to an increase of \(X\) makes the set \(C(X)\) is smaller.

Under Assumption 3, the relevant optimization of a VaR constrained investor is now of the form

\[(P2): \quad J(W, X, t) = \max_{(c, w) \in \mathcal{R}(W, X)} \mathbb{E} [V(W, C, T) | \mathcal{F}_t].\]

As in the case without consumption, we compute an asymptotic solution for this problem by exploiting the homogeneity properties of the implied HJB equation.

**Proposition 5.** Given Assumption 3 and the optimization problem \((P2)\), the first-order approximations of the optimal policies are given by

\[w^{(1)}(X, t) = (1 + \gamma)X_t^{m-n} \frac{\Lambda}{\sigma^2} + \gamma X_t^{m-n} \frac{\rho \sigma X}{\sigma} \frac{\partial g_0(X_t, t)}{\partial X},\]

\[c^{(1)}(X, t) = \frac{1 - \gamma (g_0(X_t, t) + \log A_t)}{A_t},\]

with \(A_t = e^{-\delta(T-t)} + \frac{1-e^{-\delta(T-t)}}{\delta}\) and

\[g_0(X, t) = \frac{1}{A_t} e^{-\delta(T-t)} \mathbb{E} [H_{t,T} | \mathcal{F}_t] + \frac{1}{A_t} \int_t^T e^{-\delta(s-t)} (\mathbb{E} [H_{t,s} | \mathcal{F}_t] - \log A_s) ds.\]
The function $E[H_{t,T} | \mathcal{F}_t]$ is defined as

$$E[H_{t,T} | \mathcal{F}_t] = E\left[H_{t,T}^f | \mathcal{F}_t\right] - \frac{\sigma^2}{2} \int_t^T E\left[I\{w_{\log}^f c_{\log}\} \mathcal{C}(X)\} (\hat{w}^\pm(X,s)X_s^{n_2})^2 | \mathcal{F}_t\right] ds,$$

$$E\left[H_{t,T}^f | \mathcal{F}_t\right] = r(T-t) - \int_t^T A_s^{-1} ds + \frac{1}{2} \frac{\lambda^2}{\sigma^2} \int_t^T E\left[X_s^{2(n_1-n_2)} | \mathcal{F}_t\right] ds,$$

$$\hat{w}^\pm(X,t) = w^\pm(X,t) - \frac{\lambda(X_t)}{\sigma(X_t)^2}.$$

In Proposition 5 the function $E[H_{t,T} | \mathcal{F}_t]$ is the sum of two terms. If there were no constraints, $H_{t,T}$ would degenerate to the first term $E[H_{t,T}^f | \mathcal{F}_t]$. The influence of the second term in $E[H_{t,T} | \mathcal{F}_t]$ acts like a utility loss that is caused by the presence of regulatory VaR constraints. As expected, this second term is related to the regulatory variables $\beta$ and $\nu$. Precisely, even when the VaR constraint is not binding today investors anticipate that it could bind in the future. This reduces the utility that can be expected to be obtained by applying a given optimal policy and affects the current optimal decision. Thus, the existence of VaR constraints produces a non trivial anticipative effect on the banks decision. This dynamic optimal behavior is strongly different from the one implied by a myopic portfolio optimization as for instance in Danielsson, Shin, and Zigrand (2001).

### 2.5 Partial Equilibrium: Do VaR Constraints Distort Incentives?

As mentioned in the introduction, Basak and Shapiro (2001) conclude that incentives are distorted in risk management when a VaR based regulation framework is imposed on the economy. They find that VaR regulation lets stock market volatility and the risk premium increase in down markets and decrease in up markets. Refining the assumptions of the Basak and Shapiro (2001) model, Cuoco, He, and Issaenko (2001) come to a different conclusion for the partial equilibrium: neither VaR nor any other coherent risk measure give rise to distortions. Cuoco, He, and Issaenko (2001), however, do not provide a general equilibrium analysis of VaR incentives and do not consider asset price dynamics with stochastic volatility.

Before embedding our model into general equilibrium, we summarize and briefly discuss the
resulting incentives for the bank’s portfolio policies in partial equilibrium. From Proposition 5, we obtain the following corollary characterizing the solutions of (P2) relative to the portfolio problem without VaR constraints.

**Corollary 3.** Consider the partial equilibrium problem (P2) with solutions approximated to first-order in $\gamma$. Before hitting the VaR constraints,

i) banks with $\gamma > 0$ increase optimal consumption $c_t$ and banks with $\gamma < 0$ decrease consumption.

ii) Given $\partial g/\partial X > 0$, banks with $\gamma > 0$ increase (decrease) the portfolio fraction invested in the risky asset if $\rho < 0$ ($\rho > 0$). Banks with $\gamma < 0$ will decrease (increase) their exposure in the risky asset.

iii) Given $\partial g/\partial X < 0$, banks with $\gamma > 0$ increase (decrease) the portfolio fraction invested in the risky asset if $\rho < 0$ ($\rho > 0$). Banks with $\gamma < 0$ will decrease (increase) their exposure in the risky asset.

If volatility is deterministic and hence the price process follows a geometric Brownian motion, then the presence of VaR regulation

iii) does not affect portfolio policies.

v) increases consumption $c_t$ of banks with $\gamma > 0$ and decreases consumption for banks with $\gamma < 0$.

At least two remarks are necessary. *First*, from the second part of the corollary, we see that the conclusion of Cuoco, He, and Issaenko (2001) is a peculiarity of their restrictive assumption: When prices follow a geometric Brownian motion, the VaR constraints have no anticipatory impact on portfolio strategies. Banks only adjust their portfolio holding when they hit the VaR constraint. However, VaR constraints influence consumption yet before the constraint becomes binding.
Second, from the first part of Corollary 3, VaR constraints lead either to an increase or to a decrease in the bank’s exposure to the risky asset. The crucial role is played by, a) the risk aversion coefficient $\gamma$, and b) the correlation $\rho$ between changes in the instantaneous volatility and the changes in the asset price. The correlation $\rho$ determines the direction of hedging demand against changes in $X$ (see Proposition 3).

If correlation is negative and $\gamma > 0$, VaR has an adverse effect on the riskiness of the banks portfolio decision. The bank increases its position in the risky asset. Only if correlation is strictly positive does VaR what the regulator intends it to do: decrease the exposure in the risky asset. Empirical evidence suggests that stock price movements are negatively correlated. Such evidence was already reported in Black (1976), Christie (1982), and Schwert (1989), and is commonly referred to as “leverage effect”. More recent empirical investigations in Anderson, Benzoni, and Lund (2002) find a correlation$^{10}$ of $\rho = -0.4$ for S&P 500 daily returns during the period from 1/3/1980 to 12/31/1996. If $\gamma < 0$, the above effects change signs.

To illustrate the above findings, we provide a graphical analysis of Corollary 3 for a specific numerical example. We assume a lognormal volatility with mean-reversion. More precisely, we set $\lambda(X) = \lambda X^2$, $\sigma(X) = \sigma X$, and $\sigma_X(X) = \sigma_X X$. We first graphically analyze the optimal portfolio policy when the bank is approaching the VaR constraint. Figure 3 plots the optimal portfolio strategies for the constrained and unconstrained investor given $\gamma > 0$. Panel (A) plots the portfolio strategies when the correlation between $X_t$ and the asset return is positive. For a one year investment horizon, the difference between the two investors are small. However, in Panel (C) with a five year time horizon, the portfolio fraction in the risky assets has already been substantially reduced before the bank hits the VaR constraint. Therefore the mere presence of VaR constraints, although not yet binding, leads to a reduction in the bank’s exposure to the risky asset. This follows from the fact that the bank has to take into account the current value of future constraints into today’s investment decision. Similar effects are shown in Panel (E) for a ten year time horizon. However, if we look at the graphs on the right of Figure 3, we observe that when $\rho$ is negative the effect of VaR constraints is reversed. As long as the VaR constraint does not bind, the bank is more exposed in the risky asset than in the
absence of VaR constraints. Only when the constraints are binding and the volatility increases further, the bank will finally end up with a lower exposure than in the unconstrained economy. Therefore, the effect of VaR regulation strongly depends on the sign of the correlation between changes in asset returns and changes in asset volatility. As soon as we have $\rho < 0$, i.e. the “leverage effect”, the bank increases its exposure in the risky asset. Remarkably, the increase is more substantial the longer the investment horizon of the bank is. Similar issues follow for the case $\gamma < 0$, but with opposite signs.

As noted in Cuoco, He, and Issaenko (2001), a dynamic VaR setting allows to transform a VaR limit into an equivalent Expected Shortfall limit, and vice versa (see e.g. Acerbi and Tasche (2001)). Thus, in our setting with an approximated VaR constraint up to first order, imposing an Expected Shortfall limit is equivalent to imposing a tighter confidence bound on the VaR limit. The impact of a tighter confidence bound in the model with stochastic volatility is plotted in Figure 4 where we use the model specification given above. Again we distinguish between positive (Panel (A), (C), and (E)) and negative correlation (Panel (B), (D), and (F)). In Panel (A) and (B) we assume $\nu = 0.01$ and in Panel (C) and (D) we take $\nu = 0.05$. Again, as in Figure 3, we observe for both confidence bounds an increase in the bank’s exposure when correlation is negative. In Panel (E) and (F) we plotted the differences between the portfolio policies under the two confidence levels. Intuitively, one expects portfolio policies to decrease when moving from $\nu = 0.05$ to $\nu = 0.01$. Surprisingly, this is not quite the case when correlation is negative. As shown in Panel (F), tightening the confidence level gives the bank an incentive to increase its exposure at low volatility levels. Only for a volatility larger than about 50% does a tighter confidence level lead to a decrease in the bank’s risk exposure.

3 General Equilibrium

To fully assess the impact of VaR regulation on the economy, it is not sufficient to analyze portfolio strategies in a partial equilibrium setting. Indeed, through feedback effects, changes in portfolio strategies will eventually impact other risk measurement relevant quantities, such
as interest rates, volatilities, and risk premia. We therefore extend our model to a general equilibrium setting. The previous section’s results on partial equilibrium build the basis for this analysis.

Using perturbation theory, the computation of the general equilibrium amounts to knowing the optimal policies of the log-investor in a *homogeneous economy*, avoiding to solve for the heterogeneous problem *ab initio*. Admittedly, this methodology does not provide exact results, but asymptotic approximations. However, comparative statics are exact. Further, as in the partial equilibrium case, approximations can be improved by considering higher-order terms. Finally, using perturbation theory the equilibrium expressions can be split into economically interpretable terms.

3.1 The Exchange Economy

We consider an exchange economy. Our financial market consists of two financial instruments, an instantaneous risk-free asset and a stock. The risk-free asset is available in zero net supply and the interest rate $r_t$ is determined in equilibrium together with the asset price dynamics. The risky asset is a contingent claim on aggregate endowment, $e_t$, given as

$$\frac{de_t}{e_t} = \mu_e(X,t)dt + \sigma_e(X_t)dZ^e_t,$$  \quad (23)

with $Z^e_t$ a standard Brownian motion. The state variable $X_t$ follows an Itô diffusion

$$dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dZ^X_t.$$  \quad (24)

The instantaneous correlation between the endowment process and the state variable $X_t$ is given by $E(dZ^e_t dZ^X_t) = \rho_{eX}$.

In general equilibrium, our model is populated by two banks$^{12}$, labelled with indices $I$ and $II$, facing problems $(P2)^I$ and $(P2)^{II}$, respectively. The programs $(P2)^{(i)}$, $i = \{I, II\}$, are equivalent to $(P2)$ with risk aversion index $\gamma$ replaced by $\gamma_i$. Therefore, the two banks are
heterogeneous with respect to their risk aversion only. The variables $\omega^I_t$ and $\omega^H_t = 1 - \omega^I_t$ denote the cross-sectional distribution of wealth, i.e.

$$\omega^i_t = \frac{W^i_t}{W^I_t + W^H_t}, \ i = I, II,$$

where $W^i_t$ is the current wealth of bank $i$. Compared to the partial equilibrium analysis, the state of the exchange economy is described by the state vector $\chi_t = (X_t, \omega^I_t)^\top$. The cumulative stock equals aggregate wealth and follows

$$\frac{dP_t + e_t dt}{P_t} = \alpha(\chi,t)dt + \sigma_p(\chi,t)dZ_t^X + \sigma_p(\chi,t)dZ_t^e
= \alpha(\chi,t)dt + \sigma(\chi,t)^\top dZ_t,$$

with the drift $\alpha(\chi,t)$ and the vector $\sigma(\chi,t)$ determined in equilibrium.

**Definition 2.** We call $(\alpha(\chi,t), \sigma(\chi,t), r(\chi,t), w^I_t, w^H_t, c^I_t, c^H_t)$ an asymptotic pure exchange equilibrium if

a) individual portfolio rules $w^I_t, w^H_t$ are optimal to first order, i.e., they satisfy (14).

b) the financial market clears,

$$w^I_t \omega^I_t + w^H_t \omega^H_t = 1 + O(\gamma^2). \quad (25)$$

b) the market of consumption goods clears,

$$c^I_t W^I_t + c^H_t W^H_t = e_t + O(\gamma^2). \quad (26)$$

### 3.2 Equilibrium Policies

We derive first-order asymptotic expressions for the individual policies in a heterogeneous-agent economy, where all agents solve problem (P2) with intermediate consumption. Two cases are considered. In the first benchmark case the economy is not constrained. In the sec-
ond case, the regulator constrains banks with a VaR limit as outlined in the previous sections. The functional form of the portfolio and consumption policies are the one obtained in partial equilibrium. However, the function \( g_0 \) in Proposition 5 is replaced by some corresponding equilibrium function \( g_{0e} \), which now depends on the state variables, \( X_t \) and \( \omega^I_t \), and the preference parameters \( \gamma_I \) and \( \gamma_{II} \). Since \( g_{0e} \) depends on equilibrium quantities, it is endogenously determined. Expanding \( g_{0e} \) with respect to the risk aversion parameters, we obtain

\[
g_{0e}(\chi, \gamma_I, \gamma_{II}, t) = g_{0,0}(\chi, t) + \left( \begin{array}{c} \gamma_I \\ \gamma_{II} \end{array} \right)^\top g_{0,1}(\chi, t) + O(\gamma_I^2, \gamma_{II}^2).
\]

For notational brevity we write \( g_{0e}(\chi, \gamma_I, \gamma_{II}, t) = g_{0e}(X, t) \), and \( O(\gamma^2) \) for \( O(\gamma_I^2, \gamma_{II}^2) \). The function \( g_{0,0} \) is uniquely determined by the log-investor’s value function in an economy populated by log-investors solely.

With the above specifications the portfolio policies in the general equilibrium for investor \( j, j = \{I, II\} \) up to first order are

\[
\begin{align*}
\omega^*_t &= (1 + \gamma_j) \frac{\lambda^*(\chi, t)}{\|\sigma^*(\chi, t)\|^2} + \gamma_j \xi^*_t + O(\gamma^2), \\
\gamma_j \eta^*_t &= 1 - \gamma_j \xi^*_t + O(\gamma^2),
\end{align*}
\]

where

\[
\begin{align*}
\xi^*_t &= \frac{g^*_t(\chi, t) + \log A_t}{A_t}, \\
\eta^*_t &= \frac{1}{\|\sigma^*(\chi, t)\|^2} \left( \begin{array}{c} \sigma^*_{PX}(\chi, t) \\ \sigma^*_{P\omega}(\chi, t) \end{array} \right)^\top \frac{\partial g^*_t(\chi, t)}{\partial \chi}.
\end{align*}
\]

with \( \sigma^*_{PX}(\chi, t) \) and \( \sigma^*_{P\omega}(\chi, t) \) the covariances between stock price changes and the state variables \( X \) and \( \omega^I \). In the above equations, we use ‘\( \ast \)’ as a placeholder for either ‘\( f \)’ (unconstrained case) or ‘\( c \)’ (constrained case). Before summarizing first properties of the unconstrained and constrained economy, we specify the function \( g_{0,0} \), which can be calculated explicitly, in the
Lemma 2. With an endowment and state variable process given in (23) and (24), the function \( g_{0e}^*(\chi, t) \) up to order \( O(\gamma) \) is given by

\[
g_{0e}^*(\chi, t) = g_{0,0}(X, t) + O(\gamma)
= e^{-\delta(T-t)} \frac{1}{A_t} \left( \log A_t + \mathbb{E} \left[ H_{t,T}^e | \mathcal{F}_t \right] + \frac{1}{A_t} \int_t^T e^{-\delta(s-t)} \mathbb{E} \left[ H_{t,s}^e | \mathcal{F}_t \right] ds \right) + O(\gamma),
\]

with

\[
\mathbb{E} \left[ H_{t,T}^e | \mathcal{F}_t \right] = \int_t^T \mathbb{E} \left[ \mu_e(X, s) - \frac{1}{2} \sigma_e(X, s)^2 | \mathcal{F}_t \right] ds.
\]

It is important to note that the function \( g_{0,0} \) is the same for the economy with and without VaR-constraints in equilibrium. This is due to the market clearing conditions in a homogeneous log-economy: Investment in the risky asset has be equal to 1 irrespective of any VaR-constraint. Moreover, in the log-economy, \( g_{0,0} \) is independent of cross-sectional wealth, since the portfolio fractions in the homogenous log-economy always move in the same direction. As a consequence, in a constrained economy banks will not change their portfolio policies up to first order in \( \gamma \) due to the possibility of being constrained in the future. This differs from partial equilibrium analysis, where banks adjust their asymptotic portfolio policies in the presence of VaR constraints.

As \( \xi_t^* \) will not be affected by the presence of VaR regulation, we write \( \xi_t^* = \xi_t \) and also for \( g_{0e}^* \). Also, note that \( \xi_t \) is a function of \( X \) and \( t \) only, but not of \( \omega_t \), contrary to \( \eta_t \), which depends on cross-sectional wealth. However, since \( g_{0e}(X, t) \) becomes independent of \( \omega \), equation (30) simplifies to

\[
\eta_t^* = \frac{1}{\|\sigma^*(\chi, t)\|^2} \sigma_{PX}^*(\chi, t) \frac{\partial g_{0e}(\chi, t)}{\partial X}.
\]

The individual optimal policies are given in equations (27) and (28). As follows from the next proposition, we only need to calculate \( g_{0e}(X, t) \) up to order \( O(\gamma) \) to obtain second-order asymp-
otic equilibrium quantities. We define by $\Delta^*$ a weighted aggregated risk aversion coefficient

$$\Delta^* = \gamma_I \omega_t^I + \gamma_{II} (1 - \omega_t^I).$$

The next proposition summarizes the properties of the unconstrained equilibrium.

**Proposition 6.** With an endowment and state variable process given in (23) and (24) and no VaR-constraints,

i) the equilibrium interest rate is given by

$$r(\chi^I, t) = \alpha(\chi^I, t) - \|\sigma(\chi^I, t)\|^2 \left(1 - (1 + \eta_f^n)\Delta_f^n\right) + O(\gamma^2).$$

ii) the cross-sectional wealth dynamics of investor $I$ is given by

$$d\omega_{t}^I = (\gamma_I - \gamma_{II})\omega_{t}^{I I} \left(1 - \omega_{t}^{I I}\right) \left(\xi_{t} dt + \left(1 + \eta_f^n\right)\sigma_{e}(X_{t})dZ_{t}^{e}\right) + O(\gamma^2).$$

iii) drift and volatility of the cumulative price process in (25) are given by

$$\alpha_f(\chi^f, t) = \delta + \mu_e(X_t) + (\partial_t A_t - 1)\xi_t \Delta_f + A_t \Delta_f \left(\frac{\partial\xi_t}{\partial t} + \sigma_{e}(X_t) \frac{\partial\xi_t}{\partial X_t}\right) + O(\gamma^2),$$

$$\sigma(\chi^f, t) = \begin{pmatrix} \sigma_{P_1}^f(\chi^f, t) \\ \sigma_{P_2}^f(\chi^f, t) \end{pmatrix} = \begin{pmatrix} A_t \Delta_f \frac{\partial\xi_t}{\partial X_t} \sigma_{e}(X_t) \\ \sigma_e(X_t) \end{pmatrix} + O(\gamma^2).$$

It follows from this proposition that as soon as $\xi$ is fixed, which corresponds to choosing a particular process for the endowment process, the risky asset and cross-sectional wealth dynamics are explicitly given. We then can explicitly calculate $d\xi$. The cross-sectional wealth dynamics are non-linear in general (see e.g. Trojani and Vanini (2001)). Further, since $d\omega_{t}^I$ is of order $O(\gamma)$, the drift and the variance of the equilibrium price process only involve infinitesimal generators of $X$, but not of $\omega_{t}^{I I}$. We next turn to the characterization of the constrained equilibrium. In order to obtain tractable expressions, we use the fact that the VaR horizon, $\tau$, is a short time interval, typically
a day. Therefore, we expand the resulting expressions in $\tau$ and consider terms only up to second order. We furthermore assume without loss of generality that bank II is less risk averse than bank I.

**Proposition 7.** Given an endowment and state variable process given in (23) and (24). In an economy with VaR-constraints where bank II is constrained and I not,

i) the equilibrium interest rate is given by

$$r(\chi^c, t) = \alpha(\chi^c, t) - ||\sigma(\chi^c, t)||^2 + ||\sigma(\chi^c, t)|| \frac{v(1 - \omega'Ic)}{\sqrt{\omega'IcB}} + h \left( \gamma_I; \frac{1}{\sqrt{\tau}}, \sqrt{\tau}, \tau \right) + O(\gamma^2, \tau^2),$$

with $B = \omega'Ic\nu^2 + 2 \log(1 - \beta)(\omega'Ic - 2)$ and $h(\gamma_I; \frac{1}{\sqrt{\tau}}, \sqrt{\tau}, \tau)$ given in the appendix.

ii) the cross-sectional wealth dynamics of bank I is given by

$$d\omega'Ic = \left( \mu_0^{\omega}(\chi^c, t) + \mu_\gamma^{\omega}(\chi^c, t) + \mu_\tau^{\omega}(\chi^c, t) \right) dt + \sigma_0^{\omega}(\chi^c, t) dZ_t^e + \left( \sigma_{\tau}^{\omega}(\chi^c, t) + \sigma_{\gamma}^{\omega}(\chi^c, t) \right)^T dZ_t + O(\gamma^2, \tau^2),$$

where the functions $(\mu^{\omega}(\chi^c, t), \sigma^{\omega}(\chi^c, t))$ are functions of the state variables and the regulatory parameters.

iii) drift and volatility of the cumulative price process in (25) are given by

$$\alpha^{c}(\chi^c, t) = \left( \gamma_I - \gamma_{II} \right) A_t \xi_t \left( \mu_0^{\omega}(\chi^c, t) + \left( \rho_{eX} \sigma_{X}(X_t) \frac{\partial \xi_t}{\partial X} + \sigma_e(X_t) \right) \sigma_0^{\omega}(\chi^c, t) \right)$$

$$+ \alpha^{f}(\chi^c, t) + O(\gamma^2, \tau^2),$$

$$\sigma(\chi^c, t) = \left( \begin{array}{c} \sigma_{P1}^{f}(\chi^c, t) \\ \sigma_e(X_t) + (\gamma_I - \gamma_{II}) \xi_t \sigma_0^{\omega}(\chi^c, t) \end{array} \right) + O(\gamma^2, \tau^2),$$

where $\alpha^{f}(\chi^c, t)$ and $\sigma_{P1}^{f}(\chi^c, t)$ are given in Proposition 6 with $\omega'Ic^{f}$ replaced by $\omega'Ic$. The functions $\mu_0^{\omega}(\chi^c, t)$ and $\sigma_0^{\omega}(\chi^c, t)$ are given in (A.41) and (A.42).

By inspection of Propositions 6 and 7, the cross-sectional dynamics in the constrained equilibrium do not vanish in $O(\gamma)$ as in the unconstrained economy. However, for determining
the equilibrium quantities in an $O(\gamma^2)$-asymptotic economy, terms up to order $O(\gamma)$ of the cross-sectional wealth dynamics do matter. Therefore, volatility and drift of the asset price process do change in the constrained equilibrium. However, the equilibrium interest rate changes irrespective of the cross-sectional dynamics’ order in $\gamma$.

Before discussing different model specification, it is instructive to consider a log-economy. Since in our equilibrium setting with only two banks an equilibrium cannot exist with both banks constrained, we assume that bank I is not regulated. In this economy, the only heterogeneity stems from different initial wealth levels of the investors. The corollary below summarizes the equilibrium quantities for a log-economy.

**Corollary 4.** Given an endowment and state variable process given in (23) and (24) with bank II VaR-constrained and bank I unregulated. When $\gamma_I = \gamma_{II} = 0$,

1) the equilibrium interest rate is given by

$$r^e(\chi, t) = r^f(\chi, t) + \|\sigma(\chi, t)\| \frac{\nu(1 - \omega^c t)}{\sqrt{\omega^c B}} + h\left(0; \frac{1}{\sqrt{\tau}}, \sqrt{\tau}, \tau\right) + O(\tau^2),$$

$$r^f(\chi, t) = \alpha(\chi, t) - \|\sigma(\chi, t)\|^2.$$

2) drift and volatility of the cumulative price process in (25) are given by

$$\alpha^e(\chi, t) = \alpha^f(\chi, t) = \delta + \mu_e(X_t),$$

$$\sigma^e(\chi, t) = \sigma^f(\chi, t) = \sigma_e(X_t).$$

In a log-economy, drift and volatility of the equilibrium asset price process remain unchanged. Only interest rates do change as well as the asymptotic volatility and drift of the cross-sectional wealth dynamics. Hence, regulation induces a change in wealth distribution among financial institutions even in a homogeneous log-economy. This is remarkable, since log-investors have zero dynamic hedging demand with respect to the state variables and therefore interest rates are the same independent of the existence of a stochastic opportunity set. Second, our choice of VaR limit (6) implies a wealth independent approximate VaR constraint.
This shows that an regulatory standard may well have an equilibrium impact on economic variables which are not part of the definition of the standard.

3.3 General Equilibrium: Do VaR Constraints Distort the Economy?

We next explore the effect of different model specifications \( \mathcal{M} \) of the endowment process on the general equilibrium. We choose the models:

\[
\mathcal{M} = \begin{cases} 
M_0 : & \mu_e(X) = \mu_e, \quad \sigma_e(X) = \sigma_e X_t, \quad \gamma_I = \gamma_{II} = 0; \\
M_1 : & \mu_e(X) = \mu_e, \quad \sigma_e(X) = \sigma_e X; \\
M_2 : & \mu_e(X) = \mu_e X, \quad \sigma_e(X) = \sigma_e X.
\end{cases}
\]

Furthermore, we assume that the state variable \( X \) follows a mean-reverting geometric Brownian motion. Alternatively, we could assume a square-root or Gaussian process for \( X \). However, the results do not change qualitatively.

Our main focus is the VaR impact on equilibrium interest rates, volatility, portfolio fractions and risk premia. With these model specifications at hand, we conduct a graphical analysis (Figures 5-8) of an economy, where bank II is subject to VaR regulation, whereas bank I is not constrained.

First, we discuss model \( M_0 \), i.e. the log-economy. Corollary 4 states that in a log-economy VaR regulation has no effect on the asset price dynamics under the physical probability measure. Drift and volatility are the same whether bank II is constrained or unconstrained. However, as soon as bank II becomes constrained, interest rates change even in a pure log-economy. Only in the limit,

\[
\lim_{\omega \to 1} = \alpha^e(\chi, t) - \|\sigma(\chi, t)\|^2 \frac{1 - \frac{\gamma_{II} \eta_t}{1 + \gamma_I}}{1 + \gamma_I},
\]

interest rates remain unchanged compared to the unconstrained log-economy.

Figure 5 plots the asymptotic portfolio fractions and the equilibrium interest rates as well as the asymptotic drift and volatility of the cross-sectional wealth process. In the first panel, left to point \( A \), the portfolio fractions equal one, i.e. \( w = 1 \) for both banks. To the right of
point \( A \), the regulated bank \( II \) becomes constrained and therefore has to decrease its exposure (dashed line). Part of its wealth are now shifted into the money market. This influences the equilibrium interest rates, which have to decrease due to the increased demand for bonds from bank \( II \). The bold straight line in the upper right panel shows the equilibrium interest rate. When point \( A \) is reached, the interest rates become lower than in the corresponding unconstrained economy (dotted line). Lower interest rates make an investment in the risky asset more attractive for the unregulated bank, which, to the right of point \( A \), increases its exposure (straight bold line in the upper left panel). The changes in portfolio fractions will induce changes in the cross-sectional wealth dynamics in leading order in \( \gamma \). This means that regulation causes changes in the wealth distribution regardless of the risk-appetite of regulated and non-regulated institutions. In the lower panels, we plot the drift component \( \mu_{\text{occ}}(\chi, t) \) and the volatility component \( \sigma_{\text{occ}}(\chi, t) \) of cross-sectional wealth dynamics. First, we note that due to regulation, cross-sectional wealth dynamics become stochastic in \( O(\gamma) \). Second, we note that the drift of cross-sectional wealth becomes positive in \( O(\gamma) \). Therefore, regulation favors unregulated institutions as it eventually leads to a shift of cross-sectional wealth from the regulated bank to the unregulated bank.

In Figure 6 we discuss model \( M_1 \) assuming a positive correlation \( \rho_{eX} \). As soon has bank \( II \) hits the VaR constraint at point \( A \), it has to decrease its portfolio fraction invested in the risky asset. The dotted line through point \( A \) plots the VaR constraint. This VaR constrained is determined by the interest rate, drift and volatility of the asset price process in an unconstrained economy. As soon as bank \( II \) hits point \( A \), these equilibrium quantities change to their corresponding values in the constrained economy. Therefore, the VaR constraint will change at point \( A \) to the lower dotted line. As equilibrium quantities change in a non-smooth manner, there will be a downward jump in the portfolio fraction at point \( A \). The increased demand for bonds leads to a decrease in interest rates (upper-right panel). Thus, beyond point \( A \), the riskless interest rate starts to decline. At the same time, we observe an upward jump in the asset’s drift rate and an increase its volatility. Thus, in case of positive correlation \( \rho_{eX} \), regulation induces an upward jump in equilibrium volatility. Only when \( X \) increases further,
volatility will eventually be lower than in the unconstrained economy.

Since interest rates, drift and volatility have to change in such a way that bank $I$ increases its demand for stocks (point $B$ in Figure 6), the overall effect is an increase in the Sharpe Ratio. Figure 7 plots the Sharpe Ratio for the parameter used in Figure 6. At point $A$ the Sharpe Ratio jumps to a higher level and obtains a steeper slope in $X$ than the Sharpe Ratio of the unconstrained economy. Thus, the presence of regulated and non-regulated financial institutions may help to explain in part the stylized fact of relatively high Sharpe Ratios together with low interest rates, which standard financial models fail to explain.

In Figure 8 we explore model $M_2$ with negative correlation $\rho_{eX}$. Again, we assume that bank $I$ is not regulated. Since we assume correlation to be negative, the portfolio fraction invested in the risky asset is lower for bank $II$ than for bank $I$ in the unconstrained economy. When bank $II$ becomes constrained at point $A$, it has to decrease its portfolio fraction. This leads to a decrease in interest rates at point $A$ in the upper-right panel. Compared to model $M_1$ with negative correlation, it turns out that, when bank $II$ becomes constrained, both drift and volatility of the asset price process jump downwards and remain at a lower level compared to the unconstrained economy. Eventually however, the Sharpe Ratio will increase and bank $I$ will increase its exposure in the risky asset at point $B$ as shown in the first panel of Figure 8.

We have now analyzed different models under different parameter constellations. It follows that to fully understand the impact of VaR regulation on equilibrium quantities, such an analysis has to be carefully considered at least along three dimensions:

1. Regulatory control variables: $\beta, \nu, \tau$.
2. Market factors: $X, \rho_{eX}$.
3. Market participants characteristics: $\gamma, \omega, T$.

All the above variables have different and mostly ambiguous impacts on the effectiveness of VaR regulation. In contrast to the analysis in partial equilibrium, the effect of VaR regulation is hardly predictable. The reason for this lies in the fact that risk regulation is no longer
exogenously given, but enters as an endogenous constraint into the optimization problem of the bank. The VaR constraint is calculated using interest rate, asset price drift and volatility as inputs. These quantities are determined endogenously in equilibrium and, thus, complicate the equilibrium analysis of VaR constraints by introducing highly non-linear effects and jumps. There are nevertheless some important conclusions to be drawn from the above analysis:

- Regulation leads to a change in cross-sectional wealth, which is independent of risk aversion. Wealth is shifted from the regulated to the unregulated institutions.

- For unregulated institutions, an investment in the risky asset becomes more attractive, since regulation leads to an increase in the Sharpe Ratio.

- When the regulated bank becomes constrained, equilibrium interest rates, asset price drift and volatility can jump.

- When $\rho_{eX}$ is positive and bank II becomes constrained, the Sharpe ratio seems to be mostly augmented by an increase in the drift rate. Whereas with negative correlation, the Sharpe ratio tends to be augmented by a decrease in volatility.

In summary, the exact effects of VaR regulation are difficult to establish in a fairly realistic model. Considering the costs, which the implementation of regulatory standards are causing to the financial industry, one would expect that regulation leads to more decisive statements. However, present and planned regulation fails to provide such a statement.

As a final remark, our framework can also be applied to analyze the case of a large bank with VaR limited traders that can move the market.

4 Conclusion

We analyze the implications of VaR based regulation on the investment policies of utility maximizing investors in a continuous-time setting. We extend the existing literature by allowing
for a stochastic opportunity set and intermediate consumption expenditures. In partial equi-
librium we showed that VaR regulation can pronounce the bank’s exposure in the risky asset.
This distortion of incentive is however not related to the lack of coherence of VaR. It instead
depends on the correlation between asset price changes and changes in volatility and the de-
gree of risk aversion. If correlation is negative, for a bank which is less risk-averse then the
log-investor, the presence of VaR limits increases the risky asset exposure. Furthermore, it is
impossible to remove any incentive distortions by replacing VaR with a coherent risk measure
such as Expected Shortfall: Market factors and preferences of the investors, which are both
linked to the VaR constraint, induce the distortions.
Extending our analysis to the general equilibrium, we explore the effect of VaR regulation
on equilibrium interest rates, volatility, portfolio fractions, and we analyze the effect on the
banks’ portfolio policies. Applying the general results to some well-known models, we find
that impacts of VaR regulation strongly depend on both the model used and the parameter
values in the economy. Indeed, to fully assess the effectiveness of VaR regulation, the analysis
has to be done along three dimensions, a) the market participants, b) the market factors, and
c) the regulatory control variables. Conclusions remain mixed, however, we find a tendency
for lower interest rates and an increase in the Sharpe Ratio. Furthermore, regulation leads
to changes in cross-sectional wealth, which are independent of risk aversion. Therefore, reg-
ulation distorts competition among regulated banks, and between regulated and unregulated
financial institutions. As the costs for the financial industry to implement regulatory stan-
dards are considerable, one would expect that the effect of regulation can be summarized in a
clear statement. However, within an adequate model, such a statement cannot be made. One
reason for this lies in the endogenization of regulation by using VaR as a regulatory control
mechanism. Such an endogenization cannot be circumvented by using Expected Shortfall or
other coherent risk measures, which depend on equilibrium quantities such as interest rates
and volatilities.
Table 1: *First-Order Approximation*. The table displays the upper bound for the probabilities given in Proposition 1. Reported numbers are percentage numbers. We consider the following model specification: \( dX = \left( \theta - \kappa X \right) dt + \sigma_X X dZ_t^X \), \( r(X_t) = r \), \( \lambda(X_t) = \lambda X^2 \), and \( \sigma(X) = \sigma X \). To calculate the values for different \( M \), \( X \), \( w \) and time-horizons, we assumed \( \theta = 0.2 \), \( \kappa = 0.2 \), \( \sigma_X = 0.15 \), \( \lambda = 0.05 \), and \( \sigma = 0.2 \). The unconditional mean is set to \( \theta / \kappa = 1 \).

<table>
<thead>
<tr>
<th>( X = 1 )</th>
<th>( X = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 day</td>
<td>10 days</td>
</tr>
<tr>
<td>( M = 1% )</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000</td>
</tr>
<tr>
<td>( M = 5% )</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000</td>
</tr>
<tr>
<td>( M = 10% )</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Figure 1: Comparing perturbation approach with Monte Carlo simulation. We assume \( \alpha(X_t) = 0.05X_t^2, \sigma(X_t) = 0.25X_t, \theta = \kappa = 0.8, \sigma_X = 0.2, T = 1 \).
Figure 2: VaR constraint set parameterized by consumption and portfolio value.
Figure 3: Portfolio policy under VaR and influence of investment horizon. Graphs on the left assume a positive correlation of $\rho = 0.4$. Graphs on the right assume $\rho = -0.4$. (A) and (B) assume an investment horizon of $T = 1$ year. The next two graphs assume $T = 5$, and the final two graphs assume $T = 10$. The solid bold line represents the portfolio policy of the VaR constrained bank. At the circled point, the bank runs into the VaR constrained represented by the dotted line. The dashed line represents the optimal policy in case of no constraints. The parameters were chosen as follows: $\gamma = 0.5$, $\nu = 0.01$, $\beta = 0.05$, $\tau = 10/250$, $r = 0.05$, $\lambda = 0.03$, $\sigma = 0.32$, $\kappa_X = \theta_X = 0.2$, $\sigma_X = 0.6$. The calculations were performed using standard Monte Carlo methods.
Figure 4: Portfolio policy under VaR and the influence of confidence level. Graphs on the left assume $\rho = 0.4$ and $\rho = -0.4$ on the right side. (A) and (B) assume a confidence bound of $\nu = 0.01$. (C) and (D) assume $\nu = 0.05$. (E) and (D) show the differences in optimal policies when the confidence level is tightened, i.e. we plot the differences (C)-(A) and (D)-(B). We assumed the same parameters as in Figure 3. The time horizon is set to $T = 5$ years.
Figure 5: Impact of VaR regulation in a log-economy $M_0$. We assumed the following parameter values: $\sigma_e = 0.2, \mu_e = 0.1, \sigma_X = 1, \theta = \kappa = 0.1, \delta = 0.05, T = 5, \tau = 1/250, \beta = 2.5\%, v = 5\%, \omega = 0.5, \gamma_I = \gamma_{II} = 0$. Bold lines represent quantities resulting from the constrained economy. In the panels for the interest rate, drift and volatility of the cross-sectional wealth dynamics, the dotted lines to the right of point $A$ represent the corresponding quantities in the unconstrained economy.
Figure 6: Impact of VaR regulation for $M_1$ with positive correlation. We assumed $\frac{d\epsilon_t}{\epsilon_t} = \mu_e dt + \sigma_e X_t dZ_t^e$ and the following values: $\sigma_e = 0.2, \mu_e = 0.1, \sigma_X = 1, \theta = \kappa = 0.1, \delta = 0.1, T = 2, \tau = 1/250, \beta = 5\%, v = 1\%, \omega = 0.5, \rho_{eX} = 0.4, \gamma_I = -0.1, \gamma_{II} = 0.4$. Bold lines represent quantities resulting from the constrained economy. In the panels for the interest rate, drift and volatility of the asset price process, the dotted lines to the right of point $A$ represent the corresponding quantities in the unconstrained economy.
Figure 7: Impact of VaR regulation on Sharpe Ratio for $M_1$ with positive correlation. We assumed the same parameter values as for Figure 6. The bold line is the Sharpe Ratio resulting from the constrained economy. The dotted line to the right of point $A$ represents the Sharpe Ratio in the unconstrained economy.
Figure 8: Impact of VaR regulation for $M_2$ with negative correlation. We assumed $de_t/e_t = \mu_e X_t dt + \sigma_e X_t dZ_t^e$ and the following values: $\sigma_e = 0.2, \mu_e = 0.1, \sigma_X = 1, \theta = \kappa = 0.1, \delta = 0.1, T = 2, \tau = 1/250, \beta = 2.5\%, \nu = 1\%, \omega = 0.5, \rho_{eX} = -0.4, \gamma_I = -0.1, \gamma_{II} = 0.4$. Bold lines represent quantities resulting from the constrained economy. In the panels for the interest rate, drift and volatility of the asset price process, the dotted lines to the right of point $A$ represent the corresponding quantities in the unconstrained economy.
Appendix

Proof of Proposition 1

With portfolio weights fixed at time $t = 0$, the wealth dynamics under Assumption 2 reads

$$
\log W_t = \log W_0 + \int_0^t \left( r(X_s) + w_0 \mu(X_s) - \frac{1}{2} w_0^2 \sigma^2(X_s) \right) ds + w_0 \int_0^t \sigma(X_s) dZ_s.
$$

Performing an Itô-Taylor expansion in $X$, we obtain $\log W = \log W^{(1)} + R$ with remainder

$$
R = w_0 \sigma(X_0) \int_0^t \int_0^s dZ_u ds + \int_0^t \int_0^s \left( \mathcal{L}_0 h_1(X_u) du + \mathcal{L}_1 h_1(X_u) dZ_u^X \right) ds
$$

$$
+ w_0 \int_0^t \int_0^s \left( \mathcal{L}_0 h_2(X_u) du + \mathcal{L}_1 h_2(X_u) dZ_u^X \right) dZ_u,
$$

(A.32)

where $h_1(X) = r(X) + w_0 \lambda(X) + \frac{1}{2} w_0^2 \sigma(X)^2$, $h_2(X) = \sigma(X)$, and

$$
\mathcal{L}_0 = \mu_X(X) \frac{\partial}{\partial X} + \frac{1}{2} \sigma_X^2(X) \frac{\partial^2}{\partial X^2}, \quad \mathcal{L}_1 = \sigma_X(X) \frac{\partial}{\partial X}.
$$

Using Markov's inequality we get the bound

$$
P \left( |\log W^{(1)}_t - \log W_t| \geq M \right) \leq \frac{1}{M} \mathbb{E} \left[ |\log W^{(1)}_t - \log W_t| \right] = \frac{1}{M} \mathbb{E} [ |R| ].
$$

Since the first moment of a multiple Itô integral vanishes, if it has at least one integration with respect to a component of the Wiener process, $\mathbb{E} [ |R| ]$ follows as claimed. □

Proof of Proposition 2

From Proposition 1,

$$
P \left( W_{t+\tau} - W_t \leq L \mid \mathcal{F}_t \right)
$$

$$
= P \left( Z_{t+\tau} - Z_t \leq \frac{\log(1 - \beta) - \left( r(X_t) + w_t \lambda(X_t) + \frac{1}{2} w_t^2 \sigma(X)^2 \right) \tau}{w_t \sigma(X_t)} + R \mid \mathcal{F}_t \right)
$$

$$
= P \left( W^{(1)}_{t+\tau} - W_t \leq L \mid \mathcal{F}_t \right)
$$

$$
= N \left( \frac{\log(1 - \beta) - \left( r(X_t) + w_t \lambda(X_t) + \frac{1}{2} w_t^2 \sigma(X)^2 \right) \tau}{w_t \sigma(X_t) \sqrt{\tau}} \right),
$$
where $\beta = L/W$ and $N(\cdot)$ the cumulative normal distribution function. The approximated VaR at the $\nu$-confidence level then reads

$$
\hat{\text{VaR}}^{\nu,w}_t = W_t \left( 1 - e^{(r(X_t) + w_t \lambda(X_t) - \frac{1}{2} w_t^2 \sigma(X_t)^2) \tau + w_t \lambda(X_t) \sqrt{\tau}} \right), \tag{A.33}
$$

which is equivalent to

$$
\log(1 - \beta) - \left( r(X_t) + w_t \lambda(X_t) - \frac{1}{2} w_t^2 \sigma(X_t)^2 \right) \tau - w_t \lambda(X_t) \sqrt{\tau} \leq 0.
$$

But this inequality is equivalent to the upper and lower bound $w^+_b$ stated in the proposition. \hfill \Box

**Proof of Proposition 4**

We calculate the value function for a log-investor as

$$
J_{\text{log}}(W, X, t) = \mathbb{E} [\log W_T | \mathcal{F}_t], \tag{A.34}
$$

where

$$
w_{\text{log}} = \begin{cases} 
    w^f_{\text{log}} & \text{if } w^f_{\text{log}} \leq w^+_b, \\
    w^+_b & \text{if } w^f_{\text{log}} \geq w^+_b, \\
    w^-_{\text{log}} & \text{if } w^f_{\text{log}} \leq w^-_b,
\end{cases}
$$

is the log-investor's portfolio policy in the unconstrained case. Since

$$
J_{\text{log}}(W, X, t) = \lim_{\gamma \to 0} \frac{e^{\gamma(g_0(X_t) + \gamma g_1(X, t))} W_t^\gamma - 1}{\gamma} = \log W_t + g_0(X, t), \tag{A.35}
$$

holds, equating (A.34) and (A.35) $g^f_0(X, t)$ follows. Accounting for the presence of constraints, we have

$$
g_0(X, t) = r(T - t) + \lambda \int_t^T \mathbb{E} \left[ w^f_{\text{log}}(X_s) + \mathbb{I}_{\{w^f_{\text{log}} \geq w^+_b\}} w^+_b(X_s) X_s^{n_1} | \mathcal{F}_t \right] ds \\
- \frac{1}{2} \sigma^2 \int_t^T \mathbb{E} \left[ \left( w^f_{\text{log}}(X_s) + \mathbb{I}_{\{w^f_{\text{log}} > w^+_b\}} w^+_b(X_s) \right)^2 X_s^{2n_2} | \mathcal{F}_t \right] ds.
$$
Simplifying the above term, we arrive at the decomposition for $g_0(X, t)$ as given in the theorem. □

**Proof of Corollary 1**

Plugging the bound $KX^{2j(n_1-n_2)}$ into equation (16), we can use the Euler gamma function $\Gamma(a)$ and the incomplete Gamma function, $\Gamma(b, a)$ to obtain

$$w_f^{(n)}(X, t) = X^{n_1-2n_2} \frac{\lambda}{\sigma^2(1-\gamma)} - \frac{1}{1-\gamma} + \frac{\rho \sigma X}{\sigma} X^{m-n_2} \frac{\sigma}{\sigma^2(1-\gamma)} \frac{\partial g(X, t)}{\partial X}.$$

Since $\lim_{b \to \infty} \Gamma(b, a)/\Gamma(b) = 1$ for finite $a$, the claim follows.

**Proof of Proposition 5**

The HJB equation for the control problem (P2) is

$$0 = \max_w \left\{ J_t + \mu_X(X) J_X + \frac{1}{2} \sigma_X(X)^2 J_{XX} + W (r - c + w \lambda(X)) J_W 
+ w W \rho \sigma(X) J_{WX} + \frac{1}{2} w^2 W^2 \sigma(X)^2 J_{WW} 
- \phi \left( \log(1-\beta) - \left( r - c + w \lambda(X) - \frac{1}{2} w^2 \sigma(X)^2 \right) \tau - \sigma(X) \sqrt{\tau} \right) \right\}$$

$$= \max_{w,c} (-\phi_1 Q(w, c) - \phi_2 c + \mathcal{G} J).$$

Due to the homogeneity properties of the utility function, the solution is of the form

$$J(W, X, t) = \frac{A_t}{\gamma} \left( e^{\gamma g(x,t)} W^\gamma - 1 \right). \tag{A.36}$$

The first-order conditions and slackness imply

$$w_f = -X^{n_1-2n_2} \frac{\lambda J_W}{\sigma^2 W J_{WW}} - \frac{1}{1-\gamma} \frac{\rho X^{m-n_2}}{\sigma^2(1-\gamma)} \frac{\partial g(X, t)}{\partial X},$$

$$c_f = (J_W W)^{\frac{1}{\gamma-1}} = \left( A_t e^{\gamma g(X, t)} \right)^{\frac{1}{\gamma-1}}.$$
As we are only interested in the first-order approximation, we assume \( g(X, t) = g_0(X, t) + \gamma g_1(X, t) + O(\gamma^2) \). From (A.36) the log-investor’s value function reads

\[
J(W, X, t) = A_t \left( \log(W_t) + g_0(X, t) \right).
\]

Expanding the optimal portfolio weight and optimal consumption rate up to first order we obtain \( w^{(1)}_f(X, t) \) and \( c^{(1)}_f(X, t) \) as claimed. To obtain the function \( g_0(X, t) \), we recall the wealth dynamics with consumption which is given in equation (21). Then, the \( J \) function reads

\[
J(W, X, t) = e^{-\delta(T-t)} \left( \log(W_t) + E[H_{t,T} | \mathcal{F}_t] \right) + \int_t^T e^{-\delta(s-t)} \left( \log(W_t) + E[H_{t,s} | \mathcal{F}_t] + E[\log c_s | \mathcal{F}_t] \right) ds.
\]

Equating the last two expressions for the value function we get

\[
A_t = e^{-\delta(T-t)} + \frac{1 - e^{-\delta(T-t)}}{\delta},
\]

\[
g_0(X, t) = \frac{1}{A_t} e^{-\delta(T-t)} E[H_{t,T} | \mathcal{F}_t] + \frac{1}{A_t} \int_t^T e^{-\delta(s-t)} \left( E[H_{t,s} | \mathcal{F}_t] + E[\log c_s | \mathcal{F}_t] \right) ds.
\]

We still need to determine the function \( H_{t,T} \). Its general form is given in equation (22). Similar as in Proposition 4, the structure of \( w^{\pm}(X, t) \) leads to the decomposition of \( H_{t,T} \) as given in the theorem. □

**Proof of Corollary 3**

To prove the increase in consumption \( c_t \), we have to show that \( g_f(X, t) > g_0(X, t) \). For the non-degenerated case this follows from \( E\left[H_{t,T}^{f} | \mathcal{F}_t\right] > E\left[H_{t,T} | \mathcal{F}_t\right] \). To prove the impact on the portfolio fractions, we have to show

\[
\frac{\partial g_0(X, t)}{\partial X} < \frac{\partial g_f(X, t)}{\partial X}.
\]

From the definition of \( E[H_{t,T} | \mathcal{F}_t] \) in Proposition 5, the first derivative of the Heaviside function with respect to \( X \) is positive, the first derivative of the function \( \hat{w}_T^{\pm} \) is negative and the derivative of the volatility function is positive by assumption. Then (A.37) follows. □
Proof of Lemma 2

The proof of Proposition 5 implies that the function \( g_0(\chi, t) \) must be of the form

\[
g_0(\chi, t) = \frac{1}{A_t} e^{-\delta(t-t')} E[H_t, T | F_t] + \frac{\int_{t-t'}^{T} e^{-\delta(s-t')} (E[H_{s-t} | F_t] + E[\log c_s | F_t]) ds}{A_t}
\]

with

\[
E[H_t, T | F_t] = E \left[ \int_t^T \left( r_{log} + w_{log}(a_{log} - r_{log}) - e_{log}(s) - \frac{1}{2} w_{log}^2 \sigma_{log}^2 \right) ds | F_t \right].
\]

As \( g_0(\chi, t) \) is determined in the homogenous log-economy, the above equation can be simplified considerably. First, we note that in the log-economy, market clearing implies \( w_{log} = 1 \). Further, \( c_{log} = 1/A_t \) and, as \( \gamma_I = \gamma_H = 0 \),

\[
\alpha_{log} = \delta + \mu_e(X, t), \quad \sigma_{log} = \sigma_e(X_t).
\]

Inserting the above expressions in \( g_0(\chi, t) \) proves the claim. \( \Box \)

Proof of Proposition 6

To obtain the equilibrium interest rate \( r_f(\chi, t) \) we start with the market clearing condition

\[
\omega_I \left( \frac{\alpha(\chi, t) - r(\chi, t)}{\|\sigma(\chi, t)\|^2} \right) (1 + \gamma_I) + (1 - \omega_I) \left( \frac{\alpha(\chi, t) - r(\chi, t)}{\|\sigma(\chi, t)\|^2} \right) (1 + \gamma_H) + \gamma_I \eta_I = 1.
\]

Solving for \( r(\chi, t) \) and expanding in \( \gamma \) gives the equilibrium interest rate. To determine \( d\omega_I \), recall that \( \omega_I = W_I^f/P_t \). Applying Itô’s Lemma to cross-sectional wealth, inserting the wealth differentials and dropping the \( O(\gamma^2) \) terms, we obtain the expression for \( d\omega_I^f \) up to second order in \( \gamma \). To derive the price process, we consider

\[
\frac{dP_t}{P_t} = \frac{d(P_t/e_t)}{P_t/e_t} + \frac{d e_t}{e_t} - \frac{d e_t d(P_t/e_t)}{P_t/e_t} + O(\gamma^2),
\]

with the ratio \( P_t/e_t \) given by

\[
\frac{P_t}{e_t} = \frac{1}{\omega_I^f (c_I^f - c_H^f) + c_I^f} = A_t (1 + A_t \xi_t \Delta^*) + O(\gamma^2).
\]

48
Then,
\[
\frac{dP_t}{P_t} = \frac{de_t}{e_t} + (\partial_t \log A_t + \xi_t \Delta^* \partial_t A_t) dt + (\gamma_I - \gamma_H) A_t \xi_t (d\omega_t^{I*} + \sigma_e(X_t) d\langle Z^e, \omega^{I*} \rangle_t) \\
+ A_t \Delta^* (d\xi_t + \sigma_e(X_t) d\langle Z^e, \xi_t \rangle_t) + (\gamma_I - \gamma_H) A_t d\langle \omega^{I*}, \xi_t \rangle_t + O(\gamma^2). \tag{A.39}
\]

Since the bracket processes can be written as
\[
d\langle Z^e, \omega_t^{I*} \rangle_t = \rho_e X_t \sigma_e(X_t) \partial \omega_t^{I*} \partial e_t \partial X_t dt + e_t \sigma_e(X_t) \partial \omega_t^{I*} \partial e_t \partial X_t dt,
\]
\[
d\langle Z^e, \xi_t \rangle_t = \rho_e X_t \sigma_e(X_t) \partial \xi_t \partial X_t dt,
\]
\[
d\langle \omega_t^{I*}, \xi_t \rangle_t = e_t \rho_e X_t \sigma_e(X_t) \sigma_e(X_t) \partial \xi_t \partial \omega_t^{I*} \partial e_t + \sigma_e(X_t) \partial \xi_t \partial \omega_t^{I*} \partial e_t - \sigma_e(X_t)^2 \partial \xi_t \partial \omega_t^{I*} \partial e_t - \sigma_e(X_t)^2 \partial \xi_t \partial \omega_t^{I*} \partial e_t - \sigma_e(X_t)^2 \partial \xi_t \partial \omega_t^{I*} \partial e_t,
\]
we insert the above expressions into (A.39) and (A.38). Since in an unconstrained homogeneous log-investor economy \(d\omega_t = O(\gamma) dt + O(\gamma) dZ_t^X\), the instantaneous drift and the instantaneous volatility of the cumulative price process, \(dP_t + e_t dt)/P_t\), follow as claimed. \(\Box\)

**Proof of Proposition 7**

We assume without loss of generality \(\gamma_I < \gamma_H\) and that the less risk averse investor II is binding. Market clearing implies
\[
\omega_t^{I} \left( \frac{a(x, t) - r(x, t)}{\|\sigma(x, t)\|^2} (1 + \gamma_I) + \gamma_I \eta_t \right) + (1 - \omega_t^{I}) w_t^{II} (x, t) = 1.
\]
Solving for the interest rate and expanding in \(\tau\) and \(\gamma\), we obtain the interest rate \(r(x^c, t)\) as given in the proposition with
\[
h \left( \gamma_I; \frac{1}{\sqrt{\tau}}, \sqrt{\tau}, \tau \right) = \frac{1}{\sqrt{\tau}} \theta_{\tau, 0} + \sqrt{\tau} \theta_{\tau, 1} + \tau \theta_{\tau, 2} + \gamma_I \theta_{\gamma},
\]

49
where

\[ \theta_{r,0} = \|\sigma(\chi, t)\| \frac{1 - \omega_t^{Ic}}{2 - \omega_t^{Ic}} \left( v + \sqrt{B}\omega_t^{Ic} \right), \]

\[ \theta_{r,1} = \left( 1 - \omega_t^{Ic} \right) \omega_t^{Ic} \|\sigma(\chi, t)\| \left( 2B \left( \alpha(\chi^c, t) - \frac{1}{\chi} - \frac{\|\sigma(\chi, t)\|^2}{2} \right) + v^2 \|\sigma(\chi, t)\|^2 (\omega_t^{Ic} - 2) \right)^{3/2}, \]

\[ \theta_{r,2} = \frac{v\|\sigma(\chi, t)\| (\omega_t^{Ic} - 2)}{\omega_t^{Ic}} \theta_{r,1}, \]

\[ \theta_\gamma = \frac{-\|\sigma(\chi, t)\|^2 (1 + \eta_i)}{\omega_t^{Ic} - 2} \frac{\omega_t^{Ic} (1 - \omega_t^{Ic}) v\|\sigma(\chi, t)\|^2 (2 \log(1 - \beta) (\omega_t^{Ic} - 2) + (B(1 + \eta_i) - v^2 (\omega_t^{Ic} - 2) (\omega_t^{Ic} - 2)) (B(\omega_t^{Ic} - 2))^3/2 \right) - \frac{1}{\sqrt{T}} \left( B(\omega_t^{Ic} - 2) (2N + (\omega_t^{Ic} - 2) \rho^{Ic} - 2 \log(1 - \beta)) \right) }{B(\omega_t^{Ic} - 2)^2 \omega_t^{Ic}}. \]

To obtain the price dynamics we can use (A.39). By inspection of (A.39) we see that we need to calculate \( d\omega_t^{Ic} \) only up to \( O(\gamma, \tau) \), since the expression involving \( \omega \)-terms is pre-multiplied by the first order difference \( \gamma_t - \gamma_t \). This gives

\[ d\omega_t^{Ic} = \mu_0^{\omega c}(\chi^c, t) dt + \sigma_0^{\omega c}(\chi^c, t) dZ_t^c + O(\gamma, \tau), \] (A.40)

where

\[ \mu_0^{\omega c}(\chi^c, t) = \omega_t^{Ic} \frac{(1 - \omega_t^{Ic})}{\sigma_c(X_t)^2} \left( A_t \left( EC^2 (C + \nu \sigma_c(X_t)) - D^2 \nu \sigma_c(X_t) \right) - 2C^2 \sigma_c(X_t)^2 (C + \nu \sigma_c(X_t)) \right) \]

\[ + \frac{\omega_t^{Ic} (1 - \omega_t^{Ic})}{\sigma_c(X_t)^2 \tau} \left( \theta_{r,0} - 2\theta_{r,0} d\sigma_c(X_t) + 2\sigma_c(X_t) \left( v(C + \nu \sigma_c(X_t)) + \sigma_c(X_t) \log(1 - \beta) \right) \right) \]

\[ + \frac{2(C + \nu \sigma_c(X_t)) \omega_t^{Ic} (1 - \omega_t^{Ic})^2}{C \sigma_c(X_t)^2 \sqrt{T}} \] (A.41)

\[ \sigma_0^{\omega c}(\chi^c, t) = -\frac{\omega_t^{Ic} (1 - \omega_t^{Ic})}{\sigma_c(X_t)} \left( \frac{D}{C} + \frac{C + \nu \sigma_c(X_t)}{\sqrt{T}} \right), \] (A.42)
with

\begin{align*}
C &= \sqrt{\rho_{\tau,0}^2 - 2v\rho_{\tau,0}\sigma_e(X_t) + \sigma_e(X_t)^2(v^2 - 2\log(1 - \beta))}, \\
D &= \sigma_e(X_t)^2 \left( \rho_{\tau,0} + (v\sigma_e(X_t) - \rho_{\tau,0}) \left( 1 - \frac{v(1 - \omega_{Ic}^t)}{\sqrt{\omega_{Ic}^t B}} \right) \right), \\
E &= \sigma_e(X_t)^{\frac{3}{2}} \left( 1 - \frac{v(1 - \omega_{Ic}^t)}{\sqrt{\omega_{Ic}^t B}} \right)^2 + 2\sigma_e(X_t)^2 \left( \delta + \mu_e(X_t) + \sigma_e(X_t)^2 \left( 1 - \frac{v(1 - \omega_{Ic}^t)}{\sqrt{\omega_{Ic}^t B}} \right) \right)
+ 2\rho_{\tau,0} \left( \rho_{\tau,0} - v\sigma_e(X_t) \right).
\end{align*}

We note that the \(dZ^X_t\) term for the dynamics of \(\omega_{Ic}^t\) vanishes in \(O(\gamma, \tau^2)\). To obtain the price dynamics, we go back to equation (A.39). Moving from the unconstrained to the constrained economy adds additional terms, namely

\begin{align*}
(\gamma_I - \gamma_{II})A_t\xi_t \left( d\omega_{Ic}^t + \sigma_e(X_t)d(Z^c, \omega_{Ic}^c) \right) + (\gamma_I - \gamma_{II})A_t d(\xi, \omega_{Ic}^c) + O(\gamma^2, \tau^2)
= (\gamma_I - \gamma_{II})A_t\xi_t \left( \left( \mu_{0c}(\chi^c, t) + \left( \rho_{eX}\sigma_e(X_t) \frac{\partial \xi_t}{\partial X} \right) + \sigma_e(X_t) \right) \sigma_{0c}(\chi^c, t) \right) dt + \sigma_{0c}(\chi^c, t) dZ^c
+ O(\gamma^2, \tau^2).
\end{align*}

In the unconstrained economy all terms of order \(O(\gamma^2, \tau^2)\) can be omitted, but this is not true in the constrained economy. Incorporating this term into the price dynamics, and noting from (A.39) that the price dynamics must be of the form

\[
\frac{dP_t + e_t dt}{P_t} = \alpha(\chi^c, t) dt + O(\gamma) dZ_t + \sigma_e(X_t) dZ^c_t,
\]

we obtain the drift and volatility as claimed in the proposition. Inserting these expressions together with \(r(\chi^c, t)\) into the dynamics of \(\omega_{Ic}^t\), we obtain the expressions for \(\omega_{Ic}^t\). Finally, the explicit functions for \(\mu_{0c}(\chi^c, t), \mu_{c}(\chi^c, t), \sigma_{0c}(\chi^c, t)\), and \(\sigma_{c}(\chi^c, t)\) follow. Since these are lengthy expressions they are not displayed but can be obtained by the authors. \(\square\)
Notes

1 More precisely, they simplify the structure of the dynamic optimal policies in their multiperiod model by using some approximations of a sequence of single-period optimizations problems.

2 For an overview on VaR see Jorion (1997) and Duffie and Pan (1997).

3 The VaR limit (6) takes into account the fact that successful traders typically see their VaR limit increase with wealth. Other forms of VaR limits may be considered, as for instance a constant limit or a limit defined by a sum of fixed proportions of initial wealth and a running gain part (see Cuoco, He, and Issaenko (2001)).

4 Other forms of the VaR limit lead in general to wealth dependent VaR boundaries under the above approximation procedure.

5 If higher-order approximations of the initial VaR constraint are considered, we obtain a sequence of control problems \( (P_n)_{n=1,2,...} \) which eventually converges to the original problem \( P \). We do not consider this issue further in this paper.

6 One can show that solution of problem \( P_1 \) exists, i.e. the gradient of the value function is positive and the Hessian is strictly negative definite.

7 Detemple, Garcia, and Rindisbacher (2003) combine simulation with the computation of Malliavin derivatives to solve for the optimal portfolio strategy. Related to their work is the paper by Cvitanic, Goukasian, and Zapatero (2003) who propose a method based solely on Monte Carlo simulation. Both approaches are restricted to a complete market model setting. In our exercise we follow the approach proposed by Cvitanic, Goukasian, and Zapatero (2003). For the estimation, we partition the investor’s time-horizon into 100 time steps. In each time step, we draw 100'000 random numbers using antithetic variates and calculate the paths of the state variable \( X \) in order to calculate the evolution of price and variance.

8 Other specifications produced very similar findings as those presented in Figure 1 below.

9 By \( \mathbb{I}_A \) we denote the Heaviside function which equals one if \( A \) is true and zero otherwise.

10 This value holds for a model where volatility follows a lognormal process. For other processes, similar values were found.

11 For \( \gamma < 0 \), the following conclusions would just change sign.

12 The model can easily be extended to an arbitrary number of banks.
References


