Correlated Defaults in Reduced-Form Models

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Abstract

Reduced-form models have proven to be a useful tool for analyzing the dynamics of credit spreads. However, some have recently questioned their ability to produce sufficient levels of default correlation. The key concern appears to be the assumption that defaults are independent conditional on the default intensity. I address this concern in two ways. First, I use numerical examples calibrated to recent studies to show that the magnitude of default correlation is quite sensitive to the common factor structure imposed on the default intensity. Second, I relax the conditional independence assumption and present an algorithm for constructing default processes with an arbitrary dependency structure, extending the work of Jarrow and Yu (2001). In contrast to the popular approach based on copula functions, the joint distribution of default times is completely determined by the default intensities, obviating the need to specify a copula. I discuss the calibration of such generalized reduced-form models using corporate bond prices, and provide an application to basket credit derivatives.
Reduced-form models have recently been used to study the behavior of credit spreads. In contrast to the structural models pioneered by Merton (1974), this approach treats default as a jump process with an exogenous intensity. As long as the intensity is assumed to be a linear function of affine diffusion state variables, the methodology of Duffie and Singleton (1999) can be used to econometrically identify the intensity from observed prices and spreads, much like the estimation of affine term structure models of default-free bonds. Examples of this approach include Duffee (1999) on corporate spreads and Duffie, Pedersen and Singleton (2003) on sovereign spreads.

Rarely mentioned, however, is the fact that reduced-form models can also be used to study default correlation. At the heart of reduced-form models lies the assumption that multiple defaults are independent conditional on the sample paths of the default intensities. This is the assumption that facilitates the construction of the doubly stochastic Poisson processes of default [also called the “Cox process” in Lando (1998)]. It implies that in such models, default correlation is synonymous with the correlation of the default intensities. Naturally, one could ask whether this type of models can reproduce empirically estimated default correlations.

The answer to this question is likely to depend on the specific setting. If one is interested in fitting the default correlation between two generic rated issuers, then the conditionally independent construction is a good starting point. Intuitively, such models are not unlike multi-factor models of stock returns. As long as defaults are driven by systematic risk factors and these factors are correctly specified, the default correlation between generic rated issuers will be automatically built into a model calibrated to individual default experience. On the other hand, if there is firm-specific information affecting the likelihood of default, a model based on conditional independence will probably be inadequate. Jarrow and Yu (2001) term such firm-specific information “counterparty risk,” resulting for instance from the concentrated ownership of assets, as well as industry competitive and contagion effects.

Despite the apparent scope for further research into the issue, recent studies of default correlation appear to have written off the reduced-form approach. For example, Hull and White (2001)

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1Earlier examples of reduced-form credit risk models include Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Lando (1998), and Duffie and Singleton (1999).

2The recent Fitch survey of the credit derivatives market found concentrated counterparty risk within the top ten global banks and broker dealers, with JP Morgan Chase, Merrill Lynch and Deutsche Bank ranked the top three. It also found that smaller banks were driven toward selling credit protection to enhance their thin margin on traditional loans. The potential “double whammy” begs the question whether the protection will pay off when it is needed, precisely what motivated the example on credit default swaps in Jarrow and Yu (2001).
suggest that “...the range of default correlations that can be achieved is limited. Even when there is a perfect correlation between two hazard rates, the corresponding correlation between defaults in any chosen period of time is usually very low. This is liable to be a problem in some circumstances.” Schonbucher and Schubert (2001) comment that “...the default correlations that can be reached with this approach are typically too low when compared with empirical default correlations, and furthermore it is very hard to derive and analyze the resulting default dependency structure.” Casual conversations with practitioners indicated that this belief is widely held in the industry as well.

In this paper, I present a systematic study of default correlation in reduced-form models. Staying within the confines of the conditional independence assumption, I argue in Section 1 that the generic default correlation is quite sensitive to the common factor structure imposed on individual default intensities. This is illustrated using numerical examples calibrated to two recent studies—Duffee (1999), where there are two common factors, both of which extracted from Treasury yields, and Driessen (2002), where two additional common factors capture the co-movement of corporate credit spreads. I show that the first case implies a default correlation much lower than empirical observations, while the second case implies comparable or even higher values. Therefore, the low default correlation in these models may have more to do with an inadequate common factor structure than the assumption of conditional independence.

In Section 2, I present an algorithm for constructing default processes with an arbitrary dependency structure. In traditional reduced-form models, defaults are triggered by independent unit exponential random variables that are also assumed to be independent of the default intensities. In order to capture firm-specific sources of default correlation, Jarrow and Yu (2001) partially relax this assumption to allow the intensities to depend on a subset of default triggers (“primary firms” in their terminology). In their model, the default processes of the primary firms are part of the state variables that determine the default intensities of the “secondary” firms, preserving the conditionally independent construction of the default processes. While this is a valuable first attempt, it may not work when there is no clear distinction between primary and secondary firms.

This paper is not the first to consider general default dependencies in reduced-form models. Jarrow and Yu (2001) attempt to solve a case of “looping default,” although their solution is still grounded in conditional independence. More rigorously, Kusuoka (1999) constructs two interdepen-
dent default times by transforming two independent default times under a cleverly chosen change of measure. His technique is used by Bielecki and Rutkowski (2002) to value defaultable bonds in Jarrow and Yu’s example. Nevertheless, this technique cannot be easily extended to a large number of issuers.

To accommodate general default dependencies, I introduce stochastic default intensities that depend on the common state variables and the entire set of default triggers. The definition of point processes with such general intensities is provided, for example, in Brémaud (1981). Their construction, however, is more subtle because the unit exponential default triggers can no longer be defined separately from the default intensities. To circumvent this difficulty, I use the total hazard construction of Shaked and Shanthikumar (1987) to simulate the default times. Given the joint distribution of a number of stopping times, this method can be used to synthesize the original random variables by transforming an equal number of independent unit exponentials using the associated hazard rate functions. It implies that the joint distribution of default times (and default correlations) are completely determined by the default intensities, obviating the need for a copula as in Schonbucher and Schubert (2001).

To illustrate the general methodology, I use Monte-Carlo simulation to generate the marginal distribution of default times and the default correlations. Of particular interest is how these entities change when the internal history is ignored, as in the case of traditional reduced-form models. More importantly, I focus on the calibration of the model to empirical corporate bond pricing data, which naturally leads to a discussion of industry-wide contagion and competitive effects in the setting of multiple dependent defaults. Lastly, I introduce common factors that drive part of the default correlation into the default intensity, and use the model to price basket credit default swaps. As expected, their valuation is sensitive to the assumed default dependency structure.

1 Conditionally Independent Defaults

While existing research often suggests that the default correlation generated by reduced-form models with conditionally independent defaults is too low to match empirical observations, there is so far no solid empirical support for this conjecture. Indeed, empirical tests of reduced-form models based on criteria related to default correlation are almost nonexistent.

To bridge this gap, I examine the default correlation implied by several recent empirical studies
of credit spreads using the reduced-form methodology. These studies, to be introduced in more
detail later in this section, are those of Duffee (1999) and Driessen (2002). Several observations
are noted. First, as these models are estimated across credit ratings using a large cross-section
of issuers, the average intensity for each rating is representative of what one would encounter in
a well-diversified portfolio. Second, since the intensities are estimated firm-by-firm, these models
are consistent with an interpretation based on conditional independence—that default correlation
is built into the common variation of the individual default intensities. Third, since the models are
estimated using credit spreads, the estimated risk-neutral intensity needs to be transformed into
the physical intensity required for computing default correlation.

In the following subsection, I outline the general steps for computing default correlation in the
“standard” models. I ignore much of the technical conditions required for specifying such models
and concentrate on the procedure instead.

1.1 Calibration Procedure

To start, assume the existence of two default stopping times, $\tau^1$ and $\tau^2$, with physical intensities
$\lambda^1$ and $\lambda^2$. The precise meaning of this statement is that

$$1\{t \geq \tau\} - \int_0^t \lambda_i^1 \mathbf{1}\{s \leq \tau\} ds$$

is a martingale under the physical measure.\(^3\)

The intensities are assumed to be $\mathcal{F}_t^X$-adapted where $X_t$ is a vector-valued process representing
state variables driving changes in default rates. These can be common macroeconomic factors
such as the Treasury term structure level and slope, or firm-specific characteristics such as book
to market and leverage ratios. When reduced-form models are used to fit credit spreads, one can
often infer the value of $X_t$ from bond prices rather than assuming $X_t$ to be observable. In theory,
this latent variables approach can be used with bankruptcy data to estimate the physical intensity,
although it has not been applied in this way to my knowledge.

To construct the stopping time with the given intensity, define as in Lando (1998):

$$\tau^i \equiv \inf \left\{ t : \int_0^t \lambda^i_s ds \geq E^i \right\}, \quad (2)$$

\(^3\)Although I focus on modeling correlated defaults under the physical measure, the framework is also applicable to
the construction of default times under the risk-neutral measure, in particular, for the valuation of credit derivatives.
where \( E^i \) is a unit exponential random variable independent of \( X_t \). This construction satisfies the martingale property of equation (1), and leads to the following distribution for the stopping time:

\[
\Pr(\tau^i > t | \mathcal{F}_t^X) = \exp\left(-\int_0^t \lambda^i_s ds\right).
\]  

(3)

Furthermore, as in Lando (1994), assume that \( E^1 \) is independent of \( E^2 \), so that the two default times are conditionally independent given the history of \( X_t \). This defines the class of standard reduced-form models.

The default correlation between the two stopping times is commonly defined as

\[
\rho(t) \equiv \text{Corr} \left( 1_{\{\tau^1 \leq t\}}, 1_{\{\tau^2 \leq t\}} \right).
\]  

(4)

Note that the default correlation is a function of the horizon under consideration. Utilizing the above construction, this can be rewritten as

\[
\rho(t) = \frac{E(y^1_t y^2_t) - E(y^1_t) E(y^2_t)}{\sqrt{E(y^1_t) - (E(y^1_t))^2} \sqrt{E(y^2_t) - (E(y^2_t))^2}}.
\]  

(5)

where

\[
y^i_t = \exp\left(-\int_0^t \lambda^i_s ds\right).
\]  

(6)

This equation shows that in standard reduced-form models, default correlation is completely determined by the individual default intensities.

Various reduced-form models can be distinguished for their choice of the state variables and the processes that they follow. In the models considered below, the intensity is a linear function of \( X_t \) and \( X_t \) are diffusions in the affine class as defined in Duffie and Kan (1996). Therefore, the expectations above are exponentially linear in \( X_t \), which facilitates computation.

An important issue is that the above framework requires the physical default intensity, while existing studies invariably estimate the risk-neutral intensity implicit in bond prices. The connection between risk-neutral and physical default intensities is studied in Jarrow, Lando and Yu (2001), who show that with a large number of conditionally independent default times, the functional forms of the two intensities are identical in an asymptotic sense. The empirical validation of this conjecture, however, is clouded by the fact that what one calls “credit spread” may contain tax and liquidity components. The analyses in Jarrow, Lando and Yu (2001) and Driessen (2002)
provide no conclusive findings, but do suggest that the equivalence might hold when non-default components of the credit spread can be more accurately controlled.

The overall procedure for imputing default correlation from risk-neutral default intensities is as follows:

1. Obtain the risk-neutral intensity function \( \lambda \) estimated for various credit ratings as well as the estimated physical dynamics of the state variables \( X_t \) from the empirical literature. Assume the physical intensity function \( \lambda \) to be identical to \( \lambda \).

2. Define the adjusted physical intensity function \( \lambda_{\text{adj}} \) as

\[
\lambda_{\text{adj}}^t \equiv \lambda_t - \frac{a}{t + b},
\]

where constant coefficients \( a \) and \( b \) are determined by minimizing the sum of squared differences between the model-implied conditional default rates and those inferred from historical default experience.

3. Substitute the \( \lambda_{\text{adj}} \) function for two different ratings and the physical dynamics of \( X_t \) into equation (5) to compute the default correlation between two generic rated issuers.

The static adjustment to the intensity in Step 2 is an attempt to eliminate the effect of liquidity and taxes from credit spreads and implied physical default rates. Currently there is no consensus on how to accurately estimate these components. What we do know, however, is that the tax spread is roughly constant [see Elton, Gruber, Agrawal and Mann (2001)] and liquidity spread is perhaps decreasing with maturity [see Ericsson and Renault (2001) and Perraudin and Taylor (2002)], both of which are incorporated into (7). The approach here assumes the equivalence of the risk-neutral and physical intensities, and ignores any dynamics of the liquidity spread. It implies that default rates and credit spreads would fluctuate in exactly the same manner. A casual comparison between spreads and default rates in, for example, Yu (2002, Exhibit 1), shows that this may not be a poor assumption. In any case, since the focus is on the role of the assumed common factor structure on the determination of default correlation, the accuracy of the outlined procedure should be of secondary concern.
1.2 Examples

The calibration procedure given above is applied to two examples:

1.2.1 Duffee (1999)

Duffee (1999) assumes the risk-neutral intensity to be

\[ \tilde{\lambda}_t = \alpha + \lambda^*_t + \beta_1 (s_{1t} - \bar{s}_1) + \beta_2 (s_{2t} - \bar{s}_2), \]

where \( s_{1t} \) and \( s_{2t} \) are default-free factors inferred from Treasury yields through the short rate model

\[ r_t = \alpha_r + s_{1t} + s_{2t}, \]

and \( \bar{s}_1 \) and \( \bar{s}_2 \) are the respective sample means. The firm-specific factor \( \lambda^*_t \) and coefficients \( \alpha, \beta_1 \) and \( \beta_2 \) are in turn inferred from the prices of corporate bonds issued by a given firm, taking the Treasury term structure dynamics as given.

All factors are assumed to be square-root diffusions. The independent Treasury factors are common to every firm and the firm-specific factors are independent across firms. This specification is consistent with Duffee’s firm-by-firm estimation approach. In a setting with conditionally independent default times, default correlation is generated by the dependence on the common factors through the \( \beta \) coefficients. On the other hand, the firm-specific factor \( \lambda^*_t \) should reduce the default correlation. Intuitively, this is because the firm-specific component contributes little to the covariance while increasing the variance of default rates.

The physical dynamics of each square-root diffusion can be designated by the triple \((\kappa, \theta, \sigma)\), where \( \kappa \) is the speed of mean-reversion, \( \theta \) is the long-run mean and \( \sigma \) is the volatility of the process. To calibrate the liquidity adjustments, one also needs the initial values of the state variables. For the firm-specific factors, these are taken to be their sample means. For the Treasury factors, since their sample means are not provided by Duffee, I use their long-run means instead.\(^4\) Table 1 summarizes the estimates from Duffee (1999) and the liquidity adjustments.

Using the necessary inputs from Table 1, I compute the default correlation between generic rated issuers. The results, presented in Table 2, bear two distinct patterns. First, adjusting for liquidity and tax effects appears to be very important when inferring default correlations from credit spreads. Moving from Panel B to Panel A, default correlation can increase by more than ten-fold in some instances. This is because a major part of the short-term credit spread is due to

\(^4\)The precise values of \( \bar{s}_1 \) and \( \bar{s}_2 \) are not essential to the calculation as they are constants that can be compounded into the liquidity adjustment term.
liquidity and state taxes. The adjustments to the intensity, for instance, can be more than 100 basis points at zero maturity. A deterministic reduction of the magnitude of the default intensity increases the proportion of the intensity that is stochastic, thereby increasing default correlation. This effect is the strongest for short-term default correlation between high-quality issuers. A second pattern is that default correlation increases monotonically with maturity. This is not surprising given that default over a very long horizon is almost a certainty.

To better gauge the results, I compare them with those of Zhou (2001) and Lucas (1995). Zhou’s calculation is based on a first passage time approach in the Black and Cox (1976) structural framework, while Lucas’ results are obtained from historical default data. A comparison between Table 2 and these two studies shows that the procedure used here slightly overestimates the default correlations for short horizons, while significantly underestimate the default correlations for longer horizons. For references, the results from Zhou and Lucas are reproduced here as Table 5.

The overestimation at short horizons is unlikely to be a serious concern here. For Lucas’ historical estimation, there are simply not enough observations to pin down the default correlations precisely for, say, investment-grade issuers at a one-year horizon. The quality of the historical estimates for lower-rated issuers at longer horizons, however, is much higher due to the larger sample size. Hence the underestimation at longer horizons is a major concern that demands our full attention.

One promising explanation of the underestimation seems to be the insufficient specification of the common factor structure in Duffee’s model. In the standard reduced-form framework, default correlation is attributed wholly to the common factors in the intensity function. It is critical, then, that the specification of the intensity function adequately capture the sources of common variation in yield spreads. In Duffee, however, the only common factors are related to the Treasury term structure. To the extent that other important common factors are omitted, using Duffee’s model is likely to underestimate the default correlation. Indirectly supporting this conjecture, Duffee shows that the market prices of risk associated with the firm-specific factors are statistically significant, and that the firm-specific factors are also correlated across firms, suggesting an active role for “missing” common factors.
1.2.2 Driessen (2002)

The aforementioned problems with Duffee’s specification are tackled by Driessen (2002), who assumes the risk-neutral intensity as

\[
\tilde{\lambda}_t = \alpha + \beta_r r_t + \beta_\nu \nu_t + \gamma_1 F_{1t} + \gamma_2 F_{2t} + G_t.
\]  

(9)

In this setup, \(r_t\) and \(\nu_t\) are short-rate factors in an affine Treasury term structure model, \(F_{1t}\) and \(F_{2t}\) are factors common to every firm, and \(G_t\) is a firm-specific factor. Both the common factors and the firm-specific factors are assumed to follow independent square-root diffusions, and hence can be specified by the triple \((\kappa, \theta, \sigma)\). As evidence of improvement over Duffee (1999), Driessen shows that the estimated market prices of risk for the firm-specific factors are close to zero, and that the firm-specific factors have negligible cross-firm correlations.

For the purpose of calibrating default correlation, I ignore the short-rate factors. The subsequent results show that the inclusion of the two common factors elevates the default correlation dramatically relative to the results obtained using Duffee’s model. The portion of the default correlation generated by short-rate dependence is likely to be relatively small.

Table 3 presents Driessen’s estimates and the liquidity adjustments. Since Driessen does not aggregate his estimates for the firm-specific factor by credit rating, his median estimates are used for all three ratings. I also divide the parameter estimates of the instantaneous spread by one minus the recovery rate, assumed to be 44 percent for all three ratings, to obtain estimates of the default intensity function. Furthermore, I set the initial values of all factors to be their long-run means.

The default correlations, presented in Table 4, are much higher than those in Table 2 with or without intensity adjustments. In fact, the default correlation at long horizons now appears to be too high relative to the estimates of Lucas (1995).5 The key point is clear—by incorporating more of the common variation in credit spreads, the default correlation implied by a reduced-form model with conditionally independent defaults can easily be made higher. At a minimum, the evidence here suggests that more work is needed before dismissing such models as unsuitable for describing generic default correlations.

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5 A more recent study by De Servigny and Renault (2002) find speculative-grade default correlation at the five-year horizon that could easily exceed 10 percent for some industries.
2 General Default Dependency

Having investigated default correlation in models with conditionally independent defaults, I turn to the statistical modeling of general default dependency.

The literature on general default dependency has so far focused on the copula function approach. The typical procedure is to divide the portfolio into broad asset classes based on credit rating and perhaps industry, estimate the marginal distribution of default time for a “representative” asset in each class, and then stitch the marginals into a joint distribution using a copula. The copula function is often chosen to fit characteristics of the empirical joint distribution, such as tail dependence or exceedance correlation [see Hamilton, James and Webber (2002) and Das and Geng (2002)].

While straightforward to implement, the intuition for why certain classes of copulas are needed is not clear. In contrast, the generalized reduced-form models to be introduced below are specified through default intensity functions that have clear economic interpretations, which can lead to new calibration procedures.

2.1 The Model

As a natural starting point of the analysis, I first present an overview of Jarrow and Yu (2001, JY). This is followed by an extension to general default dependencies and a brief comparison with the copula approach.

2.1.1 Jarrow and Yu (2001)

Assume an information structure given by the filtered probability space \( \left( \Omega, \{ \mathcal{F}_t \}_{t=0}^{T^*}, P \right) \), where \( P \) is the physical measure and the terminal date \( T^* \) can be \( +\infty \). The choice of the filtration \( \mathcal{F}_t \) should reflect the information available to the investor. Naturally, this can include both economy-wide and firm-specific determinants of default risk. JY define this as

\[
\mathcal{F}_t \equiv \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^I.
\]  

(10)

Here, \( \mathcal{F}_t^X \equiv \sigma (X_s, 0 \leq s \leq t) \) denotes the information generated by the common factors \( X_t \), while \( \mathcal{F}_t^i \equiv \sigma (N^i_s, 0 \leq s \leq t) \) denotes the information generated by the default process of firm \( i \), \( N^i_t \), where...
The filtration $\mathcal{F}_t$ is sometimes referred to as a “history” of the default processes and, when the set of common factors is empty, an “internal history.”

In this description of the economy, the only firm-specific information is given by the observation of a firm’s default status. This is of course a simplifying assumption, but is motivated by the ample evidence on financial contagion. JY term this setup a model of “counterparty risk,” reflecting the fact that the counterparties in a transaction may have their financial health tied to each other.

To be more concrete about the driving forces behind the default processes, JY specify a non-negative process $\lambda^i_t$, one for each firm, assumed to be adapted to $\mathcal{F}_t$. The intention is to make this the intensity of the default process, capturing the fact that defaults are driven by both common and firm-specific factors.

Following the construction in equation (2), JY define $N^i_t \equiv 1_{\{t \geq \tau^i\}}$ as a doubly stochastic Poisson process. This can be accomplished by selecting a unit exponential random variable $E^i$ that is independent of both $\mathcal{F}^X_{T^*}$ and $\mathcal{F}^{-i}_{T^*}$, where

$$\mathcal{F}^{-i}_t = \mathcal{F}^1_t \lor \cdots \lor \mathcal{F}^{i-1}_t \lor \mathcal{F}^{i+1}_t \lor \cdots \lor \mathcal{F}^I_t. \quad (11)$$

Conditional on $\mathcal{F}^X_{T^*} \lor \mathcal{F}^{-i}_{T^*}$, $N^i_t$ is a non-homogeneous Poisson process with intensity function $\lambda^i_t$, implying a distribution of $\tau^i$ given by

$$\Pr (\tau^i > t | \mathcal{F}^X_{T^*} \lor \mathcal{F}^{-i}_{T^*}) = \exp \left( - \int_0^t \lambda^i_s ds \right). \quad (12)$$

This procedure can be used to construct a single default process when all other default processes are assumed to be exogenously given. If the intention is to construct the complete set of default processes, a technical obstacle is encountered. JY illustrate this problem with a two-firm example, a simple alternative explanation of which is given as follows:

Consider two firms with intensity $\lambda^1$ and $\lambda^2$, respectively. Assume that $\lambda^2$ is dependent on $\tau^1$ and $\lambda^1$ on $\tau^2$. In this case, if one were to construct $\tau^1$ and $\tau^2$ by equation (2), it is clear that both stopping times would be functions of unit exponentials $E^1$ and $E^2$. The problem, however, is that one can no longer assume independence between $E^1$ and $\tau^2$, and $E^2$ and $\tau^1$. This causes the breakdown of equation (12).

JY circumvent this problem by modifying the information structure. They assume that a subset of firms, called “primary firms,” have intensities that depend only on the common factors, and that
the others, called “secondary firms,” have intensities that depend on both the common factors and the default status of the primary firms. In essence, for the secondary firms one recovers the doubly stochastic Poisson framework with the default status of the primary firms being the additional common factors. JY then apply equation (12) to study the valuation of defaultable bonds and default swaps in the presence of counterparty risk.

The question of general default dependency, however, continues to generate research interest. For example, Kusuoka (1999) constructs the general dependent default processes for the case of two firms by transforming two independent Poisson processes. The transformation makes use of Girsanov’s Theorem for jump processes with a judicious choice of the Radon-Nikodym derivative. Within the context of Kusuoka’s example, Bielecki and Rutkowski (2002) show that the JY assumption of primary and secondary firms is in fact unnecessary for the valuation of defaultable bonds. It is noted, however, that much of this development is contingent on the two-firm case and some of the results may not readily extend to cases with more than two firms.

2.1.2 The Total Hazard Construction

Keeping the previous notations, I study the construction of default processes $N^i_t$, $i = 1, \ldots, I$, with stochastic intensities $\lambda^i_t$ adapted to the history $\mathcal{F}_t$.

In the language of Brémaud (1981), this is a case of general stochastic intensity. Specifically, $N_t$ admits an $\mathcal{F}_t$-intensity $\lambda_t$ if the equality

$$E \left( \int_0^\infty C_s dN_s \right) = E \left( \int_0^\infty C_s \lambda_s ds \right)$$

holds for all non-negative $\mathcal{F}_t$-predictable process $C_t$, where the expectation is taken under measure $P$.\(^6\) With this definition, one maintains the interpretation of $\lambda_t$ as the hazard rate, or intensity, of jumps, because it implies that

$$E_t \left( N_{t+dt} \right) - N_t = \lambda_t dt,$$

(14)
given a left-continuous $\lambda_t$.\(^7\)

Intuitive as it is, this definition does not say whether such processes exist. For that one can turn to the constructive approach of Norros (1986) and Shaked and Shanthikumar (1987). Given a

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\(^6\)This definition is a direct extension of Watanabe’s characterization of doubly stochastic Poisson processes. For details, refer to Brémaud (1981, Chapter II).

\(^7\)One can always find a predictable version of the intensity, which is also unique. A predictable process is one that is measurable with respect to the predictable $\sigma$-field, generated by all processes with left-continuous sample paths.

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set of intensities adapted to the internal history $\mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^I$, this is a way to synthesize the default times $\tau = (\tau^1, \ldots, \tau^I)$ from a set of i.i.d. unit exponential random variables $E = (E^1, \ldots, E^I)$. The detailed procedure, as given in Shaked and Shanthikumar (1987), is presented below with the generalization to $\mathcal{F}_t$-intensities to follow.

First, noting the explicit dependence of $\lambda^i_t$ on $\mathcal{F}_t^{i-1}$, I use the notation $\lambda^i_t(t|n)$ to denote the intensity process for firm $i$ given the observed default times of $n$ other firms, $t_{k_1}, \ldots, t_{k_n}$, where it is assumed that $0 = t_{k_0} < t_{k_1} < \cdots < t_{k_n} < t < \tau^i$. I then define the total hazard accumulated by firm $i$ by time $t$ given $n$ observed defaults as

$$
\psi^i(t|n) \equiv \sum_{m=1}^{n} \Lambda^i(t_{k_m} - t_{k_{m-1}}|m-1) + \Lambda^i(t - t_{k_n}|n),
$$

(15)

where $\Lambda^i(t|m) \equiv \int_{t_{k_m}}^{t_{k_m}+t} \lambda^i(u|m) \, du$ is the total hazard accumulated by firm $i$ between the $m$th default and $t$. It is assumed that there is no observed default between $t_{k_m}$ and $t$.

A well-known result shows that the total hazards accumulated by the firms until they defaulted are i.i.d. unit exponential random variables. This suggests the definition of an inverse function that would map a set of i.i.d. unit exponentials back to the original default stopping times. Hence I define

$$
\Lambda^{-1}^i(x|n) \equiv \inf \{t: \Lambda^i(t|n) \geq x\}, \ x \geq 0.
$$

(16)

The following procedure defines a collection of stopping times $\tilde{\tau} = (\tilde{\tau}^1, \ldots, \tilde{\tau}^I)$ based on the unit exponentials $E = (E^1, \ldots, E^I)$:

1. Let

$$
k_1 = \arg \min_{1 \leq i \leq I} \{ \Lambda^{-1}^i(E^i) \},
$$

and let

$$
\tilde{\tau}^1 = \Lambda^{-1}_{k_1} \left( E^{k_1} \right).
$$

(18)

2. Assume that the values of $(\tilde{\tau}^1, \ldots, \tilde{\tau}^{m-1})$ are already given, where $m \geq 2$. Define the set $I_{m-1} = \{k_1, \ldots, k_{m-1}\}$ and $\mathcal{T}_{m-1}$ as the set of firms excluding $I_{m-1}$. Recall that $\psi^i(t|m-1)$ is the total hazard accumulated by firm $i$ given the first $m-1$ defaults, let

$$
k_m = \arg \min_{i \in \mathcal{T}_{m-1}} \{ \Lambda^{-1}^i \left( E^i - \psi^i(\tilde{\tau}^{m-1}|m-1)|m-1 \right) \},
$$

(19)
and let

$$\hat{\tau}^m = \hat{\tau}^{m-1} + \Lambda_{km}^{-1} \left( E_{km}^{\tau} - \psi_{km} \left( \hat{\tau}^{m-1} | m-1 \right) \right).$$  \hspace{1cm} (20)

3. Repeat Step 2 until $m = I$.

Norros (1986) shows that the constructed stopping times $\hat{\tau}$ and the original stopping times $\tau$ have the same distribution. Hence this procedure is a practical way to simulate the general dependent default times if their intensities are adapted to the internal history. For default times with intensities given over the larger history $\mathcal{F}_t$, the above procedure and Norros’ result can be understood as conditional on the history of the common factors.\(^8\) This suggests adding an initial step to the above construction:

0. Simulate a complete sample path of $X_t$. Draw i.i.d. unit exponentials $E = (E^1, \ldots, E^I)$ independent of the history of $X_t$.

Given this trivial extension, one can establish the following:

**Proposition 1**  
*The simulated stopping time $\hat{\tau}^i$ has $\mathcal{F}_t$-intensity given by $\lambda^i_t$, for $i = 1, \ldots, I$.***

**Proof.**  
Following Step 0-4 and as a direct result of the Shaked and Shanthikumar construction, $\hat{\tau}^i$ has $\mathcal{G}_t$-intensity given by $\lambda^i_t$, where $\mathcal{G}_t \equiv \mathcal{F}_t^{\infty} \lor \mathcal{F}_t^1 \lor \cdots \lor \mathcal{F}_t^I$. Since $\lambda^i_t$ is in fact assumed to be $\mathcal{F}_t$-adapted, it follows that the $\mathcal{F}_t$-intensity of $\hat{\tau}^i$ is given by $E \left( \lambda^i_t | \mathcal{F}_t \right) = \lambda^i_t$. \(\square\)

**2.1.3 Comparison with the Copula Function Approach**

At this point, a comparison with the copula approach can help understanding the general model. The copula approach as given by Schonbucher and Schubert (2001, SS) associates each firm with a “pseudo-intensity” $h^i_t$, assumed to be $\mathcal{F}_t^{X_i}$-adapted. The definition of the default time for firm $i$ is identical to that of equation (2) with $h^i_t$ playing the role of $\lambda^i_t$. The pseudo-intensity has the interpretation of the intensity when ignoring the observed defaults of other firms. The crucial difference

\(^8\)Recall that even the general case is still endowed with a rather special structure—that is, the set of common factors $X_t$ are assumed to be exogenously given. Therefore, simulating $X_t$ prior to the determination of the default processes is feasible. If there were two-way dependencies between $X_t$ and the default processes, of course one would not be able to separate the simulation of $X_t$ and $N_t$. 

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from the conditionally independent construction is that, instead of assuming i.i.d. unit exponentials, a copula function now governs their joint distribution. Specifically, the joint distribution of \( E = (E^1, \ldots, E^I) \) is given by

\[
F(E^1, \ldots, E^I) = C(\exp(-E^1), \ldots, \exp(-E^I)).
\]  

(21)

In this setup, default dependency is summarized by the copula function \( C \). For example, the product copula would imply the construction based on conditional independence.

Denoting \( \gamma^i_t = \exp \left( - \int_0^t h^i_t du \right) \), \( C(\gamma^i_t) = C(\gamma^1_t, \ldots, \gamma^I_t) \), and \( C_i \) and \( C_{ij} \) as first and second partial derivatives, SS show that the intensity of firm \( i \) given no observed default is

\[
\lambda^i_t = h^i_t \gamma^i_t \frac{C_i(\gamma^i_t)}{C(\gamma^i_t)},
\]

and, after the default of firm \( j \), is

\[
\lambda^i_t = h^i_t \gamma^i_t \frac{C_{ij}(\gamma^i_t)}{C_j(\gamma^i_t)},
\]

(23)

which results from replacing the copula \( C \) in (22) by the conditional copula \( C_j \). These equations jointly indicate the possibility of a jump in \( \lambda^i_t \) at the default of another firm. Given how these intensities change at default events, one can apply the total hazard construction in order to recover the original default times. This shows that the set of default times that can be constructed using the total hazard construction is no smaller than the set that can be achieved through the copula approach of SS.

In the other direction, given the \( \mathcal{F}_t \)-intensity \( \lambda^i_t \), one can easily compute \( h^i_t \) as \( E(\lambda^i_t|\mathcal{F}_t^X \lor \mathcal{F}_t^i) \). Then, equations similar to (22) and (23), which specify the jump in the intensities given other defaults, can in theory be used to solve for the copula function \( C \). Therefore, the SS copula approach and reduced-form models with general intensities are apparently equivalent ways to model default dependency. The advantage of the intensity-based approach is that the jump in the intensity given other credit events can be calibrated to historical default experience or changes in the yield spread. This is the approach taken in, for example, Collin-Dufresne, Goldstein and Helwege (2003).

### 2.2 Examples

This subsection presents numerical examples to illustrate the general model. I simulate a large number of realizations for the set of default times under consideration. This information allows
easy computation of key statistics such as the marginal distribution of default times and the default correlation for various horizons. A particular focus is on how these entities change from cases where the internal history is ignored, as in standard reduced-form models.

I start with a two-firm case considered in Jarrow and Yu (2001), then move on to a more realistic multi-firm case that can be calibrated to empirical data. In the third example, I experiment with non-trivial common factors that drive part of the default correlation, and present an application to basket credit default swaps.

2.2.1 A Two-Firm Case

The case of two stopping times with hazard rates adapted to the internal history is well-known in the study of dependent random variables. Since the dependency structure is relatively simple, the total hazard construction and the resulting distributions can be derived analytically. Therefore, this case is primarily used as a device to check the simulation algorithm.

The construction starts from the original default times, \( \tau^1 \) and \( \tau^2 \), with default intensities given by

\[
\lambda^1_t = a + (a' - a) 1_{\{t > \tau^2\}}, \\
\lambda^2_t = b + (b' - b) 1_{\{t > \tau^1\}}.
\]

With no observed defaults, the total hazard accumulated from 0 to \( t \) is \( \Lambda^1_t = at \) and \( \Lambda^2_t = bt \). If Firm 1 has defaulted at \( t_1 \), the total hazard accumulated by Firm 2 from \( t_1 \) to \( t_1 + t \) is \( \Lambda^2_{t|t_1} = b't \). Similarly, one can set \( \Lambda^1_{t|t_2} = a't \).

Using the inverse of these functions, one can construct two default times \( \hat{\tau}^1 \) and \( \hat{\tau}^2 \) from two independent unit exponential random variables \( E^1 \) and \( E^2 \) as

\[
\hat{\tau}^1 = \begin{cases} 
\frac{E^1}{a}, \\
\frac{E^1}{a} + \frac{E^2}{b} \left(1 - \frac{a}{a'}\right), \\
\frac{E^2}{a} > \frac{E^2}{b}
\end{cases},
\]

and

\[
\hat{\tau}^2 = \begin{cases} 
\frac{E^2}{b}, \\
\frac{E^2}{b} + \frac{E^1}{a} \left(1 - \frac{b}{b'}\right), \\
\frac{E^1}{a} > \frac{E^2}{b}
\end{cases}.
\]

These default times and the original default times share the same multivariate distribution known as the Freund distribution, whose density is given as

\[
f(t_1, t_2) = \begin{cases} 
ab'e^{-(a+b-b')t_1-b't_2}, & t_1 \leq t_2, \\
ab'e^{-a't_1-(a+b-a)t_2}, & t_1 > t_2.
\end{cases}
\]
The marginal densities can be derived as
\[
g_1(t_1) = \frac{ab'}{a + b - a'} \left( e^{-a't_1} - e^{-(a+b)t_1} \right) + ae^{-(a+b)t_1}, \tag{29}
\]
\[
g_2(t_2) = \frac{ab'}{a + b - b'} \left( e^{-b't_2} - e^{-(a+b)t_2} \right) + be^{-(a+b)t_2}, \tag{30}
\]
and the marginal distributions as
\[
G_1(t_1) = \Pr(\tau^1 \leq t_1) = 1 - \frac{be^{-a't_1} - (a' - a) e^{-(a+b)t_1}}{a + b - a'}, \tag{31}
\]
\[
G_2(t_2) = \Pr(\tau^2 \leq t_2) = 1 - \frac{ae^{-b't_2} - (b' - b) e^{-(a+b)t_1}}{a + b - b'}. \tag{32}
\]
From these equations, one can see that neither \(G_1\) depends on \(b'\), nor \(G_2\) on \(a'\). Intuitively, what occurs to Firm 2 after the default of Firm 1 (an increase in Firm 2’s intensity given by \(b' - b\)) should have no impact on the distribution of Firm 1’s default time. Hence, if one interprets everything under the risk-neutral measure and proceeds to valuation, the bonds issued by Firm 1 can be priced while ignoring the impact of Firm 1’s default on Firm 2. In the terminology of Jarrow and Yu (2001), one can treat Firm 2 as a “primary firm” if the specific concern is the correct pricing of bonds issued by Firm 1.\(^9\) Clearly, this is a result that is peculiar to the two-firm case. With multiple firms, generally one cannot value bonds issued by a firm while ignoring the complete set of default dependency of other firms.\(^{10}\)

Besides the marginal distributions, one can also calculate the probability that both firms will default before time \(t\):
\[
\Pr(\tau^1 \leq t, \tau^2 \leq t) = 1 - \frac{be^{-a't}}{a + b - a'} - \frac{ae^{-b't}}{a + b - b'} + \frac{a'b'e^{-(a+b)t}}{(a + b - a')(a + b)} + \frac{ab'e^{-(a+b)t}}{(a + b - b')(a + b)}. \tag{33}
\]
Equations (31)-(33) are useful in the calculation of the default correlation between the two firms.

I simulate the default times based on the prescriptions of (26) and (27) using different numbers of independent scenarios. The simulated entities are compared with their theoretical counterparts in order to examine the accuracy of the numerical procedure. To generate Figures 1 and 2, I assume that the parameters are: \(a = 0.01\), \(a' = 0.0105\), \(b = 0.02\), and \(b' = 0.021\). In this base case, a firm’s intensity will jump up by 5 percent given the default of the other firm.

\(^9\)This interesting observation and its interpretation are first given in Bielecki and Rutkowski (2002, Chapter 9).
\(^{10}\)This point will be revisited in a later example.
Figure 1 plots the marginal distribution of $\hat{\tau}^1$. It shows that using 5,000 scenarios, there is virtually no difference between the simulated and theoretical values. Figure 2 plots the default correlation as defined in equation (4). Clearly, to obtain accurate simulated values for the default correlation, many more scenarios are required. For example, even at 1 million scenarios, there are still perceptible differences between the two series. This is due to the low default correlation dictated by the choice of the model parameters. For other parameters corresponding to higher default correlation, the number of scenarios required for a given accuracy will be lower. In practice, one encounters the same problem when estimating the one-year default correlation between investment-grade issuers using historical data; there are simply not enough observations to estimate this accurately.\textsuperscript{11}

Because of the low default correlation given these parameter values, one would expect that changes in $a'$ will not have a significant effect on the marginal distribution of $\hat{\tau}^1$, but its effect on default correlation will be more dramatic. To demonstrate this, I plot the theoretical and simulated marginal distribution and default correlation using three values of $a'$, 0.01, 0.0105, and 0.011 while fixing other parameters at the base case values. Figure 3 confirms that a small increase in $a'$ can significantly increase default correlation at all horizons.

An elevation of default risk thus can affect default correlation in the model. However, the particular direction of the effect depends on the manner in which the level of the intensity is raised. For example, if one fixes the level of $b' - b$ at its base case value of 0.001 while increasing the value of $b$, default correlation will in fact decrease. This is demonstrated in Figure 4. I note that the change in the marginal distribution is still minimal, and that there is almost no effect on default correlation for short horizons. At short horizons, default correlation appears to depend only on the counterparty risk components $a' - a$ and $b' - b$.

2.2.2 A Multi-Firm Case with Intra-Industry Spillover Effects

Having validated the simulation procedure using the test case, I turn to a slightly more complicated example that could be of some practical value. In this case, for reasons that shall become clear below, I consider $I$ firms where $I - 1$ of them have identical intensities, where $I \geq 3$. I let $\tau_F$ be

\textsuperscript{11}Empirical evidence shows that the default correlation between investment-grade issuers at short horizons is very low. This has led to the practical treatment of investment-grade short-horizon defaults as independent events. For example, see Erturk (2000).
the first-to-default time, namely, \( \tau_F = \min \left( \tau^1, \ldots, \tau^I \right) \). The intensities are given by

\[
\lambda^i_t = \begin{cases} 
    b + (b' - b) 1\{t > \tau_F\}, & i = 1, \\
    a + (a' - a) 1\{t > \tau_F\}, & 2 \leq i \leq I.
\end{cases}
\]

One interpretation of this form is that it describes the default intensity of similar firms in a concentrated industry. When a firms defaults, the event triggers a change in the default intensity of all other firms in the same industry, which can be calibrated by examining bond price changes around credit events. In equation (34), the coefficient \( a' - a \) and \( b' - b \) could be negative or positive, depending on whether contagion or competitiveness effect dominates. I note that this specification considers only the first default. This is a simplifying assumption that allows one to dispense with the relatively cumbersome path-dependency in the simulation procedure.

Lang and Stulz (1992) show that there are significant industry-wide abnormal stock returns in response to a bankruptcy filing. Newman and Rierson (2002) show that European telecom credit spreads are driven by the industry-wide expected default loss and interpret their finding as a demand curve effect. One can of course argue that a large issue exhausts the industry debt capacity, causing market yield spreads to widen. However, a large issue also increases the default probability of the issuer and thus may be similar to an actual default event in its impact on industry-wide credit spreads. As a result, the two effects may be empirically indistinguishable. The specification in equation (34) allows one to conduct a comparative statics exercise of this scenario, for example, by increasing the value of \( b \).

In this example, one can also express the default time construction in a compact form. It is straightforward to show that the first-to-default time \( \hat{\tau}_F \) can be written as

\[
\hat{\tau}_F = \min \left( \frac{E^1}{b}, \frac{E^2}{a}, \ldots, \frac{E^I}{a} \right),
\]

and the individual default times are given through their respective unit exponentials as

\[
\hat{\tau}^i = \begin{cases} 
    \frac{E^1 - b\hat{\tau}_F}{b} + \hat{\tau}_F, & i = 1, \\
    \frac{E^i - a\hat{\tau}_F}{a} + \hat{\tau}_F, & 2 \leq i \leq I.
\end{cases}
\]

Although equation (36) gives the impression that the constructed default times are independent, this is not true due to the induced dependence on the common first-to-default time \( \hat{\tau}_F \). In fact, these default times are not independent even after conditioning on \( \hat{\tau}_F \).

Similar to the two-firm example, it is possible to derive the marginal distributions and default correlations analytically. In spite of the appearance of more than two firms, the default dependency
in this case is summarized by a single random variable—the first-to-default time, and the intuition from the two-firm case remains valid. For example, by substituting equation (35) into equation (36), one can see that neither the distribution of $\tilde{\tau}^1$ is affected by $a'$, nor that of $\tilde{\tau}^2$ by $b'$. Simply put, whatever happens after the first-to-default should not influence the distribution of the first-to-default time. Hence, when using simulations to generate the marginal distribution of a given default time, one can effectively ignore the dependency of all other default times.\textsuperscript{12} Of course, this is not applicable to more general cases, nor if the goal is to study default correlation.

Even without the aid of analytical expressions, the effect of the first-to-default on the marginal distributions and default correlations can be explained intuitively. According to its definition in equation (35), $\tilde{\tau}_F$ has an intensity equal to $b+(I-1)a$. Therefore, as the number of firms increases, the first-to-default is expected to occur more and more rapidly. In the limit as $I \to \infty$, firms will have independent default times with intensities equal to either $a'$ or $b'$. To confirm this intuition, I assume that $a = b = 0.01$ and $a' = b' = 0.011$ and examine how the outputs of the model vary with the number of firms, $I$. Using the symmetry of the example, I focus on the marginal distribution of $\tilde{\tau}^1$ and the default correlation between $\tilde{\tau}^1$ and $\tilde{\tau}^2$. To emphasize changes in the marginal distribution, I plot $-\log (1 - G_1(t))/t$ instead of $G_1(t)$.

Figure 5 shows two findings. First, as the number of firms increases, the marginal distribution converges to $1 - e^{-0.011t}$. This is associated with a default time with constant intensity equal to $a' = 0.011$. Second, as $I$ increases, the default correlation converges to zero. Thus the previous intuition has been confirmed.

Next, I study the effect of an increase in $b$ on the marginal distribution of other default times, with $I = 10$. I let $a = 0.01$, $a' = 0.011$, $b' - b = 0.001$ and let $b$ be equal to 0.01, 0.02, or 0.04.\textsuperscript{13} In the upper left panel of Figure 6, I plot $-\log (1 - G_2(t))/t$ as a function of $t$ for different $b$ values. By symmetry, this is the marginal distribution of the default time of Firm 2-10. To interpret the results, I note that if this were the marginal distribution under the pricing measure, it would have implied roughly a 1 basis point rise in the yield spread on a 30-year bond when $b$ quadruples from

\textsuperscript{12}This does not imply that one can compute the distribution of $\tilde{\tau}^1$ by taking expectation over $\exp \left(-\int_0^t \lambda_s ds\right)$, treating the first-to-default time as exponentially distributed with intensity $b+(I-1)a$. This would give incorrect results because the unit exponentials used to construct the first-to-default time and $\tilde{\tau}^i$ are not independent of each other.

\textsuperscript{13}As shown earlier, the choice of $b' - b$ is irrelevant for the marginal distribution of other default times. This is confirmed by the simulation results to follow.
0.01 to 0.04, assuming zero recovery and constant interest rates. In comparison, Newman and Rierson (2002) find that a $16 billion issue by Deutsche Telekom increased all telecom spreads by 12 basis points, a much larger impact. This example shows that the increase in spreads is unlikely to come from the change in credit quality of a single issuer. Instead, a large debt issue must directly alter the perception of credit risk in the entire industry. With the parameters used in this example, if the large debt issue were to be treated as a default event, its impact on industry-wide spreads would be exactly 10 basis points.

It is also noted in the upper right panel of Figure 6 that different $b$ values do not seem to change the simulated default correlation between Firm 1 and 2. In contrast, if one maintains the value of $b$ at 0.01 while increasing $b'$ from 0.001 to 0.004, the marginal distribution is not affected at all while the level of default correlation increases significantly. This is illustrated in the lower panels of Figure 6.

Given the intricate default dependency in this example, it would be interesting to see whether a construction based on conditional independence can provide reasonable approximations of entities such as the marginal distribution of default times. Returning to an earlier comment, I calculate the marginal distribution of $\tau^1$ with the formula

$$G_1^{\text{incorrect}}(t) = E \exp \left( - \int_0^t \left( b + (b' - b) 1_{\{u > \tau_F\}} \right) du \right),$$

assuming that $\tau_F$ is exponentially distributed with intensity $b + (I - 1)a$. Implicitly, I have assumed that $\tau_F$ is independent of the unit exponential $E^1$ used to construct $\tau^1$. This is, of course, incorrect. However, one might intuitively expect that the approximation in equation (37) will become more accurate as the number of firms increases. Judging from the results presented in Figure 7, this approximation works well for cases with more than 10 firms.

### 2.2.3 A Case with Common Factors

By now perhaps one has realized that neither of the previous examples considers the effect of common factors. Of course, this is done deliberately in order to concentrate on the role of default dependency without being overly burdened with realism. Building on the previous examples, however, one can now deal with this issue in a straightforward manner. After all, as shown in Section 1, common factors alone can account for a major part of empirically estimated default correlation. This suggests that the inclusion of non-trivial $X_t$ is a necessity.
For the purpose of managing the expected default loss on a fairly diverse credit-risky portfolio, one can envision a rating-specific intensity process similar to those in Duffee (1999) and Driessen (2002), with common factors capturing economy-wide variations in default risk. Augmenting this standard setup with industry-wide effects, one can combine a state-dependent intensity with the previous example.

For example, as a toy model one can specify the default intensity of an issuer as

$$\lambda_{l,k}^t = a_l + b_l F_t + \delta_k 1\{t > \tau_{k,F}^t\},$$

(38)

where \(l\) and \(k\) are, respectively, the credit rating and the industry affiliation of the issuer, \(F_t\) is a common factor, and \(\tau_{k,F}^t\) is the first-to-default time within industry \(k\). This form of the intensity implies that defaults within the same industry are correlated through two channels: 1) dependence on the common factor; 2) spillover effect from the first-to-default. With empirically motivated choices of the coefficients, one can study what percentage of the default correlation is attributed to each source.

For simplicity, I ignore credit rating and focus on a single industry. To simulate the default times, I set an overall horizon of, say, \(N = 100\) years, and choose an interval of one year for discretization. Following the total hazard construction, I first generate a discrete-time sample path of length \(N\) for the process \(F_t\), summarized by \(\{F_j\}_{0 \leq j \leq 100}\). Then, for each independent unit exponential \(E_i\), I find the integer \(n_i\) such that

$$\sum_{j=0}^{n_i-2} (a + bF_j) < E_i \leq \sum_{j=0}^{n_i-1} (a + bF_j).$$

(39)

The smallest such integer \(n = \min(n_1, \ldots, n_I)\) is defined as the first-to-default time. In this procedure, a simple predictable process is used to approximate \(F_t\), thus turning the integral in the definition of the total hazard into a summation, resulting in equation (39).

Having identified the first-to-default, one can show that the default time \(\tau_i\) can be determined by choosing an integer \(m_i\) such that

$$\sum_{j=n}^{m_i-2} (a + bF_j + \delta) < E_i - \sum_{j=0}^{n-1} (a + bF_j) \leq \sum_{j=n}^{m_i-1} (a + bF_j + \delta),$$

(40)

and then setting \(\tau_i\) to \(m_i\). If nothing less than 100 can be found to satisfy this inequality, then I record the respective default time as being greater than 100 years. I repeat the above procedure
many times in order to generate the marginal distribution and default correlation functions over the domain $[0, 100]$.

As an illustration of this procedure, assume that there are 10 firms in the industry. I take $F_t$ to be the first common factor in Driessen’s model. This is a square-root diffusion with parameters $\kappa = 0.03$, $\theta = 0.005$, and $\sigma = 0.016$. In the base case I take $a = 0.004$ and $b = 5.707$, which correspond to Driessen’s Baa intensity parameters. In addition, I assume that $\delta = 0.002$. This represents roughly a five percent jump over the long-run mean of the default intensity.

Figure 8 presents a typical sample path of the default intensity $\lambda_t = a + bF_t + \delta 1_{\{t>\tau_F\}}$. In this particular case the first-to-default occurs in year 3, and the sample path henceforth splits into two, the dotted path corresponding to no jump and the solid one a jump of $\delta = 0.004$.

To study the effect of the common factor, I set $\delta = 0$ and change $b$ while maintaining the long-run mean of the default intensity at the base case value (which means adjusting the value of $a$ simultaneously). The results are presented in the upper panels of Figure 9. In the case where $b = 0$, I choose $a = 0.032535$. One can see that there is zero default correlation between the firms and that the marginal distribution reflects a constant intensity equal to $a$. As $b$ is increased, the survival (default) probability increases (decreases) slightly due to the effect of Jensen’s inequality. The elevation of default correlation, however, is much more noticeable.

In the lower panels of Figure 9, I also verify the impact of industry-wide contagion on the outputs of the model. Specifically, I move away from the base case by increasing $\delta$ from 0 to 0.004. As indicated by the plot, this increases default probability for all maturities. Default correlation, on the other hand, initially increases with $\delta$ but decreases with $\delta$ at longer maturities. This is because at longer maturities the contagion effect is sure to have kicked in, while the factor sensitivity of the intensity has not changed.

In all of the simulations I have assumed that the starting value of $F_t$ is its long-run mean of 0.0050. Each scenario involves 100 draws of standard normal random variables and 10 draws of unit exponential random variables. The results, for each set of model parameters, are obtained using one million independent scenarios. Although this may appear complex, in practice it usually takes less than a minute on a PC. Extensions of (38), including to multi-factor specifications, thus seem computationally feasible and may lend themselves to econometric estimations.

To end the numerical illustrations, I use the current setup to study the effect of default de-
pendency on the valuation of basket credit default swaps. To stay focused, I make a number of simplifying assumptions on the structure of the default swap. First, this is a digital swap on a portfolio of 30 names that pays one dollar at the maturity of the swap if there are \( n \) or more defaults before the swap expires. Second, the buyer of the swap pays \( c_n \) dollars at the initiation of the swap. A constant interest rate of 5 percent per annum is assumed, as is the absence of the swap seller’s credit risk. Given these assumptions, the rate for the \( n \)th-to-default swap is

\[
c_n = e^{-rT} E \left( 1_{\{\tau(n) \leq T\}} \right) = e^{-rT} Pr \left( \tau(n) \leq T \right),
\]

where \( T \) denotes the maturity of the swap and \( \tau(n) \) the \( n \)th default time. As the example addresses valuation, the expectation and the default intensity both pertain to the equivalent martingale measure.

Figure 10 demonstrates how the correlation structure can affect the valuation of basket default swaps. With the overall mean of the intensity fixed, an increase in the dependence on the common factor increases the probability that a large number of defaults are observed prior to the expiration of the swap. This effect is captured in the upper panel where the value of \( b \) is altered. A second way to inject more default correlation is to increase the parameter \( \delta \), which describes the extent of industry contagion. As seen in the lower panel, an increase in \( \delta \) raises the price of the basket default swap for all \( n \). The effect is relatively more important at larger \( n \).

### 3 Conclusion

I examine the properties of correlated defaults in intensity-based models. The study of default correlation is a research topic that has solicited a great deal of interest from both academic and industry perspectives. The purpose of this paper is to present a systematic approach to the problem, integrating existing methodologies while offering new insights and new results.

In the first part of this paper, I turn to the popular belief that reduced-form models based on conditional independence cannot generate empirically observed levels of default correlation. Besides noting that this is an unsubstantiated claim, I undertake the simple procedure of computing default correlations implied from existing empirical studies of intensity-based credit risk models. This exercise suggests that the root of the problem is an insufficient specification of the common factor structure of the default intensity. In fact, with just two common factors driving the intensity, the
standard approach implies too much, rather than too little, default correlation.

In the second part of the paper, I develop an intensity-based framework for describing arbitrary default dependency structures. This is motivated by the practical need to incorporate firm-specific information that may affect credit derivatives valuation and/or credit risk management decisions. The ease with which one can introduce such information into the default intensity is a feature that is superior to existing models of correlated defaults. With the ever increasing activity in the credit default swap market, the day when one can back out an “implied default dependency” is perhaps not too far from reality.

The general framework presented here is an extension of the Jarrow and Yu (2001) counterparty risk model. Using the so-called “total hazard construction” from the literature on reliability theory, I construct default times of arbitrary dependency starting from a set of independent unit exponential random variables. I then use simulation examples of progressive complexity to illustrate this approach and discuss its relation with the standard construction based on conditional independence. One such example, using the notion of first-to-default to capture industry-wide contagion or competitive effects, seems particularly relevant to practical applications.
References


Table 1: **Estimates from Duffee (1999).** The Treasury factors $s_1$ and $s_2$ and the firm-specific factors are specified through the triple $(κ, θ, σ)$. The intensity parameters are $α$, $β_1$ and $β_2$. The liquidity adjustment parameters are $a$ and $b$. $\overline{λ}_t$ denotes the sample mean of the firm-specific factor. $adj_n$ denotes the liquidity adjustment to the conditional default rate (CDR) at the $n$-year maturity. The firm-specific factor and the intensity function are specified for three rating categories, Aa, A and Baa.

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Panel A: Adjusted intensity

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Panel B: Unadjusted intensity

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Table 2: **Default correlation inferred from Duffee (1999).** Values are in percentages, given for three rating categories, Aa, A and Baa, and horizons of one, two, five and ten years. Panel A includes liquidity and tax adjustments, while Panel B does not.
Table 3: Estimates from Driessen (2002). The common factors $F_1$ and $F_2$ and the firm-specific factors are specified through the triple $(\kappa, \theta, \sigma)$. The intensity parameters are $\alpha$, $\gamma_1$ and $\gamma_2$. The liquidity adjustment parameters are $a$ and $b$. $\text{adj}_n$ denotes the liquidity and tax adjustment to the conditional default rate (CDR) at the $n$-year maturity. The firm-specific factor and the intensity function are specified for three rating categories, Aa, A and Baa.

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Panel A: Adjusted intensity

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Panel B: Unadjusted intensity

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Table 4: Default correlation inferred from Driessen (2002). Values are in percentages, given for three rating categories, Aa, A and Baa, and horizons of one, two, five and ten years. Panel A includes the liquidity and tax adjustments, while Panel B does not.
Panel A: Zhou (2001)

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Panel B: Lucas (1995)

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Table 5: Default correlations from Zhou (2001) and Lucas (1995).

Figure 1: Simulated versus theoretical marginal distributions for $\tau^1$ under different number of simulation scenarios. The dotted line is the theoretical distribution. The solid line is the simulated distribution. The parameter values are: $a = 0.01$, $a' = 0.0105$, $b = 0.02$, and $b' = 0.0210$. 

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Figure 2: Simulated versus theoretical default correlations using different number of simulation scenarios. The dotted line is the theoretical default correlation. The solid line is the simulated default correlation. The parameter values are: $a = 0.01$, $a' = 0.0105$, $b = 0.02$, and $b' = 0.0210$. 
Figure 3: Marginal distribution for $\tau_1$ and default correlation given different values for $a'$. The left hand side panels are theoretical values. The right hand side panels are simulated values using 1 million scenarios. The values of other parameters are: $a = 0.01$, $b = 0.02$, and $b' = 0.0210$. 
Figure 4: Marginal distribution for $\tau^1$ and default correlation given different values for $b$, fixing $b' - b$ at 0.001. The left hand side panels are theoretical values. The right hand side panels are simulated values using 1 million scenarios. The values of other parameters are: $a = 0.01$, $a' = 0.0105$, and $b' - b = 0.001$. 
Figure 5: Simulated marginal distribution and default correlation as the number of firms increases from 2 to 100. One million scenarios are used in the simulation. The parameters of the model are $a = b = 0.01$ and $a' = b' = 0.011$. 
Figure 6: Simulated marginal distribution of the default time of Firm 2 and the default correlation between Firm 1 and Firm 2. I assume that $I = 10$ firms, $a = 0.01$, and $a' = 0.011$. In addition, for the upper panels $b' - b = 0.001$, and $b = 0.01, 0.02$, or 0.04. For the lower panels I assume $b = 0.01$ and $b' - b = 0.001, 0.002$, or 0.004. I use one million simulation scenarios. By symmetry, $G_2(t)$ is the marginal distribution of the default times of Firm 2-10.
Figure 7: Marginal distributions of $\hat{\tau}_1$. Each “correct” series are based on one million simulation scenarios. The “incorrect” series are computed from equation (37). The number of firms, $I$, is equal to 2, 5, 10, or 100. The parameters of the model are: $a = b = 0.01$ and $a' = b' = 0.011$. 
Figure 8: A typical sample path of the intensity as specified in equation (38). The deviation of the solid path from the dotted path is due to a jump of magnitude 0.004, triggered by the first-to-default at year 3. Except for $\delta$, the parameters are their base case values. The discrete-time sample path has an interval of one year.
Figure 9: The effect of the common factor and industry-wide contagion on the marginal distribution and default correlation functions. In the upper panels I vary the value of $b$ while keeping the long-run mean of the intensity at 0.032535. In the lower panels I change the value of $\delta$. The parameters are otherwise equal to their base case values. To generate the plots I use one million simulation scenarios.
Figure 10: **The effect of the common factor and industry-wide contagion on the prices of basket credit default swaps.** In the upper panels I vary the value of $b$ while keeping the long-run mean of the intensity at 0.032535. In the lower panels I change the value of $\delta$. The parameters are otherwise equal to their base case values. To generate the plots I use one million simulation scenarios.