Asymptotics and calibration of local volatility models

H Berestycki\(^1\), J Busca\(^2\) and I Florent\(^3\)

\(^1\) CAMS, Ecole des Hautes Études en Sciences Sociales, 54 bd Raspail, 75270 Paris Cedex 06, France
\(^2\) Ceremade, Université Paris Dauphine, Pl. Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France
\(^3\) HSBC-CCF, 103 av. des Champs-Elysées, 75419 Paris Cedex 08, France

E-mail: hb@ehess.fr, busca@ceremade.dauphine.fr and igor.florent@ccf.com

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Abstract

We derive a direct link between local and implied volatilities in the form of a quasilinear degenerate parabolic partial differential equation. Using this equation we establish closed-form asymptotic formulae for the implied volatility near expiry as well as for deep in- and out-of-the-money options. This in turn leads us to propose a new formulation near expiry of the calibration problem for the local volatility model, which we show to be well posed.

In the Black–Scholes–Merton model \([4, 24]\), it is assumed that the price of a non-dividend paying stock \(S_t\) follows the log-normal stochastic differential equation

\[
\text{d}S_t = S_t(\mu \text{d}t + \sigma \text{d}W_t),
\]

where \(t\) is time, \(\mu\) and \(\sigma\) are constants and \(W_t\) is a standard Brownian motion. The parameter \(\sigma\) is called the volatility of the stock \(S_t\). It is well-known that the price \(C(S_t, t; K, T)\) of a European call option written on \(S_t\) with strike \(K\) and maturity \(T\) satisfies the linear parabolic partial differential equation

\[
\frac{\partial C}{\partial t} + \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rSCS - rC = 0 \quad \text{in } (0, +\infty) \times (0, T)
\]

\[
C(S, T) = (S - K)_+,
\]

where \(r\) is the risk-free short-term interest rate. Such options are commonly traded on markets, however \(\sigma\) is not directly observable. Therefore it is common practice to start from the observed prices and invert the closed-form solution to (2) in order to find that constant \(\sigma\)—called implied volatility—for which the solution to (2) agrees with the market price at today’s value of the stock. It is widely observed that calls having different strikes and otherwise identical have different implied volatilities. This phenomenon, usually referred to as the smile effect, clearly violates the Black–Scholes–Merton model, since in this framework a constant \(\sigma\) is supposed to determine the dynamics of the underlying stock \(S_t\) through (1) regardless of options, strikes and maturities.

To overcome this difficulty, the model has to be extended. One widely used approach is to consider that the volatility \(\sigma\) also follows a stochastic diffusion process. Another case is when the volatility \(\sigma\) is not a constant any more but rather a (deterministic) function of the underlying asset and the time. Actually this can be seen as a particular case of the previous approach. These types of models are called local volatility models. For them, the dynamics of the underlying asset is governed by the stochastic differential equation

\[
\text{d}S_t = \mu S_t \text{d}t + \sigma(S_t, t)S_t \text{d}W_t,
\]

where \(\mu\) is the drift term, \(\sigma(S_t, t)\) is the volatility function, and \(W_t\) is a standard Brownian motion.
computationally difficult, especially near expiry or far from the money.

Secondly, one wants to recover the value of the parameter of the model from market data—the calibration problem. One remarkable result, due to Dupire [13], states the following: should the call options corresponding to all possible strikes and maturities be priced on the market in a consistent manner, the local volatility \( \sigma(S, t) \) would be uniquely determined by the relation

\[
(Dupire’s \ formula) \quad \sigma(K,T) = \sqrt{\frac{C_T + r K C_K}{K^2 C_{KK}}},
\]

where \( C_K, C_{KK} \) are the first- and second-order derivatives of the call price with respect to its strike and \( C_T \) the derivative with respect to its maturity. For the reader’s convenience, we present in the appendix a proof of this result using PDE methods.

It turns out that in practice, this approach has two severe shortcomings. Firstly, there is but a finite set of observations. Hence some interpolation is needed in order to use (4). It is nowadays generally acknowledged (see for instance [8, 26]) that it is by no means obvious to figure out how to interpolate the data set in such a way that the radicand in (4) remains positive and finite. Further, the result is overly sensitive to the (arbitrary) choice of the interpolation—especially for short maturities—this resulting in poor robustness of the method.

Secondly, a (related) difficulty lies in the intrinsic indeterminacy of formula (4) in the regions \( \{T - t \ll 1\}, \{\ln(S_t/K)\gg 1\}, \{T - t \gg 1\} \), where it assumes the form \( 0 \).

This is the reason why several other approaches have been proposed in the recent years. We shall not attempt to give any comprehensive survey of this broad subject, but rather focus on the results that are the closest to our point of view. Let us mention in this respect the approach by Lagnado and Osher [20], further extended by Jackson et al [18] and Berestycki and Crépey [3] (see also [9]). The basic idea is to introduce a regularized cost functional in the form

\[
J(\sigma) = \varepsilon \int |\nabla \sigma|^2 + \sum_{i,j} ((C - C^*)(S_0, t_0; K_i, T_j))^2,
\]

where \( C^* \) are observed prices, to be minimized over an appropriate functional space. Here \( \varepsilon \) represents the trade-off between accuracy and smoothness of the minimizer, this follows the lines of Tychonov’s method [28].

Avellaneda et al [1] and Bodurtha and Jermakyan [5] also propose other interesting minimization-like methods that we do not detail here. Let us mention finally the works by Bouchouev and Isakov [6, 7] who propose a different approach based on a representation formula that results in an integral equation for the local volatility.

The purpose of this paper is to propose a completely new point of view. We intend to show that there exists an explicit link between the implied and the local volatilities, in the form of a quasilinear degenerate parabolic PDE. We then examine several consequences that we can derive from this equation. Firstly, a particularly important one is an asymptotic formula for the implied volatility near expiry. Furthermore, using this limit, we then show that some new formulation of the calibration problem is well posed. Another use of the equation allows us to give asymptotics of the implied volatility for deep out-of-the-money options. Most of the results stated here have been announced in [2].

In a broader perspective we also hope that this nonlinear PDE approach will prove useful for the challenging problem of calibrating the local volatility and understanding its qualitative properties.

We adopt throughout the paper the reduced variables 
\( x = \ln(S/K) + r t, \quad \tau = T - t, \) so that from (2) the transformed prices

\[
v(x, \tau) = e^{\tau} C(S, T - \tau; K, T)/K
\]

formally satisfy

\[
v_t + \frac{1}{2} \sigma^2(x, \tau)(v_{xx} - v_x) \quad \text{in} \quad \Omega_T = \mathbb{R} \times (0, T),
\]

abusing somewhat the notation (see for instance [5]). We shall assume that

\[
\sigma \in \text{BUC}(\Omega_T), \quad 0 < \varepsilon \leqslant \sigma(x, \tau) \leqslant \bar{\sigma} < \infty,
\]

BUC being the space of (globally) bounded uniformly continuous functions and \( \varepsilon, \bar{\sigma} \) are constants. In order to make precise statements let us now specify the technical conditions that we impose, and state some definitions of functional spaces that we shall need.

We require that \( v \in C(\Omega_T) \cap W^{2,1, p}(\Omega), \Omega = \mathbb{R} \times (0, +\infty) \), satisfy the equation in (6) pointwise almost everywhere in \( \Omega \) (strong solution). As is classical when studying parabolic problems, we make use of the following anisotropic Sobolev spaces:

\[
W^{2,1, p}(\Omega) = \left\{ w \left| \int_\Omega |w_{xx}|^p + |w_x|^p + |w|^p < \infty \right\}
\]

\[
W^{2,0, \infty}(\Omega) = \{w||w||_{\infty} + |w_x|_{\infty} < \infty\}
\]

endowed with their natural norms, together with their local versions \( W^{2,1, p}_{loc} \) and \( W^{2,0, \infty}_{loc} \).

Let us denote by \( u \) the solution to (6) corresponding to \( \sigma \equiv 1 \), i.e. satisfying

\[
u_t = \frac{1}{2} (u_{xx} - u_x) \quad \text{in} \quad \Omega,
\]

\[
u(x, 0) = (e^\varepsilon - 1)_+.
\]

The explicit solution to (9) is readily seen to be

\[
u(x, \tau) = e^{\tau}N \left( \frac{x}{\sqrt{\tau}} + \frac{1}{2} \sqrt{\tau} \right) - N \left( \frac{x}{\sqrt{\tau}} - \frac{1}{2} \sqrt{\tau} \right),
\]

where

\[
N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-\frac{y^2}{2}} \, dy.
\]
Proposition 1 (existence). Under assumption (7), there exists a unique solution \( v \) in the class
\[
W^{2,1,p}(\Omega) \cap C(\overline{\Omega}) \cap \{ w \mid \exists C, \beta > 0, \forall (x, \tau) \in \Omega \mid |w(x, \tau)| \leq Ce^{\beta x^2} \}
\]
for all \( 1 < p < \infty \) to
\[
v_t - \frac{1}{2} \sigma^2(x, \tau) (v_{xx} + v_x) = 0 \quad \text{a.e. in} \quad \Omega = \mathbb{R} \times (0, +\infty),
\]
\[
v(x, 0) = (e^x - 1)_+.
\]

Furthermore, \( v \) satisfies
\[
0 < v(x, \tau) < e^x \quad \forall (x, \tau) \in \Omega, \quad v_x(\tau, \tau) > 0 \quad \text{in} \quad \Omega_T.
\]

A trivial yet crucial observation is that \((x, \tau) \rightarrow u(x, \tau(\lambda)) \) (\( \lambda > 0 \)) satisfies (12) with \( \sigma \equiv \sqrt{\lambda} \). This implies that if one defines \( \varphi \) such that
\[
\varphi(x, \tau) = u(x, \tau \varphi^2(x, \tau)),
\]
for all \((x, \tau) \in \Omega\), then \( \varphi \) is clearly the implied volatility of the corresponding call option, in the sense described above. That (14) uniquely determines \( \varphi(x, \tau) \geq 0 \) follows easily from the fact that \((e^x - 1)_+ \leq v(x, \tau) < e^x \) (see (13) above), \( u(x, 0) = (e^x - 1)_+ \), \( \lim_{\tau \to \infty} u(x, \tau) = e^x \) (see (10)), together with \( u_T > 0 \) in \( \Omega \).

Our main result gives the implied volatility as the unique solution to a well-posed degenerate quasilinear parabolic problem.

Theorem 1. Under assumption (7) suppose that the implied volatility \( \varphi \) is defined by (14) where \( v \) and \( u \) are solutions to (12) and (9) respectively. Then:

(i) The implied volatility \( \varphi \) lies in \( W^{2,1,p}(\Omega) \) for all \( 1 < p < \infty \) and satisfies
\[
2 \tau \varphi \varphi_T + \varphi^2 - \sigma^2(x, \tau) \left( \frac{1 - x \varphi}{\varphi} \right)^2 - \sigma^2(x, \tau) \tau \varphi \varphi_x + \frac{1}{2} \sigma^2(x, \tau) \tau^2 \varphi^2_x = 0
\]
a.e. in \( \Omega \).

(ii) In the limit \( \tau \to 0 \), the implied volatility is the harmonic mean of the local volatility, namely
\[
\lim_{\tau \to 0} \frac{1}{\varphi(x, \tau)} = \int_0^1 \frac{ds}{\sigma(sx, 0)}
\]
uniformly in \( x \in \mathbb{R} \).

(iii) Conversely, if \( \varphi \in W^{2,1,p}(\Omega) \) (for some \( p > 1 \)) satisfies (15) and (16) then \( \varphi \equiv \varphi \).

Remark. The small time-to-maturity asymptotic in theorem 1 part (ii) is an interesting connection with a classical result by Varadhan [29], although it does not readily follow from it. In [29] the fundamental solution of the operator \( \partial_t - \frac{1}{2} \sigma^2(x) \partial^2_x \), is considered, namely
\[
v_t = \frac{1}{2} \sigma^2(x) v_{xx}
\]
\[
v(x, 0) = \delta(x)
\]
(here and throughout the paper \( \delta(x) \) is the Dirac mass at the origin), with \( \sigma \) satisfying (7). It is then proved that
\[
-2\tau \ln v(x, \tau) \to \left( \int_0^1 \frac{ds}{\sigma(s)} \right)^2
\]
as \( \tau \to 0 \). This corresponds to the Riemannian metric associated with the inverse of the diffusion coefficient. Denoting by \( U \) the fundamental solution of \( \partial_t - \frac{1}{2} \partial^2_x \), namely \( U(x, \tau) = (2\pi \tau)^{-1/2} \exp \left( -\frac{x^2}{2\tau} \right) \), we can rephrase the result as follows. If \( \phi \) is such that \( v(x, \tau) = U(x, \tau \phi(x, \tau)^2) \), noting that by (7) \( \phi \) is bounded, we deduce that
\[
\frac{1}{\phi(x, \tau)} \to \left( \int_0^1 \frac{ds}{\sigma(s)} \right)^{1/2}
\]
as \( \tau \to 0 \). Hence this problem gives rise to the same change of metric as in theorem 1 (ii). However, it is not straightforward to derive from (18) the asymptotics in theorem 1 (ii), since we are dealing with different initial conditions and composition with a different function.

We also refer the reader to [14] for other large-deviation results.

We next establish asymptotics for deep in- and out-of-the-money options.

Theorem 2. Suppose \( \sigma \) satisfies (7) and
\[
\lim_{\tau \to \infty} \sigma(x, \tau) = \sigma_+(\tau) \quad \text{respectively} \quad x \to -\infty, \sigma_-(\tau) \quad \text{locally uniformly in} \ \tau, \quad \text{with} \ \sigma_+ \ \text{continuous}.
\]

Then
\[
\lim_{x \to \pm \infty} \varphi(x, \tau) = \left( \frac{1}{\tau} \int_0^\tau \sigma_+^2(s) \frac{ds}{\sigma(s)} \right)^{1/2}.
\]

We point out that theorem 1 (ii) and theorem 2 clarify in particular the indeterminacy in Dupire’s formula in the regions \( \{ T - t \ll 1 \} \) and \( \{ |\ln(S_t/K)| \gg 1 \} \) that we have mentioned earlier.

It results from theorem 1 (ii) that \( \varphi \) can be extended up to \( \tau = 0 \) as a continuous function. That the limit of \( \varphi(x, \tau) \) at \( \tau = 0 \) exists at all is by no means obvious from (14).

As a matter of fact, the value of the limit is quite specific to the problem under consideration, as the following proposition makes clear. Specifically, if the payoff function is strictly convex in the initial variable \( S \), then the asymptotic behaviour is dominated by the local volatility.

Proposition 2. If one replaces \((e^x - 1)_+ \) in (9) by any smooth function \( \xi(x) \) satisfying \( \xi_x - \xi > 0, \) that is, \( \xi_{SS} > 0, \) then
\[
\lim_{x \to 0^\pm} \varphi(x, \tau) = \sigma(x, 0).
\]

This result is actually much simpler than theorem 1. The main reason is that here \( u_T > 0 \) throughout \( \overline{\Omega}_T \), so that the implicit function theorem applied to (14) gives bounds for \( \phi \) together with its derivatives near \( \tau = 0 \). In other words, it is the very degeneracy of the original problem near \( \tau = 0 \) that allows such an ‘instantaneous averaging’ as that in (16) to take place.

A important feature of (16) is the following. Suppose that for \( x \neq 0 \) \( \sigma \) vanishes on some interval \( \omega \) between 0 and \( x \). Then \( \varphi(x, 0) = 0 \). This is consistent with the probabilistic
point of view. Indeed, in this limiting case the stock price process starting at \( x \) will never cross \( a \) and so never reaches the convexity region \( (x = 0) \). This feature was absent from other approximation formulae that were proposed, like weighted linear means (see for instance [10] “the poor man’s model”).

We refer to the book by Rebonato [26] for a very interesting and detailed discussion on the qualitative properties of local and implied volatilities.

The asymptotics in theorem 1 (ii) exhibits a linear relation between the inverse of the local and implied volatilities. This leads us to propose a new regularization of the calibration problem. Instead of \( (5) \) we introduce the following penalized functional:

\[
J^\varepsilon(\sigma) = \varepsilon \int \left| \nabla \left( \frac{1}{\sigma} \right) \right|^2 + \sum_{i,j} \left( \frac{1}{\phi} - \frac{1}{\phi^*} \right) (x_i, \tau_j)^2, \tag{20}
\]

where \( \phi^* \) are observed implied volatilities, to be minimized over a suitable functional space. We suspect that this minimization problem is well posed, at least for short time-over a suitable functional space. We refer to the book by Rebonato [26] for a very interesting and detailed discussion on the qualitative properties of local and implied volatilities.

\[
\phi^*(x) = \frac{1}{2}. \tag{21}
\]

We assume that these volatilities are consistent, i.e. that there exists \( \sigma_0(x, \tau) \) for which the solution to \( (12) \) with \( \sigma(x) = \sigma_0(x, 0) \) asymptotically replicates market prices, i.e. such that \( \lim_{\tau \to 0} \phi(x, \tau) = \phi(x, \tau) \equiv \phi^*(x) \). It follows that \( J^\varepsilon(\xi_0)_{\varepsilon=0} = 0 \), with \( \xi_0 = \sigma_0^{-1} \). This means that, by assumption, we have a solution to the exact asymptotic calibration problem. As a consequence, there are in fact infinitely many of them, as can easily be seen from the argument in the proof of theorem 3 below. The whole point is to choose one of these solutions in a stable way. This is question that the following result addresses.

**Theorem 3.**

(i) For all \( \varepsilon > 0 \) there exists a unique solution of the minimization problem

\[
\inf_{\xi \in H^2(\mathbb{R})} J^\varepsilon(\xi), \tag{22}
\]

denoted by \( \xi^\varepsilon \).

(ii) When \( \varepsilon \to 0 \), \( \xi^\varepsilon \) converges uniformly in \( \mathbb{R} \) to a solution \( \hat{\xi} \) of the exact asymptotic calibration problem, i.e. such that

\[
\hat{\xi}(x, 0) = \int_0^1 \xi(x_s) \, dx = \xi^*(x). \tag{23}
\]

1. **Proofs.**

In the proof of theorem 1 we shall need the following series of lemmas.

**Lemma 4 (maximum principle).** Suppose \( a, b, c \) are globally bounded continuous coefficients defined in \( \Omega_T \), and assume that \( a(x, \tau) \geq \alpha > 0 \). Suppose that

\[
z \in W^{2,1,\nu}_{\text{loc}}(\Omega_T) \cap C(\partial \Omega_T) \text{ satisfies}
\]

\[
z_t - a(x, \tau)z_{xx} + b(x, \tau)z_x + c(x, \tau)z \geq 0 \quad \text{a.e. in } \Omega_T
\]

\[
\lim_{\tau \to 0} z(x, \tau) \geq 0 \quad \text{in } D'(\mathbb{R})
\]

\[
z(x, \tau) \geq -Ce^{\beta(x^2)} \quad \text{for some } \beta, C > 0.
\]

Then \( z \geq 0 \) in \( \Omega_T \).

**Proof.** This is a straightforward adaptation of (a particular case of) theorem 9, chapter 2, section 4 in [15] to the strong solutions framework and to the distributional initial data.

**Lemma 5.** For any \( \psi \in W^{2,1,\nu}_{\text{loc}}(\Omega_T) \) define

\[
w(x, \tau) = u(x, \tau) \psi^2(x, \tau) \]

where

\[
u(x, \tau) = u(x, \tau)^2 \psi^2(x, \tau) + \psi^2 - \frac{\alpha^2}{4} \left( 1 - x \frac{\psi}{\psi} \right)^2.
\]

**Proof.** A straightforward computation gives

\[
w_t - \frac{1}{2} \sigma^2(w_{xx} - w_x) = u_t(x, \tau \psi^2(x, \tau) \left( 2 \tau \psi \psi_t + \psi^2 - \frac{\alpha^2}{4} \left( 2 - 2 \tau \psi \psi_x + 2 \tau (\psi_x^2 + \psi_{xx}^2)
\right.
\]

\[
+ 4 \tau \psi \psi_x \frac{u_t}{u_x} (x, \tau \psi^2) + 4 \tau^2 \psi^2 \psi_x^2 \frac{u_t}{u_x} (x, \tau \psi^2) \right)
\]

Besides, one easily derives from \( (10) \) the relations

\[
\frac{u_x}{u_t}(x, \xi) = \frac{1}{2} - \frac{x}{\xi}
\]

\[
\frac{u_{xx}}{u_t}(x, \xi) = \frac{1}{2\xi} + \frac{x^2}{2\xi^2} - \frac{1}{8}
\]

Combining the above identities gives the result.

**Lemma 5 applied to \( w \equiv v \), hence to \( \psi \equiv \psi \), yields statement (i) in theorem 1.**

An important tool in the subsequent argument is the fact that this problem satisfies a certain comparison principle that
we now state. For this purpose, let us denote by \( H \) the quasilinear operator
\[
H[\psi] = H(x, \tau, \psi, D\psi, D^2\psi) = \left(1 - \frac{\psi_x}{\psi}\right)^2 + \tau \psi_{xx} - \frac{1}{2} \tau^2 \psi_x^2 \psi_z^2,
\]
and define \( I(0, T) \) to be the class of those functions \( \psi \in C^{2,1}(\Omega) \) for which the "associated local volatility"
\[
\sigma[\psi](x, \tau) = \left(\frac{\tau^2 \psi_x^2}{H(x, \tau, \psi, D\psi, D^2\psi)}\right)^{1/2}
\]
is well defined, continuous in \( \Omega_T \), and satisfies there
\[
\sigma \leq \sigma[\psi](x, \tau) \leq \sigma;
\]
and define \( \Omega_{\psi} \) as the set of those functions \( \psi \in C^{2,1}(\Omega) \) for which
\[
\sigma \leq \sigma[\psi](x, \tau) \leq \sigma.
\]
Furthermore, we require the growth condition at zero
\[
\tau \psi^2(x, \tau) \to 0 \quad \text{as} \quad \tau \to 0.
\]
We have the following useful result.

**Lemma 6 (comparison principle).** Given \( \psi, \tilde{\psi} \in I(0, T) \), suppose that
\[
\sigma[\psi](x, \tau) \leq \sigma[\tilde{\psi}](x, \tau)
\]
for all \( (x, \tau) \in \Omega_T = \mathbb{R} \times (0, T) \). Then \( \psi \leq \tilde{\psi} \) there.

Note that, due to the degeneracy of (25), it is not necessary to prescribe any initial ordering condition on \( \psi, \tilde{\psi} \).

**Proof.** Let us define \( u(x, \tau) = u(x, \tau \psi^2(x, \tau)) \) and \( \bar{u}(x, \tau) = u(x, \tau \tilde{\psi}^2(x, \tau)) \) in \( \Omega_T \). By (5), (25) and (27), (29), (30), the functions \( u, \bar{u} \) satisfy
\[
u_x - \frac{1}{\sigma[\psi]} \bar{u}^2 \tau \psi_x (\bar{u}_{xx} - u_x) = 0 \quad \text{in} \quad \Omega_T
\]
and
\[
u_x - \frac{1}{\sigma[\tilde{\psi}]} \bar{u}^2 \tau \psi_x (\bar{u}_{xx} - u_x) = 0 \quad \text{in} \quad \Omega_T
\]
The difference function \( w(x, \tau) = \bar{u}(x, \tau) - u(x, \tau) \) then satisfies
\[
w_x - \frac{1}{\sigma[\tilde{\psi}]} \tau \psi_x (w_{xx} - w_x) = \left(\frac{\sigma[\tilde{\psi}]}{\sigma[\psi]} - 1\right) \bar{u}_x \quad \text{in} \quad \Omega_T
\]
and
\[
w_x - \frac{1}{\sigma[\psi]} \tau \psi_x (w_{xx} - w_x) \quad w(x, 0) = 0.
\]
By (13) and (31), the right-hand side is non-negative. It then follows from the maximum principle (lemma 4) that \( w \geq 0 \) in \( \Omega_T \). Since \( \bar{u}_x > 0 \) this implies \( \psi \leq \tilde{\psi} \) in \( \Omega_T \).

**Proof of theorem 1 (ii).** Note that the existence of the limit is by no means obvious from (15) since this equation degenerates near \( \tau = 0 \). This implies loss of a priori estimates for \( \psi \) in this region. To circumvent this serious difficulty, the essential idea is to define suitable sub and supersolutions of (12) from the formal limiting solution of (15) and prove that actual convergence takes place through the comparison principle.

For the sake of clarity we first treat the case that \( \sigma \) satisfies the additional regularity assumption
\[
\sigma \in C^{2,1}(\Omega_T) \quad \text{and} \quad \sigma_{xx} \in L^\infty(\Omega_T).
\]

We next define
\[
\bar{v}(x, \tau) = \psi^0(x)(1 + \kappa \tau),
\]
and, respectively, \( \phi(x, \tau) = \psi^0(x)(1 - \kappa \tau) \) in \( \Omega_T \), for \( \kappa > 0 \), \( T < 1/\kappa \). Let us compute the associated local volatilities \( \sigma[\psi], \sigma[\tilde{\psi}] \) in the sense of lemma 6, see (28). A simple computation yields
\[
H[\bar{v}] = \left(1 - \frac{\psi_0^2}{\bar{v}}\right)^2 + \tau \psi_0^2 \psi_x^2 + O(\tau^2).
\]
It follows from here and (28) that
\[
\sigma[\bar{v}](x, \tau) = \sigma(x, \tau)
\]
and
\[
\sigma[\bar{v}](x, \tau) = \sigma(x, \tau)
\]
for all \( (x, \tau) \in \Omega_T \). Hence, taking first \( \kappa \) large enough and then \( \delta \) small enough we have \( \psi \in I(0, \delta) \) with \( \sigma[\psi] \geq \sigma \in \Omega_\delta \). Similarly, changing \( \kappa \) by \( -\kappa \), one sees that \( \psi \in I(0, \delta) \) and \( \sigma[\psi] \leq \sigma \in \Omega_\delta \). Clearly \( \psi \in I(0, \delta) \) as well (note that (30) comes from the initial condition \( v(x, 0) = (e^x - 1)_+ \), with \( \sigma[\psi] \equiv \sigma \), this resulting from uniqueness (see lemma 4). An application of the comparison principle (lemma 6) then yields
\[
\phi(x, \tau) \leq \psi(x, \tau) \leq \bar{v}(x, \tau)
\]
for all \( (x, \tau) \in \Omega_T \). In particular, the desired result (ii) in theorem 1 holds, in the case (35) is met.

We finally remove assumption (35) by an approximation procedure. Let us first observe that assumption (7) gives the existence of a sequence \( \sigma^\epsilon \) satisfying \( \sigma^\epsilon \in C^{2,1}(\Omega_{\delta/2}) \) for any \( \alpha \in (0, 1), \sigma^\epsilon \geq \sigma, \sigma^\epsilon \to \sigma \) uniformly in \( \Omega_{\delta/2} \). Indeed, take \( \sigma(x, \tau) = \sigma(x, \tau) \) for all \( \tau \geq 0 \) as an extension of \( \sigma \) and define for any \( \eta > 0 \) \( \rho^\epsilon \) (\( \eta \) to be chosen) a standard mollifier (see (73)). Define then \( \sigma^\epsilon \) by
\[
\sigma^\epsilon = \rho^\epsilon \ast (\sigma + \epsilon).
\]
Now clearly
\[ -\omega(\eta) + \varepsilon \leq \sigma^\varepsilon - \sigma \leq \omega(\eta) + \varepsilon \]  
(43)
in \Omega_{3/2}, if \( \omega \) denotes a modulus of uniform continuity for \( \sigma \) throughout \( \Omega_b \). We can then choose \( \eta = \eta(\varepsilon) \) in such a way that \( \omega(\eta) < \varepsilon \), so that \( \sigma^\varepsilon \geq \sigma \) in \( \Omega_{3/2} \) and \( \sigma^\varepsilon \rightarrow \sigma \) uniformly in \( \Omega_{3/2} \).

Let us now define \( \psi^0(x) \) as the harmonic mean of \( \sigma^\varepsilon(x,0) \) i.e.
\[ \psi^0(x) = \left( \int_0^1 \frac{ds}{\sigma^\varepsilon(sx,0)} \right)^{-1} \]  
(44)
and, correspondingly, \( \overline{\psi}(x,\tau) = \psi^0(x)(1+\kappa \tau) \) together with \( v^\varepsilon(x,\tau) = u(x,\tau \overline{\psi}(x,\tau)^2) \).

Repeating the computations in the first step, we get that \( v^\varepsilon \in \mathcal{I}(0,\delta) \) and
\[ \sigma[\overline{\psi}^\varepsilon] \geq \sigma^\varepsilon \geq \sigma \]  
(45)
in \( \Omega_3 \) by fixing \( \kappa = \kappa(\varepsilon) \) large enough, and \( \delta = \delta(\varepsilon) \) small enough. Lemma 6 thus implies \( \varphi \leq \overline{\psi}^\varepsilon \in \Omega_f \). Similarly, one constructs \( \varphi \leq \sigma, \psi^0, \psi^\varepsilon = (1-\kappa \tau) \), and shows that \( \varphi(x,\tau) \leq \varphi(x,\tau) \) in \( \Omega_3 \).

Summing up, we get that for all \( \varepsilon > 0 \) there exists \( (x,\delta) = (x,\delta)(\varepsilon) \) such that
\[ \left( \int_0^1 \frac{ds}{\sigma^\varepsilon(sx,0)} \right)^{-1} (1-\kappa \tau) \leq \varphi(x,\tau) \leq \left( \int_0^1 \frac{ds}{\sigma^2(sx,0)} \right)^{-1} (1+\kappa \tau) \]  
(46)
for all \( (x,\tau) \in \mathbb{R} \times (0,\delta) \). This yields
\[ \left( \int_0^1 \frac{ds}{\sigma^\varepsilon(sx,0)} \right)^{-1} \leq \liminf_{\tau \to 0} \varphi(x,\tau) \leq \limsup_{\tau \to 0} \varphi(x,\tau) \leq \left( \int_0^1 \frac{ds}{\sigma^2(sx,0)} \right)^{-1} \]  
(47)
for all \( \varepsilon > 0 \), so that
\[ \lim_{\tau \to 0} \varphi(x,\tau) = \left( \int_0^1 \frac{ds}{\sigma(sx,0)} \right)^{-1}. \]  
(48)
That this limit is uniform in \( x \in \mathbb{R} \) is easily seen through (46).

Proof of proposition 2. We apply identity (26) to \( u = v \) solution of (6), hence to \( \psi = \varphi \), the implied volatility. In this case, \( u_\varepsilon(x,0) = \frac{1}{2}(u_{xx}(x,0) - u_{\tau\tau}(x,0)) = \frac{1}{2}(\xi_{xx} - \xi_\tau) > 0 \), so that the terms \( \frac{\xi_{xx}}{\psi} \) and \( \frac{\xi_\tau}{\overline{\psi}} \) in (26) remain bounded as \( \tau \to 0 \). Furthermore, applying the implicit function theorem to (14), and using \( u_\varepsilon \geq \alpha > 0 \), we get that \( \phi, \psi, \varphi \), \( \psi_\tau \) remain bounded as \( \tau \to 0 \). Hence, sending \( \tau \) to 0 in (26) clearly yields \( \psi(x,0) = \sigma(x,0) \).

Proof of theorem 2. By an obvious symmetry in the argument, we treat only the case \( x \to +\infty \). As in the proof above, we shall construct for a given \( T > 0 \) a sub- and supersolution \( \overline{\psi}, \underline{\psi} \in \mathcal{I}(0,T) \) that have the required behaviour at infinity. To this purpose, we shall need auxiliary functions whose relevant properties are summarized in the following lemmas. We refer the reader to the appendix for the proof.

Lemma 7. Given \( A > 0, \eta \in (0,0.1), \kappa > 1/(1-\eta) \), there exists \( \overline{\psi} \in C^2(\mathbb{R}) \) satisfying the following properties:

(i) \( \overline{\psi} \in W^{2,\infty}(\mathbb{R}) \) with \( \|\overline{\psi}\|_{W^{2,\infty}} \) independent of \( A \)

(ii) \( \overline{\psi}(z) \geq \frac{1}{\kappa} \lim_{z \to -\infty} \overline{\psi}(\xi) \forall z \in \mathbb{R} \)

(iii) \( \overline{\psi}(z) \geq 2\kappa \forall z \in (-\infty, A) \)

(iv) \( \left[ \frac{\varphi(z)}{\overline{\psi}(z)} \right] \leq \frac{1}{\kappa} \overline{\psi} \in \mathbb{R} \)

(v) \( \frac{\varphi(z)}{\overline{\psi}(z)} \to 0, \overline{\psi}(z) \to 0, \frac{\varphi(z)}{\overline{\psi}(z)} \to 0 \) as \( z \to +\infty \).

By assumption, \( \sigma(x,\tau) \to \sigma^\varepsilon(x,\tau) \) as \( x \to +\infty \), uniformly in \( \tau \in (0, T) \). Hence, given \( \eta \in (0,0.1) \) there exists \( A \) such that
\[ \frac{\sigma^\varepsilon(x,\tau)}{\sigma(x,\tau)} \geq \sqrt{1-\eta} \forall (x,\tau) \in (A, +\infty) \times (0, T). \]  
(49)
Besides, using (7) we get \( \kappa > 1 \) for which
\[ \frac{1}{\kappa} \leq \frac{\sigma^\varepsilon(x,\tau)}{\sigma(x,\tau)} \leq \kappa \]  
in \( \mathbb{R} \times (0, T) \).

Let us define the quadratic mean limiting volatility
\[ \Sigma^\varepsilon(\tau) = \left( \frac{1}{\tau} \int_0^\tau \sigma^2(s) \, ds \right)^{1/2} \]  
(51)
and, for \( \varepsilon > 0 \)
\[ \overline{\psi}(x,\tau) = \Sigma^\varepsilon(\tau) \overline{\psi}(\xi x), \]  
(52)
where \( \overline{\psi} \) is given in lemma 7, the values of \( \eta, \kappa > 0 \) being defined as above, and \( A, \varepsilon \) to be fixed. A simple computation yields
\[ H(\overline{\psi})(x,\tau) = \left( 1 - e^{-\varepsilon \overline{\psi}(\xi x)} \right)^2 + \tau \Sigma^2 \overline{\psi} \]  
(53)
Clearly by (iv) in lemma 7 we can choose \( \varepsilon = \varepsilon(\|\overline{\psi}\|_{W^{2,\infty}}, T, \Sigma^\varepsilon) > 0 \) independent of \( A \) such that
\[ \frac{1}{\kappa^2} \leq H(\overline{\psi}) \leq 2 \]  
(54)
in \( \Omega_T \); now by (v) in lemma 7 there is \( B > 0 \) for which 
\[ x \geq B, \; \tau \in (0, T) \] 
implies \( H[\varphi](x, \tau) < \frac{1}{1 - \eta} \). Setting 
\( A = \varepsilon \max(B, \bar{A}) \), we see that (54) holds for all \( x \) and 
\[ H[\varphi](x, \tau) < \frac{1}{1 - \eta} \] 
(55)

for all \( z = \varepsilon x \geq A, \tau \in (0, T) \).

We next compute the local volatility associated with \( \varphi \) in the sense of proposition 6, that is
\[ \sigma^2[\varphi](x, \tau) = \frac{(\tau \Sigma_e(\tau) \varepsilon \varphi(\varepsilon x)^2)}{H[\varphi](x, \tau)} \] 
(56)
\[ = \sigma^2_e(\tau) \frac{\varphi(\varepsilon x)^2}{H[\varphi](x, \tau)}. \] 
(57)

Since \( \varphi(\varepsilon x) \geq \frac{1}{1 - \eta} \) for \( z = \varepsilon x \geq A \) and \( \varphi(\varepsilon x) \geq 2\kappa \) for 
\( z \in (-\infty, A) \), it follows from (49), (50), (54), (55) that 
\[ \sigma(x, \tau) = \sigma[\varphi] \] 
(58)

for all \( (x, \tau) \in \mathbb{R} \times (0, T) \). By applying our comparison principle (lemma 6) we get that \( \varphi \leq \varphi \) in \( \Omega_T \). By (ii) in lemma 7 and (52) we get 
\[ \limsup_{x \to +\infty} \varphi(x, \tau) \leq \frac{1}{1 - \eta} \Sigma_e(\tau), \] 
(59)

for all \( \eta \in (0, 0.1) \); this yields \( \limsup_{x \to +\infty} \varphi(x, \tau) \leq \Sigma_e(\tau) \).

We finally claim that 
\[ \liminf_{x \to +\infty} \varphi(x, \tau) \geq \Sigma_e(\tau). \] 
(60)

To this aim, we resort to an appropriate subsolution \( \varphi \) of the problem. Since its construction is quite similar to that of \( \varphi \), the rest of the argument is only sketchy.

Given \( \eta \in (0, 0.1) \), we have \( \bar{A} \) for which
\[ \frac{\sigma_e(\tau)}{\sigma(x, \tau)} \leq \sqrt{1 + \eta} \] 
\forall (x, \tau) \in (\bar{A}, +\infty) \times (0, T). \] 
(61)

We shall make use of an auxiliary function defined in the following lemma.

**Lemma 8.** Given \( A \geq 0, \eta \in (0, 0.1), \kappa > 1/1 + \eta, \) there exists \( \psi \in C^2(\mathbb{R}) \) satisfying the following properties:

(i) \( \psi \in W^{2,\infty}(\mathbb{R}) \) with \( ||\psi||_{W^{2,\infty}} \) independent of \( A \)
(ii) \( \psi(z) \leq \frac{1}{1 + \eta} = \lim_{z \to +\infty} \psi(\varepsilon) \forall z \in \mathbb{R} \)
(iii) \( \psi(z) \leq \frac{1}{\kappa} \forall z \in (-\infty, A) \)
(iv) \( \frac{\psi(z)}{\varphi(\varepsilon x)} \leq \frac{1}{\kappa} \forall z \in \mathbb{R} \)
(v) \( \frac{\psi(z)}{\varphi(\varepsilon x)} \to 0, \psi'(z) \to 0, \psi''(z) \to 0 \) as \( z \to +\infty \).

We set
\[ \frac{\psi(x, \tau)}{\Sigma_e(\tau)} = \Sigma_e(\tau) \psi(\varepsilon x), \] 
(62)

with the values of \( \eta, \kappa > 0 \) defined previously. A choice of \( \varepsilon \) and \( A \) similar to the above argument ensures that
\[ H[\varphi](x, \tau) \geq \frac{1}{1 + \eta} \] 
(63)

for \( x \geq A \) and \( \frac{1}{2} \leq H \leq 2 \) in \( \Omega_T \). Now, by (50), (61), (ii), (iii) in lemma 8, it follows that the local volatility associated with \( \psi \)
\[ \sigma^2[\psi](x, \tau) = \frac{(\tau \Sigma_e(\tau) \varepsilon \psi(\varepsilon x)^2)}{H[\psi](x, \tau)} \] 
(64)
\[ = \sigma^2_e(\tau) \frac{\psi(\varepsilon x)^2}{H[\psi](x, \tau)} \] 
(65)
satisfies
\[ \sigma[\varphi] \leq \sigma(x, \tau) \] 
(66)
for all \( (x, \tau) \in \Omega_T \). By applying our comparison principle (lemma 6) we get that \( \varphi \leq \varphi \) in \( \Omega_T \), and sending \( \eta \to 0 \), that \( \liminf_{x \to +\infty} \varphi(x, \tau) \geq \Sigma_e(\tau) \), hence the result.

**Proof of theorem 3.** That the infimum of \( J^e \) is achieved over \( H^1 \) follows fairly directly from convexity. The uniqueness of the minimizer follows from the Euler equation:
\[ -\varepsilon \xi^e + \sum_i (\zeta_i(x_i) - \zeta^{+}_i)^2 \frac{1}{1(x_i)} = 0. \] 
(67)

Indeed, if \( \xi \) and \( \tilde{\xi} \) are two solutions to (67) (with corresponding quantities \( \zeta, \tilde{\zeta} \) respectively), multiplication by \( \xi - \tilde{\xi} \) results in
\[ \varepsilon \int (\xi - \tilde{\xi})^2 + \sum_i (\zeta - \tilde{\zeta})^2 = 0. \]

Finally, it is not difficult to infer from (67) uniform \( H^1 \) bounds and to pass to the limit \( \varepsilon \to 0 \). \hfill \Box

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**Appendix**

**Sketch of the proof of (4).** Let us sketch briefly the proof of this well known result [13]. Since \( \frac{\partial^2}{\partial K^2}(S - K)_+ = \delta(S - K), \) \( \delta \) being the Dirac mass at zero, \( G(S, t; K, T) = \frac{\partial}{\partial K} C(S, t; K, T) \) is the Green function for the operator \( L = \bar{\partial} + \frac{\partial^2}{\partial S^2}(S, t)S^2 \bar{\partial}_{S} + rS \bar{\partial}_{S} - rC, \) i.e. satisfies the backward parabolic equation
\[ G_t + LG = 0 \] 
(68)
\[ G(S, t; K, T) = \delta(S - K). \]

It is then classical (see [15]) to infer from the Green identity that \( G \) also satisfies the dual forward equation
\[ G_T = L^*G \] 
(69)
\[ G(S, t; K, t) = \delta(S - K). \]
where
\[ L^* = \frac{\partial^2}{\partial K^2} \left( \frac{1}{2} K^2 \sigma^2(K, T) \right) - r \partial_K(K) - r. \]  
(70)
Integrating twice (69) with respect to \( K \) and integrating by parts results in
\[ C_T = \frac{1}{2} K^2 \sigma^2(K, T) C_K K - r K C_K \]
\[ C(S, T; K, T) = (S - K)_+, \]
so that
\[ \sigma(K, T) = \sqrt{2 \left( \frac{C_T + r K C_K}{K^2 C_K} \right)}. \]
(72)

**Proof of proposition 1.** In the case \( \sigma \) is Hölder continuous, namely \( \sigma \in C^{2,\alpha}(\Omega_T) \) (\( 0 < \alpha < 1 \)), it is theorem 12 chapter 1 section 7 in [15]. We shall construct the solution in the general case by an approximation procedure. Take a standard mollifier \( \rho^\varepsilon(x, \tau) \), that is, a function satisfying
\[ \rho^\varepsilon \geq 0, \quad \rho^\varepsilon \in C^\infty_0, \quad \text{supp} \rho^\varepsilon \subset B_\varepsilon(0), \]
(73)
Define \( \tilde{\sigma}(x, -\tau) = \sigma(x, \tau) \) for all \( \tau \geq 0 \) and set for all \( \varepsilon > 0 \)
\[ \sigma^\varepsilon = \rho^\varepsilon \ast \tilde{\sigma}. \]
By (7) clearly \( \sigma^\varepsilon \in C^{2,\alpha}(\Omega_T) \) for all \( \alpha \in (0, 1) \).
For each \( \varepsilon > 0 \) by theorem 12 chapter 1 section 7 in [15] there exists a unique solution to the approximate problem
\[ v^\varepsilon_T = \frac{1}{2} (\sigma^\varepsilon)^2(x, \tau) (v^\varepsilon_T + v^\varepsilon_T) = 0 \quad \text{in} \quad \Omega_T \]
\[ v^\varepsilon(x, 0) = (e^\varepsilon - 1)_+ \]
(74)
in the class \( W^{2,1}_b(\Omega_T) \cap \{ u | \forall C, \beta > 0 \forall (x, \tau) \in \Omega_T \}
\[ \{ u(x, \tau) \} \leq C e^{B\varepsilon}. \]
(75)

Further, noting that \( \psi(\tilde{\sigma}) \) has its first time derivative and second space derivative lie actually in a local Hölder space. Applying the maximum principle (lemma 4), we have
\[ 0 \leq v^\varepsilon(x, \tau) \leq e^\varepsilon \]
(76)
in \( \Omega_T \). Since the \( \sigma^\varepsilon \) are uniformly continuous, uniformly in \( \varepsilon \), by [30] and (75) we have uniform interior estimates in \( W^{2,1}_b(\Omega_T) \). To see this, note that (75) together with the initial condition control the boundary terms. Hence \( v^\varepsilon \) converges locally uniformly to \( v \), viscosity solution to the limiting equation. By [30] again \( \varepsilon \in W^{2,1}_b(\Omega_T) \) and satisfies the limiting equation pointwise a.e. (strong solution).

Now from (75) clearly \( 0 \leq \tilde{v}(x, \tau) \leq e^\varepsilon \), which implies \( 0 < v(x, \tau) < e^\varepsilon \) by the strong maximum principle (see [15]). Let us next take the notation \( z^\varepsilon = v^\varepsilon \) and observe that it satisfies
\[ z^\varepsilon_T = \frac{1}{2} (\sigma^\varepsilon)^2(x, \tau) (z^\varepsilon_T + z^\varepsilon_T) + 2 \frac{\sigma^\varepsilon}{\sigma^\varepsilon}(x, \tau) z^\varepsilon \]
\[ z^\varepsilon(x, 0) = \frac{1}{2} (\sigma^\varepsilon)^2(0, 0) \delta(x), \]
\[ \delta(x) \] being the Dirac mass at \( x = 0 \). It follows that \( z^\varepsilon \geq 0 \) in \( \Omega \) by the maximum principle (lemma 4). This implies in the limit \( \varepsilon \to 0 \) in \( \Omega \), hence \( v(x, \tau) \geq v(x, 0) = (e^\varepsilon - 1)_+ \).

**Proof of lemma 7.** Take
\[ \psi(z) = \begin{cases} \frac{1}{1 - \eta} \left( 1 + 4 \kappa \right) & \text{if } z \leq 0 \\ \frac{1}{1 - \eta} \left( 1 + 4 \kappa N \left( - \frac{1}{4 \kappa} \ln \left( \frac{z}{A} \right) \right) \right) & \text{if } z > 0 \end{cases} \]
(77)
where \( N \) is defined in (11). It is easily seen that \( \psi \in C^2(\mathbb{R}) \) and satisfies (i) and (ii). Then
\[ \frac{\partial \psi}{\partial z}(z) \leq \frac{1}{1 - \eta} \left( 1 - \frac{2}{\sqrt{\pi}} \kappa N \right) \leq \frac{1}{2}. \]
Further, noting that \( N \left( - \frac{1}{4 \kappa} \ln \left( \frac{z}{A} \right) \right) \geq 1/2 \) for \( z \leq A \) yields (iii).

**Proof of lemma 8.** One may follow the lines of the argument above, with
\[ \psi(z) = \begin{cases} \frac{1}{1 + \eta} \left( 1 + 4 \kappa \right) & \text{if } z \leq 0 \\ \frac{1}{1 + \eta} \left( 1 - \frac{1}{4 \kappa} N \left( - \frac{1}{4 \kappa} \ln \left( \frac{z}{A} \right) - z_0(\kappa) \right) + \frac{1}{4 \kappa} \right) & \text{if } z > 0 \end{cases} \]
(78)
where \( z_0(\kappa) \) solves \( (1 - \frac{1}{4 \kappa}) N(z(\kappa)) = \frac{1}{8 \kappa} \).

**References**


