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Équations aux dérivées partielles/*Partial Differential Equations*
(Probabilités/*Probability Theory*)

Fully nonlinear stochastic partial differential equations

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Abstract. In this Note, we propose a new theory of "stochastic viscosity solutions" for fully nonlinear stochastic partial differential equations. This theory allows to handle a large class of equations which covers in particular various applications such as models of phase transitions and front propagation in random media and pathwise stochastic control. These applications will be detailed in a subsequent note. © Académie des Sciences/Elsevier, Paris

Équations aux dérivées partielles stochastiques complètement non-linéaires

Résumé. *Dans cette Note, nous proposons une théorie nouvelle de « solutions de viscosité stochastiques » pour les équations aux dérivées partielles stochastiques non-linéaires. Cette théorie permet de traiter une classe générale d'équations aux dérivées partielles stochastiques complètement non-linéaires qui contient en particulier diverses applications comme des modèles de transition de phase ou de propagation de fronts dans des milieux aléatoires et le contrôle stochastique trajectoriel. Ces applications seront détaillées dans une note ultérieure.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Nous introduisons dans cette Note une théorie nouvelle pour la résolution d'équations aux dérivées partielles stochastiques, éventuellement dégénérées, complètement non-linéaires de la forme :

$$\begin{cases} \mathrm{d}u = F(\mathrm{D}^2u, \mathrm{D}u)\mathrm{d}t + \sum_{i=1}^{m} H_i(\mathrm{D}u) \circ \mathrm{d}W_i & \text{dans } \mathbb{R}^N \times]0, +\infty[, \\ u = u_0 & \text{sur } \mathbb{R}^N \times \{0\}, \end{cases} \tag{0.1}$$

Note présentée par Pierre-Louis Lions.

où le signe $\circ$ correspond à la formulation de Stratonovich, $u_0 \in \mathrm{BUC}(\mathbb{R}^N)$, W est un mouvement brownien m-dimensionnel standard ($N, m \geq 1$) et $H \in \mathrm{C}^3(\mathbb{R}^N; \mathbb{R}^m)$. La fonction F est supposée continue et elliptique dégénérée, i.e. vérifiant pour tous $p \in \mathbb{R}^N$, $X, Y \in \mathrm{S}^N$ (l'espace des matrices $N \times N$ symétriques) :

$$F\,(X,p) \leq F\,(Y,p) \quad \text{si} \quad X \leq Y. \tag{0.2}$$

Notre théorie permet de traiter des équations plus générales que (0.1) où F et H peuvent également dépendre de (x,t,ω), où le « bruit blanc » $\mathrm{d}W_i(t)$ est remplacé par $\mathrm{d}W_i(t,x)$ et W_i est régulier en x, et à des classes de fonctions H moins régulières. Ces extensions sont détaillées dans [14] et nous nous restreignons à (0.1) pour simplifier la présentation.

Des équations du type (0.1) apparaissent dans des applications très variées (modèles de propagation de fronts et de changement de phase dans des milieux aléatoires, contrôle stochastique trajectoriel, équation de Zakaï en filtrage et en contrôle stochastique avec information partielle, modèles de taux en Finance mathématique...). Quelques applications principales sont détaillées dans [12].

Le résultat essentiel de cette Note porte sur la convergence d'approximation de (0.1) du type suivant :

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t} = F(\mathrm{D}^2 u^\epsilon, \mathrm{D}u^\epsilon) + \displaystyle\sum_{i=1}^m H_i(\mathrm{D}u^\epsilon)\dot{\zeta}_\epsilon^i & \text{dans} \quad \mathbb{R}^N \times]0,T[, \\ u^\epsilon|_{t=0} = u_0^\epsilon & \text{sur} \quad \mathbb{R}^N, \end{cases}$$

où u_0^ϵ converge uniformément vers u sur $\mathbb{R}^N$, ζ_ϵ est régulière sur $[0,T]$ (à valeurs dans $\mathbb{R}^m$) et converge uniformément sur $[0,T]$ vers W p.s. quand ϵ tend vers 0. On suppose que F vérifie la condition (classique dans la théorie des solutions de viscosité (*voir* [3], [2]...) : il existe $G \in \mathrm{C}(\mathrm{S}^{2N} \times \mathbb{R}^N)$ elliptique dégénérée telle que, pour tout $p \in \mathbb{R}^N$, $G\left(\begin{pmatrix} \lambda I & -\lambda I \\ -\lambda I & \lambda I \end{pmatrix}, p\right) = 0$ $(\forall \lambda > 0)$ et $F(X,p) - F(Y,p) \leq G(Z,p)$ si $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq Z$.

THÉORÈME. – *Sous ces conditions, u^ϵ converge uniformément sur $\mathbb{R}^N \times [0,T]$ p.s. et la limite est indépendante de l'approximation choisie ζ_ϵ.*

Nous donnons également une notion de solution de (0.1) satisfaite par la limite et prouvons dans certains cas que cette notion caractérise la limite.

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0. Introduction

In this Note we present a new theory for solutions of parabolic, possibly degenerate, second-order, stochastic partial differential equations, which, written in the Stratonovich sense, have the form:

$$\begin{cases} \mathrm{d}u = F(\mathrm{D}^2 u, Du)\mathrm{d}t + \displaystyle\sum_{i=1}^m H_i(\mathrm{D}u) \circ \mathrm{d}W_i & \text{in } \mathbb{R}^N \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{R}^N \times \{0\}. \end{cases} \tag{0.1}$$

Here, u_0 is bounded uniformly continuous on $\mathbb{R}^N$, $W = (W_1, \ldots, W_m)$ is a standard m-dimensional Brownian motion in time and, hence, $\mathrm{d}W = (\mathrm{d}W_1, \ldots, \mathrm{d}W_m)$ is the "usual" m-dimensional white noise in time, and $H = (H_1, \ldots, H_m) \in \mathrm{C}^3(\mathbb{R}^N; \mathbb{R}^m)$. The function F is assumed to be continuous and degenerate elliptic, i.e., satisfying, for all $X, Y \in \mathrm{S}^N$ (the space of $N \times N$ symmetric matrices) and $p \in \mathbb{R}^N$,

$$\text{if } X \leq Y, \quad \text{then } F(X,p) \leq F(Y,p). \tag{0.2}$$

Our theory extends to (x, t, ω)-dependent equations of the form:

$$\mathrm{d}u = F(\mathrm{D}^2u, \mathrm{D}u, x, t, \omega)\mathrm{d}t + \sum_{i=1}^{m} H_i(\mathrm{D}u, x, t, \omega) \circ \mathrm{d}W_i(t,x) \ \text{ in } \mathbb{R}^N \times (0, \infty), \tag{0.3}$$

where now, $\mathrm{d}W(t,x) = (\mathrm{d}W_1(t,x), \ldots, \mathrm{d}W_m(t,x))$ is a m-dimensional white noise in time and regular in x. Here to keep the presentation of the main ideas simple we only consider (0.1), and we refer to Lions and Souganidis [14] for the general case.

Equations (0.1) and (0.3) can be thought of as special cases of

$$\mathrm{d}u = F(\mathrm{D}^2u, \mathrm{D}u, x)\mathrm{d}t + \sum_{i=1}^{m} H_i(\mathrm{D}u, x) \circ \mathrm{d}\alpha_i(t,x) \ \text{ in } \mathbb{R}^N \times (0, \infty), \tag{0.4}$$

where, for each $i = 1, \ldots, m$, α_i is a continuous function of time and regular in x, provided one makes sense of the differentiation operator. We refer to [12], [13], [14] for a detailed discussion of this.

Finally our theory also extends to Equations (0.1) and (0.3) with less smooth Lipschitz continuous Hamiltonians H_i. We refer to [12] for the exact assumptions on H_i and the relevant theory.

The class of stochastic p.d.e. as in (0.1) and (0.3) is an important one, since it arises in a number of applications like, for example, asymptotic limits of p.d.e. with rapid oscillations in time, phase transitions and front propagation in random media, and pathwise stochastic control theory. In spite of their importance, very little was known for such equations until now, with the exception of the uniformly elliptic linear theory, i.e., when both F and H_i are linear and F is uniformly elliptic (*see*, for example, Watanabe [16]) and some uniformly elliptic quasilinear cases (*see*, for example, Pardoux [15]). As far as the applications are concerned results about asymptotic limits were known again only for linear uniformly elliptic problems (*see* Watanabe [16], Kushner and Huang [10]), and about phase transitions under very strong regularity assumptions (*see* Funaki [6]).

The main difficulty about equations like (0.1) and (0.3) is the well-known fact, even in the deterministic case, that there are no global smooth solutions in general. Moreover, the fully nonlinear character of the equations seems to make them inaccessible to the classical martingale theory employed for the linear case. Finally, even when smooth solutions may exist, the equations cannot be described in a pointwise sense, because of the lack of differentiability of the Brownian motion. In the deterministic case the lack of regularity was overcome with the introduction by Crandall and Lions [4] of the notion of viscosity solutions – we refer to the "User's Guide" by Crandall, Ishii, and Lions [3], the books of Barles [2] and Fleming and Soner [7] and the lecture notes [BCESS] for an up-to-date overview of the theory of viscosity solutions and their applications in the deterministic setting.

Here we overcome the aforementioned difficulties by (i) proving (see Section 1) that the solutions of all possible deterministic approximations of (0.1) converge a.s. uniformly in (x, t) to the same limit, hence providing an extension of the deterministic theory to the class of stochastic p.d.e. (0.1) and (ii) introducing (see Section 2) a notion of weak solution for (0.1), which can be thought as stochastic viscosity solution, which is satisfied by the limits of (i) and showing that, in some cases, the weak solution is unique in its class. As mentioned earlier our theory of (0.1) extends to nonsmooth Hamiltonians. We refer to the companion paper [12] for this as well as a discussion of the possible applications.

1. Convergence

We consider the following class of approximations to (0.1):

$$\begin{cases} u_t^\varepsilon = F(\mathrm{D}^2 u^\varepsilon, \mathrm{D}u^\varepsilon) + \sum_{i=1}^m H_i(\mathrm{D}u^\varepsilon)\dot{\zeta}_\varepsilon^i(t) & \text{in } \mathbb{R}^N \times (0,T), \\ u^\varepsilon = u_0^\varepsilon & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.1}$$

where $u_0^\varepsilon \to u_0$ in $\mathrm{BUC}(\mathbb{R}^N)$, the smooth function $\zeta_\varepsilon = (\zeta_\varepsilon^1, \dots, \zeta_\varepsilon^m) : [0,T] \times \Omega \to \mathbb{R}^m$ is such that, as $\varepsilon \to 0$ and for all $T > 0$,

$$\zeta_\varepsilon \to W \quad \text{uniformly in } (0,T) \text{ and a.s.} \tag{1.2}$$

Such approximations to $W(t)$ arise by either simply molifying W or suitably scaling a function ζ, for example $\zeta_\varepsilon(t) = \frac{1}{\varepsilon}\zeta\left(\frac{t}{\varepsilon^2}\right)$, which satisfy appropriate mixing conditions. (*See*, for example, [4].) Note that in case we consider the more general problem (0.4) we need to consider approximations converging uniformly to α.

To obtain the convergence of $(u^\varepsilon)_{\varepsilon>0}$ we need the following assumption on F:

$$\begin{cases} \text{There exists } G \in \mathrm{C}(\mathrm{S}^{2N} \times \mathbb{R}^N) \text{ degenerate elliptic such that, for all } p \in \mathbb{R}^N, \\ \text{(i)} \quad G\left(\begin{matrix} \lambda I & -\lambda I \\ -\lambda I & \lambda I \end{matrix}, p\right) = 0, \quad \text{for all} \quad \lambda > 0, \\ \text{and} \\ \text{(ii)} \quad F(X,p) - F(Y,p) \le G(Z,p), \\ \text{for all } X, Y \in \mathrm{S}^N \text{ and } Z \in \mathrm{S}^{2N} \text{ such that } \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le Z. \end{cases} \tag{1.3}$$

Although (1.3) appears complicated, it turns out that it is satisfied by a large class of F's, and, in particular, F's which can be written as maxmin or minmax of linear operators. We refer to Ishii and Lions [9] for a discussion of this matter in the context of deterministic equations.

Our result is:

THEOREM 1.1. – *Let* $(\zeta_\varepsilon)_{\varepsilon>0}$ *and* $(\xi_\eta)_{\eta>0}$ *satisfy* (1.2). *Assume that* F *satisfies* (0.2) *and* (1.3) *and that, for each* $i = 1, \dots, m$, $H_i \in \mathrm{C}^2(\mathbb{R}^N)$. *If* $\|u_0^\varepsilon - v_0^\eta\|_{\mathrm{C}(\mathbb{R}^N)} \to 0$, *as* $\varepsilon, \eta \to 0$, *then* $\lim_{\varepsilon,\eta\to 0} \|u^\varepsilon - v^\eta\|_{\mathrm{C}(\mathbb{R}^N \times [0,T])} = 0$, *for every* $T > 0$, *a.s. in* ω, *where* u^ε *and* v^η *solve* (1.1) *with initial data* u_0^ε *and* v_0^η *respectively. In particular, each family* $(u^\varepsilon)_{\varepsilon>0}$ *is Cauchy in* $\mathbb{R}^N \times [0,T]$ *and a.s. in* ω, *hence, converges uniformly to* $u \in \mathrm{BUC}(\mathbb{R}^N \times [0,T])$ *a.s. in* ω.

2. The notion of weak solution

Consider the stochastic Hamilton–Jacobi equation:

$$\begin{cases} \mathrm{d}v = \sum_{i=1}^m H_i(\mathrm{D}v) \circ \mathrm{d}W_i & \text{in } \mathbb{R}^N \times (t, \infty), \\ v = \phi & \text{on } \mathbb{R}^N \times \{t\}. \end{cases} \tag{2.1}$$

If $\phi \in C^2(\mathbb{R}^N) \cap C^{0,1}(\mathbb{R}^N)$ and $H_i \in C^3(\mathbb{R}^N)$, the method of characteristics yields an a.s. smooth, short time solution of (2.1), which we denote by $S(s,t)\phi$. Indeed, the characteristics associated with (2.1) are given by:

$$\begin{cases} dX = \sum_{i=1}^m D_p H_i(P) \circ dW_i, & X_t = x, \\ dP = 0, & P_t = p. \end{cases} \tag{2.2}$$

A straightforward integration yields, for $s > t$:

$$X(s) = x + \sum_{i=1}^m D_p H_i(p)(W_i(s) - W_i(t)) \quad \text{and} \quad P(s) = p. \tag{2.3}$$

It follows that, a.s. in ω, there exists $T(\phi,\omega) > 0$ such that the X-characteristic given by (2.2) is invertible in $[t, t+T(\omega))$ with the inverse denoted $X^{-1}(x,s)$. It is now a straightforward computation to check that, for $(x,s) \in \mathbb{R}^N \times [t, t+T(\phi,\omega))$,

$$\begin{aligned} S(s,t)\phi(x) = \phi(X^{-1}(x,s)) + \sum_{i=1}^m (W_i(s) - W_i(t))[H_i(D\phi(X^{-1}(x,s))) \\ - D_p H_i(D\phi(X^{-1}(x,s))) \cdot D\phi(X^{-1}(x,s))]. \end{aligned} \tag{2.4}$$

The definition is:

DEFINITION 2.1. – A function $u : \mathbb{R}^N \times [0,T] \times \Omega \to \mathbb{R}$ is a *subsolution* (resp. *supersolution*) of (0.1) if $u(\cdot,\cdot,\omega) \in \mathrm{BUC}(\mathbb{R}^N \times [0,T])$ a.s. and $(t,\omega) \mapsto u(\cdot,t,\omega) \in \mathrm{BUC}(\mathbb{R}^N)$ is $\mathcal{F}_t$-measurable, and for all $t \in [0,T)$, all $\phi \in C^2(\mathbb{R}^N) \cap C^{0,1}(\mathbb{R}^N)$, and all $g \in C^1(\mathbb{R})$, if $u(\cdot, t+\cdot, \omega) - S(\cdot,t)\phi(\cdot) - g(\cdot)$ attains a local maximum (resp. minimum) at (x_0,h_0) for $h_0 \in (0, T(\omega))$, then, at (x_0,h_0),

$$g'(h_0) \le F(D^2 S(h_0,t)\phi(x_0), DS(h_0,t)\phi(x_0)), \tag{2.5}$$

(resp.

$$g'(h_0) \ge F(D^2 S(h_0,t)\phi(x_0), DS(h_0,t)\phi(x_0)). \tag{2.6}$$

A function $u : \mathbb{R}^N \times [0,T] \times \Omega \to \mathbb{R}$ is a *solution* of (0.1), if it is both a supersolution and subsolution.

To explain and motivate the definition a few remarks are in order.

Remark 1. – Comparing u to $S(\cdot,t)\phi$ has the effect of "removing" the stochastic part of (0.1) by inverting the characteristics. Indeed, let us, for example, consider the linear problem:

$$du = u_x \circ dW + u_{xx} \ \text{in } \mathbb{R} \times (0,\infty), \quad u = u_0 \ \text{in } \mathbb{R} \times \{0\}.$$

For $u_0 \in C^2(\mathbb{R})$, its solution is given, using Ito's formula, by $\widetilde{u}(x,t) = u_0(x + W_t, t)$, where $\widetilde{u}_t = \widetilde{u}_{xx}$. We also have, for all $t,\ h > 0$, that $u(x, t+h) = \widetilde{u}(x + W_{t+h} - W_t, t)$, and, similarly, $S(h,t)\phi(x) = \phi(x + W_{t+h} - W_t)$. Hence, looking at maxima of $u(\cdot, \cdot + t) - S(\cdot,t)\phi(\cdot)$ amounts to looking at maxima of $\widetilde{u}(x,t) - \phi(x)$.

Remark 2. – It is a straightforward calculation to check that the above definition agrees with the classical definition of viscosity solution when dealing with a deterministic problem.

Remark 3. – In Definition 2.1 the stochastic nature of W plays no important role. As a matter of fact the whole theory can be applied to (0.4), for any continuous function αt, with $\alpha(0) = 0$. The stochastic nature of W comes in full play when dealing with x-dependent p.d.e. like (0.3).

Remark 4. – The classical theory of deterministic viscosity solutions can deal with equations with measurable, integrable time dependence (*see* Lions and Perthame [11] and Ishii [8]). The equations studied here fall, however, dramatically beyond the scope of this theory, since $\mathrm{d}W \notin \mathrm{L}^1$.

Remark 5. – The definition introduced here clearly depends on having short time smooth solutions of the stochastic Hamilton–Jacobi equation. This depends on having smooth, at least C^2, Hamiltonians H_i, which is, of course, a strong assumption. We discuss in [12] a possible way to relax this requirement.

We continue now stating a number of results about the weak solutions. For their proofs we refer to [13].

PROPOSITION 2.1. – *Let u be such that, for all t and a.s. in ω, $u(\cdot, t, \omega) \in \mathrm{C}^{2,\alpha}(\mathbb{R}^N)$. Then u is a classical solution of* (0.1) *if and only if it is a weak solution of* (0.1).

Note that we need to assume more than C^2, which is enough for the deterministic problem. This is due to the fact that classical solutions of (0.1) are defined using Ito's formula and not in a pointwise sense. The additional regularity is necessary to control the error estimates.

Our next result is about the existence of weak solutions. We have:

THEOREM 2.1. – *The function u obtained by Theorem* 1.1 *is a weak solution of* (0.1).

We continue with a stability result, which as in the determinstic case, plays a fundamental role in both the theory and the applications.

PROPOSITION 2.2. – *Let $(u_n)_{n\in\mathbb{N}}$ be a family of weak subsolutions (resp. supersolutions) of $\mathrm{d}u_n = F_n(\mathrm{D}^2u_n, \mathrm{D}u_n)\mathrm{d}t + \sum_{i=1}^m H_{i,n}(\mathrm{D}u_n) \circ \mathrm{d}W_i^n$. Assume that, as $n \to \infty$, $W^n \to W$ locally uniformly in t and a.s., $F_n \to F$ and $H_{i,n} \to H_i$ locally uniformly and $u_n \to u$ locally uniformly, and a.s. Then $u(x,t,\omega)$ is a subsolution (resp. supersolution) of* (0.1).

This proposition is proved using the definition and the fact that, if v_n, v are smooth solutions of $\mathrm{d}v_n = \sum_{i=1}^m H_{i,n}(\mathrm{D}v_n) \circ \mathrm{d}W_i^n$ and $\mathrm{d}v = \sum_{i=1}^m H_i(\mathrm{D}v) \circ \mathrm{d}W_i$ given by the method of characteristics, then $v_n \to v$ uniformly in (x,t) and a.s. in ω. (*See* [13] for the details.)

We now turn to the issue of the uniqueness of the weak solution. We can prove two such results. The first one applies to the stochastic Hamilton–Jacobi equation:

$$\begin{cases} \mathrm{d}u = \sum_{i=1}^m H_i(\mathrm{D}u) \circ \mathrm{d}W_i & \text{in } \mathbb{R}^N \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{R}^N \times \{0\}. \end{cases} \tag{2.7}$$

The second is about equations of the form:

$$\begin{cases} \mathrm{d}u = F(\mathrm{D}^2u, \mathrm{D}u)\mathrm{d}t + \sigma^{\mathrm{T}}\mathrm{D}u \circ \mathrm{d}W & \text{in } \mathbb{R}^N \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{2.8}$$

which are fully nonlinear but with linear dependence on the stochastic part, σ being a vector in $\mathbb{R}^N$.

We have:

THEOREM 2.3. – *For each $u_0 \in \mathrm{BUC}(\mathbb{R}^N)$,* (2.7) *admits a unique weak solution.*

Sketch of the Proof. – 1. Using Trotter–Kato type formula on the short time smooth solutions and the definition, we can prove that, if u is a weak subsolution and v is a weak supersolution of (2.7), then $z(x,y,t) = u(x,t) - v(y,t)$ is a weak solution of

$$\mathrm{d}z = \sum_{i=1}^{m}[H_i(\mathrm{D}_x z) - H_i(-\mathrm{D}_y z)] \circ \mathrm{d}W_i.$$

2. The uniqueness now follows from the observation that the definition implies that the function $\sup_{x,y}[u(x,t) - u(y,t) - \lambda|x-y|^2]$ is decreasing in time, since $\lambda|x-y|^2$ is the smooth solution of the doubled equation starting with $\lambda|x-y|^2$.

THEOREM 2.4. – *For each* $u_0 \in \mathrm{BUC}(\mathbb{R}^N)$, (2.8) *admits a unique weak solution.*

Sketch of the Proof. – The result follows from the classical theory of viscosity solutions of fully nonlinear second-order p.d.e. after the observation that, if u is a weak solution of (2.8), then the function

$$\widetilde{u}(x,t) = u(x - \sigma^{\mathrm{T}}t, t)$$

is a viscosity solution of

$$\widetilde{u}_t = F(\mathrm{D}^2\widetilde{u}, \mathrm{D}\widetilde{u}) \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

We conclude with a result about the short time behavior of the weak solutions (for the proof we refer to [13]). We have:

PROPOSITION 2.5. – *Assume* $F \in \mathrm{C}^3(\mathrm{S}^N \times \mathbb{R}^N)$ *and let* u *be a smooth weak solution of* (0.1). *Then, a.s. in* ω,

$$u(\cdot,t) = u_0 + tF(\mathrm{D}^2u_0, \mathrm{D}u_0) + \sum_{i=1}^{m} W_i(t)H_i(\mathrm{D}u_0)$$
$$+ \frac{1}{2}(W_i(t))^2(\mathrm{D}^2u_0 DH_i(\mathrm{D}u_0), \mathrm{D}H_i(\mathrm{D}u_0)) + \mathrm{O}(t^2 + t \max_{\substack{i=1,\dots,m\\ 0\le s\le t}} |W_i(s)| + \max_{i=1,\dots,m} |W_i(t)|^3).$$

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