Exotic options in general exponential Lévy models

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1 Introduction

Recently, financial models of the type

\[ S(t) = \exp(X_t), \quad (1) \]

where \( X \) is a Lévy process, have attracted a lot of interest, both among academics and the industry. Indeed, these models are a natural generalization of the well-known Black-Scholes model, the flaws of which are long recognized in various domains of application. In particular from the point of view of option pricing, there is in most markets a total disbelief that the “volatility” associated to the Black-Scholes model is constant with respect to strike prices (a phenomenon known as the smile effect).

Empirical studies show that models such as (1) are able to take account of this phenomenon. One may thus decide to use such a model to value other options; in general this will be done numerically by Monte Carlo simulations, or a finite differences scheme, but in particular cases, one can derive analytical formulas for some transform of the option.

Here we will be interested in two types of options. The first one is barrier and lookback options; these options have a fixed maturity and a payoff that involves the maximum (or the minimum) of the asset price over the period. The valuation of such options has been examined in [5], under restrictive conditions on \( X \). Here we take the probabilistic approach detailed in [15] and obtain a completely general result.

The second type of options is American and Russian options; those have a random maturity and their payoffs involve either the terminal value or the maximum of the asset over the period. The valuation problem has been taken up in [1, 2] in special cases. In particular in [2] the authors developed independently from us a probabilistic method very close to ours. Our method is quite general and only requires a slight restriction on the regularity of the resolvent of \( X \), which in fact is the same as in [5, 2].

Our results are quite general, and the price we pay is that the formulas are quite involved. In a special and interesting case, namely when \( X \) is spectrally negative, they can however be greatly simplified, and we shall present such simplifications.
2 Framework and notations

On a fixed probability space \((\Omega, \mathcal{F}, P)\) we denote by \(X\) a real Lévy process started at 0 and by \(\mathcal{F}_t\) the right-continuous completion of the filtration generated by \(X\). To \(X\) we associate a Markov family of probability measures \(P^x, x \in \mathbb{R}\). More generally, if \(Y\) is a Markov process on \((\Omega, \mathcal{F}, P)\), we denote by \(P_Y^0 = y\) the family of probability measures associated to different starting points.

\(X\) is characterized by its Lévy exponent \(\phi\), which is defined by

\[ E[e^{iuX_t}] = e^{-t\phi(u)}, \quad u \in \mathbb{R}, \]

and is given by the Lévy-Khintchine formula

\[ \phi(u) = iau + \sigma^2u^2 - \int_{\mathbb{R}\setminus\{0\}} \left( e^{iux} - 1 - iux1_{|x|\leq 1} \right) d\nu(x), \]

where \(a \in \mathbb{R}\), \(\sigma \in \mathbb{R}\), and \(\nu\) a Radon measure on \(\mathbb{R}\setminus\{0\}\) such that \(\int(x^2 \wedge 1)d\nu(x) < \infty\).

For a general process \(Y\), we denote by \(T_k(Y)\) the first passage time of \(Y\) above \(k\): \(T(x) = \inf\{t: Y_t > x\}\), and by \(\hat{T}_k(Y)\) the first passage time of \(Y\) below \(k\): \(\hat{T}_k(Y) = \inf\{t: Y_t < k\}\). Note that in general, \(X_{T_k(X)} > k\) and \(X_{\hat{T}_k(X)} < k\) with positive probability.

\(M\) denotes the past maximum of \(X\), that is \(M_t = \sup_{s \leq t}(0 \vee X_s)\), and \(I\) is the past infimum \(I_t = \inf_{s \leq t}(0 \wedge X_s)\). We call \(R\) the reflected Lévy process at its maximum: \(R_t = M_t - X_t\); recall that \(R\) is a Feller process under \(P\) relative to the filtration \(\mathcal{F}_t\). Lastly, we call \(L\) the Markov local time of \(R\) at 0, \(\tau\) its right-continuous inverse, and \(\eta\) the Itô measure of excursions of \(R\) away from 0.

We suppose that the price of a basic asset of interest (e.g. the price of a stock, an index, or an exchange rate) is modeled by (1):

\[ S(t) = \exp(X_t). \]

Such a model is customarily called an exponential Lévy model.

In the following, we will be interested in deriving option prices based on \(S\); we assume for simplicity that interest rates are zero. Moreover, we assume that \(P\) is a martingale measure for \(S\), and that \(P^x\) is our pricing measure, where \(x = \ln(S_0)\). The preceding assumption actually means that we consider a market model of \(S\); we make no assumptions on the dynamics of \(S\) under a historical measure. In general, the law of \(X\) is determined by a set of parameters which are calibrated to some market data, such as liquid options prices.

The price process of a financial flow \(H\) paid at a fixed time \(T > 0\), based on \(S\), is then given by:

\[ E_x[H|\mathcal{F}_T]. \]

On the other hand, \(H\) may be paid out at any stopping time \(\sigma\) in a family \(\Sigma\), and may depend on \(\sigma\); then the price process is given by

\[ \sup_{\sigma \in \Sigma} E_x[H(\sigma)|\mathcal{F}_T] \]

(clearly this is the lowest price at which one would accept to sell the product).
In this context, European options, i.e. options with payoff \( H = (S_T - K)^+ \) (call) or \((K - S_T)^+\) (put), where \(T\) and \(K\) are fixed, are easily priced thanks to Fourier inversion methods (see [7] or [9]).

When \(H\) is not of the type above, the option with payoff \(H\) is called “exotic”. In the following, we will deal with specific such \(H\), and we will always assume \(t = 0\). Because of the strong Markov property of \(X\) (or, equivalently, of \(S\)), this is not a restriction.

3 Barrier and lookback options, and the Wiener-Hopf factorization

3.1 The products

In this section, we are concerned with barrier and lookback options. These options have a fixed maturity \(T\) and a payoff that depends on the maximum (or minimum) of the asset price on \([0, T]\). Specifically, given a strike price \(K\) and a barrier level \(H\), different types of barrier options are defined by the payoffs:

- **Up-and-In Call**: \(UIC(H, K) = (S_T - K)^+ \mathbb{1}_{\sup_{t \leq T} S_t > H}, H > K;\)
- **Up-and-Out Call**: \(UOC(H, K) = (S_T - K)^+ \mathbb{1}_{\sup_{t \leq T} S_t < H}, H > K;\)
- **Down-and-In Call**: \(DIC(H, K) = (S_T - K)^+ \mathbb{1}_{\inf_{t \leq T} S_t < H}, H \leq K;\)
- **Down-and-Out Call**: \(DOC(H, K) = (S_T - K)^+ \mathbb{1}_{\inf_{t \leq T} S_t > H}, H < K.\)

Replacing \((S_T - K)^+\) with \((K - S_T)^+\), we obtain the corresponding Put options.

In the sequel we shall focus on the Up-and-In Call option, but our method is easily adapted to the other cases.

On the other hand, given a strike price \(K\), different lookback options have the payoffs:

- **Fixed-strike lookback Call**: \(LBCall^{fi}(K) = (\sup_{t \leq T} S_t - K)^+\) and Put: \(LBP ut^{fi}(K) = (K - \inf_{t \leq T} S_t)^+\)
- **Floating-strike lookback Call**: \(LBCall^{fl}(K) = (S_T - \lambda \inf_{t \leq T} S_t)^+\) and Put: \(LBCall^{fl}(K) = (\lambda \sup_{t \leq T} S_t - S_T)^+\)

To fix ideas, we will focus on the fixed strike lookback call and the floating strike lookback put.

In order to derive the price of barrier and lookback options, we will see that we need to know the law of \((T_h(X), X_{T_h(X)})\) for fixed \(h\). This law is characterized in terms of the Wiener-Hopf factorization, which we now recall.

3.2 The Wiener-Hopf factorization and the Pecherskii-Rogozin identity

Let us briefly recall the essentials of the Wiener-Hopf factorization; for more details, we refer to [4]. It is well-known that the ladder process \((\tau, X \circ \tau)\), is
a two-dimensional Lévy process and that its Laplace exponent \( \kappa \), defined by 
\[
\mathbb{E} \left[ e^{-\sigma(t) - \beta X \sigma(t)} \right] = e^{-t \kappa(\alpha, \beta)},
\]
is given by Fristedt’s formula
\[
\kappa(\alpha, \beta) = C \exp \left( \int_0^\infty \frac{dt}{t} \int_0^\infty (e^{-t} - e^{-\alpha t - \beta x}) \mathbb{P} \left[ X_t \in dx \right] \right)
\]
where the constant \( C > 0 \) depends only on the normalization of the local time \( L \). \( \kappa \) is called the ladder exponent of \( X \).

On the other hand, we recall the well-known Wiener-Hopf factorization for Lévy processes: for every \( q > 0 \) there exists two functions \( \phi_+^q \) and \( \phi_-^q \) such that:

1. Both \( \phi_+^q \) and \( \phi_-^q \) are characteristic functions; besides, \( \phi_+^q \) is analytic in the half-plane \( \{ z \in \mathbb{C}, \Re(z) \leq 0 \} \) and \( \phi_-^q \) is analytic in the half-plane \( \{ z \in \mathbb{C}, \Re(z) \geq 0 \} \).

2. For all \( u \in \mathbb{R} \), the following identity holds (the Wiener-Hopf factorization)
\[
\frac{q}{q + \phi(u)} = \phi_+^q(u) \phi_-^q(u).
\]

The functions \( \phi_+^q \) and \( \phi_-^q \) are called the Wiener-Hopf factors of \( X \). They have the following probabilistic interpretation: Let \( \theta \) be an exponential random variable with parameter \( q \), independent of \( X \). Then \( \phi_+^q \) is the characteristic function of \( M_\theta \), and \( \phi_-^q \) is the characteristic function of \( X_\theta - M_\theta \).

The Wiener-Hopf factors are related to the ladder exponent \( \kappa \) and the dual ladder exponent \( \hat{\kappa} \) (i.e. the ladder exponent of the dual process \( \hat{X} = -X \)) by
\[
\phi_+^q(u) = \frac{\kappa(q, 0)}{\kappa(q, -iu)} \quad (u \leq 0), \quad \phi_-^q(u) = \frac{\hat{\kappa}(q, 0)}{\kappa(q, -iu)} \quad (u \geq 0).
\]

It turns out that the joint law of \( (T_h(X), X_{T_h(X)}) \), can be characterized entirely in terms of the function \( \kappa \):

**Theorem 1 (The Pecherskii-Rogozin identity)** Set \( T_h = T_h(X) \). For every \( \alpha, \beta, q > 0 \), we have:
\[
\int_0^\infty e^{-qh} \mathbb{P} \left[ e^{-\alpha T_h(X) - \beta (X_{T_h(X)})} \right] dk = \frac{1}{q - \beta} \left( 1 - \frac{\kappa(\alpha, \beta)}{\kappa(\alpha, q)} \right)
\]

Formula (3) can be written in terms of the Wiener-Hopf factor \( \phi_+^q \):
\[
\int_0^\infty e^{-qh} \mathbb{P} \left[ e^{-\alpha T_h(X) - \beta (X_{T_h(X)})} \right] dh = \frac{1}{q - \beta} \left( 1 - \frac{\phi_+^q(iq)}{\phi_+^q(i\beta)} \right)
\]

Since \( \hat{T}_h(X) = T_{-h}(-X) \), similar identities, involving \( \hat{\kappa} \) or, \( \phi_-^q \) can easily be deduced for the joint law of \( (\hat{T}_h(X), X_{\hat{T}_h(X)}) \).

### 3.3 Valuation

For the valuation of barrier and lookback options, we proceed in two steps.
3.3.1 Step 1: reduction to digital options

Our first step is to reduce the problem of the valuation of barrier and lookback options to the valuation of simpler options. The digital barrier options are defined by their payoffs:

- digital Up-and-In Call: \( \text{BinUIC}(H, K) = 1_{S_T > K} 1_{\sup_{t \leq T} S_t > H} \);
- digital Up-and-In Put: \( \text{BinUIP}(H, K) = 1_{S_T < K} 1_{\sup_{t \leq T} S_t > H} \),

and so on.

The payoffs of the barrier and lookback options can be expressed in terms of digital options: for example,

\[
\text{UIC}(H, K) = \int_{-\infty}^{\infty} \text{BinUIC}(H, k) \, dk
\]

\[
\text{LBCall}^I(K) = \int_{-\infty}^{\infty} \text{BinUIC}(h, 0) \, dh
\]

\[
\text{LBPut}^I(K) = \int_{0}^{\infty} \text{BinUIP}(k, k/K) \, dk
\]

By Fubini’s theorem, all we need to do now is to derive the price of a digital option.

3.3.2 Step 2: pricing of the digital options

Let us now derive the price of a digital Up-and-In Call option, i.e. the probability \( P_{x}[S_T > K, \sup_{t \leq T} S_t \geq H] \). The same method can be applied to obtain the price of the Up-and-In Put. Set:

\[
\pi(T, h, k) = P_{x}[X_T > k, T(h) < T]
\]

where \( k = \ln(K) \) and \( h = \ln(H) \). By the Markov property,

\[
\pi(h, k) = E_{x}[1_{T(h) < T} \beta(T(h), X_{T(h)})]
\]

where \( \beta \) is the function defined by

\[
\beta(t, x) = P_{x}[X_{T-t} > k].
\]

Hence the price \( \pi(h, k) \) is characterized by the formulas (3) or (4). Precisely, it follows from quite straightforward computations that

\[
\int_{0}^{\infty} dh \int_{0}^{\infty} dT \int_{-\infty}^{\infty} dk \, e^{-(\beta+i\lambda)k} e^{-qT} e^{i\lambda k} \pi(h + x, k)
\]

\[
= \frac{1}{i\lambda (q + \phi(\lambda))} \frac{1}{\beta - i\lambda} \left( 1 - \frac{\phi^+_\lambda(i\beta)}{\phi_-^\lambda(-\lambda)} \right)
\]

(5)

Of course this is equivalent to the results of [5], where the authors have derived them by means of an analytic method, for which they needed extra assumptions on \( X \).
4 American options. Russian options and path decompositions

4.1 The products

In this section we are concerned with options whose lifetime is a random time determined by the buyer of the option. We will only deal with perpetual options, i.e. there is no fixed finite bound on this time. Note however that by the same method, one can also deal with such options when the exercise time is bounded by an independent exponential time. Such options are called “Canadized options”, and were introduced in [6] (see also [2]).

Given a strike price $K$ and a discounting constant $a > 0$ (e.g. the dividend yield of the asset), the perpetual American put option entitles the buyer to exercise at any finite stopping time $\sigma$, and to get at this time:

$$e^{-a\sigma} (K - S_\sigma)^+.$$ 

As we shall see, the appropriate mathematical tool to derive the price of this option is again the Wiener-Hopf factorization.

The Russian option with strike price $K$ entitles the buyer to exercise at any finite stopping time $\sigma$, and to get at this time:

$$e^{-a\sigma} \max \left( K, \sup_{t \leq \sigma} S_t \right),$$

where $a > 0$ is a constant.

In order to derive the price of the perpetual American put and Russian option, a first step is to give an equivalent formulation to the optimization problem. This has been done in [14] for the American put and in [1] for the Russian option, and we will recall it below.

The next step is to compute the value function for the new optimization problem. In the case of the American put, as already mentioned, this can be done in terms of the Wiener-Hopf factorization. For the Russian option, however, we need another mathematical tool, which we now introduce.

4.2 Path decompositions and a description of the Ito measure

In the valuation problem for the Russian option, we will need the Laplace transform $E_{R_0 = r} \left[ e^{-aT_k(R) - bR_{k(R)}} \right]$ where $k \geq r$. In order to obtain this, we will rely on excursion theory and we will need a slight extra assumption on the regularity of $X$:

From now on, we assume that the resolvent measures of $X$ are absolutely continuous with respect to Lebesgue measure.

As is well-known, under this condition $X$ can “creep across levels”, that is,

$$\mathbb{P} \left[ X_{T(x)} = x \right] > 0, \quad x > 0.$$ 

This will enable us to give a description of the excursion measure $n$ of $R$ away from 0; precisely, $n$ can be disintegrated with respect to the height of the excursion, and the canonical process can be described under this disintegrated
measure. For the Brownian motion, this is the famous Williams description of Ito’s measure.

Let us motivate this decomposition in the present framework. As mentioned above, we will need to compute \( E_{R_0 = r} \left[ e^{-a T_k(R) - b R_T(n)} \right] \) in relation to the valuation problem of the Russian option. Since

\[
E_{R_0 = r} \left[ e^{-a T_k(R) - b R_T(n)} \right] = e^{-br} E_{R_0 = 0} \left[ e^{-a T_k(R) - b R_T(n)} \right]
\]

we suppose \( r = 0 \). Note that \( T_k(R) \) occurs during the first excursion of \( R \), the height of which is \( \geq k \). This means that

\[
E \left[ e^{-a T_k(R) - b R_T(n)} \right] = E \left[ \sum_{t < L_{\infty}} e^{-a (\tau_{t-} + T_k(e_t)) - b e_t (T_k(e_t))} 1_{\sup_{s \leq \tau_{t-}} R_s < k, \sup_{s \in [\tau_{t-}, \tau_t]} R_s \geq k} \right]
\]

where \( e_t \) denotes the excursion over \( [\tau_{t-}, \tau_t) \): \( e_t(u) = R_{\tau_{t-} + u}, u < \Delta \tau_t \). By the compensation formula, the foregoing equals

\[
E \left[ \int_0^{L_{\infty}} e^{-a \tau_t} 1_{\sup_{s \leq \tau_t} R_s < k} \, dt \right] = \int_0^{\infty} E \left[ e^{-a \tau_t}; A_t \right] \, dt
\]

where \( A_t \) is the event

\( A_t = \{ \text{No excursion ending before } \tau_t \text{ has height } \geq k \} \)

Now \( \tau \) is a subordinator and can be written \( \tau_t = \delta t + \sum_{s \leq t} \Delta \tau_s \), so we have

\[
E \left[ e^{-a \tau_t}; A_t \right] = e^{-a \delta t} E \left[ \prod_{s \leq t} 1_{\Delta \tau_s > 0} 1_{\sup_{s \leq t} e_t < k} e^{-a \Delta \tau_s} \right]
\]

Since excursions are independent of one another, we can formally take the product outside of the expectation, so that

\[
E \left[ e^{-a \tau_t}; A_t \right] = e^{-a \delta t} \prod_{s \leq t} E \left[ 1_{\Delta \tau_s > 0} 1_{\sup_{s \leq t} e_t < k} e^{-a \Delta \tau_s} \right]
\]

Now because \( \Delta \tau_s \) is the lifetime of the excursion \( e_t \), this can be computed again if we can describe the excursion measure. Of course, the argument above is not rigorous since the last product does not make sense a priori, but it can easily be made so by first taking only excursions with height \( \geq \epsilon > 0 \), which are denumerable, and then passing to the limit as \( \epsilon \to 0 \).
4.2.1 Conditioning to stay positive and a description of the excursion measure

The Lévy process “conditioned to stay positive” was introduced in [3] as a path transformation of the original process $X$. Roughly speaking, this transformation consists of pasting together positive excursions of $X$, while deleting the negative ones. Following [3], we shall denote this process by $X^\uparrow$.

It was also shown in [3] that $X^\uparrow$ can also be obtained as a superharmonic transform of the Lévy process $X$ killed when it enters $(-\infty, 0]$. Now, fix $k > 0$ and consider the first excursion of $R$ with height $> k$, denote it by $e^k$. Hence, $e^k$ is the excursion straddling $T_k(R)$. By the strong Markov property at $T_k(R)$, we have that, given $R_{T_k(R)} = r$, the process $(RT_k(R), t \geq 0)$ is independent of $F_{T_k(R)}$ and has the same law as $R$ started at $r$. Hence, conditionally on $R_{T_k(R)}$, the part of the excursion after $T_k(R)$ is independent of $F_{T_k(R)}$ and has the same law as $R$, started at $r$, killed when first reaches 0. Since $M$ is constant over this time-interval, we see that $(e^k(t), t \geq T_k(e^k))$ has the same semigroup as $-X$. Now, the piece of the excursion from time $T_k$ on is a killed Lévy process. Let us denote $\gamma$ the a.s. unique time at which it reaches its maximum. Then by the results of [13], the pieces before and after $\gamma$ are conditionally independent given $R_{\gamma-} \lor R_{\gamma} = m$, and are both Markovian; moreover our assumption about the resolvent measures of $X$ implies that $X$ is continuous at $\gamma$. Lastly, [13] specifies their respective semi-groups: and the pre-maximum piece has the same semi-group as $(-X)^1$, while the post-maximum one has a semi-group specified by

$$
\tilde{p}_h f(x) = \frac{1}{h_m(x)} \mathbb{E}_{-x}[f(-X_t)h_m(-X_t); t < T_m]
$$

where $T_k = \inf\{t > 0, -X_t \geq k\}$ and $h_k$ is the function $h_k(x) = \mathbb{P}_{-x}[T_k = \infty]$.

By considering $t < k$, we see moreover that the bit of the excursions between $T_i$ and $T_k$ has the same law as $-X$, conditioned to reach $k$ before 0, that is, has the same semi-group as $(-X)^1$.

Thus, the pieces up to $T_k$ and between $T_k$ and $\gamma$ are independent conditionally on the value at $T_k$ and have the same law; thus they can be pasted together into a single piece with that common law.

From the preceding, we deduce the following description of the excursion process:

**Theorem 2** Conditionally on $\sup_s e(s) = h > 0$, the excursion process splits into two independent parts:

- the pre-maximum part $(e(s), s < \gamma)$, which is distributed as $((-X)^1, t < T_k)$, conditionally on $(-X)^1_{T_k} = h$;
- the post-maximum part $(e(\gamma + s), 0 < s < V - \gamma)$, which is Markovian, with semi-group given by (7), where $V$ is the lifetime of the excursion $e$.

In order to end the description of the Itô measure of excursions, it remains to specify the law of the height $h(e) = \sup_{s \leq V} e(s)$ under $\mu$. In fact, it turns out that $\mu(h(e) > x)$ can be characterized in terms of the Wiener-Hopf factors.
We can now finish to compute (6) by
\[
n \left( e^{-aT_k(e)} - be(T_k(e)) \right) = \int_k^\infty n(h(e) \in dx)n \left( e^{-aT_k(e)} - be(T_k(e)) \right) h = x
\]

Rather than going on with presenting this in a general setting, which would lead us to quite complicated formulas, we prefer to stop here our general discussion; in section 5 we shall give more details on a simpler and more illustrative situation, when \( X \) is spectrally negative; in that case one can obtain formulas that are relatively simple.

4.3 Valuation

4.3.1 Step 1: reformulation of the optimization problems

The valuation problem for American and Russian options is a priori an optimal stopping problem. Denote by \( Am(x, K) \) the price of the perpetual American put option with strike price \( K \):
\[
Am(x, K) = \sup_\sigma \mathbb{E}_x \left[ e^{-a\sigma} (K - S_\sigma)^+ \right]
\]
and by \( Ru(x, m) \) the price of the Russian option:
\[
Ru(x, m) = \sup_\sigma \mathbb{E}_x \left[ e^{-a\sigma} \max \left( e^m, \sup_{t \leq \sigma} S_t \right) \right].
\]

It turns out that those optimal stopping problem have solutions as passage times.

First, in the case of the perpetual American put, there exists \( k_a^* \) such that \( \hat{T}_{k_a^*}(X) \) maximizes \( \mathbb{E}_x \left[ e^{-a\sigma} (K - S_\sigma)^+ \right] \), and moreover \( k_a^* \) is given by
\[
e^{k_a^*} = K \mathbb{E} \left[ e^{\theta} \right],
\]
where \( \theta \) is an exponential variable with parameter \( a \); it follows that:
\[
Am(x, K) = \mathbb{E}_x \left[ e^{-a\hat{T}_{k_a^*}(X)} (K - S_{\hat{T}_{k_a^*}(X)})^+ \right]
\]
(see [14]).

For the Russian option, there exists \( k_r^* \) such that \( T_{k_r^*}(R) \) maximizes \( \mathbb{E}_x \left[ e^{-a\sigma} \max \left( e^m, \sup_{t \leq \sigma} S_t \right) \right] \), and \( k_r^* \) is given by
\[
k_r^* = \arg \max_{k \geq 0} \mathbb{E} \left[ e^{-aT_k(R) + (R_{T_k(R)} - k)} \right]
\]
(8)
and the price of the Russian option is then given by
\[
Ru(x, m) = e^x \mathbb{E}_{R_0 = m - x} \left[ e^{-aT_{k_r^*}(R) + R_{T_{k_r^*}(R)}} \right]
\]
(9)
(see [1, 2]).
4.3.2 Step 2: computation of the value function

In the case of the perpetual American put, the optimal stopping level \( k^*_a \) can be computed explicitly in terms of the Wiener-Hopf factors. In fact, we have

\[
\mathbb{E} \left[ e^{I_0} \right] = \int_0^\infty ae^{-at} \mathbb{E} \left[ e^{I_0} \right] dt = \phi_a^*(-i)
\]

so that

\[
k^*_a = \ln(k) + \ln(\phi_a^*(-i)).
\]

In the following, we denote \( T^* = \hat{T}_{k^*_a}(X) \) in order to simplify the notation. Now, the price of the perpetual American put is characterized by the identity (3) or (4). Specifically, first notice that \( k^*_a \leq \ln(k) \), so that we have in fact

\[
Am(x, K) = \mathbb{E}_x \left[ e^{-aT^*}(K - S_{T^*}) \right]
\]

\[
= KE_x \left[ e^{-aT^*} \right] - KE_x \left[ e^{-aT^* + X_{T^*}} \right]
\]

Note that we may furthermore suppose that \( x > k^*_a \), for otherwise, \( T^* = 0 \) and we are in a trivial situation. Set \( \beta = k^*_a - k \), where \( k = \ln(K) \). Then by simply taking Laplace transforms in the last formula above, we obtain

\[
\int_{-\infty}^{x-\beta} e^{\lambda k} Am(x, e^k) dk = e^{-(1+\lambda)(\beta-x)} \frac{1 - \phi_a^*(-i(1 + \lambda))}{1 + \lambda} - e^{-\lambda(\beta-x)} \frac{1 - \phi_a^*(-i(1 + \lambda))}{\phi_a^*(-i)}
\]

In the case of the Russian option, it is much more difficult to obtain \( k^*_r \) explicitly; however, we have computed the function

\[
\rho(x, m, k) = \mathbb{E}_{R_0=m-x} \left[ e^{-aT_k(R) + R_k(n)} \right]
\]

as explicitly as possible to facilitate the determination of the optimal level \( k^*_r \) and, once this level has been found, the computation of \( Ru(x, m) \). In the next section, we shall see that one can do much better if \( X \) has only negative jumps.

5 Spectrally negative processes

We have developed method above, which can be applied to any Lévy process (with a restriction on the resolvent measures in the case of the Russian options). Our tools are part of the fluctuation theory for such processes. For the generality that we have achieved, we pay the price that the formulas obtained are quite complicated, and may be of little use in practice. For example, inverting a 3-dimensional Fourier-Laplace transform as in the case of the digital option may not be fast enough on a present-time computer, for application on a trading floor.

It is well-known that fluctuation theory simplifies a lot when \( X \) has only one-sided jumps. To fix ideas, we consider the case when \( X \) is spectrally negative, that is \( X \) has only negative jumps.
From a modeling point of view, the case of spectrally negative $X$ is quite interesting; in fact if we calibrate today a Merton model (see [12]) on European option prices for a common index such as STOXX50 or S & P, we often find that the mean size of the jumps is negative while the variance is quite small (this could reflect a “fear” of the market of a sudden huge drop-down of the price of a stock or index).

5.1 The Wiener-Hopf factorization

When $X$ is spectrally negative, one great advantage is that the Wiener-Hopf factors can be computed explicitly in terms of the Lévy exponent $\phi$. Indeed, it is known that if $\theta$ is an exponential variable with parameter $q$, independent of $X$, then $M_\theta$ has an exponential distribution with parameter $\Phi(q)$, where $\Phi$ is the inverse of the function $\lambda \mapsto -\phi(-i\lambda)$ (this inverse is known to exist on $[l, \infty)$ for $l$ large enough). From this, we have:

$$\phi^+_q(u) = \frac{\Phi(q)}{\Phi(q) - iu}$$

and one can deduce $\phi^-_q$ from identity (2).

Moreover, the process of first passage times for $X$ is then a subordinator with Laplace exponent given by $\Phi$, as is easy to see from the facts that $e^{\lambda X_t + \phi(-i\lambda) t}$ is a martingale ($\lambda \geq 0$) and that $X_{T_h(X)} = h$ a.s.

From these observations, one can deduce the following characterization of the price of a digital Up-and-In call option:

$$\int_0^\infty dT \int_{-\infty}^\infty dk \ e^{-qT} e^{i\lambda k} \pi(h, k) = \frac{e^{i\lambda h}}{i\lambda} \left( q + \phi(\lambda) \right) e^{-(h-z)\Phi(q)}$$

which is a lot simpler than (5).

The valuation problem for the perpetual American put is also simplified. The optimal stopping level $k^*_a$ can be computed explicitly:

$$e^{k^*_a} = K \frac{a}{a + \phi(-i)} \Phi(a) - 1,$$

and the price is then characterized by

$$\int_{-\infty}^{x-\beta} e^{\lambda k} Am(x, e^k)dk$$

$$= \frac{e^{-(1+\lambda)(\beta-x)}}{1 + \lambda} \left( \frac{1 - a(\Phi(a) - 1 - \lambda)}{\Phi(a)(a + \phi(-i - i\lambda))} \right)$$

$$- \frac{e^{-\lambda(\beta-x)}}{\lambda} \left( \frac{1 - a + \phi(-i)}{\Phi(a) - 1} - \frac{\Phi(a) - 1 - \lambda}{a + \phi(-i - i\lambda)} \right)$$

Lastly, the pricing of the Russian option becomes also a lot more tractable. In fact, for a spectrally negative process the problem can be solved in a reasonably simple form. This problem has also been solved independently in [2], by a method very close to ours. To present this, we first need to recall that there exists a family of scale functions $W^{(q)}$, characterized by their Laplace transform

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)dx = \frac{1}{\phi(-i\lambda) - q}.$$
Moreover, the function $W = W^{(0)}$ solves the two-sided exit problem: for every $a < x < b$, the probability that $X$, starting at $x$, exits $[a,b]$ through $b$ is given by

$$P_x \left[ T_{aT_1(X)} \wedge T_{bT_1(X)} > a \right] = \frac{W(x-a)}{W(b-a)}.$$  

Lastly, $W$ bears the following relationship to $n$:

$$a(h(e) > x) = \frac{W'(x)}{W(x)}$$

(see [4]). Of course since $\phi - q$ is the Lévy exponent of the Lévy process $X$ killed at an independent exponential time of parameter $q$, $W^{(0)}$ gives the quantities corresponding to this process.

We also recall that in the case of a spectrally negative process $X$, its supremum process $M$ is continuous and is a local time for $R$ at 0. We may then take $L = M$ and in this case $\tau = T(X)$ is simply the process of first passage times of $X$. The computation of the first term in formula (6) is then also easier (see [2] for more details). Indeed, it can be written

$$\mathbb{E} \left[ \int_0^{M_\infty} e^{-aT_1(X)} 1_{\sup_{s \leq T_1(X)} R_s < k} dt \right].$$

Since $X$ has no positive jumps, we have $X_{T_1(X)} = t$ a.s. and on the other hand, the event $\{ \sup_{s \leq T_1(X)} R_s < k, t < M_\infty \}$ is $\{\text{No excursion ending before } t \text{ has a height } \geq k\}$. Setting $\mathbb{P}^a = \exp(\Phi(a)X_t - at)\mathbb{P}$ on $\mathcal{F}_t$, we obtain

$$\mathbb{E}^a \left[ \int_0^{M_\infty} e^{-\Phi(a)x} 1_{\sup_{s \leq T_1(X)} R_s < k} dt \right] = \int_0^{\infty} e^{-t\Phi(a)\mathbb{P}^a} \left[ \sup_{s \leq T_1(X)} R_s < k, t < M_\infty \right] dt$$

$$= \int_0^{\infty} e^{-t\Phi(a)} e^{-t \frac{W'(k)}{W(k)} dt} = \frac{W_a(k)}{\Phi(a)W_a(k) + W'_a(k)}$$

where $W_a$ is the scale function under $\mathbb{P}^a$. Then to compute the second term in (6) we can follow the previous method involving the disintegration of $n$. The computations are elementary but lengthy and one can check that the result agrees with the formula given in [2]. In the end, one has

$$\mathbb{E}_{R_{a \equiv a}} \left[ e^{-aT_k(R) - bR_k(R)} \right]$$

(11)

$$= e^{-b \bar{W}^{(a)}(k-r)} - W^{(a)}(k-r) \frac{(a - \phi(-ib))W^{(a)}(k)e^{-bk} + b\bar{W}^{(a)}(b)(k)}{W^{(a)}(k)}$$

where

$$\bar{W}^{(a,b)}(x) = 1 + (a - \phi(-ib)) \int_0^x e^{-byW^{(a)}(y)} dy.$$  

Inserting this into formulas (8) and (9), the authors of [2] prove that

$$k^*_x = \inf \{ x : \bar{W}^{(a,0)}(x) \leq aW^{(a)}(x) \}$$

and that the corresponding value of the Russian option is given as

$$Ru(x, m) = e^m \bar{W}^{(a,0)}(k^* - (m - x))$$

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To conclude on this subject, we have presented our results in terms of excursion theory, because it is a natural tool to study the fluctuations of a Lévy process (in fact, it generalizes the renewal theory for random walks). But in the case of spectrally negative processes, we could have adopted a more elementary presentation based solely on martingale theory. In fact, by using some “good” martingales, all the identities that we used can be obtained without appealing to excursion theory. For the first passage times of $X$ above a level, the relevant martingale is the well-known exponential martingale already mentioned $e^{\lambda X_t + t\phi(-i\lambda)}$; for the first passage times of $R$, the appropriate martingale is based on the scale functions $W^{(a)}$ and $W^{(a,b)}$, see [16].

6 Related questions of interest

6.1 Generalized barrier options

In [10], a formula was derived in the Black-Scholes framework for double barrier options, i.e., options that are killed if the asset price ever goes out of a strip during the lifetime of the product. It would be of interest to derive formulas for this option in the setting of an exponential Lévy model. Presumably, the method used here will lead to the result. In particular, in [11], the author defined a Lévy process confined in an interval and described the resulting process.

Moreover in [8], the authors studied even more general options, where the barrier constraint was required to hold only in a limited period of time before maturity. This study was made possible by the knowledge of occupation times of Brownian motion. It would also be of interest to try and generalize these results to the present framework.

6.2 Multidimensional products

In this work, we have only been concerned by one-dimensional models. However, exotic products traded in the industry are more and more based on several assets, especially on the OTC equity market. Since these products are also in general affected by the smile effect, it would be interesting to study multidimensional exponential Lévy models and the pricing of options in such a framework.

In particular, an interesting question is how to model the dependence between the components, when the underlying processes are Lévy processes and not Brownian motion.

References


