Arbitrage and pricing in a general model with flows

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Abstract

In this paper we study a fundamental issue in the theory of modeling of financial markets. We consider a model where any investment opportunity is described by its cash flows. We allow for a finite number of transactions in a finite time horizon. However, we let a transaction be held at a random moment. This places our model closer to the real world situation than discrete-time or continuous-time models. Moreover, our model creates a general framework to consider markets with different types of imperfections: proportional transaction costs, frictions on the numeraire, etc.

We develop an analog of the fundamental theorem of asset pricing. We show that lack of arbitrage is essentially equivalent to existence of a Lipschitz continuous discount process such that expected value of discounted cash flows of any investment is non-positive. We address a question of contingent claim pricing and hedging.

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1 Introduction

Financial mathematics is mostly engaged in design of models of the financial world. In addition to being ”good copies” of reality these models should be tractable by mathematics. There are several reasons to develop models. We will mention two of them: better understanding of ”the nature of the market” and pricing of contingent claims. A key notion to this questions is lack of arbitrage. It has always had an economic meaning of inability to acquire positive gains without any risk. However, it required major refinements while incorporating in the models of the market. Simple discrete-time models used the most intuitive definition: a self-financing strategy with zero initial capital is an arbitrage opportunity if it produces a non-negative final value with probability one and positive final value with probability greater than zero (Bingham, Kiesel [1]). During its history financial mathematics developed a lot of theories that addressed different aspects of non-ideal market. These were lack of numeraire, transaction costs, short-sale constraints, partial knowledge, etc. But there has been a strong need for a general framework that would cover all those aspects and allow to research into a ”nature” of arbitrage and pricing.
A remarkable step in this direction was made by Jouini and Napp in [3]. They adopted a model where all investment opportunities were described by their cash flows. For instance, in such a model, the investment which consists of buying, in a perfect financial model, at date $t_1$ one unit of risky asset whose price process is given by $(S_t)_{t \in \mathbb{R}_+}$ and selling at date $t_2 > t_1$ this unit, is described by a process $(\Phi_t)_{t \in \mathbb{R}_+}$ which is null outside $\{t_1, t_2\}$ and which satisfies $\Phi_{t_1} = -S_{t_1}$, $\Phi_{t_2} = S_{t_2}$. Each investment $(\Phi_t)_{t \in \mathbb{R}_+}$ is null everywhere but a finite number of points. The market is defined as a positive convex cone of such investments.

The framework proposed by Jouini and Napp is quite general. It allows for a wide class of models: with perfect numeraire, with different rates of borrowing and lending, with convex cone constraints on the quantities of assets held by investors (e.g. short-sale constraints) and with proportional transaction costs. Moreover, it is much closer to the real world than classical discrete-time models. It generalizes the notion of discrete-time. Each investment consists of a finite number of cash flows in deterministic moments, but different investments can have cash flows in different moments. It reflects what happens in real world. No investor can make an infinite number of transactions in finite time. However, one can argue that it is possible to make transactions in finite number of stopping times, for instance when the price of a stock reaches certain limit. A small progress in this direction was made by Napp in [5]. She modified a model in such a way that one could consider stopping times with countable number of values.

In both papers the analog of the Fundamental Theorem of Asset Pricing was presented. It stated that there is no free-lunch if there exists an adapted discount process $g_t$ with a uniformly bounded and strictly positive modification such that the expected value of discounted cash flows of any investment is non-positive.

In the present paper we make further generalizations of Jouini’s approach. We construct a model that goes along with the following real world rules:

- at each moment an agent has full knowledge of what has happened up to this moment,
- an agent can make only a finite number of transactions in finite time horizon,
- each transaction consists of a random cash flow occurring in a random moment; the decision of choosing the moment is made according to the knowledge available to the agent.

We satisfy these rules by allowing for random cash flows in any bounded stopping time. Therefore we exploit advantages of discrete-time and continuous-time models. Continuous models express our perception of time, but continuous trading is a huge simplification to what one can observe. On the contrary discrete-time models go along our requirements for trading (there must be a brake between subsequent transactions). However, it allows for transactions only in a finite set of moments in a finite horizon.

For the constructed model we derive an analog of the Fundamental Theorem of Asset Pricing in the spirit of Jouini and Napp ([3]). We obtain that the absence of free lunch (a stronger version of no-arbitrage condition) is equivalent to the
existence of a discount process $g_t$ such that the expected value of discounted cash flows of any investment is non-positive. The process $g_t$ has uniformly bounded and Lipschitz continuous trajectories, a natural property when looking from economic point of view.

Last sections of the paper are devoted to contingent claim pricing. We present how theorems from previous sections can be used to obtain links between arbitrage pricing (in the spirit of Harrison and Kreps [2]) and hedging (a super-replication cost). We prove that the upper limit of the no-arbitrage interval of option prices is equal to the super-replication cost. A paper by Napp ([5]) provides a much wider coverage of the pricing issues in the model with flows, but with a different approach.

In section 2 we define a model of the market. Following two sections are devoted to design of a framework that enables us to define a notion of arbitrage and free lunch. In section 3 we design a Banach space $\mathcal{M}$ and prove some of its properties. Section 4 is based on a seminar by Schwartz [7] and defines a space $L^1_\mathbb{P}(\Omega, \mathcal{M})$ of integrable functions and its dual. Section 5 provides a representation of investments as elements of $L^1_\mathbb{P}(\Omega, \mathcal{M})$ and defines conditions of no-arbitrage and no-free-lunch. We prove there a Fundamental Theorem of Asset Pricing, i.e. existence of discount processes (our proof is based on the proof by Jouini and Napp [3]). Contingent claim pricing and hedging is addressed in section 6.

2 A model

The model presented in this paper is a generalization of the model proposed by Jouini and Napp in [3]. We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ representing all states of the world. The probability distribution $\mathbb{P}$ is the real world probability. The knowledge about the world is encoded in the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfying usual conditions (completeness and right continuity) with $\mathcal{F}_0$ being trivial (we have perfect knowledge of the world at time zero). In this probabilistic framework we define investments. Each investment opportunity is described by its cash flows. Every cash flow occurs in a random moment defined by a bounded stopping time. A positive cash flow means obtaining money, negative - paying.

**Definition 2.1.** An investment opportunity $(\Phi_t)_{t \in \mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ adapted process that can be written in the form

$$\Phi_t = \sum_{i=1}^{n} \Phi_{\tau_i} 1_{\tau_i}(t)$$

for some $n > 0$ and a sequence of bounded stopping times $\tau_1, \tau_2, \ldots, \tau_n$ such that $\Phi_{\tau_i} \in L^1(\Omega, \mathcal{F}_{\tau_i}, \mathbb{P})$ for every $i \in \{1, \ldots, n\}$. We denote by $\Delta$ the set of all investments.

The market is defined by available investment opportunities. We assume that each investment is infinitely divisible, i.e. if there exists investment $(\Phi_t)_{t \in \mathbb{R}_+}$ then there exists investment $(\lambda \Phi_t)_{t \in \mathbb{R}_+}$ for any non-negative $\lambda$. Moreover, if
there are two different investment opportunities \((\Phi_t)_{t \in \mathbb{R}_+}, (\Psi_t)_{t \in \mathbb{R}_+}\) an agent can submit to both of them at the same time, so there exists an investment that is the sum of \((\Phi_t)_{t \in \mathbb{R}_+}\) and \((\Psi_t)_{t \in \mathbb{R}_+}\).

**DEFINITION 2.2.** A market \(J\) is a positive convex cone of investment opportunities, i.e.

1) \(\Phi_1, \Phi_2 \in J \Rightarrow \Phi_1 + \Phi_2 \in J,\)

2) \(\Phi \in J, a \in \mathbb{R}_+ \Rightarrow a\Phi \in J.\)

**EXAMPLE 2.1.** We will construct a cone \(J\) representing prefect market with \(N\) assets whose price processes are given by \((S^i_t)_{t \in \mathbb{R}_+}, i = 1, \ldots, N.\) We require that for any \(i = 1, \ldots, N\) and \(\tau\) - bounded stopping time \(\tau \in L^1(\Omega, \mathcal{F}_\tau, \mathbb{P})\). We assume that there are no short-sale constraints or any other frictions. Define

\[
\Phi^0_{\tau, \sigma} = \theta(t) (1_{\tau}(t)S_\tau - 1_{\sigma}(t)S_\sigma),
\]

where \(\tau, \sigma\) are bounded stopping times and \(\theta \in L^\infty(\Omega, \mathcal{F}_{\tau \wedge \sigma}, \mathbb{P}).\) Denote by \(J\) a positive convex cone generated by all investments of the form (1). The set \(J\) defines a market.

Because of the generality of our model we cannot use classical approach to define a notion of absence of free lunch. In the following sections we develop a theory that enables us to formulate this definition.

### 3 A Banach space \(\mathcal{M}\)

The aim of this section is to design a Banach space \(\mathcal{M}\) fulfilling a few special requirements. We enlist some of them:

- for every \(t \in \mathbb{R}_+\) there exists an element \(\delta_t \in \mathcal{M},\)
- \(\|\delta_s - \delta_t\|_\mathcal{M} > 0\) for \(s \neq t,\)
- the space \(\mathcal{M}\) supports finite linear combinations of \(\delta_t,\)
- it is possible to decide ”easily” whether all coefficients in linear combination of \(\delta_t\) are positive,
- for \(t_n \to t, a_n \to a\) we have \(a_n \delta_{t_n} \to a \delta_t.\)

We start with a definition of a linear normed space \(M.\) Let

\[
M = \{\mu: \mathbb{R}_+ \to \mathbb{R} : \text{ supp } \mu \text{ is finite}\}.
\]

It is easy to see that \(M\) is actually a linear space. We shall equip it with a norm. First denote by \(D\) a set of bounded by 1 and Lipschitz continuous functions with constant 1, i.e.

\[
D = \{f: \mathbb{R}_+ \to \mathbb{R} : \forall t, s \in \mathbb{R}_+ |f(t)| \leq 1, |f(t) - f(s)| \leq |t - s|\}.
\]
Then define a functional on $M$ by

$$
\|\mu\|_M = \sup \left\{ \left| \sum_{t \in \mathbb{R}_+} f(t)\mu(t) \right| : f \in \mathcal{D} \right\}.
$$

**Lemma 3.1.** $(M, \| \cdot \|_M)$ forms a normed linear space.

**Proof.** First we show that $\| \cdot \|_M$ is well-defined for all elements of $M$. Fix $\mu \in M$. Then $\sum_{t \in \mathbb{R}_+} |\mu(t)| < \infty$ since we sum over a finite number of non-zero points. Obviously $\|\mu\|_M \leq \sum_{t \in \mathbb{R}_+} |\mu(t)|$.

We shall show that $\| \cdot \|_M$ is a norm:

1. $\|\mu\|_M = 0$ if and only if $\mu = 0$,
2. $\|\lambda\mu\|_M = |\lambda|\|\mu\|_M$,
3. $\|\mu + \nu\|_M \leq \|\mu\|_M + \|\nu\|_M$.

In (1) left implication is clear. To show the opposite one let $\mu \neq 0$. Thus $\mu$ has a representation $\mu = \sum_{k=1}^{K} \alpha_k \delta_{t_k}$ for some natural $K$, $(\alpha_k) \subset \mathbb{R} \setminus \{0\}$, $0 < t_1 < t_2 < \cdots < t_K$. Put

$$
f(t) = \begin{cases} 
(t_2 - t_1) \wedge 1 & t \leq t_1 \\
(t_2 - t_1) \wedge 1 & t_1 < t \leq t_2 \\
0 & t > t_2
\end{cases},
$$

where $a \wedge b = \min(a, b)$. Observe that $f \in \mathcal{D}$ and $\left| \sum_{t \in \mathbb{R}_+} f(t)\mu(t) \right| = |\alpha_1| \left[ (t_2 - t_1) \wedge 1 \right]$. Thus $\|\mu\|_M \geq |\alpha_1| \left[ (t_2 - t_1) \wedge 1 \right] > 0$. Condition (2) results from the fact that $\left| \sum_{t \in \mathbb{R}_+} f(t)\lambda\mu(t) \right| = |\lambda| \left| \sum_{t \in \mathbb{R}_+} f(t)\mu(t) \right|$. Proof of (3) is also simple. Fix $f \in \mathcal{D}$.

$$
\left| \sum_{t \in \mathbb{R}_+} f(t)(\mu(t) + \nu(t)) \right| \leq \left| \sum_{t \in \mathbb{R}_+} f(t)\mu(t) \right| + \left| \sum_{t \in \mathbb{R}_+} f(t)\nu(t) \right|
\leq \sup_{h \in \mathcal{D}} \left| \sum_{t \in \mathbb{R}_+} h(t)\mu(t) \right| + \sup_{h \in \mathcal{D}} \left| \sum_{t \in \mathbb{R}_+} h(t)\nu(t) \right|
= \|\mu\|_M + \|\nu\|_M.
$$

Once we have defined the space $M$ we shall need to calculate norms of some simple elements. We denote by $\delta_t$ a function in $M$ which is null in all points but $t$ and equals 1 in $t$, so $\delta_t(s) = 1_{t=s}$.

**Lemma 3.2.** Let $\alpha, \beta \geq 0$, $t, s \in \mathbb{R}_+$, $t \neq s$.

1. $\|\alpha\delta_t + \beta\delta_s\|_M = \alpha + \beta$

2. $\|\alpha\delta_t - \beta\delta_s\|_M = \begin{cases} 
|\alpha - \beta| + |t - s|(\alpha \wedge \beta) & |t - s| \leq 2 \\
\alpha + \beta & |t - s| > 2
\end{cases}$

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Proof. Part (1) is obvious by taking \( f = 1 \in \mathcal{D} \). Part (2) is more complicated. Set \( \epsilon = |t - s| \). If \( \epsilon > 2 \) then there exists a function \( f \in \mathcal{D} \) such that \( f(t) = 1 \) and \( f(s) = -1 \). This function reaches supremum in definition of the norm and yields \( \|\alpha \delta_t - \beta \delta_s\|_M = \alpha + \beta \).

Let \( \epsilon \leq 2 \). Assume that \( \alpha \geq \beta \). We can give an equivalent characterization of the norm \( \|\alpha \delta_t - \beta \delta_s\|_M \). We are only interested in possible values in \( s, t \) for functions in \( \mathcal{D} \). We introduce a set \( \mathcal{D}_t = \{(a, b) : |a| \leq 1, |b| \leq 1, |a - b| \leq \epsilon\} \).

Then \( \|\alpha \delta_t - \beta \delta_s\|_M = \sup_{(a, b) \in \mathcal{D}_t} |a \alpha - b \beta| \). The function \( b \mapsto a \alpha - b \beta \) is decreasing, so

\[
F(a_0) := \sup_{(a, b) \in \mathcal{D}_t} |a \alpha - b \beta| = \max \left\{ |a \alpha - [(a_0 - \epsilon) \vee -1] \beta|, |a_0 \alpha - [(a_0 + \epsilon) \wedge 1] \beta| \right\}.
\]

Hence \( \|\alpha \delta_t - \beta \delta_s\|_M = \sup_{a \in [-1, 1]} F(a) \). First consider a function \( a \mapsto a \alpha - [(a - \epsilon) \vee -1] \beta \). It is increasing since \( \alpha \geq \beta \). So the extremal points (we look at the absolute value of this function) are for \( |a| = 1 \).

\[
\begin{align*}
a &= 1 & \mapsto & \alpha - (1 - \epsilon) \beta = \alpha - \beta + \epsilon \beta, \\
a &= -1 & \mapsto & |\alpha + \beta| = \alpha - \beta.
\end{align*}
\]

In a similar way we work with a function \( a \mapsto a \alpha - [(a + \epsilon) \wedge 1] \beta \) which is also increasing. Thus

\[
\begin{align*}
a &= 1 & \mapsto & \alpha - \beta, \\
a &= -1 & \mapsto & |\alpha - (-1 + \epsilon) \beta| = \alpha - \beta + \epsilon \beta.
\end{align*}
\]

The conclusion is that \( \|\alpha \delta_t - \beta \delta_s\|_M = \alpha - \beta + \epsilon \beta \).

Having the above lemma we can see that \( M \) is not complete. Consider a sequence \( \mu_n = \sum_{i=1}^{n} \frac{1}{2} \delta_{2i} \). It is a Cauchy sequence: for \( n > m \) \( \|\mu_n - \mu_m\|_M = \sum_{i=m}^{n} \frac{1}{2} \). But it does not have a limit in \( M \). A good candidate for the limit would be \( \mu_\infty \), but it would need to have an infinite support.

**Definition 3.3.** Denote by \( (\mathcal{M}, \|\cdot\|_\mathcal{M}) \) the completion of \( M \) with the norm generated by \( \|\cdot\|_M \).

Now we shall concentrate on the space \( \mathcal{M}' \) of continuous functionals on \( \mathcal{M} \). We will see that there exists \( 1 - 1 \) correspondence between \( \mathcal{M}' \) and the set of all Lipschitz continuous bounded functions.

**Lemma 3.4.** Let \( \mu^* \in \mathcal{M}' \). A function \( f(t) = \langle \mu^*, \delta_t \rangle \) is bounded and Lipschitz continuous i.e. there exists a constant \( C \) such that \( |f(t)| \leq C \) and \( |f(t) - f(s)| \leq C |t - s| \). In particular we can take \( C = \|\mu^*\|_{\mathcal{M}'} \).

**Proof.** To prove boundedness we exploit continuity of \( \mu^* \). For \( t \in \mathbb{R} \),

\[
|f(t)| = |\langle \mu^*, \delta_t \rangle| \leq \|\mu^*\|_{\mathcal{M}'} \|\delta_t\|_M = \|\mu^*\|_{\mathcal{M}'}.
\]

Now fix \( t, s \in \mathbb{R}_+ \). In a similar way we obtain

\[
|f(t) - f(s)| = |\langle \mu^*, \delta_t - \delta_s \rangle| \leq \|\mu^*\|_{\mathcal{M}'} \|\delta_t - \delta_s\|_M.
\]
From lemma 3.2 we get that $\|\delta_t - \delta_s\|_M = |t - s| \wedge 2 \leq |t - s|$, which finishes the proof with $C = \|\mu^*\|_{M'}$.

**Lemma 3.5.** Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a bounded and Lipschitz continuous function i.e. $|f(t)| \leq C$ and $|f(t) - f(s)| \leq C|t - s|$ for some constant $C$. Then there exists exactly one continuous linear functional $\mu^*$ on $\mathcal{M}$ such that $\langle \mu^*, \delta_t \rangle = f(t)$. Moreover, $\|\mu^*\|_{M'} \leq C$.

**Proof.** We can restrict ourselves to the case $C = 1$. To see it consider a function $\tilde{f}(t) = \frac{f(t)}{C}$ that is bounded by 1 and Lipschitz continuous with parameter 1. Let $\tilde{\mu}^*$ be a linear functional linked with $\tilde{f}(t)$. Then $\mu^* = C\tilde{\mu}^*$ is a linear functional such that $\langle \mu^*, \delta_t \rangle = f(t)$ and $\|\mu^*\|_{M'} \leq C$. From uniqueness of $\mu^*$ we get uniqueness of $\mu^*$.

Now we have $C = 1$. We define $\mu^*$ on a space spanned by $(\delta_t)_{t \in \mathbb{R}_+}$ (the space $\mathcal{M}$) as $\langle \mu^*, \delta_t \rangle = f(t)$. We shall prove that $\mu^*$ is indeed a continuous linear functional on $\mathcal{M}$. Linearity is obvious. Continuity requires more consideration. A function $f$ is an element of the set $\mathcal{D}$. Thus for any $\mu \in \mathcal{M}$

$$|\langle \mu^*, \mu \rangle| = |\sum_{t \in \mathbb{R}_+} \mu(t)f(t)| \leq \sup_{h \in \mathcal{D}} |\sum_{t \in \mathbb{R}_+} \mu(t)f(t)| = \|\mu\|_{\mathcal{M}}.$$ 

Hence $\|\mu^*\|_{M'} \leq 1$. We can extend $\mu^*$ to the whole of $\mathcal{M}$ as a continuous linear functional with the norm 1. This extension is unique since $\mathcal{M}$ is a completion of $\mathcal{M}$.

**4 Random variables on $\mathcal{M}$**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X$ a mapping defined on $\Omega$ with values in the Banach space $\mathcal{M}$. $X$ is a simple random variable if it is measurable and admits a finite number of values, i.e. there exists $N \in \mathbb{N}$, sequence $(\mu_n)_{n=1,\ldots,N} \subset \mathcal{M}$ and $N$ disjoint measurable sets $(A_n)_{n=1,\ldots,N} \subset \Omega$ such that $\bigcup_{n=1}^N A_n = \Omega$ and $X = \sum_{n=1}^N 1_{A_n} \cdot \mu_n$. A mapping $X$ is strongly measurable if there exists a sequence of simple random variables converging to $X$ a.s. in the norm of $\mathcal{M}$.

**Definition 4.1.** A space $L^1_b(\Omega, \mathcal{M})$ consists of all strongly measurable random variables $X$ for which the functional

$$\|X\|_{L^1_b(\Omega, \mathcal{M})} = \mathbb{E}\|X\|_{\mathcal{M}}$$

is finite. The space $L^1_b(\Omega, \mathcal{M})$ with the norm $\|\cdot\|_{L^1_b(\Omega, \mathcal{M})}$ forms a Banach space.

Following Schwartz ([7]) we construct a dual space to $L^1_b(\Omega, \mathcal{M})$. A mapping $\Phi : \Omega \to \mathcal{M}'$, where $\mathcal{M}'$ is dual to $\mathcal{M}$, is called *-weakly measurable if for any $x \in \mathcal{M}$ a function $\omega \mapsto \langle \Phi(\omega), x \rangle$ is measurable as a function from $\Omega$ to $\mathbb{R}$. Let $\mathcal{L}^\infty(\Omega, \mathcal{M}')$ be a set of all *-weakly measurable mappings for which the function $\Phi \mapsto \inf\{K \geq 0 : \|\Phi\|_{\mathcal{M}'} \leq K \text{ a.s.}\}$ is finite. Define an equivalence relation in the set $\mathcal{L}^\infty(\Omega, \mathcal{M}')$: $\Phi \sim \Psi$, if $\forall x \in \mathcal{M}$ $\langle \Phi, x \rangle = \langle \Psi, x \rangle$ a.s.
**DEFINITION 4.2.** Define a linear space

\[ L^\infty_s(\Omega, \mathcal{M}') = L^\infty(\Omega, \mathcal{M}') / \sim \]

with the functional

\[ \| \Phi \|_{L^\infty(\Omega, \mathcal{M}')} = \inf_{\phi \sim \Phi} \| \phi \|_{L^\infty(\Omega, \mathcal{M}')} . \]

Note, that

\[ \| \Phi \|_{L^\infty(\Omega, \mathcal{M}')} = \inf \{ K \geq 0 : \| \Phi \|_{\mathcal{M}'} \leq K \text{ a.s.} \} . \]

**THEOREM 4.3.** ([7]) The space \( L^\infty_s(\Omega, \mathcal{M}') \) with the functional \( \| \cdot \|_{L^\infty(\Omega, \mathcal{M}')} \) forms a Banach space. Moreover, it is dual to \( L^1_s(\Omega, \mathcal{M}) \). Every \( \Psi \in L^\infty(\Omega, \mathcal{M}') \) defines a linear functional on \( L^1_s(\Omega, \mathcal{M}) \) as

\[ L^1_s(\Omega, \mathcal{M}) \ni X \mapsto \langle \Psi, X \rangle_{L^\infty(\Omega, \mathcal{M}'), L^1(\Omega, \mathcal{M})} = \mathbb{E} \langle \Psi, X \rangle_{\mathcal{M}', \mathcal{M}} . \]

As a final result we will prove a simple technical lemma.

**LEMMA 4.4.** Let \( \tau \) be a non-negative bounded random variable \((\tau < K \text{ a.s. for some } K \in \mathbb{R})\), \( Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \). Then \( Y \delta_\tau \in L^1_s(\Omega, \mathcal{M}) \).

**Proof.** We will construct a sequence of simple random variables \( X_n \in L^1_s(\Omega, \mathcal{M}) \) with the limit equal to \( Y \delta_\tau \). Let \( Y_n \) be a sequence of simple random variables converging to \( Y \) a.s. Fix \( n \in \mathbb{N} \). Set \( A_k = \{ \tau \in [\frac{k-1}{n}, \frac{k}{n}) \} \) for \( k = 0, \ldots , n-1 \). Put

\[ X_n = \sum_{k=0}^{n-1} Y_n 1_{A_k} \delta_{\frac{k}{n}} . \]

Then we have a pointwise convergence of \( X_n \) to \( Y \delta_\tau \) from lemma 3.2.

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**5 Absence of free lunch and its equivalent characterization**

In this section we will establish a definition of no free lunch for the model presented in section 2. Moreover, we will derive an analog of Fundamental Theorem of Asset Pricing – an equivalent characterization of the absence of free lunch. It will be done in the spirit of Jouini’s and Napp’s paper [3].

First, we will find a representation of any investment as an element of the space \( L^1_s(\Omega, \mathcal{M}) \). Denote by \( \Lambda \) the subset of \( L^1_s(\Omega, \mathcal{M}) \) of all random variables that can be written in the form

\[ \gamma_1 \delta_{\tau_1} + \cdots + \gamma_m \delta_{\tau_m} \]

for some \( m \in \mathbb{N} \), a sequence \( \tau_1, \ldots , \tau_m \) of bounded stopping times and real valued random variables \( \gamma_1, \ldots , \gamma_m \) such that \( \gamma_i \in L^1(\Omega, \mathcal{F}_{\tau_i}, \mathbb{R}) \), \( i = 1, \ldots , m \). The subset of all non-negative elements in \( \Lambda \) will be denoted by \( \Lambda_+ \)

\[ \Lambda_+ = \{ \gamma \in \Lambda : \gamma = \sum_{i=1}^m \gamma_i \delta_{\tau_i}, \gamma_i \geq 0 \text{ a.s.} \} . \]
LEMMA 5.1. For any $\gamma \in \Lambda$ with representation $\gamma = \sum_{i=1}^{m} \gamma_i \delta_{\tau_i}$

$$\|\gamma\|_{L^{1}(\Omega, \mathcal{M})} \leq \sum_{i=1}^{m} \|\gamma_i\|_{L^{1}(\Omega, \mathbb{R})}.$$ 

Proof. It suffices to prove the lemma for $\gamma$ of the form $\gamma = \gamma_1 \delta_{\tau_1} + \gamma_2 \delta_{\tau_2}$.

$$\|\gamma\|_{L^{1}(\Omega, \mathcal{M})} = \mathbb{E}\|\gamma\|_{\mathcal{M}} \leq \mathbb{E} (|\gamma_1|\|\delta_{\tau_1}\|_{\mathcal{M}} + |\gamma_2|\|\delta_{\tau_2}\|_{\mathcal{M}})$$

$$= \mathbb{E} |\gamma_1|\|\delta_{\tau_1}\|_{\mathcal{M}} + \mathbb{E} |\gamma_2|\|\delta_{\tau_2}\|_{\mathcal{M}}$$

$$= \mathbb{E} |\gamma_1| + \mathbb{E} |\gamma_2|.$$

Having all necessary tools we will establish an isomorphism between $\Lambda$ and the set of all investments $\Delta$

$$\Delta \ni \sum_{i=1}^{m} \phi_{\tau_i} 1_{\tau_i = t} \iff \sum_{i=1}^{m} \phi_{\tau_i} \delta_{\tau_i} \in \Lambda.$$ 

Note that both representations, in $\Lambda$ and $\Delta$, are not unique. From now on we will treat both representations equivalently; an investment is a stochastic process or an element of $L^{1}(\Omega, \mathcal{M})$ depending on the context.

In the future considerations we will also need the following notation. Let $(h_t)_{t \in \mathbb{R}_+}$ be a stochastic process and $\Phi = \sum_{i=1}^{m} \phi_{\tau_i} 1_{\tau_i = t}$ - an element of $\Delta$. Then by $\mathbb{E} \left[ \sum_t \Phi_t h_t \right]$ we mean the number $\sum_{i=1}^{m} \mathbb{E} [\phi_{\tau_i} h_{\tau_i}]$.

We now come to the notion of absence of arbitrage and free lunch. The first one is proper for simplest models. In the more complicated framework one has to introduce stronger condition which is usually called "no free lunch”. We will need it to obtain a reasonable characterization of markets without possibility of getting free lunch.

DEFINITION 5.2. There is no arbitrage on the market $J$ iff $J \cap \Lambda_+ = \{0\}$.

DEFINITION 5.3. There is no free lunch on the market $J$ iff $(J - \Lambda_+) \cap \Lambda_+ = \{0\}$, where the closure is in the norm of $L^{1}(\Omega, \mathcal{M})$.

THEOREM 5.4. Let $J \subset \Lambda$ be a positive convex cone and $\mu$ be a non-null element of $\Lambda$. If $\mu \notin (J - \Lambda_+)$ (the closure is in the norm of $L^{1}(\Omega, \mathcal{M})$), then there exists a (not necessarily adapted) measurable process $(h_t)_{t \in \mathbb{R}_+}$ such that

(i) $\mathbb{P}(0 \leq h_t \leq M^h \forall t \in \mathbb{R}_+) = 1$ (boundedness),

(ii) $\mathbb{P}(|h_t - h_s| \leq M^h |t - s| \forall t, s \in \mathbb{R}_+) = 1$ (Lipschitz continuity),

(iii) $\mathbb{E} \left[ \sum_t \mu_t h_t \right] > 1$,

(iv) $\mathbb{E} \left[ \sum_t \Phi_t h_t \right] \leq 0$ for every $\Phi \in J$.

The constant $M^h$ depends on $\mu$ and $J$. 

Proof. Let $C = J - \Lambda_+$. We apply Hahn-Banach separation theorem (see theorem IV.6.3’ in Yosida [8]) in the space $L^1_p(\Omega, \mathcal{M})$ to the closed convex set $C$ ($0 \in C$) and the element $\mu$. We find a linear functional $f \in L^\infty_s(\Omega, \mathcal{M}^t)$ such that $f(\mu) > 1$ and $f \leq 1$ on $C$. We recall the properties of $f$ (see section 4)

\[
\begin{align*}
\text{a) } & \langle f, \nu \rangle_{(\mathcal{M}', \mathcal{M})} \text{ is measurable for any element } \nu \in \mathcal{M} \\
\text{b) } & \|f\|_{L^p(\Omega, \mathcal{M}')} = \inf \{ K \geq 0 : \|f\|_{\mathcal{M}'} \leq K \text{ a.s.} \}
\end{align*}
\]

We claim that $f \leq 0$ on $C$ since $C$ is a positive cone.

Let $h_t = \langle f, \delta_t \rangle_{(\mathcal{M}', \mathcal{M})}$ for $t \in \mathbb{R}_+$. Then by a) $h_t$ is a random variable for all $t \in \mathbb{R}_+$. From lemma 3.4 and b) we deduce that all trajectories of $(h_t)_{t \in \mathbb{R}_+}$ are Lipschitz continuous and bounded with the constant independent of $\omega \in \Omega$, i.e. $\forall t, s \in \mathbb{R}_+, |h_t(\omega) - h_s(\omega)| \leq M^h|t - s|$ and $|h_t(\omega)| \leq M^h$ for almost all $\omega \in \Omega$. Then a simple argument shows that $(h_t)_{t \in \mathbb{R}_+}$ is a measurable process (see for example remark 1.14 in Karatzas, Shreve [4]).

We claim that for $t \in \mathbb{R}_+$, $h_t \geq 0$ a.s. Assume the contrary. Thus there exists $s \in \mathbb{R}_+$ such that $\mathbb{P}(h_s < 0) > 0$. Let $V = \{h_s < 0\} = \{\langle f, \delta_s \rangle_{(\mathcal{M}', \mathcal{M})} < 0\}$ and $L^1_p(\Omega, \mathcal{M}) \ni \Psi : \omega \mapsto -1_V(\omega)\delta_s$. Obviously $\Psi \in -\Lambda_+$, so $\Psi \in C$. Hence

\[
\langle f, \Psi \rangle_{(L^\infty_p(\Omega, \mathcal{M}'), L^1_p(\Omega, \mathcal{M}))} = \mathbb{E} \{\langle f, \Psi \rangle_{(\mathcal{M}', \mathcal{M})}\} \leq 0.
\]

But

\[
\mathbb{E} \{\langle f, \Psi \rangle_{(\mathcal{M}', \mathcal{M})}\} = \mathbb{E} \{-1_V \langle f, \delta_s \rangle_{(\mathcal{M}', \mathcal{M})}\} > 0,
\]

since $\langle f, \delta_s \rangle_{(\mathcal{M}', \mathcal{M})} < 0$ on $V$. We have obtained a contradiction. We will exploit continuity of trajectories of $h_t$ to show that $\mathbb{P}(h_t \geq 0 \ \forall \ t \in \mathbb{R}_+) = 1$. From above reasoning we obtain that $h_t$ truncated to a countable dense set $D \subset \mathbb{R}_+$ has almost surely non-negative trajectories. Then by continuity we extend this property to $h_t$ on $\mathbb{R}_+$.

For the proof of (iv) we recall a definition

\[
\mathbb{E} \left[ \sum_t \Phi_t h_t \right] = \mathbb{E} \langle f, \Phi \rangle_{(\mathcal{M}', \mathcal{M})}.
\]

Now we come to the main part of this section - an analog of the Fundamental Theorem of Asset Pricing. To establish it we will need a technical assumption which is satisfied in most of the models. It assures that one can always find an investment that enables him to transfer some money from any moment to one of the specified moments in the future with positive probability. It is a kind of roulette or lottery condition.

**Definition 5.5.** A market $J$ satisfies a roulette condition if there exists a sequence of bounded stopping times $(\sigma_n)_{n \in \mathbb{N}}$ such that for each bounded stopping time $\tau$ for each subset $A \in \mathcal{F}_\tau$ with positive probability we can find such an investment $\Phi$ in $J$ that

\[
\begin{align*}
&\bullet \ \mathbb{P}(\Phi_s = 0 \ \forall \ s < \tau) = 1, \\
&\bullet \ \Phi_\tau = 0 \text{ a.s. on } A^c,
\end{align*}
\]
• \( \Phi_\tau \) is arbitrary on \( \Lambda \),
• \( \mathbb{P}(\Phi_s \geq 0 \quad \forall \ s > \tau) = 1 \),
• there exists \( n \in \mathbb{N} \) with \( \mathbb{P}(\Phi_{\sigma_n} > 0) > 0 \).

**THEOREM 5.6.** Assume that the market \( J \) satisfies a roulette condition. There is no free lunch in \( J \) if there exists a measurable process \( (g_t)_{t \in \mathbb{R}_+} \) such that

(i) \( \mathbb{P}(0 \leq g_t \leq M^g \forall t \in \mathbb{R}_+) = 1 \) (boundedness),
(ii) \( \mathbb{P}(|g_t - g_s| \leq M^g|t - s| \forall t, s \in \mathbb{R}_+) = 1 \) (Lipschitz continuity),
(iii) \( \mathbb{E}[g_\tau | \mathcal{F}_\tau] > 0 \) a.s. for any bounded stopping time \( \tau \),
(iv) \( \mathbb{E}[\sum_t \Phi_t g_t] \leq 0 \) for \( \Phi \in J \).

Every process \( g_t \) satisfying above conditions is called a discount process for the market \( J \).

**Proof.** Assume that there exists a discount process \( g_t \) for the market \( J \). We have to show that \( J - \Lambda_+ \cap \Lambda_+ = \{0\} \). We will prove it by contradiction. Let \( X^n \in (J - \Lambda_+ \cap \Lambda_+) \) be a sequence with the limit \( X \in \Lambda_+ \setminus \{0\} \) in the norm of \( L_p^1(\Omega, \mathcal{M}) \). We will construct a continuous linear functional \( \Psi \) on \( L_p^1(\Omega, \mathcal{M}) \) such that \( \Psi(1_A \delta_t) = \mathbb{E} 1_A g_t \) for \( A \in \mathcal{F}_t \). Let \( H \) be a linear subspace of \( L_p^1(\Omega, \mathcal{M}) \) spanned by random variables \( 1_A \delta_t \) for \( A \in \mathcal{F}_t, t \in \mathbb{R}_+ \). On \( H \) we set \( \Psi(1_A \delta_t) = \mathbb{E} 1_A g_t \). Linearity of this function is clear. We have only to show that \( \Psi \) is continuous. Let \( Y \in H \). We can write \( Y = \sum_{k=1}^K \alpha_k 1_A \delta_{t_k} \) for some \( K, A_k \in \mathcal{F}_{t_k}, t_k \in \mathbb{R}_+, \alpha_k \in \mathbb{R}, k = 1, \ldots, K \). Hence

\[
\Psi(Y) = \sum_{k=1}^K \alpha_k \Psi(1_A \delta_{t_k}) = \sum_{k=1}^K \alpha_k \mathbb{E}[1_A \delta_{t_k}] = \mathbb{E} \sum_{k=1}^K \alpha_k 1_A \delta_{t_k} = \int_{\Omega} \sum_{k=1}^K \alpha_k 1_A(\omega) g_{t_k}(\omega) d\mathbb{P}(\omega).
\]

For almost all \( \omega \in \Omega, g_t(\omega) \) as a function of \( t \in \mathbb{R}_+ \) is Lipschitz continuous with constant \( M^g \) and is bounded by \( M^g \). Fix \( \omega \in \Omega \). Then by lemma 3.5 \( g_t(\omega) \) defines a continuous linear functional on \( \mathcal{M} \) with norm bounded by \( M^g \). Since \( \sum_{k=1}^K \alpha_k 1_A(\omega) \delta_{t_k} \in \mathcal{M} \) we obtain

\[
\left|\sum_{k=1}^K \alpha_k 1_A(\omega) g_{t_k}(\omega)\right| \leq M^g \left\| \sum_{k=1}^K \alpha_k 1_A(\omega) \delta_{t_k} \right\|_{\mathcal{M}} = M^g \| Y(\omega) \|_{\mathcal{M}}.
\]

Thus

\[
\left|\Psi(Y)\right| = \left| \int_{\Omega} \left| \sum_{k=1}^K \alpha_k 1_A(\omega) g_{t_k}(\omega) \right| d\mathbb{P}(\omega) \right| \leq \int_{\Omega} \left| \sum_{k=1}^K \alpha_k 1_A(\omega) g_{t_k}(\omega) \right| d\mathbb{P}(\omega) \leq \int_{\Omega} M^g \| Y(\omega) \|_{\mathcal{M}} d\mathbb{P}(\omega) = M^g \| Y \|_{L_p^1(\Omega, \mathcal{M})}.
\]
We extend $\Psi$ to the whole of $L^1_+(\Omega,\mathcal{M})$ as a continuous linear functional (see Yosida [8] IV.5.1). Observe that $\Psi(1_A\delta_\tau) = \mathbb{E}1_Ag_\tau$ for any bounded stopping time $\tau$ and $A \in \mathcal{F}_\tau$. Let $r_n$ be a sequence of stopping times admitting finite number of values and converging to $\tau$ almost surely. By lemma 3.2 $1_A\delta_{r_n} \rightarrow 1_A\delta_\tau$. Thus $\Psi(1_A\delta_{r_n}) \rightarrow \Psi(1_A\delta_\tau)$. On the other hand $\mathbb{E}1_Ag_{r_n} \rightarrow \mathbb{E}1_Ag_\tau$ by the dominated convergence theorem $(g_t)$ is bounded by $M^g$. Then $\Psi(1_A\delta_\tau) = \mathbb{E}1_Ag_\tau$. Similar argument shows that for any $\Theta \in L^1(\Omega,\mathcal{F}_\tau,\mathbb{P})$ we have $\Psi(\Theta\delta_\tau) = \mathbb{E}\Theta g_\tau$.

Hence it is clear that $\Psi(X^n) = \mathbb{E}\left[\sum_t X^n_tg_t\right]$ and $\Psi(X) = \mathbb{E}\left[\sum_t X_tg_t\right]$. By continuity of $\Psi$ we obtain $\Psi(X^n) \rightarrow \Psi(X)$. Therefore $\Psi(X) \leq 0$ since $\Psi(X^n) \leq 0$. However, $X \in \Lambda_+ \setminus \{\}$, so $\Psi(X) > 0$ (from condition $\mathbb{E}[g_\tau | \mathcal{F}_\tau] > 0$ a.s. for any bounded stopping time $\tau$). This yields a contradiction.

Suppose now that there is no free lunch in $J$. Let $\mathcal{G}$ be a set of equivalence classes of measurable processes $g_t$ with $\mathbb{P}(|g_t - g_s| \leq M^g|t-s|, 0 \leq g_t \leq M^g \forall t, s \in \mathbb{R}_+, 1 \leq |g_t| \leq 0$ a.s. Let $\mathcal{G}$ be the family of equivalence classes of subsets of $\Omega$.

We will show that for any bounded stopping time $\tau$ there exists a process $g^\tau \in \mathcal{G}$ such that $g^\tau_T > 0$ a.s. Let $\mathcal{H}$ be a family of equivalence classes of subsets of $\Omega$.

\[ \mathcal{H} = \{A \in \mathcal{F} : \exists h \in \mathcal{G} \{h_\tau \neq 0\} = A \text{ a.s.}\}. \]

We claim that there exists a set of positive measure in $\mathcal{H}$. Fix $\mu = \delta_\tau \in \Lambda_+ \setminus \{\}$. We apply theorem 5.4 with $\mu = \delta_\tau$ to obtain a process $h \in \mathcal{G}$ such that $\mathbb{E}h_\tau > 1$. Hence the set $\{h_\tau \neq 0\}$ must have a positive measure. A simple argument shows that the family $\mathcal{H}$ is closed under countable sums. Let $A_n \in \mathcal{H}$ be a sequence of sets. Denote by $g^n$ processes for which $A_n = \{g^n_\tau \neq 0\}$. Put $h = \sum_{n=1}^{\infty}(2^nM^g)^{-1}g^n$. Then $h \in \mathcal{G}$ and $\{h_\tau \neq 0\} = \bigcup A_n$ a.s. Hence there exists $S^* \in \mathcal{H}$ such that $\mathbb{P}(S^*) = \sup\{\mathbb{P}(S) : S \in \mathcal{H}\}$. If $\mathbb{P}(S^*) < 1$ then by theorem 5.4 applied to $\mu = 1_{\Omega \setminus S^*}\delta_\tau$ we obtain a process $h' \in \mathcal{G}$ with $\mathbb{E}1_{\Omega \setminus S^*}h'_\tau > 1$. Considering a process $h + h' \in \mathcal{G}$ we have a contradiction.

Using above results we can find a process $g \in \mathcal{G}$ such that for every $n \in \mathbb{N}$ $g_{\sigma_n} > 0$ a.s.: 

\[ g = \sum_{i=1}^{\infty} \frac{1}{2^iM^g\sigma_n}g_{\sigma_n}. \]

We shall prove now that $\mathbb{E}[g_\tau | \mathcal{F}_\tau] > 0$ a.s. for any bounded stopping time $\tau$. Assume that it is not true, so there exists $\tau$ such that $\mathbb{P}(\mathbb{E}[g_\tau | \mathcal{F}_\tau] = 0) > 0$. Let $B = \{\mathbb{E}[g_\tau | \mathcal{F}_\tau] = 0\}$. By the roulette condition we can find an investment $\Phi \in J$ null before $\tau$, non-negative after $\tau$, null on $B^c$ and such that $\mathbb{P}(\Phi_{\sigma_n} > 0) > 0$ for some $n$. $\Phi$ can be written as $\sum_{i=1}^{\infty} \Phi_{\tau_i}1_{\tau_i} + \Phi_{\tau_1}1_{\tau_1} + \Phi_{\sigma_n}1_{\sigma_n}$ for some $m$. Then

\[ \mathbb{E}\left[\sum_t \Phi_t g_t\right] = \mathbb{E}\left[\sum_{i=1}^{m} \Phi_{\tau_i} g_{\tau_i} + \Phi_{\tau} g_{\tau} + \Phi_{\sigma_n} g_{\sigma_n}\right] \]

\[ = \mathbb{E}\left[\sum_{i=1}^{m} \Phi_{\tau_i} g_{\tau_i} + \Phi_{\tau} \mathbb{E}[g_\tau | \mathcal{F}_\tau] + \Phi_{\sigma_n} g_{\sigma_n}\right] \geq \mathbb{E}g_{\sigma_n} \Phi_{\sigma_n} > 0, \]

since $\Phi_{\tau}$ is $\mathcal{F}_\tau$ measurable by definition. This yields a contradiction. \hfill \blacksquare
Remark. Notice that the roulette condition was required only to obtain right implication, i.e. to construct a discount process for the market $J$.

Discount processes do not have to be adapted. A mild discount process (Lipschitz continuous) must incorporate knowledge of the future (one can construct a simple example). However, from economic point of view, discount factors should be known at each moment, thus they should be adapted. If we ease our requirements for mildness of the discount process we can construct a continuous and adapted discount process.

**COROLLARY 5.7.** Assume that the market $J$ satisfies a roulette condition. If there is no free lunch in $O$ then there exists an RCLL adapted process $g_t$ such that $\mathbb{E}[\sum_t \Phi_t g_t] \leq 0$ for $\Phi \in J$ and $\mathbb{P}(0 < g_t \leq M^g \forall t \in \mathbb{R}_+) = 1$. If, in addition, the filtration is quasi-left continuous ($\mathcal{F}_{\tau-} = \mathcal{F}_\tau$ for any previsible stopping time $\tau$), there exists a continuous process $g_t$ with above properties.

**Proof.** Let $f_t$ be a discount process for the market $J$ obtained in theorem 5.6. We stress that in this proof we use the assumption that the filtration satisfies usual conditions and the fact that the process $f_t$ is measurable.

We define $g_t$ as an optional projection of $f_t$. Notice that $g_t$ satisfies condition (iv) of theorem 5.6. By theorem VI.7.10 in Rogers, Williams [6] $g_t$ has RCLL trajectories. Since $g_t = \mathbb{E}[f_t | \mathcal{F}_t]$ we obtain $0 < g_t \leq M^g$ a.s. for $t \in \mathbb{R}_+$. By the right continuity of trajectories of $g_t$ we have $\mathbb{P}(0 \leq g_t \leq M^g \forall t \in \mathbb{R}_+) = 1$.

We only have to prove that $\mathbb{P}(0 < g_t \forall t \in \mathbb{R}_+) = 1$. Let $\gamma_n = \inf \{t \in [0,n] : g_t = 0\} \land n$ (with convention $\inf \emptyset = \infty$). Random variables $\gamma_n$ are stopping times. Then $g_{\gamma_n} = \mathbb{E}[f_{\gamma_n} | \mathcal{F}_{\gamma_n}] > 0$ a.s. from the fact that $g_t$ is an optional projection and $f_t$ is a discount process. Thus $\mathbb{P}(\exists t \in [0,n] : g_t = 0) = 0$. Consequently $\mathbb{P}(\exists t \in \mathbb{R}_+ : g_t = 0) = 0$, which completes the proof.

Assume now that the filtration is quasi-left continuous. Let $z_t$ be a previsible projection of $f_t$. Then $z_t$ is LCR (left continuous with right limits). Observe that for any previsible stopping time $\tau$

$$g_{\tau} 1_{\tau<\infty} = \mathbb{E}[f_{\tau} 1_{\tau<\infty} | \mathcal{F}_\tau] = \mathbb{E}[f_{\tau} 1_{\tau<\infty} | \mathcal{F}_{\tau-}]$$

for filtration $\mathcal{F}_t$ is quasi-left continuous. Then $g_t$ is also a previsible projection. So it is indistinguishable from $z_t$. Thus $g_t$ is continuous (it is indistinguishable from a continuous process).

6 Pricing and hedging

In this section we take up a problem of option pricing and hedging. We shall place ourselves in the context of the model of section 2. The market is described by a positive convex cone $J$ of investment opportunities. Each investment is an adapted process. We assume that the market satisfies roulette condition and there is no free lunch on it. Thus we are given a set $\mathcal{G}_J$ of discount processes for $J$, i.e. for any $g \in \mathcal{G}_J$

- $\mathbb{P}(0 \leq g_t \leq M^g \forall t \in \mathbb{R}_+) = 1$, 


Throughout this section we will use the following definition of contingent claim

**DEFINITION 6.1.** A contingent claim is a pair \((\tau, Y)\), where \(\tau\) is a bounded stopping time and \(Y\) is a non-negative random variable from \(L^1(\Omega, \mathcal{F}_\tau, \mathbb{R}_+)\).

It is a slight generalization of a widely used notion. We allow a claim to be executed at a random moment. From further results it will be clear that we can also consider contingent claims of the form of investments with non-negative cash flows i.e. finite sums of contingent claims from definition 6.1. It enables us to model, apart from classical European options, corporate bonds and more complicated contracts. However, we cannot deal with American options.

There are plenty of methods for determining a price of the contingent claim. We will concentrate on two of them, as old as finance. We will answer the following questions

- which prices of the option do not lead to arbitrage?
- what is the minimal price that enables hedging of the option?

### Arbitrage pricing

A very natural approach to option pricing is based on the notion of no free lunch. A fair price is a price that does not generate free lunch. No-one, neither buyer nor seller, gets profit without a risk. As we will see later, the set of such prices is an interval, a common result in financial mathematics.

To consider options we have to define an investment representing buying and selling of the option. Let

\[
\Psi_i^{(\xi,C,\tau,Y)} = \xi(C_{1_{t=0}} - Y_{1_{t=\tau}}),
\]

where \(\xi, C \in \mathbb{R}\). Then \(\Psi^{(-1,C,\tau,Y)}\) denotes the investment consisting of buying of the option \((\tau, Y)\) at moment 0 for the price \(C\). A claim \(Y\) is paid back at the random moment \(\tau\). In a natural way, selling is opposite to buying, thus it is represented by \(\Psi^{(1,C,\tau,Y)}\).

**DEFINITION 6.2.** An option price \(C\) is called fair if the market \(J^{opt}\) generated by \(J\) and investments \(\Psi^{(\pm 1,C,\tau,Y)}\) does not admit free lunch.

Hence, the set of discount processes \(G^{opt}\) is not empty. It is clear that \(G^{opt} \subset G_J\). Let \(g \in G^{opt}\). The inequality

\[
\mathbb{E} \sum_{t \in \mathbb{R}_+} \Psi_i^{(\pm 1,C,\tau,Y)} = \pm (C \mathbb{E} g_0 - \mathbb{E} [g_\tau Y]) \leq 0
\]

results in \(C \mathbb{E} g_0 - \mathbb{E} [g_\tau Y] = 0\). Thus \(C\) is a fair price iff there exists \(g \in G_J\) such that

\[
C = \frac{\mathbb{E} [g_\tau Y]}{\mathbb{E} g_0}.
\]
LEMMA 6.3. Let \( C_1 \leq C_2 \) be fair prices of the contingent claim \((\tau, Y)\). Then any price from interval \([C_1, C_2]\) is fair.

Proof. Let \( g_1, g_2 \) be discount processes for prices \( C_1, C_2 \). Take any number \( C \in [C_1, C_2] \). We can find a real \( a \in [0, 1] \) such that \( C = aC_1 + (1 - a)C_2 \). Define \( g_t = a \frac{g_1}{g_0} + (1 - a) \frac{g_2}{g_0} \). Then \( C = \frac{\sum g_t Y}{\sum g_t} \). It is clear that \( g \in \mathcal{G}_J \).

A remarkable result is a simple consequence of the above lemma. It is a characterization of the interval of fair prices. However, it is unspecific about the ends of the interval. We do not know if these prices lead to arbitrage or not.

COROLLARY 6.4. Let \( C^h = \sup_{g \in \mathcal{G}_J} \frac{\sum g_t Y}{\sum g_t} \) and \( C^l = \inf_{g \in \mathcal{G}_J} \frac{\sum g_t Y}{\sum g_t} \). Any price from the open interval \((C^l, C^h)\) is fair. Moreover, any price from outside of the closed interval \([C^l, C^h]\) leads to free lunch.

Proof. For any \( C \in (C^l, C^h) \) we can find fair prices \( C_1 \leq C_2 \) such that \( C \in [C_1, C_2] \). By lemma 6.3 \( C \) is a fair price. The second part of the lemma is obvious.

Hedging

An agent selling an option wants to hedge it. He wants to know the minimal amount of money that would enable him to hedge a contingent claim. Does it have anything to do with fair prices? To address any of these questions we have to specify what it means to hedge a contingent claim. At the moment zero we obtain a certain amount of money. Then we invest the money (we subscribe to investment opportunities available on the market) to get a cash flow in the moment \( \tau \) that covers our obligation \( Y \). This intuitive notion is specified in the following definition. We denote by \( \Psi^C \) the investment \( \Psi(-1, C, \tau, Y) \) - buying of the option.

DEFINITION 6.5. A price \( C \) is a hedging price for \((\tau, Y)\) if \( \Psi^C \in \mathcal{J} - \Lambda_+ \). A minimal hedging price is denoted by \( C^s \) and called a seller price.

If \( C \) is a hedging price, then there exists a sequence of investments that in the limit gives \( \Psi^C \) and some positive cash flows, an available consumption. It is still an open question if a seller price exists. Let \( h = \inf \{ b \geq 0 : b \text{ is a hedging price} \} \).

We will show that \( h \) is in fact a minimal hedging price. Obviously if \( C \) is a hedging price then \( C \geq h \). We shall prove that \( h \) is a hedging price. Let \( b_n \) be a sequence of hedging prices converging to \( h \). Then \( \Psi^{b_n} \) is a Cauchy sequence in \( L^2(\Omega, M) \) with limit \( \Psi^h \). From \( \Psi^{b_n} \in \mathcal{J} - \Lambda_+ \) we obtain that \( \Psi^h \in \mathcal{J} - \Lambda_+ \).

THEOREM 6.6. Let \( C^s \) be a minimal hedging price and \( C^h \) the upper limit of the fair price interval from lemma 6.4. Then \( C^s = C^h \).
Proof. First we will prove the inequality $C^s \geq C^h$. We have shown that $\Psi^{C^s} \in \mathcal{J} - \Lambda_+$. Thus for all discount processes $g \in \mathcal{G}$ we have $\mathbb{E} [Y g_T] - C^s \mathbb{E} g_0 \leq 0$. Hence

$$C^s \geq \frac{\mathbb{E} [Y g_T]}{\mathbb{E} g_0} \quad \text{for every } g \in \mathcal{G}$$

and finally $C^s \geq \sup_{g \in \mathcal{G}} \frac{\mathbb{E} [Y g_T]}{\mathbb{E} g_0} = C^h$.

If $C^h = \infty$ then $C^s = \infty$. We will show that $C^h$ is a hedging price. Assume, by contradiction, that $C^h$ is not a hedging price. Thus $\Psi^{C^h} \notin \mathcal{J} - \Lambda_+$. By theorem 5.4 there exists a process $h$ satisfying conditions (i), (ii) and (iii) for discount process (see theorem 5.6) and such that

$$\mathbb{E} [Y h_T] - C^h \mathbb{E} h_0 > 1.$$  \hfill (2)

Definition of $C^h$ grants existence of a sequence of discount processes $g^n \in \mathcal{G}$ such that $C^h - \epsilon_n = \frac{\mathbb{E} [Y g^n_T]}{\mathbb{E} g^n_0}$ for a sequence of positive numbers $\epsilon_n$ converging to zero. Let $\bar{g}^n = \frac{g^n}{\mathbb{E} g^n_0} + h$. We can easily check that $\bar{g}^n$ satisfies conditions (i), (ii) and (iii) for discount process since both $h$ and $g^n$ do. Condition (iv) results from the fact that $h_T \geq 0$ a.s. and $g^n_T > 0$ a.s. for any bounded stopping time $\tau$. Thus $\bar{g}^n \in \mathcal{G}$ and

$$C^h \geq \frac{\mathbb{E} [Y \bar{g}^n_T]}{\mathbb{E} \bar{g}^n_0}.$$  

On the other hand, from (2) we obtain

$$C^h \mathbb{E} \bar{g}^n_0 = C^h + C^h \mathbb{E} h_0$$

$$= (C^h - \epsilon_n) + \epsilon_n + C^h \mathbb{E} h_0$$

$$= \frac{\mathbb{E} [Y g^n_T]}{\mathbb{E} g^n_0} + \epsilon_n + C^h \mathbb{E} h_0$$

$$< \frac{\mathbb{E} [Y g^n_T]}{\mathbb{E} g^n_0} + \epsilon_n + \mathbb{E} [Y h_T] - 1$$

$$= \mathbb{E} [Y \bar{g}^n_T] + \epsilon_n - 1$$

$$< \mathbb{E} [Y \bar{g}^n_T] - \frac{1}{2}$$

for $n > N$, where $N = \min \{ n : \epsilon_n < 1/2 \}$. Thus combining last two results

$$\mathbb{E} [Y \bar{g}^n_T] \leq C^h \mathbb{E} \bar{g}^n_0 < \mathbb{E} [Y \bar{g}^n_T] - \frac{1}{2},$$

for $n > N$, which yields a contradiction. 

References


