Non-Parametric Estimation of an Implied Volatility Surface [1]

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Non-Parametric Estimation of an Implied Volatility Surface

Abstract

Given standard diffusion-based option pricing assumptions and a set of traded European option quotes and their pay-offs at maturity, we identify a unique and stable set of diffusion coefficients or volatilities. Effectively, we invert a set of option prices into a state- and time-dependent volatility function. Our problem differs from the standard direct problem in which volatilities and maturity pay-offs are known and the associated option values are calculated. Specifically, our approach, which is based on a small parameter expansion of the option value function, is a finite difference-based procedure. This approach builds on previous work which has followed Tikhonov's treatment of integral equations of the Fredholm or convolution type. An implementation of our approach with CBOE S&P 500 option data is also discussed.

In this paper, we address the general problem of inverting option prices into a state- and time-dependent volatility function. Specifically, we build on the foundation of research by Rubinstein[24], Shimko[25], Derman and Kani[11] and DuPire [13]. Like Rubinstein, we use an optimization-based method with a finite set of option prices; like Shimko, Derman-Kani and Dupire we work with the continuous-time specification. Taken together, these works leave open two important questions regarding their volatility surface estimates: uniqueness and stability.[1]

These two characteristics are important in estimating time- and state-dependent option volatility, and the volatility surface estimation problem belongs to the mathematical class of "ill-posed" problems. The ill-posedness problem is seen in finance in the context of multicollinearity and regression, estimation of nonnormal interest rate process parameters, extraction of a term structure from a set of bond prices and asset pricing factor identification. Ill-posedness implies that small changes in data inputs can generate large changes in parameter estimates and estimates of parameter significance. In the option valuation and hedging context, the effect of ill-posedness on derivative estimates that are used as hedging parameters is especially problematic.

Our solution to this problem, built on Tikhonov's regularization approach, suggests alternative iterative discrete linear system-based estimation procedures.[2] To implement

[1]Contemporaneous with and subsequent to our model development, Andersen and Brotherton-Ratcliffe[5] have refined the Shimko, Derman-Kani and Dupire interpolation-based approach. Avellaneda et. al. [6], Bouchouev[8], Brown-Toft[9], Derman, Kani and Zou[12], Jackwerth-Rubinstein[18] and Lagnado-Oscher[21] all have implemented regularization-based non-linear estimation approaches. Among this second set of works, Avellaneda et. al. is closest in spirit to our work in that they also estimate true local volatility surface. Furthermore, their regularizer is analogous to the one that we use in estimation. The other works either explicitly or implicitly estimate volatility curves for particular maturities and then interpolate local volatilities from these estimates. As these works are, effectively, curve fitting implementations, the regularizers used are defined by second derivatives (the Laplacian) alone. For surface estimation, such regularizers are not sufficient to control poles (spikes) in the parameter estimate surface. For some related discussion, also see Silverman[26].

[2]Work following Whaba[30] is also related. We minimize a least squares goodness of fit criterion that is smoothed by Sobolev norms that are related to the parameter vector. Our treatment of a complicated nonlinear model, which involves solution of a parabolic partial differential equation subject to boundary conditions, contributes to the statistics literature. Fisher, Nychka and Zervos[14], and Adams and Van Deventer[1] smooth a deterministic term structure. Aït-Sahalia[2] has developed a kernel-based estimator for spot interest rate-contingent volatilities. Aït-Sahalia and Lo[3] adapt this approach to the volatility surface estimation problem under particular stationarity assumptions.

a linearization, we propose a finite difference-based specification. The value and derivative estimates generated by this procedure are unique, stable and converge to their continuous-time limits.

This paper contains seven sections and an appendix. In the first section, we outline definitions, assumptions and other preliminaries. The second section introduces our formulation of option implied volatility surface estimation as an inverse problem for parabolic partial differential equations. In the third section, we recast our inverse problem by the method of small parameter. This method allows replacement of the original nonlinear volatility identification problem with a sequence of linear problems. The fourth section describes our iterative optimization-based procedure. The fifth section treats our particular implementation, which is a finite difference-based numerical procedure. The sixth section illustrates an application of our procedure for CBOE S&P 500 options. The seventh section outlines extensions to this work. Finally, the appendix describes the construction of a particular function that is used to expedite estimation.

1 Definitions, Assumptions and Other Preliminaries

In our procedure, we use the following definitions:

S	Underlying or spot price
t	Calendar time (in years)
T	Option maturity date (in years)
τ	Time to option maturity (in years), (T-t)
$C(t, S)$	Call option value at the underlying price (S) and at a time (t)
$P(t, S)$	Put option value at the underlying price (S) and at a time (t)
K	Option exercise or strike price
r	Continuously compounded domestic interest rate
y	Continuously compounded yield on the underlying
$\alpha(t, S)$	Instantaneous drift of the proportional value change of the underlying

$\sigma(t,S)$ Instantaneous standard deviation of the spot price at price (S) and time (t)

$X(t)$ Standard Wiener process

We first assume that the spot price follows a diffusion process:[3]

$$\frac{dS}{S} = (\alpha\,(t,S) + r - y)\,dt + \sigma\;(t,S)\,dX\,(t) \tag{1}$$

Next, we define the forward price relative to an option exercise price:

$$F_K = e^{(r-y)\tau}\;\frac{S}{K}$$

Substituting this scaled forward price into equation (1), the associated forward price diffusion process is the following:

$$dF_K = \alpha\,(t,S)\,F_K\,dt + \;\sigma\,(t,S)\;F_K\,dX\,(t) \tag{2}$$

Analogous to the relative forward price, we create a no-arbitrage portfolio for an exercise price-standardized call option:

$$\Pi = \lambda\,F_K\; + C_K(t,F_K),\;\; where\;\; C_K(t,F_K) = e^{r\tau}\;\frac{C(t,S)}{K}$$

The no-arbitrage condition requires that $d\Pi = 0$ at any time t. By Îto's Lemma, we have

$$d\Pi = \left(\frac{\partial C_K}{\partial t}\;+\;\frac{1}{2}\sigma^2\,(t,S)\;F_K^2\,\frac{\partial^2 C_K}{\partial F_K^2}\right)\,dt\;+\;\left(\lambda + \frac{\partial C_K}{\partial F_K}\right)\,dX\,(t)$$

With the forward position, λ, equal to $\frac{-\partial C_K}{\partial F_K}$, the portfolio becomes riskless, implying that

$$\frac{\partial C_K}{\partial t}\;+\;\frac{1}{2}\sigma^2\,(t,S)\;F_K^2\,\frac{\partial^2 C_K}{\partial F_K^2}\;=\;0 \quad with\; C_K\,(0,F_K) = Max\,(\,0,\;F_K - 1)\;and \tag{3}$$

$$C_K\,(t,0) = 0$$

[3]Direct numerical option valuation methods have been implemented for such general diffusion-volatility functions, $\sigma(t,S)$. See Cox and Rubinstein[10], Nelson and Ramaswamy[23], Heath, Jarrow and Morton[17] and Amin and Bodurtha[4].

Equation (4) states the Black-Scholes-Merton call option valuation equations on the basis of forward values, time and exercise price units. An analogous partial differential equation holds for puts.

Without loss of generality, we set time t to zero so that $\tau = T$, defining $\bar{T}$ as the maximum option maturity. All option maturities are scaled to a new time variable, v, by the maximum option maturity, $v = T/\bar{T},\ v \in [0,1]$.

We then change the t time variable in equation (4) to the scaled time to maturity variable:

$$\frac{\partial C_K}{\partial v} = \frac{\bar{T}}{2}\sigma^2\left(v,S\right)\ F_K^2\,\frac{\partial^2 C_K}{\partial F_K^2}, \quad with\, C_K\left(0,F_K\right) = Max\left(\,0,\ F_K - 1\right)\ and \qquad (4)$$
$$C_K\left(v,0\right) = 0$$

In equation (5), the time argument of the volatility function has been changed from t to v without noting the function change. Therefore, when the final volatility estimates are recovered, these estimates must be transformed back from scaled time to maturity (v) to calendar time (t). This abuse and subsequent abuses of the underlying value function definition are tolerated to ease the notation burden. Additionally, we emphasize that the volatility, $\sigma(t,S)$, enters equation (5) with a spot price argument, not an F_K argument, with $S = e^{(y-r)\bar{T}v}\ \frac{K}{F_K}$.

Analogous to the call case, put option valuation equations are derived. With put value P_K, equation (5) becomes

$$\frac{\partial P_K}{\partial v} = \frac{\bar{T}}{2}\sigma^2\left(v,S\right)\ F_K^2\,\frac{\partial^2 P_K}{\partial F_K^2}, \quad with\, P_K\left(0,F_K\right) = Max\left(\,0,\ 1 - F_K\right)\ and \qquad (5)$$
$$P_K\left(v,0\right) = 1$$

Note that the underlying (F_K), call (C_K) and put (P_K) values are unitless and the exercise price index is arbitrary. Essentially, our valuation problem requires solving two fundamental parabolic partial differential equations – equations (5) and (6) – in a variable (F_K) which is defined on the range $[0,\infty)$. Since the K index is arbitrary, we can ease notation by introducing a new variable, Z, with a range $(-\infty, \infty)$ such that $Z = \ln(F_K)$. Additionally, we define $U(v, Z) = C_K(v, F_K)$.

Substituting into equation (5), we find

$$\frac{\partial U(v,Z)}{\partial v} = \frac{\bar{T}\,\sigma^2(v,S)}{2}\left[\frac{\partial^2 U(v,Z)}{\partial Z^2} - \frac{\partial U(v,Z)}{\partial Z}\right],\ with \qquad (6)$$
$$U(0,Z) = Max\,(\,0,\ e^z - 1)$$

With $W(v, Z) = P_K(v, F_K)$, we substitute in the associated put equation (6):

$$\frac{\partial W(v,Z)}{\partial v} = \frac{\bar{T}\,\sigma^2(v,S)}{2}\left[\frac{\partial^2 W(v,Z)}{\partial Z^2} - \frac{\partial W(v,Z)}{\partial Z}\right],\ with \qquad (7)$$
$$W(0,Z) = Max\,(\,0,\ e^z - 1)$$

Equations (7) and (8) provide a parsimonious representation of the direct option valuation problem. Given a time and spot price dependent volatility function, we numerically integrate the equations to solve for the associated European call and put option values. Recovering the associated exercise price dependent option quotes requires two steps.

First, we map the $U(v, Z)$ and $W(v, Z)$ option prices into respective forward price-exercise price option prices $C_K(t, F_K)$, and $P_K(t, F_K)$. Second, we calculate the actual quotes, $C(t, S)$ and $P(t, S)$, in the spot price-time space. Given any exercise price, this mapping is one-to-one, with $S(0)$ mapping to $F_K(T)$ and $Z(T)$, $S(T)$ mapping to $F_K(0)$ and $Z(0)$, and the corresponding elements of the spot and exercise price-adjusted forward price sets mapping accordingly.

In concluding this section, we again note our toleration of multiple abuses of the underlying value function definition. These abuses are tolerated to ease the notation burden. Nevertheless, these multiple definitions must be remembered when we subsequently return to the original notation.

2 The Implied Volatility Surface Inverse Problem

Equations (7) and (8) define no arbitrage pricing relations for the European calls and puts in our sample, respectively. We have data on current prices and rates (time 0 quotes) as well as maturity date option exercise values (final or boundary conditions). However, we do not know the subsequent time- and state-contingent diffusion coefficients, which are determined by the time- and state-varying volatilities. To find these quantities, we solve an inverse problem.

Our identification of the spot price process volatility parameters involves two steps. First, we solve for a single volatility function in the Z space, defining the associated parameter set elements as $\sigma(v, Z)$. Second, we map these volatilities into the spot price-time space, $\sigma(t, S)$. As is clear from the discussion of the direct option valuation problem in section one, this mapping is one-to-one as long as a reference strike price K is chosen. This choice allows direct calculation of the volatility surface as a function of two variables. This surface consistently values all current time options and hedge parameters across all strike prices and maturities.

3 Determining $\sigma(t, S)$ by the Method of Small Parameter

We assume that a volatility function has the following form:

$$\sigma^2(v, Z) = \sigma_0^2 + \sum_{k=1}^{\infty} a_k(v, Z) \tag{8}$$

Now, consider a family of volatility functions,

$$\sigma_\varepsilon^2(T, Z) = \sigma_0^2 + \sum_{k=1}^{\infty} \varepsilon^k a_k(v, Z),\ where\ 0 \le \varepsilon \ \le 1 \tag{9}$$

In this "ε-parametric" function family, the epsilon has neither financial nor "physical" meaning. It is only a temporary parameter that builds a homotopy for decomposition purposes. This technique is widely used in classic applied mathematics.

As will become clear, our solution does not depend on epsilon in any way. From this equation, we see that $\varepsilon = 0$ corresponds to the Black-Scholes-Merton constant volatility case, while $\varepsilon = 1$ is the solution to our volatility surface estimation problem. Under this volatility definition, we create an "ε-analogue" to equation (7):

$$\frac{\partial U^{\varepsilon}(v,Z)}{\partial v} = \frac{\bar{T}\,\sigma_{\varepsilon}^{2}(v,S)}{2}\left[\frac{\partial^{2}U^{\varepsilon}(v,Z)}{\partial Z^{2}} - \frac{\partial U^{\varepsilon}(v,Z)}{\partial Z}\right],\ with$$
$$U^{\varepsilon}(0,Z) = Max\,(\,0,\ e^{z}-1) \qquad (10)$$

We expand the option value function, $U^{\varepsilon}(v,Z)$, into a formal power series with respect to ε:

$$U^{\varepsilon}(v,Z) = \sum\nolimits_{n=0}^{\infty} U_{n}(v,Z)\,\varepsilon^{n} \qquad (11)$$

We next substitute equations (9) and (11) into equation (10), and defining $\tilde{a}_0 = \bar{T}\sigma_0^2 \big/ 2$ and $\tilde{a}_k = \bar{T}a_k(v,Z)\big/2$, for $k = 1, \ldots, n, \ldots$

$$\begin{aligned}\frac{\partial U_0(v,Z)}{\partial v} + \sum_{n=1}^{\infty}\frac{\partial U_n(v,Z)}{\partial v}\varepsilon^n &= \tilde{a}_0\left[\frac{\partial^2 U_0(v,Z)}{\partial Z^2} - \frac{\partial U_0(v,Z)}{\partial Z}\right] \\ &+ \tilde{a}_0\sum_{n=1}^{\infty}\left[\frac{\partial^2 U_n(v,Z)}{\partial Z^2} - \frac{\partial U_n(v,Z)}{\partial Z}\right]\varepsilon^n \qquad (12) \\ &+ \sum_{n=1}^{\infty}\sum_{k=1}^{n}\tilde{a}_k\left[\frac{\partial^2 U_{n-k}(v,Z)}{\partial Z^2} - \frac{\partial U_{n-k}(v,Z)}{\partial Z}\right]\varepsilon^{n+k-1}\end{aligned}$$

Equating equivalent powers of ε terms, and incorporating the associated initial or boundary conditions yields the following system of equations:[4]

$$\frac{\partial U_0(v,Z)}{\partial v} = \tilde{a}_0\left[\frac{\partial^2 U_0(v,Z)}{\partial Z^2} - \frac{\partial U_0(v,Z)}{\partial Z}\right],$$
$$with\ U_0(0,Z) = Max\left(0,\ e^{Z}-1\right) \qquad (13.0)$$

$$\frac{\partial U_1(v,Z)}{\partial v} = \tilde{a}_0\left[\frac{\partial^2 U_1(v,Z)}{\partial Z^2} - \frac{\partial U_1(v,Z)}{\partial Z}\right] + \tilde{a}_1\left[\frac{\partial^2 U_0(v,Z)}{\partial Z^2} - \frac{\partial U_0(v,Z)}{\partial Z}\right],$$
$$with\ U_1(0,Z) = 0 \qquad (13.1)$$

$$\vdots$$

[4]For a systematic treatment of these types of differential equations (and more general ones) as well as a priori estimates for them, see Kreiss and Lorenz[20].

$$\frac{\partial U_n(v,Z)}{\partial v} = \tilde{a}_0 \left[\frac{\partial^2 U_n(v,Z)}{\partial Z^2} - \frac{\partial U_n(v,Z)}{\partial Z}\right] + \sum_{k=1}^{n} \tilde{a}_k \left[\frac{\partial^2 U_{n-k}(v,Z)}{\partial Z^2} - \frac{\partial U_{n-k}(v,Z)}{\partial Z}\right], \quad with\ U_n(0,Z) = 0 \qquad (13.n)$$

and so on.

In this equation system (13), all equations [except equation (13.0)] differ only in the forcing or non-homogeneous term. At each step of the iterations over these equations (starting from n=1), the forcing term is obtained from the previous equation calculation. To complete this procedure, we imply $\tilde{a}_k$ from equation (13.k) for $k = 1, \ldots, n, \ldots$

A full discussion of the convergence issue regarding the series $\sum_{n=0}^{\infty} U_n(v,Z)$ is beyond the scope of this work. Nevertheless, a remark on this issue is in order. The smallness of $\tilde{a}'s$ in a proper sense guarantees the convergence of the series $\sum_{n=0}^{\infty} U_n(v,Z)$ and its derivatives.

4 An Iterative Optimization-based Estimation Procedure

Generally, we define $i = 1, n$ as the traded strike prices, K_i, and define $j = 1, m$ as the available option maturities, $\left(T_1,\ T_2,\ \ldots,\ T_{m-1}, T_m;\ T_m\ = \bar{T}\right)$. For ease of notation, we assume that the number of traded options, n, is the same for each maturity.

Thus, we can denote $v_j = T_j \big/ \bar{T}$, $Z_{i,j} = \ln\left(e^{(r-y)\bar{T}v_j}\ \frac{S}{K_i}\right)$; and $U^{i,j}\ = e^{r\,\bar{T}v_j}\ \frac{C^{i,j}(0,S)}{K_i}$. $Z_{i,j}$ is the natural logarithm of the time zero and T_j maturity forward price weighted by strike price, K_i. $C^{i,j}(0,S)$ is the time-zero call option price with strike price K_i and maturity T_j; and $U^{i,j}$ is the K_i strike price-weighted and T_j maturity-future value of this call.

To solve equation system (13.1),we must provide an input, $\tilde{a}_0$. Our estimate of this parameter is the minimizer of the following (least squares) function:

$$M_0(\tilde{a}_0) = \sum_{j=1}^{m} \sum_{i=1}^{n} \left|U_0(v_j, Z_{i,j}) - U^{i,j}\right|^2 \qquad (14)$$

Because there are, for any given $\tilde{a}_0$, closed-form expressions for $U_0^{i,j}(v, Z_{i,j})$, we can minimize equation (14) with a Newton-Raphson search for the root of the first derivative of $M_0(\tilde{a}_0)$. Given this estimate, we first calculate both $\frac{\partial U_0(v,Z)}{\partial Z}$ and $\frac{\partial^2 U_0(v,Z)}{\partial Z^2}$ and then solve equation (13.1).

The solution to equation (13.1) has the following form:[5]

$$U_1(v,Z) = \int_0^v \int_{-\infty}^{\infty} \frac{\exp\left[\frac{-((z-\xi)-\tilde{a}_0(v-\kappa))^2}{4\tilde{a}_0(v-\kappa)}\right]}{\sqrt{4\pi\tilde{a}_0(v-\kappa)}} F_1(\kappa,\xi)\, d\xi\, d\kappa$$
$$where, F_1(\kappa,\xi) = \tilde{a}_1(\kappa,\xi)\left[\frac{\partial^2 U_0(\kappa,\xi)}{\partial\xi^2} - \frac{\partial U_0(\kappa,\xi)}{\partial\xi}\right] \tag{15}$$

For the $\tilde{a}_1$ volatility perturbation function, we define the associated minimand as

$$M_1^{\alpha}(\bullet) = \sum_{j=1}^{m}\sum_{i=1}^{n} \left|U_1(v_j, Z_{i,j}) - (U^{i,j} - U_0(v_j, Z_{i,j}))\right|^2 + \alpha_1\Omega(\bullet) \tag{16}$$

In this equation, α_1 is called a regularizing parameter, and $\Omega(\bullet)$ is a nonnegative functional (called a stabilizing functional, roughness penalty or smoother) that satisfies certain conditions. Among possible stabilizing functionals, we are primarily interested in two cases

$$\Omega(\tilde{a}_1) = \int_0^1 \int_{-\infty}^{\infty} \sum_{k_1+k_2=0}^{p} \left(\frac{\partial^{k_1+k_2}}{\partial v^{k_1}\partial Z^{k_2}}\tilde{a}_1(v,Z)\right)^2 dZ\, dv \tag{17.0}$$

$$\Omega(F_1) = \int_0^1 \int_{-\infty}^{\infty} \sum_{k_1+k_2=0}^{p} \left(\frac{\partial^{k_1+k_2}}{\partial v^{k_1}\partial Z^{k_2}}F_1(v,Z)\right)^2 dZ\, dv \tag{17.1}$$

Theses stabilizing functionals are squares of the $H^p - norm$, where p is an appropriately chosen nonnegative integer. Though other functionals may be used, the ones defined in equation (17) guarantee good numerical properties.

Assuming that we know the regularizing parameter, α_1, we solve for $\tilde{a}_1$ as the minimizer of optimand (16). Therefore, the only remaining issue, theoretically, is to determine α_1. We determine the regularizing parameter by the discrepancy method of Tikhonov and Arsenin[28].

[5]For example, see Tikhonov and Samarskii[29].

Thus, we define a discrepancy function as follows:

$$\rho_1(\alpha_1) = \sum_{j=1}^{m}\sum_{i=1}^{n} \left| U_1(v_j, Z_{i,j}) - \left(U^{i,j} - U_0(v_j, Z_{i,j})\right)\right|^2 - \delta^2 \tag{18}$$

The δ parameter is a bound on the least-squares error of the traded option quotes and is nonnegative. More precisely, define $\bar{U}^{i,j}$ as the actual call option value at time zero and spot price, $S(0)$. Following our definition of the quoted trade option prices, $U^{i,j}$, the actual $\bar{U}^{i,j}$ are also scaled by strike price, K_i and maturity date, v_j.

We do not expect the quoted trade prices and the actual option values to coincide for several reasons:[6] Nonsynchronous option and spot quotes, bid-ask costs and other factors introduce valuation errors, $\bar{U}^{i,j} - U^{i,j}$. We identify δ as a bound on the size of the least squares error:

$$\sum_{j=1}^{m}\sum_{i=1}^{n} \left|\bar{U}^{i,j} - U^{i,j}\right|^2 \le \delta^2 \tag{19}$$

With the δ parameter given, our solution for the unique volatility function, $\tilde{a}_1$, is the minimizer of optimand (16). We choose the regularizing parameter of discrepancy function equation (18) such that

$$\rho_1(\alpha_1) = 0 \tag{20}$$

Note that the regularizing parameter, α_1, is the root of the discrepancy function, equation (18).

The general theory of Tikhonov and Arsenin[28] guarantees that a unique solution to our problem exists. For the $\Omega(\tilde{a}_1)$ $[\Omega(F_1)]$ regularizer, the choice of α_1 by the discrepancy method delivers the $\tilde{a}_1(v,Z)$ $[F_1(v,Z)]$ with the minimum $H^p - norm$ among the set of all possible $\tilde{a}_1(v,Z)$ $[F_1(v,Z)]$, which give the same least-squares error. The $\tilde{a}_1(v,Z)$ $[F_1(v,Z)]$ associated with this minimum norm is unique.

[6]The option market "efficiency" literature provides relevant benchmarks. E.g. Galai[15] and Klemkosky and Resnick[19] for CBOE equity options and Bodurtha and Courtadon[7] for PHLX American currency option prices, among many.

In our finite difference-based numerical implementation, we work with the H^2-norm. In our application, we work with the computationally tractable $H^0 - norm$. These low-order norms are sufficient for our purpose, a rigorous implementation requires stabilizing functionals, $H^p - norms$, of the fourth order or higher.

5 A Finite Difference-Based Numerical Implementation

To use our algorithmic approach, we solve two separate minimization problems.[7] We first calculate the Black-Scholes-Merton model implied volatility for a fixed number of option quotes. We find the minimizer by a Newton-Raphson search for the zeros of the optimand derivative with respect to $\tilde{a}_0$. Although this initial estimate can be regularized, we have found that the regularized procedure converges quickly to the usual estimate.

Based on a closed-form expression for our $U_1(v_j, Z_{i,j})$ terms and their partial derivatives, we can use a numerical integration procedure to discretize them. We obtain a system of linear equations by equating the appropriate $\tilde{a}_1(v_q, Z_r)$ or $F_1(v_q, Z_r)$-associated partial derivatives to zero. (Note that q and r are discretization indices.)

For computational convenience, we introduce a final change of variables:

$$\xi = Z - \tilde{a}_0 v \tag{21}$$

This change corresponds to the introduction of a "moving frame of coordinates" in classical mechanics. Additionally, we denote $\bar{a}_k(v, \xi) = \tilde{a}_k(v, Z)\ for\ k = 1,\ \ldots,\ n,\ \ldots,$ and $H_0(v, \xi) = U_0(v, Z),\ H_1(v, \xi) = U_1(v, Z),\ \ldots,\ \ H_n(v, \xi) = U_n(v, Z)$.

System of equations (13.0), (13.1), ..., (13.n), and so on is equivalent to the following system:

[7] Minimization of the stage zero optimand (14) is not necessary, the procedure may start from any reasonable $\tilde{a}_0$. Using the lognormal implied volatility of the shortest maturity at-the-money option as the estimate of $\tilde{a}_0$ is an alternative. A state- and time-dependent local volatility surface may also be used as the initial estimate.

$$\frac{\partial H_0(v,\xi)}{\partial v} = \tilde{a}_0 \frac{\partial^2 H_0(v,\xi)}{\partial \xi^2}, \tag{22.0}$$

$$with\ H_0(0,\xi) = Max\left(0,\ e^{\xi}-1\right) = U_0(0,Z) = Max\left(0,\ e^{Z}-1\right)$$

$$\frac{\partial H_1(v,\xi)}{\partial v} = \tilde{a}_0\ \frac{\partial^2 H_1(v,\xi)}{\partial \xi^2} + \bar{a}_1(v,\xi)\left[\frac{\partial^2 H_0(v,\xi)}{\partial \xi^2} - \frac{\partial H_0(v,\xi)}{\partial \xi}\right],$$

$$with\ H_1(0,\xi) = 0 \tag{22.1}$$

$$\vdots$$

$$\frac{\partial H_n(v,\xi)}{\partial v} = \tilde{a}_0\ \frac{\partial^2 H_n(v,\xi)}{\partial \xi^2} + \sum_{k=1}^{n} \bar{a}_k(v,\xi)\ \left[\frac{\partial^2 H_{n-k}(v,\xi)}{\partial \xi^2} - \frac{\partial H_{n-k}(v,\xi)}{\partial \xi}\right],$$

$$with\ H_n(0,\xi) = 0 \tag{22.n}$$

and so on.

We discretize our problem with time steps Δv and state spacing $\Delta\xi$. We introduce $v_0 = 0,\ \ v_{j+1} = v_j + \Delta v$ for $j = 0, 1, 2, \ldots, M-1$, where $M = 1/\Delta v$.

Next, we let L be a positive number that corresponds to the truncation of the $\xi - line$. (I.e. $\xi \in [-L,\ L]$.) Therefore, we also introduce artificial boundary conditions at $\xi = \pm L$. For large enough L, the specification $H_1(v, \pm L) = 0$ for $0 \le v \le 1$ is a satisfactory approximation. We now define $\xi_0 = -L,\ \ \xi_{i+1} = \xi_i + \Delta\xi$ for $i = 1, \ldots, N$, where $\Delta\xi = \frac{2L}{N+1}$ and $\xi_{N+1} = L$. $\xi_1, \xi_2,\ \ldots, \xi_n$ are internal grid points that belong to $(-L, L)$.

To discretize equation (22.1), we use an explicit finite difference scheme with a first-order forward difference formula for $\frac{\partial H_1}{\partial v}$, and a second-order central difference formula for $\frac{\partial^2 H_1}{\partial \xi^2}$. These discrete approximations, which are accurate to the first and second order, respectively, result in a trinomial method.

For a stable trinomial system, we require the risk-neutral probability of an up or down step P_0, to satisfy the following inequality:

$$P_0 = \frac{\tilde{a}_0 \Delta v}{(\Delta \xi)^2} \leq \frac{1}{2} \tag{23}$$

Our discretization of equation (22.1) thus becomes

$$\begin{aligned} H_1(i, j+1) &= P_0 \; H_1(i-1, j) + P_m \; H_1(i, j) + P_0 \; H_1(i+1, j) \\ & + \Delta v \, \bar{a}_1(i, j) \frac{\partial^2 H_0}{\partial \xi^2}(i, j), \quad \forall \, i = 1, ..., N \; and \, j = 0, ...M-1 \end{aligned} \tag{24}$$

The probability of no movement is $P_m = 1 - 2P_0$. Choosing appropriate space and time steps, we can set inequality (23) as the equality. In this case, the probability of no movement is zero, and the trinomial method reduces to the binomial method.

The updated $j+1st$ time step value is calculated from P_0, and $\frac{\partial^2 H_0}{\partial \xi^2}(i, j) = \frac{\partial^2 H_0}{\partial \xi^2}(v_j, \xi_i)$, as well as the previous (jth) step $H_1(i, j) = H_1(v_j, \xi_i)$, and $\bar{a}_1(i, j) = \bar{a}_1(v_j, \xi_i)$. At all steps, the $\frac{\partial^2 H_0}{\partial \xi^2}(i, j)$, terms are identified from the calculation of $H_0(i, j)$.

With boundary conditions $H_1 = 0$ at $\xi = \pm L$, the equation (24) system may be written in matrix form:

$$H_1[j+1] = A \, H_1[j] + F_1[j], \;\; H_1[j] = \begin{bmatrix} H_1(1, j) \\ H_1(2, j) \\ \vdots \\ H_1(N, j) \end{bmatrix}, where \tag{25}$$

$$F_1[j] = \begin{bmatrix} \Delta v \, \bar{a}_1(1, j) \frac{\partial^2 H_0}{\partial \xi^2}(1, j) \\ \Delta v \, \bar{a}_1(2, j) \frac{\partial^2 H_0}{\partial \xi^2}(2, j) \\ \vdots \\ \Delta v \, \bar{a}_1(N, j) \frac{\partial^2 H_0}{\partial \xi^2}(N, j) \end{bmatrix}, \quad A = \begin{pmatrix} P_m & P_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ P_0 & P_m & P_0 & 0 & \dots & 0 & 0 & 0 \\ 0 & P_0 & P_m & P_0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & P_0 & P_m & P_0 \\ 0 & 0 & 0 & 0 & \dots & 0 & P_0 & P_m \end{pmatrix}$$

To fully specify the matrix representation of the system, we use $H_1[0] = 0$, which corresponds to the initial condition in equation (22.1), and

$$H_1[1] = F_1[0]\,,\ H_1[2] = A\ H_1[1] + F_1[1] = A\ F_1[0] + F_1[1] \tag{26}$$

$$H_1[3] = A\ H_1[2] + F_1[2] = A^2\ F_1[0] + A\,F_1[1] + F_1[2]$$

$$\vdots$$

$$\begin{aligned} H_1[M] &= A\,H_1[M-1] + F_1[M-1] \\ &= A^{M-1}F_1[0] + A^{M-2}F_1[1] + \ldots + AF_1[M-2] + F_1[M-1] \end{aligned}$$

$$Then, H_1 = \bar{A}\,F_1,\ where H_1 = \begin{bmatrix} H_1[1] \\ H_1[2] \\ \vdots \\ H_1[M] \end{bmatrix},\ F_1 = \begin{bmatrix} F_1[0] \\ F_1[1] \\ \vdots \\ F_1[M-1] \end{bmatrix}, \tag{27}$$

$$\bar{A} = \begin{pmatrix} I & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ A & I & 0 & 0 & \ldots & 0 & 0 & 0 \\ A^2 & A & I & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ A^{M-2} & A^{M-3} & A^{M-4} & A^{M-5} & \ldots & A & I & 0 \\ A^{M-1} & A^{M-2} & A^{M-3} & A^{M-4} & \ldots & A^2 & A & I \end{pmatrix}$$

In equation (27), H_1 and F_1 are $(NxM)x1$ column vectors, $\bar{A}$ is an $(NxM)x(NxM)$ lower triangular matrix, and I is the (NxN) identity matrix. The determinant of the $\bar{A}$ matrix is one. The coordinates of vector H_1 correspond to the option quotes that are indexed across maturities and strike prices. To insure this mapping, Δv and $\Delta\xi$ must be sufficiently small.

With a complete set of option quotes, we can easily solve system (27). Unfortunately, the option quote set is not complete in the coordinate space of vector H_1. Therefore, because vector H_1 is under-determined, the associated system is also under-determined. To resolve the under-determination of system (27), we use two steps: projection and regularization.

For the projection step, we recall that m is the number of quoted option maturities and n is the number of quoted option strike prices for each maturity. We then let $P: R^{MxN} \to R^{mxn}$ be the standard projection of the Euclidean space R^{MxN} onto its (mxn) dimensional subspace.

Denoting $H_p^1 = P \bullet H_1$, we define $\bar{A}_p$ to be the $(mxn)x(MxN)$ rectangular matrix that is obtained by eliminating the rows of matrix $\bar{A}$ that correspond to the nonquoted options. Rearranging equation (27), we now have

$$H_p^1 = \bar{A}_p F_1 \tag{28}$$

Because the equation (28) system has more variables than equations, the system possesses an infinite number of solutions (or no solution at all.) To identify a reasonable solution, we next implement a Tikhonov regularization.

To define the regularizer, or roughness penalty, we introduce a general column vector on which a set of forward-difference operators act:

$$G = \begin{bmatrix} G[0] \\ G[1] \\ \vdots \\ G[M-1] \end{bmatrix}, \ and \tag{29}$$

$$G[j] = \begin{bmatrix} G(j,1) \\ G(j,2) \\ \vdots \\ G(j,N) \end{bmatrix} \ are \ (MxN)x1 \ and \ (Nx1) \ vectors, \ respectively.$$

We illustrate the use of this vector with the first-order forward difference of G over the state variable:

$$D_\xi\left(G\right)=\frac{1}{\Delta\xi}\begin{bmatrix} G\left(0,2\right)-G\left(0,1\right)\\ G\left(0,3\right)-G\left(0,2\right)\\ \vdots\\ G\left(0,N\right)-G\left(0,N-1\right)\\ -G\left(0,N\right)\\ G\left(1,2\right)-G\left(1,1\right)\\ \vdots\\ G\left(1,N\right)-G\left(1,N-1\right)\\ -G\left(1,N\right)\\ \vdots\\ G\left(M-1,2\right)-G\left(M-1,1\right)\\ \vdots\\ G\left(M-1,N\right)-G\left(M-1,N-1\right)\\ -G\left(M-1,N\right) \end{bmatrix} \tag{30}$$

The corresponding first-order forward difference operator is

$$\bar{D}_\xi=\frac{1}{\Delta\xi}\begin{pmatrix} D_\xi & 0 & 0 & \dots & 0\\ 0 & D_\xi & 0 & \dots & 0\\ 0 & 0 & D_\xi & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & 0 & D_\xi \end{pmatrix},\ D_\xi=\begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0\\ 0 & -1 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -1 & 1\\ 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix} \tag{31}$$

Note that $\bar{D}_\xi$ is a bi-diagonal $(MxN)\,x\,(MxN)$ matrix. At the boundary, a homogeneous Dirichlet condition is assumed (at least from the right.) This operator corresponds to $\frac{\partial}{\partial\xi}$.

Analogously, we can define the second-order forward difference operator over the state variable:

$$\bar{D}_{\xi^2}=\frac{1}{\Delta\xi^2}\begin{pmatrix} D_{\xi^2} & 0 & \dots & 0\\ 0 & D_{\xi^2} & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & D_{\xi^2} \end{pmatrix},\ D_{\xi^2}=\begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0\\ 0 & 1 & -2 & 1 & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1\\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

Here, D_{ξ^2} is (NxN) dimensional, and $\bar{D}_{\xi^2}$ is a $(MxN)\,x\,(MxN)$ block-tri-diagonal matrix. This operator corresponds to $\frac{\partial^2}{\partial\xi^2}$.

We now identify the first-order forward difference operator over time:

$$\bar{D}_v = \frac{1}{\Delta v}\begin{pmatrix} -D_v & D_v & 0 & \dots & 0 & 0 \\ 0 & -D_v & D_v & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -D_v & D_v \\ 0 & 0 & 0 & \dots & 0 & -D_v \end{pmatrix}, \; D_v = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{33}$$

In this equation, D_v is an (NxN) identity matrix. $\bar{D}_v$ is a $(MxN)\,x\,(MxN)$ matrix which has only two non-zero diagonals, and which corresponds to $\frac{\partial}{\partial v}$.

Defining the forward second difference operator over time yields

$$\bar{D}_{v^2} = \frac{1}{\Delta v^2}\begin{pmatrix} D^1_{v^2} & D^2_{v^2} & D^1_{v^2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & D^1_{v^2} & D^2_{v^2} & D^1_{v^2} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & D^1_{v^2} & D^2_{v^2} & D^1_{v^2} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & D^1_{v^2} & D^2_{v^2} & D^1_{v^2} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & D^1_{v^2} & D^2_{v^2} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & D^1_{v^2} \end{pmatrix} \tag{34}$$

$$D^1_{v^2} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \; and\, D^2_{v^2} = \begin{pmatrix} -2 & 0 & \dots & 0 \\ 0 & -2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -2 \end{pmatrix}$$

Here, $\bar{D}_{v^2}$ is an $(MxN)\,x\,(MxN)$ block tri-diagonal matrix with three (NxN) blocks, $D^1_{v^2}$, $D^2_{v^2}$, and, again, $D^1_{v^2}$. For these three matrices, a homogenous "boundary condition" is imposed at $v \geq 1$. This matrix operator corresponds to $\frac{\partial^2}{\partial v^2}$.

Finally, we define the mixed forward difference operator over both time and state:

$$\bar{D}_{v\xi} = \frac{1}{\Delta v \Delta \xi}\begin{pmatrix} D_{v\xi} & -D_{v\xi} & 0 & \dots & 0 & 0 \\ 0 & D_{v\xi} & -D_{v\xi} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & D_{v\xi} & -D_{v\xi} \\ 0 & 0 & 0 & \dots & 0 & D_{v\xi} \end{pmatrix}, \; D_{v\xi} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{35}$$

In this equation, $D_{v\xi}$ is an (NxN) bi-diagonal matrix, and $\bar{D}_{v\xi}$ is an $(MxN)\,x\,(MxN)$ block bi-diagonal matrix which corresponds to $\frac{\partial^2}{\partial v \partial \xi}$.

Returning to our optimand (16), we define our regularizer as a function of the difference operators and the general column vector, G.

$$\begin{aligned}\Omega(G) &= \Delta v \Delta \xi \Big[(G;G) + \left(\bar{D}_{\xi}G;\bar{D}_{\xi}G\right) + \left(\bar{D}_{\xi^2}G;\bar{D}_{\xi^2}G\right) \\ &\quad + \left(\bar{D}_{v}G;\bar{D}_{v}G\right) + \left(\bar{D}_{v^2}G;\bar{D}_{v^2}G\right) + \left(\bar{D}_{v\xi}G;\bar{D}_{v\xi}G\right)\Big]\end{aligned} \tag{36}$$

or,

$$\begin{aligned}\Omega(G) &= \Delta v \Delta \xi \Big[(G;G) + \left(\bar{D}'_{\xi}\bar{D}_{\xi}G;G\right) + \left(\bar{D}'_{\xi^2}\bar{D}_{\xi^2}G;G\right) \\ &\quad + \left(\bar{D}'_{v}\bar{D}_{v}G;G\right) + \left(\bar{D}'_{v^2}\bar{D}_{v^2}G;G\right) + \left(\bar{D}'_{v\,\xi}\bar{D}_{v\,\xi}G;G\right)\Big]\end{aligned}$$

The $(U;V)$ operation is the L_2 scalar product of the $(MxN)\,x\,1$ dimensional vectors U and V.

Corresponding to equations (17.0) and (17.1), we propose two respective choices for the function to be regularized:

$$G = \begin{bmatrix} \bar{a}_1(1,0) \\ \vdots \\ \bar{a}_1(N,0) \\ \bar{a}_1(1,1) \\ \vdots \\ \bar{a}_1(N,M-1) \end{bmatrix} \tag{37.0}$$

$$G = F_1 \tag{37.1}$$

As implied by Kreiss' discrete energy estimates, these types of regularizers guarantee the convergence of our procedure.[8] To simplify notation, we introduce a regularizing matrix:

$$B = I + \bar{D}'_{\xi}\bar{D}_{\xi} + \bar{D}'_{\xi^2}\bar{D}_{\xi^2} + \bar{D}'_{v}\bar{D}_{v} + \bar{D}'_{v^2}\bar{D}_{v^2} + \bar{D}'_{v\,\xi}\bar{D}_{v\,\xi} \tag{38}$$

[8] See Kreiss and Lorenz [20] and its list of related works.

Restating the regularizer (36), we have

$$\Omega(G) = \; \Delta v \Delta \xi \, (B\ G; G) = \Delta v \Delta \xi \ G' \, B \ G \tag{39}$$

To solve for a unique $\bar{a}_1 \; = [\bar{a}_1(1,0)\,,\,...,\,\bar{a}_1(N,0)\,,\,\bar{a}_1(1,1)\,,\,...,\,\bar{a}_1(N,M-1)\ ]$, we define the discrete analogue to optimand (16). This discrete optimand is the sum of the squared pricing errors, which are squared deviations from equation (28), and the regularizer, (39):

$$M_1^{\alpha}(F_1) \; = \left(\bar{A}_p F_1 - \left(H_p - H_p^0\right); \bar{A}_p\, F_1 - \left(H_p - H_p^0\right)\right) \; + \alpha_1\, \Omega(G) \tag{40}$$

H_p is the $[(mxn)\,x\,1]$ quoted option price vector, which is obtained by eliminating the option prices (rows) that correspond to nonquoted options. Similarly, H_p^0 is the $[(mxn)\,x\,1]$ Black-Scholes-Merton model price vector that corresponds to the quoted options. Simplifying notation, we denote the difference between quoted option prices and the Black-Scholes-Merton model prices as follows: $\bar{H}_p^1 = H_p - H_p^0$.

Substituting this definition into equation (40),

$$M_1^{\alpha}(F_1) \; = \left(\left(\bar{A}_p' \bar{A}_p\, F_1 - 2\bar{A}_p' \bar{H}_p^1\right) \, ; F_1\right) \; + \; \left(\bar{H}_p^1; \bar{H}_p^1\right) + \alpha_1 \Omega(G) \tag{41}$$

In equations (37.0) and (37.1), we defined two potential regularizers. We now treat the $G = F_1$ case in detail.

Optimand (41) is minimized when its gradient with respect to F_1 is zero. Equating this gradient to zero yields

$$\bar{A}_p' \bar{H}_p^1 = \left(\bar{A}_p' \bar{A}_p + \alpha_1\, \Delta v \Delta \xi\, B\right) F_1 \tag{42}$$

This equation system forms an $(MxN)\,x\,(MxN)$ linear system in $(MxN)\,x\,(MxN)$ variables. Clearly, equation (42) corresponds to the Euler equation of minimand (41). Because $\bar{a}_1$ enters linear equation system (42) only in F_1 and B is a symmetric positive-definite matrix, the solution is feasible. We now state a transformation that yields very efficient solution.

We digress to the equation (27) matrix $\bar{A}$. The lower diagonal blocks of matrix $\bar{A}$ may be diagonalized: $A^k = UD^kU$, for $k = 0, 1, \ \ldots, \ M - 1$.

U is an orthogonal matrix consisting of the eigenvectors of A^k, and D^k is a diagonal matrix consisting of the eigenvalues of A^k.[9]

$$\bar{A} = \bar{U}'C\bar{U}, \ \bar{U} = \begin{pmatrix} U & 0 & 0 & \ldots & 0 \\ 0 & U & 0 & \ldots & 0 \\ 0 & 0 & U & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & U \end{pmatrix} \tag{43}$$

$$C = \begin{pmatrix} I & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ D & I & 0 & 0 & \ldots & 0 & 0 & 0 \\ D^2 & D & I & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ D^{M-2} & D^{M-3} & D^{M-4} & D^{M-5} & \ldots & D & I & 0 \\ D^{M-1} & D^{M-2} & D^{M-3} & D^{M-4} & \ldots & D^2 & D & I \end{pmatrix}$$

Since only a subset of options are traded, we will work only with a submatrix of matrix $\bar{A}$. Analogous to the equation (28) projection, we have $\bar{U}_p$ as the appropriate submatrix of $\bar{U}$.

$$\bar{U}_p' = \begin{pmatrix} 0 & & 0 & U_1' & 0 & & 0 & 0 & \ldots & 0 \\ 0 & & 0 & 0 & 0 & & 0 & U_2' & \ldots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ldots & \vdots \\ 0 & & 0 & 0 & 0 & & 0 & 0 & \ldots & U_m' \end{pmatrix} \tag{44}$$

Noting that matrix U is orthogonal and that the inverse of matrix C exists (and is much simpler than C) and requires much less storage, matrix multiplication and inversion transform equation system (42):

$$\bar{U}_p\bar{H}_p^1 = \left(\bar{U}_p\bar{U}_p' + \tilde{\alpha}_1 \left(C'\right)^{-1} \ \bar{U}B\bar{U}'C^{-1}\right) C\bar{U}F_1 \tag{45}$$

Equation (42) indicates the motivation for our transformations. The matrix product that multiplies the transformed regularizing parameter, $\tilde{\alpha}_1 = \alpha_1 \Delta v \Delta \xi$, is unaffected

[9] Analytic expressions are available for matrices U and D. See Smith[27].

by the set of options that trade on a particular day. Given P_0, M and N, the matrix product is fixed.

Form the Choelsky decomposition of the original norm-related matrix B. The upper block-triangular matrix of the decomposition is matrix S. Define $L = (C')^{-1}\ \bar{U}S'$, and $\phi_1 = C\ \bar{U}F_1$:

$$\bar{U}_p\bar{H}_p^1 = \left(\bar{U}_p\bar{U}_p' + \tilde{\alpha}_1 LL'\right)\phi_1 \tag{46}$$

To fully solve equation (46), we may apply the generalized singular value decomposition (GSVD) of $\bar{U}_p'$ and L'.[10] Let $W_1\,\bar{U}_p'X = D_1$ and $W_2\,L'X = D_2$. D_1 and D_2 are diagonal matrices, and W_1 and W_2 are orthogonal matrices with appropriate dimensions, and X is a $(MxN)\,x\,(MxN)$ invertible matrix. Also, let $\Psi_1 = X^{-1}\,\phi_1$. Matrix multiplication and inversion yield a diagonal equation system:

$$D_1'W_1\bar{H}_p^1 = \left(D_1'D_1 + \tilde{\alpha}_1 D_2'D_2\right)\ \Psi_1 \tag{47}$$

Given an $\tilde{\alpha}_1$, the solution of this diagonal system is trivial.

To choose α_1, we must solve for the root of the discrepancy function, equation (16). Calculating the discrepancy function requires calculation of H_1. Since the solution to equation (47) is a transformation of H_1, Ψ_1, we recover as follows: $H_1 = U'\,X\Psi_1$. The solution to our problem is the root of $\rho_1(\alpha_1)$:

$$\rho_1(\alpha_1) = \sum_{j=1}^{m}\sum_{i=1}^{n}\left|H_1(v_j,\xi_i) - \left(H^{i,j} - H_0(v_j,\xi_i)\right)\right|^2 - \delta^2 \tag{48}$$

Since the equation is linear in F_1, these parameter estimates may be concentrated out of the search for the discrepancy function root. Therefore, the only parameter that enters the discrepancy function (nonlinearly) is α_1. The search for α_1 may follow usual bisection, secant, or other similar methods. To recover the volatility surface, we scale

[10] See Golub and Van Loan[16]. In the special case of the $H^0 - norm$ based regularizer, the equation (38) B matrix is the identity matrix and the equation system may be solved using the standard singular value decomposition (SVD) method. In section six, we implement the $H^0 - norm$ and SVD based estimator.

the forcing term vector by the associated gamma, $F_1 = U' C^{-1} X \Psi_1$. The determination of the volatility surface, $\bar{a}_1$, from F_1 is trivial.

For the alternative regularizer of equation (17.0.1), $G = \bar{a}_1$, the only change in the solution procedure is a slight modification of $\bar{A}_p$ matrix in gradient equation (42). The augmented $\bar{A}_p$ is obtained by multiplying each of the MxN columns of the original $\bar{A}_p$ matrix by the appropriate second partial derivative with respect to the state variable.[11]

Calculation of discrepancy function (48) can be expedited. As specified in equation (48), the European option values must be calculated at each step of the search over the regularizing parameter, α_1. Instead, the European values can be broken into two parts, a numerically integrated function and a forcing adjustment function. We now outline this respecification of equation (48).

The European option value satisfies the following partial differential equation and boundary conditions:

$$\frac{\partial H_0(v,\xi)}{\partial v} = \tilde{a}_0 \frac{\partial^2 H_0(v,\xi)}{\partial \xi^2}, \qquad with\ H_0(0,\xi) = Max\left(0,\ e^{\xi} - 1\right) \tag{49}$$
$$H_0(v,-L) = 0, and\, H_0(v,L) = Max\left(0,\ e^{L} - 1\right)$$

We let $\varphi(v,\xi)$ be a continuous function of two variables. The function is defined over the ranges $0 \leq v \leq 1$ and $-L \leq \xi \leq L$, and is sufficiently smooth for $v > 0$. We require the function to have three attributes: $\varphi(0,\xi) = Max\left(0,\ e^{\xi} - 1\right)$, $\varphi(v,-L) = 0$, and $\varphi(v,L) = Max\left(0,\ e^{L} - 1\right)$. The construction of $\varphi(v,\xi)$ relies on a particular cut-off or buffer function. Our construction of this function is undertaken in the appendix.

Given the function $\varphi(v,\xi)$, we define the value difference $\bar{H}_0(0,\xi) = H_0(0,\xi) - \varphi(v,\xi)$. We substitute for $H_0(0,\xi)$ in the second-order partial differential equation part of equation (50), and find

$$\frac{\partial \bar{H}_0(v,\xi)}{\partial v} = \tilde{a}_0 \frac{\partial^2 \bar{H}_0(v,\xi)}{\partial \xi^2} + F_0(v,\xi)$$

[11] The first column is multiplied by $\Delta v\, \frac{\partial^2 H_0}{\partial \xi^2}(0,1)$, the second column is multiplied by $\Delta v\, \frac{\partial^2 H_0}{\partial \xi^2}(0,2)$, ..., and the MxN column is multiplied by $\Delta v\, \frac{\partial^2 H_0}{\partial \xi^2}(M-1,N)$.

$$where, F_0\left(v,\xi\right) = \ \tilde{a}_0 \frac{\partial^2 \varphi\left(v,\xi\right)}{\partial \xi^2} \ - \ \frac{\partial \varphi\left(v,\xi\right)}{\partial v} \tag{50}$$

Additionally, we transform the boundary conditions of equation (50):

$$\frac{\partial \bar{H}_0\left(v,\xi\right)}{\partial v} = \tilde{a}_0 \frac{\partial^2 \bar{H}_0\left(v,\xi\right)}{\partial \xi^2} + F_0\left(v,\xi\right), \qquad with\ \bar{H}_0\left(0,\xi\right) = 0$$
$$\bar{H}_0\left(v,-L\right) = 0, and\, \bar{H}_0\left(v,L\right) = 0 \tag{51}$$

We next denote $F_0\left(i,j\right) = F_0\left(v_j,\xi_i\right)$ and $\bar{H}_0\left(i,j\right) = \bar{H}_0\left(v_j,\xi_i\right)$. Analogous to equation (27), equation (51) can be written as follows:

$$\bar{H}_0 = \bar{A}\,F_0,\ where \tag{52}$$

$$\bar{H}_0 = \begin{bmatrix} \bar{H}_0\left[1\right] \\ \bar{H}_0\left[2\right] \\ \vdots \\ \bar{H}_0\left[M\right] \end{bmatrix},\ F_0 = \begin{bmatrix} F_0\left[0\right] \\ F_0\left[1\right] \\ \vdots \\ F_0\left[M-1\right] \end{bmatrix},$$

$$\bar{H}_0\left[j\right] = \begin{bmatrix} \bar{H}_0\left(1,j\right) \\ \bar{H}_0\left(2,j\right) \\ \vdots \\ \bar{H}_0\left(N,j\right) \end{bmatrix},\ for\ j = 1,\ldots,M,\ and$$

$$F_0\left[j\right] = \begin{bmatrix} F_0\left(1,j\right) \\ F_0\left(2,j\right) \\ \vdots \\ F_0\left(N,j\right) \end{bmatrix}\ for\ j = 1,\ldots,M-1$$

Corresponding to our previous development of equation (28), we now treat an (mxn) dimensional projection (or subvector) of the full (MxN) dimensional vector $\bar{H}_0$. The superposition of $\bar{A}$ with this projection results in $\bar{A}_p$:

$$\bar{H}_p^0 = \bar{A}_p F_0 \tag{53}$$

Analogously, we define the (mxn) vector F_p^0 which is obtained by eliminating the coordinates of vector F_0, that correspond to nonquoted options. H_p is the $[(mxn)\,x\,1]$

vector that is obtained by eliminating the rows that correspond to nonquoted options from the option quote vector.

The discrepancy function [(48)] is equal to the following:

$$\rho_1(\alpha_1) = \left(H_p^1 - \left(H_p - H_p^0\right); H_p^1 - \left(H_p - H_p^0\right)\right) - \delta^2 \tag{54}$$

This discrepancy function definition may be redefined by transformations that are similar to those made in equations (43) - (47):

$$\begin{aligned} \rho_1(\alpha_1) &= \left(D_1(\Psi_0 + \Psi_1) + W_1\left(\Psi_p^0 - H_p\right); D_1(\Psi_0 + \Psi_1) + W_1\left(\Psi_p^0 - H_p\right)\right) - \delta^2 \\ & where,\ \Psi_0 = X^{-1}C\bar{U}F_0 \end{aligned} \tag{55}$$

The transformation of equation (48) into equation (55) relies on the orthogonality of matrix W_1, and the invariance of the Euclidean norm with respect to orthogonal transformations. Additionally, the GSVD definition implies that $\bar{U}_p'X = W_1'D_1$.

Solving for the root of equation (55) is more efficient than solving for the root of the same discrepancy function [equation (48)]. As outlined in our discussion of equation (48), the first adjustment to the volatility surface estimate, $\bar{a}_1$, is recovered algebraically.

Finally, we can make higher-order adjustments to the estimated volatility function. Analogous to the development of the first-order expansion, we define $M_k^\alpha(\bullet)$ and $\rho_k(\alpha_k)$. The solution is the root of $\rho_k(\alpha_k)$. Because determination of $\bar{a}_1$ embodies many of the calculations required to calculate $\bar{a}_k$, the estimation of higher-order volatility surface adjustments is relatively quick.[12] The convergence of $\sum_{k=0}^{\infty}\bar{a}_k$ requires that the $\bar{a}_k's$ be sufficiently small. Our choice of the regularizer (17.0), for instance, delivers $\bar{a}_k's$ with the minimal $H^p - norm$ that fits the residual least squares criterion to the required discrepancy. Given $p \geq 4$ and Sobolev inequalities, convergence of this sum in the $H^p - norm$ implies convergence in the maximum-norm.

[12] Corresponding to equation (42), replace $\bar{H}_p^1$ by $\bar{H}_p^k = H_p - \sum_{r=1}^{k-1} H_p^r$ (which are available from the $k-1$ step calculations) and Ψ_1 by $\Psi_k = X^{-1}\phi_k$, with $\phi_k = C\bar{U}F_k$ and $F_k = \sum_{r=1}^{k}\bar{a}_k(v,\xi)\left[\frac{\partial^2 H_{k-r}(v,\xi)}{\partial\xi^2} - \frac{\partial H_{k-r}(v,\xi)}{\partial\xi}\right]$. Analogous adjustments are made to $\rho_k(\alpha_k)$.

6 An Example

Table 1 lists the 68 CBOE S&P 500 (SPX) option transactions that occurred following the market open at 8:31 am until 8:48 am on November 20, 1996.[13] The table reports trade time, price and volume. Additionally, the average and maximum bid-ask spreads that prevailed for each option in our sample during the observation period are tabulated. Because multiple transactions of a single contract occurred within the bid-ask spread during the trading interval, these data together highlight the importance of an approximate option pricing scheme such as ours.

Over the observed trading interval, the average S&P 500 price was 743.13. Trade prices both began and ended the training interval near the minimum price of 742.28. The maximum price was 744.27.

The discount rates were 5.438% for the December maturity and 5.5625% for the March maturity. The associated stock index dividend yields were imputed from CME S&P 500 futures prices. These dividend yields were 1.722% for the December maturity and 1.802% for the March maturity. The January and February maturity discount rates and SPX dividend yields were interpolated from the December and March values.

Across the 68 option trades, the Black-Scholes model average sum of squared errors was 2.11. To improve on the Black-Scholes model prices, we have implemented the $H^0 norm$ regularizer-based version of our model. A lattice-based explicit finite difference scheme is implemented with 18 variable time steps for each the four segments of the option maturity set.

Figure 1 illustrates our estimation results. In Panel A, we depict the local volatility surface estimates with a 150 regularizer (α) weight. This set of estimates has a 64% pricing error relative to the Black-Scholes model pricing error (1.35 relative to 2.11.) The minimum local volatility estimate ($\underline{\sigma} = 10.25\%$) is associated with the at-the-money strike price and the time step just prior to the December option maturity. The maximum local volatility estimate is associated with out of the money put strike prices

[13] All market prices were kindly supplied by Hull Trading.

($\overline{\sigma} = 16.75\%$.) Across the local volatility estimates, a clear SPX price-related local volatility pattern is manifest: a volatility trough at the money and a peak for out of the money puts.

Figure 1 - Panel B depicts the estimated volatility surface as the regularization constraint is relaxed to a 33.4 (α). In this case, the fit is improved significantly relative to Black-Scholes (with a 39% relative pricing error). However, the estimated surface is neither sufficiently smooth nor provides a fully satisfactory fit.[14]

To further address the model fitting issue, the Figure 2 - Panels (A, B and C) plot individual option pricing errors across three maturity classes. In depicting these plots, we to emphasize that the implied volatility "prices" that are graphed vary across both time and state. Such implied volatility heterogeneity across option strikes violates the assumptions underlying the standard Black-Scholes-Merton model. However, these misspecified implied volatility prices have been adopted as an alternative depiction relative option values. Nevertheless, the misspecified estimates provide a useful rescaling of actual option prices.

In Figure 2 - Panel A, the shortest 30 day maturity options values are graphed. Panel B plots the 58 day maturity option values, and panel C plots the longer 93 and 121 day maturity option values. Across the Figure 2 panels, transformed option prices, associated bids and asks, and our non-parametric model values are plotted in implied volatility terms. Uniformly across the Figure panels, we observe underpricing of low strike price options. Therefore, our example implemenation has not fully eliminated this well-known S&P 500 option pricing bias (underpricing out-of-money puts). To go further in eliminating this option strike price-related valuation bias, a state-dependent volatility surface prior should be used instead of the flat Black-Scholes volatility surface prior which we impose. Largely, this generalization follows respecification of the constant $sigma0^2$

[14]In practice, the usual target for average sum of squared model errors is between the average or maximum sum of squared bid-ask spreads. As reported at the bottom of Table I - Panel B, these bid-ask spread based errors are 0.23 and 0.31, respectively.

prior to depend on state and time ($sigma0^2(v,Z)$) in equation (8), and redefinition of the associated equation (25) $\bar{A}$ matrix.

We have also noted that the estimated volatility surface in Figure 1 - Panel B is not sufficiently smooth. The roughness of our estimated volatility surface will be corrected by implementation of the $H^2 norm$ based regularizer that has been fully specified in our model development. Unfortunately, this higher order norm-based method requires solution of a large scale GSVD that is beyond our current computing power. Since the $H^0 norm$ based specification calculates in seconds, the associated estimates provide a useful set of real-time process control parameters. Furthermore, we strongly conjecture that increased scientific computing power and improved methods will soon make the $H^2 norm$ and even higher-order norm based implementations readily tractable. We note also that implementation of such higher order surface restrictions are straight-forward extensions of our approach, and that such specifications will yield well-behaved derivative hedge parameter estimates (e.g. local volatility-based deltas and gammas.)

7 Conclusion

The matrix transformations that we have introduced for the $\bar{A}_p$ valuation operator matrix results in a sparse system. Therefore, our volatility surface estimates may be computed in a reasonable amount of time. Additionally, fixing the time and state step sizes and identifying a suitable starting value set implies that the majority of day to day volatility surface estimation calculations may be done once.

In our application, we have used a forward first-order in time and central second-order in state explicit finite-difference scheme. This specification is embodied in our definition of the equation (25) $\bar{A}$ matrix. This scheme is both most basic and reduces to the binomial and trinomial methods that are most often used in valuing financial derivatives. Nevertheless, higher-order one-step and multi-step explicit, implicit or predictor-corrector semi-discretization methods will be more efficient computationally. Such methods result in the equation (25) tridiagonal $\bar{A}$ matrix becoming denser band-matrices. Otherwise, the analysis is the same.

In our estimation example, we have illustrated the tractability of the simplest version of our model with S&P 500 option trade data. The resulting set of estimates indicate the need to implement our full model which imposes regularization constraints up to and including the $H^2 norm$.[15]

The linearization-based estimation procedure detailed in this paper readily generalizes for multiple state variable cases. In these cases, a multivariate normal kernel replaces the univariate normal kernel of equation (15). Additionally, the forcing term in equation (15) and the stabilizing function of equation (17.1) become more complex. Nonetheless, even with multiple state variables, the required analysis of these functions and computational effort are quite tractable. Furthermore, our framework can be extended to incorporate American and exotic option quotes. Finally, an optimization-based framework, such as ours, leads naturally to time series analysis of the volatility surfaces.

[15] Another approach follows Avellanede et. al. and restricts the range of volatility estimates.

8 Appendix - Constructing the $\varphi(v,\xi)$ Function

In this appendix, we construct a function $\varphi(v,\xi)$ which is a continuous and at least twice differentiable function of two variables, v and ξ. The ranges of these variables are $0 \leq v \leq 1$ and $-L \leq \xi \leq L$, respectively. The function must also satisfy the following conditions: $\varphi(0,\xi) = Max\left(0,\, e^{\xi}-1\right)$, $\varphi(v,-L) = 0,\ and\ \varphi(v,L) = Max\left(0,\, e^{L}-1\right)$. In our case, the function will be infinitely differentiable for $v > 0$.

We now recall the well-known cut-off or buffer function, $\kappa(\xi)$.[16] To construct this function, we first define the following infinitely differentiable function:

$$h(\xi) = \begin{cases} 0 & \xi \leq 0 \\ e^{-1/\xi} & \xi > 0 \end{cases} \tag{A-1}$$

Next, we construct a special infinitely differentiable function, $g(\xi)$, such that $g(\xi) = 0$ for $\xi \leq 0$ and $g(\xi) = 1$ for $\xi \geq 1$:

$$g(\xi) = \frac{h(\xi)}{h(\xi) + h(1-\xi)} \tag{A-2}$$

As $g'(\xi) \geq 0$, we have $0 \leq g(\xi) \leq 1$.

We now define the cut-off function:

$$\kappa(\xi) = g(2\xi+2)\, g(-2\xi+2) \tag{A-3}$$

Clearly, $\kappa(\xi)$ is an infinitely differentiable function which satisfies $0 \leq k(\xi) \leq 1$ on $-\infty \leq \xi \leq \infty$, $\kappa(\xi) = 0$ for $|\xi| \geq 1$, and $\kappa(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$.

We now define the function of interest:

$$\varphi^{L}(v,\xi) = \begin{cases} \left[1-\kappa\left(\frac{\xi}{vL}\right)\right] Max\left(0, e^{\xi}-1\right) & for\ v > 0 \\ Max\left(0, e^{\xi}-1\right) & for\ v = 0 \end{cases} \tag{A-4}$$

In verifying the properties of this function, we first examine its values at $\xi = \pm L$, and then verify its continuity and differentiability. We analyze the function for $\frac{|\xi|}{vL} \geq 1$, and have $\kappa\left(\frac{\xi}{vL}\right) = 0$ and $\varphi^{L}(v,\xi) = Max\left(0, e^{\xi}-1\right)$. Therefore, if $\xi = L \geq vL$, then $\varphi^{L}(v,L) = e^{L}-1$. Similarly, if $\xi = -L \leq -vL$, then $\varphi^{L}(v,L) = 0$.

[16] See Mankres[22].

It is obvious that the function $\varphi^L(v,\xi)$ is infinitely differentiable for $v=0$ and $\xi \neq 0$. To show the smoothness of the function at $(v \neq 0,\ \xi = 0)$, we have

$$\frac{\partial \varphi^L(v,0)}{\partial \xi} = \lim_{\substack{\xi_n \to 0 \\ \xi_n \neq 0}} \frac{\varphi^L(v,\xi_n)}{\xi_n} = \lim_{\substack{\xi_n \to 0 \\ \xi_n \neq 0}} \frac{\left[1-\kappa\left(\frac{\xi}{vL}\right)\right] Max\left(0, e^{\xi}-1\right)}{\xi_n} \qquad \text{(A-5)}$$

We partition the sequence $\{\xi_n\}_{n=1}^{\infty}$ into two subsequences, $\left\{\xi_n^{(1)}\right\}_{n=1}^{\infty}$ and $\left\{\xi_n^{(2)}\right\}_{n=1}^{\infty}$, which consist of the positive and negative parts, respectively. Substituting in equation $(A-5)$, we have

$$\lim_{\xi_n^{(1)} \to 0} \frac{\left[1-\kappa\left(\frac{\xi_n^{(1)}}{vL}\right)\right] Max\left(0, e^{\xi_n^{(1)}}-1\right)}{\xi_n^{(1)}} = \lim_{\xi_n^{(1)} \to 0} \left[1-\kappa\left(\frac{\xi_n^{(1)}}{vL}\right)\right] \lim_{\xi_n^{(1)} \to 0} \frac{e^{\xi_n^{(1)}}-1}{\xi_n^{(1)}} = 0, \qquad \text{(A-6)}$$

$$and \lim_{\xi_n^{(2)} \to 0} \frac{\left[1-\kappa\left(\frac{\xi_n^{(2)}}{vL}\right)\right] Max\left(0, e^{\xi_n^{(2)}}-1\right)}{\xi_n^{(2)}} = \lim_{\xi_n^{(2)} \to 0} \left[\frac{\left[1-\kappa\left(\frac{\xi_n^{(2)}}{vL}\right)\right]}{\xi_n^{(2)}} \bullet 0\right] = 0$$

Therefore, $\varphi^L(v,\xi)$ is smooth at $(v,0)$ with respect to ξ for all $v>0$. We can show the existence of all higher order derivatives with respect to ξ by a similar argument. For $v>0$, the differentiability of $\varphi^L(v,\xi)$ with respect to v is obvious.

The continuity of $\varphi^L(v,\xi)$ at all $(v,\xi) \neq (0,0)$ is shown as follows: Let $v_n \to 0$ and $\xi_n \to 0$ be two sequences of nonzero numbers in $(0,1]$ and $[-L,L]$, respectively.

$$\varphi^L(v_n,\xi_n) = \left[1-\kappa\left(\frac{\xi_n}{v_n L}\right)\right] Max\left(0, e^{\xi_n}-1\right) \qquad \text{(A-7)}$$

Recognizing that $\left|1-\kappa\left(\frac{\xi_n}{v_n L}\right)\right| \leq 1$ implies that $\lim_{n \to \infty} \varphi^L(v_n,\xi_n) = 0$. Therefore, $\varphi^L(v,\xi)$ is continuous at $(0,0)$.

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Table I: SPX Option Trades - November 20, 1996 8:30-8:48am

Panel A: December 20,1996 Maturity and Less Than 750 Strike Price

						Bid-Ask Spread	
Maturity	Strike Price	Put/Call	Time	Price	Volume	Average	Maximum
20-Dec-96	675	P	8:40:48	1.00	10	0.158	0.250
20-Dec-96	675	P	8:43:17	1.00	5	0.158	0.250
20-Dec-96	675	P	8:43:17	1.00	5	0.158	0.250
20-Dec-96	675	P	8:43:34	0.88	5	0.158	0.250
20-Dec-96	675	P	8:43:34	0.88	5	0.158	0.250
20-Dec-96	680	P	8:37:42	1.25	1	0.133	0.188
20-Dec-96	700	P	8:36:49	2.19	20	0.175	0.250
20-Dec-96	715	P	8:41:33	3.50	5	0.313	0.375
20-Dec-96	715	P	8:43:03	3.50	5	0.313	0.375
20-Dec-96	715	P	8:43:04	3.50	5	0.313	0.375
20-Dec-96	715	P	8:43:16	3.50	1	0.313	0.375
20-Dec-96	715	P	8:43:17	3.38	10	0.313	0.375
20-Dec-96	720	P	8:37:00	4.50	10	0.304	0.375
20-Dec-96	720	P	8:39:19	4.38	1	0.304	0.375
20-Dec-96	720	P	8:48:22	4.25	10	0.304	0.375
20-Dec-96	725	P	8:40:25	5.13	10	0.240	0.375
20-Dec-96	725	P	8:41:04	5.13	10	0.240	0.375
20-Dec-96	725	P	8:44:20	5.00	1	0.240	0.375
20-Dec-96	725	P	8:44:44	5.00	1	0.240	0.375
20-Dec-96	730	P	8:34:55	6.13	1	0.333	0.375
20-Dec-96	730	P	8:38:33	5.88	10	0.333	0.375
20-Dec-96	730	P	8:38:51	5.88	1	0.333	0.375
20-Dec-96	730	P	8:44:29	6.13	33	0.333	0.375
20-Dec-96	730	P	8:45:52	6.00	1	0.333	0.375
20-Dec-96	730	P	8:46:35	6.38	5	0.333	0.375
20-Dec-96	730	P	8:47:24	6.38	2	0.333	0.375
20-Dec-96	735	P	8:41:14	7.38	21	0.434	0.500
20-Dec-96	735	P	8:43:35	7.00	10	0.434	0.500
20-Dec-96	735	P	8:43:35	7.00	10	0.434	0.500
20-Dec-96	735	P	8:43:41	7.00	5	0.434	0.500
20-Dec-96	735	P	8:45:25	7.38	2	0.434	0.500
20-Dec-96	740	C	8:31:44	14.25	1	0.673	0.750
20-Dec-96	740	C	8:48:07	14.75	5	0.673	0.750
20-Dec-96	745	C	8:37:08	11.25	25	0.675	0.750
20-Dec-96	745	C	8:40:23	11.63	10	0.675	0.750
20-Dec-96	745	C	8:46:25	11.88	5	0.675	0.750
20-Dec-96	745	P	8:38:02	10.88	51	0.702	0.750
20-Dec-96	745	P	8:41:50	10.75	25	0.702	0.750
20-Dec-96	745	P	8:43:01	10.63	30	0.702	0.750
20-Dec-96	745	P	8:45:37	11.00	25	0.702	0.750

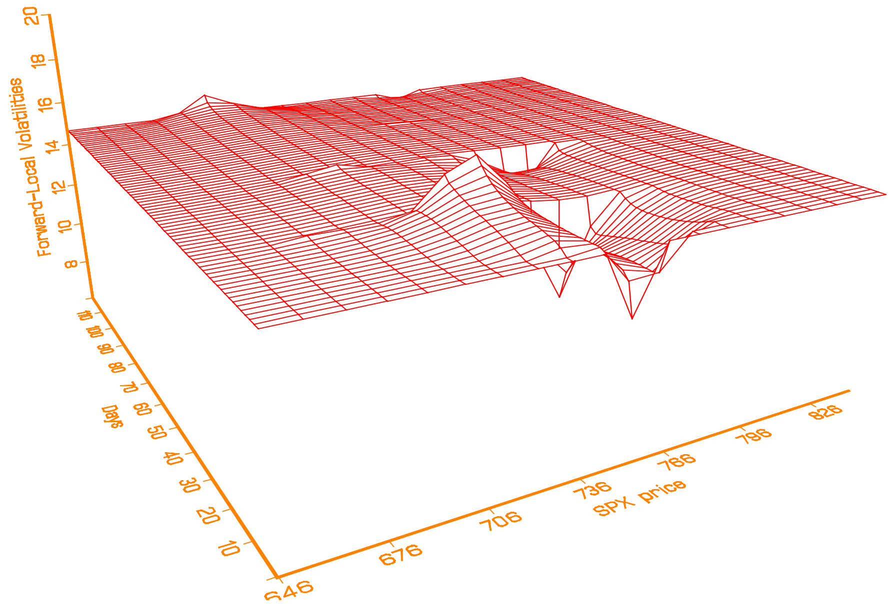
Table I: Table I: SPX Option Trades - November 20, 1996 8:30-8:48am

Panel B: Remaining Maturities and Strike Prices

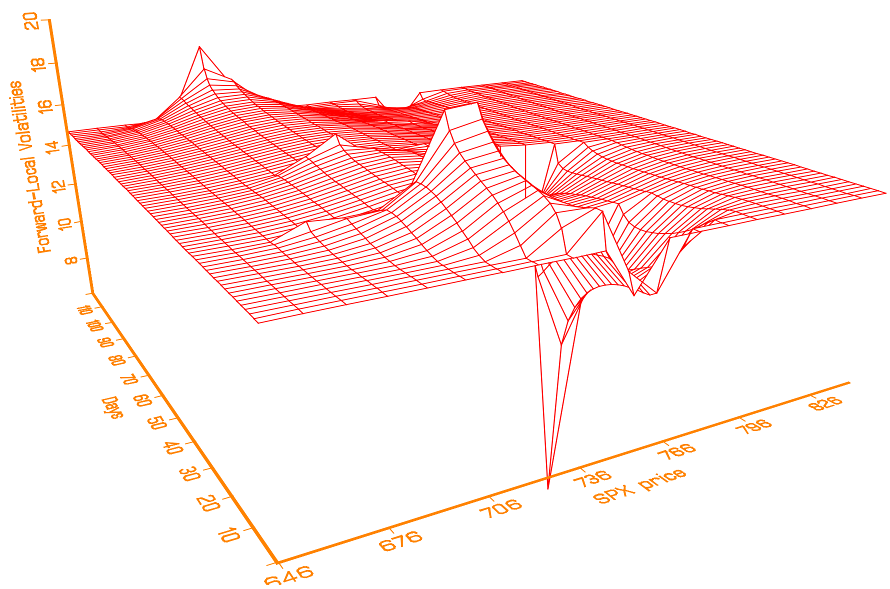
						Bid-Ask Spread	
Maturity	Strike Price	Put/Call	Time	Price	Volume	Average	Maximum
20-Dec-96	750	C	8:41:38	9.00	2	0.385	0.500
20-Dec-96	750	C	8:47:21	8.63	1	0.385	0.500
20-Dec-96	750	P	8:38:03	13.00	3	0.730	0.750
20-Dec-96	750	P	8:38:03	13.00	2	0.730	0.750
20-Dec-96	750	P	8:41:27	13.00	150	0.730	0.750
20-Dec-96	750	P	8:43:18	13.00	1	0.730	0.750
20-Dec-96	750	P	8:43:19	12.75	2	0.730	0.750
20-Dec-96	755	C	8:32:09	6.25	1	0.402	0.750
20-Dec-96	755	P	8:46:40	15.75	3	0.750	0.750
20-Dec-96	760	C	8:32:14	4.75	1	0.333	0.375
20-Dec-96	760	C	8:37:19	4.75	10	0.333	0.375
20-Dec-96	760	C	8:41:33	4.75	1	0.333	0.375
20-Dec-96	760	C	8:48:13	4.75	20	0.333	0.375
17-Jan-97	700	P	8:42:22	4.88	3	0.375	0.375
17-Jan-97	740	P	8:44:06	13.25	30	0.543	0.750
17-Jan-97	750	C	8:33:11	15.00	2	0.526	0.750
17-Jan-97	750	C	8:35:32	15.00	8	0.526	0.750
17-Jan-97	750	P	8:32:31	18.00	2	0.750	0.750
17-Jan-97	760	C	8:36:43	10.00	10	0.438	0.750
17-Jan-97	775	C	8:39:41	5.00	1	0.283	0.375
17-Jan-97	775	C	8:45:04	5.00	25	0.283	0.375
18-Jan-97	770	C	8:43:35	6.63	15	0.496	0.500
18-Jan-97	770	C	8:45:30	6.63	10	0.496	0.500
21-Feb-97	750	P	8:42:39	21.00	2	0.633	1.000
21-Mar-97	690	P	8:36:06	8.63	2	0.455	0.500
21-Mar-97	690	P	8:36:16	8.63	3	0.455	0.500
21-Mar-97	700	C	8:37:23	61.00	1	1.000	1.000
21-Mar-97	775	C	8:36:27	13.50	6	0.547	0.750
Average						0.440	0.521
Sum of Squares						0.232	0.312

Figure 1: CBOE SPX Option Local Volatility Surface Estimates - November 20, 1996

Panel A: α=150, error=1.30, $\underline{\sigma}$=0.102, $\overline{\sigma}$=0.168



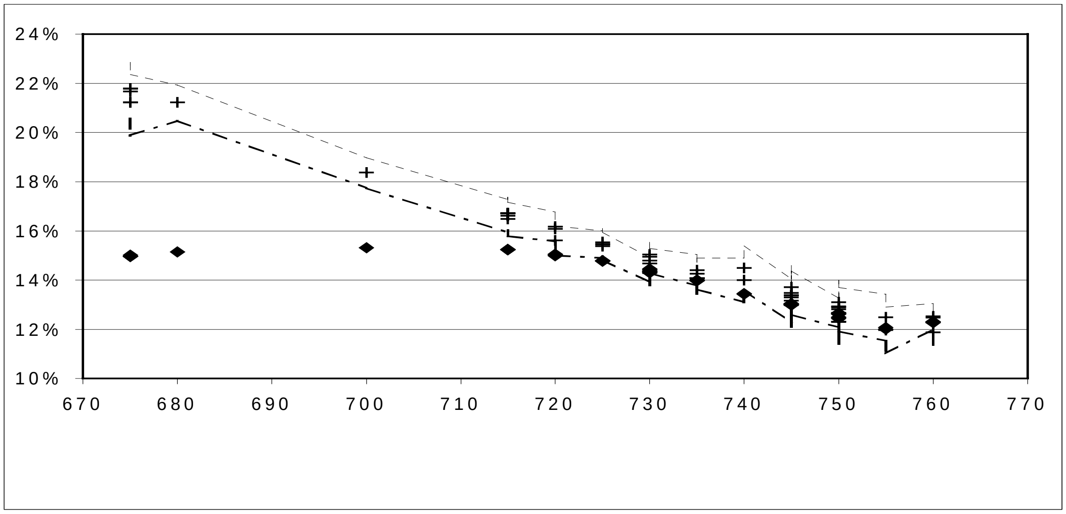
Panel B: α=33.4, error=0.82, $\underline{\sigma}$=0.002, $\overline{\sigma}$=0.187



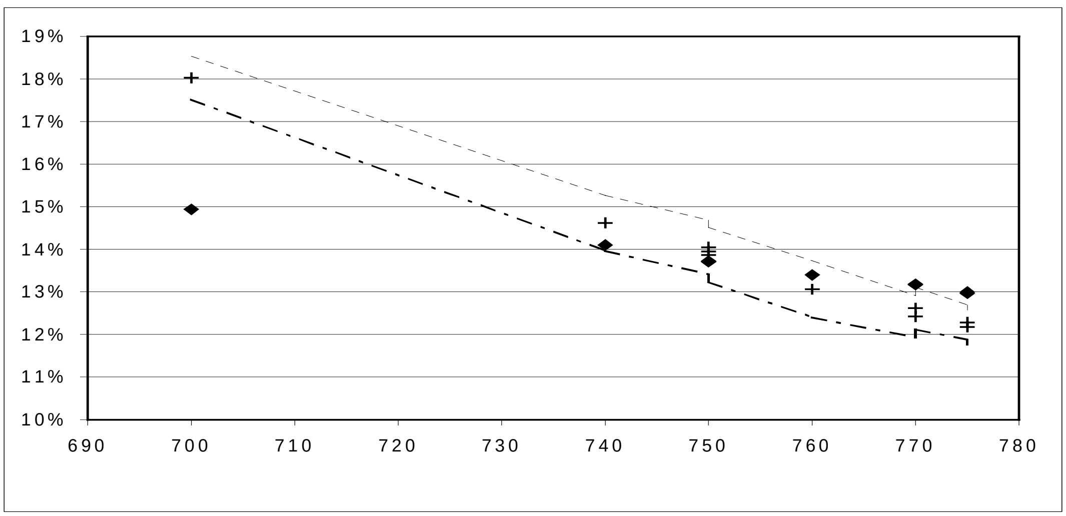
α is the regularizing parameter value. Error is the average sum of square pricing errors, and the benchmark Black-Scholes model error is 2.11. $\underline{\sigma}$ is the minimum local volatility estimate and $\overline{\sigma}$ is the maximum local volatility estimate.

Figure 2: Model Calibration - November 20, 1996
(in terms of misspecified Black-Scholes Model Implied Volatility "Prices")

Panel A: Options with 30 Days to Maturity



Panel B: Options with 58 Days to Maturity



Panel C: Options with 93 or 121 Days to Maturity