# Stability of stationary states of non-local equations with singular interaction potentials


Klemens Fellner

DAMTP, CMS, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, United Kingdom
Email: K.Fellner@damtp.cam.ac.uk
On leave from: Faculty of Mathematics, University of Vienna
Nordbergstr. 15, 1090 Wien, Austria
Email: Klemens.Fellner@univie.ac.at

Gaël Raoul

CMLA, ENS Cachan, CNRS, PRES UniverSud,
61 Av. du Pdt. Wilson, 94235 Cachan Cedex, France
Email: gael.raoul@cmla.ens-cachan.fr



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**Abstract**

We study the large-time behaviour of a non-local evolution equation for the density of particles or individuals subject to an external and an interaction potential. In particular, we consider interaction potentials which are singular in the sense that their first derivative is discontinuous at the origin.

For locally attractive singular interaction potentials we prove under a linear stability condition local non-linear stability of stationary states consisting of a finite sums of Dirac masses. For singular repulsive interaction potentials we show stability of stationary states of uniformly bounded solutions under a convexity condition.

Finally, we present numerical simulations to illustrate our results.


# 1 Introduction

In this paper, we study the following non-local interaction equation:

$$\partial_t \rho = \nabla_x \cdot (\rho \, \nabla_x [W * \rho + V]), \tag{1}$$

where $\rho(t,x)$ denotes a density of particles/individuals at position $x \in \mathbb{R}^d$ and time $t \geq 0$ subject to an interaction potential $W(x)$ and an external potential $V(x)$.

Equations like (1) model the many particles limit of various phenomena appearing, for instance, in biology and physics. We refer to [CT, Vil, MCO] for reviews on this type of equations. Moreover, it is known (see e.g. [CDFLS, BCL, Rao1, FR]) that the behaviour of the solution depends crucially on the regularity/singularity of the interaction potential $W$ at $x = 0$. One can distinguish the following three main classes :

*Regular interaction potentials* appear, for instance, in simplified models of granular media (see e.g. [CMV, LT, BCP]) with $W$ typical being convex. In cell-biomechanical models (see [CE, KPSV, PSV]) with $W$ may be a locally repulsive double-well potential like $W(x) = x^4 - x^2$. Another example is the quadratic Morse potential $W(x) = -C_a\, e^{-x^2/l_a} + C_r\, e^{-x^2/l_r}$, which is used, for instance, in models of flocking and swarming. Previous results on regular potentials can be found e.g. in [AGS, CMV, BDiF, Rao1, FR].

*Interaction potentials with an attractive singularity at* $x = 0$ appear also in models of swarms and collective behaviour (see e.g. [BV, BCM00]) but describe also chemotaxis (see [BDP, BCC]) with typically $W(x) = -\frac{1}{2\pi}\log|x|$ in 2D. This type of potentials has been studied e.g. in [CDFLS, Lau, CR, BB, BCC, BL].

*Interaction potentials with a repulsive singularity at* $x = 0$ appear mainly in swarming models (see [CHOB, MEK, TBL]) with the attractive-repulsive Morse potentials $W(x) = -C_a\, e^{-|x|/l_a} + C_r\, e^{-|x|/l_r}$ being a typical example. Related problems can be found in physics, see e.g. Lennard-Jones type potentials [The]. We refer to e.g. [Rao1, FR] for previous results.

In this article we shall focus on the one-dimensional case

$$\partial_t \rho = \partial_x \left( \rho\, \partial_x (W * \rho) + V \right), \qquad x \in \mathbb{R}. \tag{2}$$

Notice that (2) conserves the total mass $\int_{\mathbb{R}} \rho(x)\, dx = 1$, which w.l.o.g. shall be assumed to be normalised. The solution $\rho(t,x)$ can then be interpreted as a probability density and a change of variables introducing the pseudo-inverse of the distribution function $\int_{-\infty}^{x} d\rho$, i.e.

$$u(z) = \inf \left\{ x \in \mathbb{R} : \int_{(-\infty,x]} d\rho(x) > z \right\} \qquad z \in [0,1],$$

transforms equation (2) for non-negative measure solutions $\rho(t,x)$ into the following integral equation for the non-decreasing functions $u(t,z)$ (see [LT, BDiF])

$$\partial_t u(t,z) = \int_0^1 W'\left(u(\xi) - u(z)\right) d\xi - V'(u(z)), \qquad \forall z \in [0,1]. \tag{3}$$

We remark that solutions of (3) with regular or singular attractive interaction potentials are known to concentrate to measure see e.g. [BDiF, CDFLS]. In such cases, equation (3) is advantageous both for the stability analysis and for numerical simulations as atomic parts of measure solutions $\rho(x)$ correspond to constant parts of the pseudo-inverse $u(z)$.

Notice also that in absence of a confining potential $V$ the symmetry of $W$ implies that the centre of mass $\int_{\mathbb{R}} x\, \rho(x)\, dx$ is conserved by eq. (2), or equivalently, that $\int_0^1 u$ is preserved by (3) :

$$\frac{d}{dt} \int_{\mathbb{R}} x \rho(t,x)\, dx = \frac{d}{dt} \int_0^1 u(t,z)\, dz = 0. \tag{4}$$

Throughout this article we shall suppose the following basic assumptions on $\rho_{in}$, $V$ and $W$ :

**Assumptions 1: symmetry and support**

*Symmetry:* The interaction potential $W(x) = W(-x)$ is symmetic for all $x \in \mathbb{R}$.

*Confinement:* One of the two following conditions shall be satisfied :

$$\exists C_1 > 0 : \quad \forall x \geq C_1 \quad V'(x) \geq \|W'\|_\infty,\, V'(-x) \leq \|W'\|_\infty, \tag{5}$$

or

$$V = 0, \quad \exists C_1,\, C_2 > 0 : \quad \forall x \geq C_1 \quad W'(x) \geq C_2\, x,\, W'(-x) \leq -C_2\, x. \tag{6}$$

*Compactly supported initial data:* We assume initial data $\rho_{in} \in M^1(\mathbb{R})$ with compact support $\mathrm{supp}(\rho_{in}) \subset [-C, C]$ for a constant $C < \infty$. In case $V \neq 0$ we assume moreover that $C \leq C_1$ with $C_1$ as in (5).

The second set of assumptions specifies the regularity/singularity of interaction potential $W$ at $x = 0$, which is crucial for the properties and asymptotics of the solutions :

**Assumptions 2: regularity**

The external potential $V$ and the interaction potential $W$ shall satisfy

$$V \in C^2(\mathbb{R}), \qquad W \in C^2(\mathbb{R}/\{0\}),$$

and moreover that there exist a constant $W'(0^+) > 0$ such that

$$x \mapsto \tilde{W}(x) := W(x) - W'(0^+)|x| \in C^2(\mathbb{R}), \tag{7}$$

where we distinguish the following three cases:

**2A** The interaction potential $W$ is called *regular* iff $W'(0^+) = 0$,

**2B** The interaction potential $W$ is called *singular attractive* iff $W'(0^+) > 0$,

**2C** The interaction potential $W$ is called *singular repulsive* iff $W'(0^+) < 0$. In this case we moreover assume initial data $\rho_{in} \in W^{2,\infty}(\mathbb{R})$.

The existence theory of (2) for regular (Assumption 2A) interaction potentials $W$ (see e.g. [AGS, BDiF] and the precise statements are recalled in Proposition 2.1) constructs probability measures as solutions via limits of the Jordan-Kinderlehrer-Otto scheme after interpreting (2) as a gradient flow on Wasserstein spaces associated to the energy:

$$E(t) := \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \rho(t,x)\rho(t,y)W(x-y)\,dx\,dy + \int_{\mathbb{R}} \rho(t,x)V(x)\,dx. \tag{8}$$

For singular attractive interaction potentials (Assumption 2B) it is well known that classical solutions of (2) may blow up in finite time (see [BB, BCL]). Recently in [CDFLS] a Wasserstein gradient flow theory of measure-valued solutions was developed for interaction potentials $W$, which are (amongst other assumptions) $\lambda$-convex (i.e. $W - \frac{\lambda}{2}x^2$ is convex for some $\lambda < 0$). This includes singular attractive interaction potentials as in Assumption 2B.

A major point in [CDFLS] introduced a modified equation, which gives sense to $\nabla W$ at the singularity. Here, this corresponds to setting $W'(0) = 0$ (which can be seen as a reminiscence of the symmetry $W(x) = W(-x)$) and the following modified version of (2)

$$\partial_t \rho(t,x) = \partial_x \left[\rho(t,x)\left(\int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x)\right)\right], \tag{9}$$

where we write (with a slight abuse of notation) $\rho(t,y)\,dy$ instead of $d\rho(t,\cdot)(y)$. The corresponding pseudo-inverse equation reads then as:

$$\partial_t u(t,z) = \int_{\{\xi\in[0,1]:\, u(\xi)\neq u(z)\}} W'\left(u(\xi)-u(z)\right)\,d\xi - V'(u(z)). \tag{10}$$

Finally, for singular repulsive (Assumption 2C) interaction potentials, there exists a unique solution of (2) subject to initial data $\rho_{in} \in W^{2,\infty}$. The solution $\rho$ is then uniformly bounded (see [Rao1] and the precise statements are recalled in Proposition 2.1).

The main objective of this article is to study the stability of stationary states of (2) (or its generalisation (9)) for singular interaction potential $W$. We shall have to distinguish the cases with attractive singularity at $x = 0$ (Assumption 2B) and repulsive singularity at $x = 0$ (Assumption 2C).

As preliminaries we will recall in Section 2 in Proposition 2.1 previous existence results from [CDFLS, AGS, BDiF, Rao1] for all interaction potentials $W$ (satisfying the Assumptions 1 and either 2A, 2B or 2C). Moreover, Proposition 2.2 generalise a large-time estimate on $\rho$ from [Rao1] to singular interaction potentials.

In Section 3 we study singular attractive interaction potentials $W$ satisfying Assumption 2B. In Subsection 3.1, we show in this case that stable stationary states are generically finite sums of Dirac masses, that is :

$$\bar{\rho}(x) = \sum_{i=1}^{n} \bar{\rho}_i \delta_{\bar{u}_i}(x), \qquad \bar{\rho}_i > 0, \quad \sum_{i=1}^{n} \bar{\rho}_i = 1, \tag{11}$$

and the corresponding pseudo-inverse writes as :

$$\bar{u}(z) = \sum_{i=1}^{n} \bar{u}_i \, \mathbb{1}_{I_i}, \qquad I_i = [\sum_{j<i} \bar{\rho}_j, \sum_{j\le i} \bar{\rho}_j), \quad |I_i| = \bar{\rho}_i, \tag{12}$$

Moreover, Proposition 3.2 recalls a criterion from [FR] for $\bar{\rho}$ as given in (11) to be a stationary state of (9).

Our first main result proves local non-linear stability of stationary states $\bar{\rho}$ for singular attractive interaction potentials under the following condition of linear stability of stationary states $\bar{\rho}$ of (2) against all perturbations $u = \bar{u} + v$, which shift the positions $\bar{u}_i$ of Dirac masses, i.e. (see [FR])

$$\textit{Shifts}: \quad \left\{ v(z) = \sum_{i=1}^{n} v_i \, \mathbb{1}_{I_i}(z) : \begin{array}{c} v_1, \dots, v_n \in \mathbb{R}, \\ \text{if } V = 0 \text{ then } \sum_{i=1}^{n} v_i \bar{\rho}_i = 0 \end{array} \right\}, \tag{13}$$

and a stationary state $\bar{\rho}$ of (10) as given in (11) or (12), respectively, is said to be *linearly stable with respect to shifts* if and only if the matrix $M \in M_n(\mathbb{R})$ defined by:

$$M_{ij} := \left\{ \begin{array}{ll} \bar{\rho}_i W''(\bar{u}_j - \bar{u}_i), & \text{if } i \neq j, \\ -\sum_{k\neq i} \bar{\rho}_k W''(\bar{u}_k - \bar{u}_i) - V''(\bar{u}_i), & \text{if } i = j. \end{array} \right. \tag{14}$$

has a strictly positive spectrum $\sigma(M)$ in the sense that for some $\nu > 0$, either

$$(\mathbf{SS1}) \qquad V \neq 0: \quad \sigma(M) \subset \{z \in \mathbb{C} : \mathcal{R}e(z) > \nu\}.$$

or, in the case $V = 0$, the spectrum $\sigma(M|_H)$ of $M$ restricted onto the hyperspace $H = \{(w_i)_{i=1,\dots,n} : \sum_{i=1}^{n} w_i = 0\}$ is strictly positive

$$(\mathbf{SS2}) \qquad V = 0: \quad \sigma(M|_H) \subset \{z \in \mathbb{C} : \mathcal{R}e(z) > \nu\}.$$

Notice that if $V = 0$, the conservation law (4) will permit only stability w.r.t. perturbations $\rho$ of $\bar{\rho}$, which leave the centre of mass unchanged :

$$\int_{\mathbb{R}} x d\rho(x) = \int_{\mathbb{R}} x d\bar{\rho}(x). \tag{15}$$

We are now able to state our first main results, which shows that a stationary state $\bar{\rho}$ as given in (11) or (12), respectively, satisfying the condition (**SS1**) or (**SS2**) if $V = 0$ is locally non-linear stable in the Wasserstein $W_\infty$-norm.

**Theorem 1.1** (Local non-linear stability for singular attractive potentials)**.** *Assume that $\rho_{in}$, $V$, $W$ satisfy the Assumptions 1 and 2B and also that $V$, $\tilde{W} \in C^{2,\alpha}(\mathbb{R})$ (see* (7)*) for some $\alpha > 0$. Let $\bar\rho = \sum_{i=1}^n \bar\rho_i \delta_{\bar u_i}$ be a stationary state of* (2) *that satisfies* **(SS1)** *or* **(SS2)** *if $V = 0$. If $V = 0$ assume moreover that $\rho_{in}$ and $\bar\rho$ have the same centre of mass $\int_{\mathbb{R}} x\, \rho_{in}(x)\, dx = \int_{\mathbb{R}} x\, \bar\rho(x)\, dx$. Let $\rho \in AC_{loc}([0,\infty), \mathcal{P}_2(\mathbb{R}))$ be the solution of* (9) *with initial data $\rho_{in}$ as stated in Proposition 2.1.*

*Then, there exit constants $C$, $\varepsilon > 0$ (depending only on $V$, $W$ and $\bar\rho$) such that if:*

$$W_\infty(\rho_{in}, \bar\rho) = \|u_{in} - \bar u\|_\infty \le \varepsilon,$$

*then for $t \ge 0$,*

$$W_\infty(\rho(t,\cdot), \bar\rho) = \|u(t,\cdot) - \bar u\|_\infty \le C(1 + t^{n-1})\, e^{-\nu t}, \tag{16}$$

*where $\nu$ is defined in* **(SS1)** *or* **(SS2)***, respectively.*

**Remark 1.1.** *In a previous article [FR] we have shown local non-linear stability for regular interaction potentials $W$ under the conditions* **(SS1)** *or* **(SS2)** *if $V = 0$ and under the second condition* **(SR)**

$$(\mathbf{SR}) \qquad 0 < m_i := \sum_{j=1}^n W''(\bar u_j - \bar u_i)\bar\rho_j + V''(\bar u_i) \qquad \forall i = 1, \ldots, n,$$

*which implies* linear stability with respect to reallocations*, i.e. all perturbations of the form*

$$\textit{Reallocations :} \qquad \left\{ v(z) : \begin{array}{c} \int_{I_i} v(\xi)\, d\xi = 0\,,\ i = 1, \ldots, n \\ \textit{if } V = 0 \textit{ then } \int_0^1 v(z)\, dz = 0 \end{array} \right\}. \tag{17}$$

*Then, similar to Theorem 1.1, we obtained for sufficiently small initial data $W_\infty(\rho_{in}, \bar\rho) = \|u_{in} - \bar u\|_\infty \le \varepsilon$ the asymptotic stability:*

$$W_\infty(\rho, \bar\rho) = \|u(t) - \bar u\|_\infty \le C(1 + t^{n-1})\, e^{-\eta t}, \quad \eta := \min\{\nu, m_1, \ldots, m_n\},$$

*with a rate $\eta := \min\{\nu, m_1, \ldots, m_n\}$ which combines* **(SS1)** *or* **(SS2)** *if $V = 0$ and* **(SR)**.

*Note that the non-linear stability proof in [FR] is based on Taylor expansions and does not apply to singular interaction potentials like in Assumption 2B, for which* (9) *is not even linearisable around stationary states $\bar\rho$.*

*Finally we remark that the condition of linear stability with respect to reallocations is not needed for singular attractive potentials. One could say that singular attraction always ensures stability with respect to reallocations.*

The proof of Theorem 1.1 is detailed at the end of section 3. It shows in a first step how the singular attractive interaction potential forces the solution to consist entirely of Dirac masses within finite time. In a second step these Dirac masses converge towards the stationary state due to the condition **(SS1)** of **(SS2)** if $V = 0$. See also Figure 1 for a numerical example to illustrate these two steps.

The second part of this papers considers singular repulsive interaction potentials $W$ satisfying Assumption 2C in Section 4. In Subsection 4.1 Proposition 4.1 shows for any $V$, $W$ satisfying Assumptions 1 and 2C that the solutions $\rho(t,\cdot)$ of (2) converges, up to extraction of a subsequence, to a stationary state $\bar\rho$ of (2). Notice that the uniform bound $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ and the uniformly bounded support imply that $\bar\rho$ is not measure valued, i.e. $\bar\rho \in L^1 \cap L^\infty(\mathbb{R})$.

The following second main result of this paper shows that eq. (2) admits a unique globally attractive stationary state $\bar\rho$ provided some convexity assumptions on $V$ and $W$ :

**Theorem 1.2.** *Assume that $\rho_{in}$, $V$ and $W$ satisfy the Assumptions 1 and 2C. Let $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap Lip_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$ be the solution of (2) with initial data $\rho_{in} \in W^{2,\infty}(\mathbb{R})$ as given in Proposition 2.1. Assume moreover that either*

- $V'' > C > 0$ *and* $W|_{(0,\infty)}$ *is convex,*
- *or* $V = 0$ *and* $W''|_{(0,\infty)} > C > 0$.

*Then, there exists a unique (up to a shift in $x$ if $V = 0$) stationary state $\bar\rho \in L^1 \cap L^\infty(\mathbb{R})$ of (2), and:*

$$W_\infty(\rho(t,\cdot), \bar\rho) = \|u(t,\cdot) - \bar u\|_\infty \to 0.$$

Notice that in general the shape of the stationary states of Theorem 1.2 cannot easily be determined for given $W$ and $V$. One exception is the example $W(x) = x^2 - |x|$ and $V = 0$ when $(W' * \rho)(x) = 2x + 1 - 2\int_{-\infty}^{x} \rho(y)\,dy)$ and thus $\bar\rho(x) = 1$ on the support of $\bar\rho$. In Section 5 we will present numerical examples to illustrate several cases.

Moreover, we mention the interesting weak limit of Dirac-type stable stationary states of (2) for regular interaction potentials (satisfying Assumption 2A) towards the stable stationary state $\bar\rho \in L^1 \cap L^\infty(\mathbb{R})$ of (9) with a singular repulsive interaction potential $W$ (satisfying Assumption 2C) (see [FR]).

The proof of Theorem 1.2 is shown at the end of section 4.

In Section 5 finally, we present numerical simulations using mainly an implicit Euler discretisation of the pseudo-inverse equation (3).

The first example illustrate the proof of Theorem 1.1.

The second example show convergence to a stationary state consisting of two separated continuous parts for a singular repulsive interaction potential $W$ and a repulsive-confining external potential $V$.

Finally, we show the behaviour of solutions of (3) with interaction potentials $W$, which are more singular than supposed in Assumption 2C, for instance, $W(x) \sim -|x|^\alpha$ for $\alpha < 1$.

## 2 Preliminary

The following proposition recalls the existence theories for (2) and (9) and shows that the support of $\rho(t,\cdot)$ is uniformly bounded in time. Thanks to this result we shall only consider compactly supported solutions of (2) or (9) throughout this paper.

**Proposition 2.1** (Existence and compact support [CDFLS, BDiF, Rao1])**.** *Assume $V$, $W$ and $\rho_{in}$ satisfy Assumption 1. Moreover,*

*if Assumption 2A is satisfied and $W$ is* regular*, then there exists a unique solution $\rho(t,x) \in Lip_{loc}([0,\infty),\mathcal{P}_\infty(\mathbb{R}))$ of (2) subject to the initial data $\rho_{in}$ (see [AGS, BDiF, CDFLS]),*

*if Assumption 2B is satisfied and $W$ is* singular attractive at $x=0$*, then there exists a unique solution $\rho(t,x) \in AC_{loc}([0,\infty),\mathcal{P}_2(\mathbb{R}))$ of (9) subject to the initial data $\rho_{in}$ (see [CDFLS]).*

*if Assumption 2C is satisfied and $W$ is* singular repulsive at $x=0$*, then there exists a unique solution $\rho(t,x) \in L^\infty(\mathbb{R}_+\times\mathbb{R}) \cap Lip_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R})$ of (2) subject to the initial data $\rho_{in} \in W^{2,\infty}(\mathbb{R})$ (see [Rao1]).*

*Moreover, there exists a constant $C>0$ such that for all times $t\geq 0$,*

$$\operatorname{supp}(\rho(t,\cdot)) \subset [-C,C], \qquad t\geq 0.$$

The next proposition (which is a generalisation of a result of [Rao1]) provides an estimate on the long-time behaviour of the solution, which excludes, for instance, that (2) admits a periodic limit cycle. This result underlines the analysis of stationary states in order to determine the asymptotic behaviour of solutions of (2). (For an analogue result for singular repulsive potentials see Proposition 4.1.)

**Proposition 2.2** (Asymptotic control of the solution)**.** *Let $\rho_{in}$, $V$, $W$ satisfy the Assumptions 1 and either 2A, 2B, or 2C. Let $\rho$ be the solution of (9) with initial data $\rho_{in}$ given by Proposition 2.1. Then,*

$$\int_{\mathbb{R}} \rho(t,x)\Big(\int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x)\Big)^2 dx \to 0 \quad as \quad t\to\infty. \tag{18}$$

**Remark 2.1.** *If $V$ and $W$ are convex with one of them being strictly convex, then $x\mapsto \int W'(x-y)\rho(t,y)\,dy + V'(x)$ is strictly increasing and (18) implies the convergence of $\rho(t,\cdot)$ to a single Dirac mass $\delta_{\bar{x}}$, where the position $\bar{x}$ is determined either by $V(\bar{x}) = \min_{y\in\mathbb{R}} V(y)$ or by $\bar{x} = \int_{\mathbb{R}} x\,\rho_{in}(x)\,dx$ if $V=0$.*

*Here we do not assume convexity of $V$ or $W$. Thus (18) does not imply the convergence of $\rho(t,\cdot)$ to a single stationary state.*

*Proof of Proposition 2.2.* The case of regular potentials (Assumption 2A) has already been shown in [Rao1]. Here, we prove Proposition 2.2 in the cases where Assumption 2B or Assumption 2C are satisfied.

*Step 1: We estimate the first and second time derivative of the energy E (see* (8)*)*

As e.g. in [CDFLS, Rao1], we compute:

$$\frac{dE}{dt}(t) = -\int_{\mathbb{R}} \rho(t,x) \left( \int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right)^2 dx \leq 0. \tag{19}$$

Then, we calculate $\frac{d^2E}{dt^2}$:

$$\begin{aligned}\frac{d^2E}{dt^2} = {} & 2\int_{\mathbb{R}} \rho(t,x) \left( \int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right)^2 \times \\ & \qquad\qquad \partial_x \left( \int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right) dx \\ & - 2\int_{\mathbb{R}} \rho(t,x) \left( \int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right) \int_{y\neq x} W'(x-y) \times \\ & \qquad\qquad \partial_y \left( \rho(t,y) \left( \int_{z\neq y} W'(y-z)\rho(t,z)\,dz + V'(y) \right) \right) dy\,dx.\end{aligned}$$

We compute the first term and integrating the second by part to get (with $\tilde{W}(x) = W(x) - W'(0^+)|x| \in W^{2,\infty}(\mathbb{R})$ as defined in (7)) :

$$\begin{aligned}\frac{d^2E}{dt^2} = {} & 2\int_{\mathbb{R}} \rho(t,x) \left( \int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right)^2 \times \\ & \qquad \left[ (W'(0^-) - W'(0^+))\rho(t,x) + \int_{y\neq x} \tilde{W}''(x-y)\rho(t,y)\,dy + V''(x) \right] dx \\ & - 2(W'(0^+) - W'(0^-)) \int \rho(t,x)^2 \left( \int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right)^2 dx \\ & - 2\int_{\mathbb{R}} \rho(t,x) \left( \int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right) \int_{y\neq x} \tilde{W}''(x-y) \times \\ & \qquad\qquad \left( \rho(t,y) \left( \int_{z\neq y} W'(y-z)\rho(t,z)\,dz + V'(y) \right) \right) dy\,dx.\end{aligned}$$

Since $V$, $\tilde{W} \in W^{2,\infty}(\mathbb{R})$ and $\rho$ is uniformly compactly supported, we get :

$$\frac{d^2E}{dt^2} = -8W'(0^+) \int_{\mathbb{R}} \rho(t,x)^2 \left( \int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right)^2 dx + O(1),$$

Note that if $W$ is regular (Assumption 2A) we have $\frac{d^2E}{dt^2} = O(1)$. Here, in the cases where $W$ satisfies either Assumption 2B or Assumption 2C, we have

$$\frac{d^2E}{dt^2} \leq C \quad \text{if 2B holds} \quad \text{or} \quad \frac{d^2E}{dt^2} \geq -C \quad \text{if 2C holds} \tag{20}$$

for some constants $C < \infty$. However, these estimates (20) on $\frac{d^2E}{dt^2}$ are sufficient to conclude as in the following.

*Step 2: We show that* $\frac{dE}{dt} \to 0$.

We shall only detail the singular attractive case Assumption 2B. The singular repulsive case Assumption 2C is shown in a similar way.

Since the energy $E$ is non-increasing (19) and uniformly bounded from below by

$$E \geq \min_{x\in \mathrm{supp}(\rho)\subset[-C,C]} W(x) + \min_{x\in \mathrm{supp}(\rho)\subset[-C,C]} V(x),$$

there exists a limit $\lim_{t\to\infty} E$ and for $\varepsilon > 0$ a time $T > 0$ such that for all $t \geq T$ :

$$\int_t^\infty -\frac{dE}{dt} < \varepsilon.$$

Moreover, we can Taylor expand using (20) :

$$\int_t^\infty \frac{dE}{dt} \leq \int_t^{t+\left|\frac{1}{2C}\frac{dE}{dt}(t)\right|} \frac{dE}{dt}(t) + C(s-t)\,ds \leq -\frac{1}{4C}\left|\frac{dE}{dt}(t)\right|^2,$$

and thus, for any $t \geq T$,

$$\left|\frac{dE}{dt}(t)\right| \leq 2\sqrt{C\varepsilon},$$

which shows $\frac{dE}{dt} \to 0$ and finishes the proof of Proposition 2.2. □

# 3 Singular attractive interaction potentials

## 3.1 Stable stationary states for singular attractive $W$

The following Proposition 3.1 recalls that compactly supported stationary states $\bar{\rho}$ of (9) with the $\mathrm{supp}(\bar{\rho})$ containing an accumulation point satisfying (21) are unstable in the below sense. As a consequence are stable stationary states of (9) generically finite sums of Dirac masses.

**Proposition 3.1** (Instability of non-Dirac type stationary states for singular attractive interaction potentials [Rao1]). *Assume that $\rho_{in}$, $V$ and $W$ satisfy the Assumptions 1 and*

*2B. Let $\bar\rho \in M^1_+(\mathbb{R})$ be a compactly supported stationary state of* (9)*. If* $\operatorname{supp}(\bar\rho)$ *has an accumulation point $x_0$ such that :*

$$\exists C > 0,\ \exists \eta > 0 : \quad \forall \gamma \in (0,\eta), \quad \frac{1}{\gamma}\int_{x_0}^{x_0+\gamma} \bar\rho(y)\, dy \geq C, \tag{21}$$

*(or the same estimate with $-\eta < \gamma < 0$), then is $\bar\rho$ locally unstable in the sense that for any $\varepsilon > 0$ there exists $\rho^\varepsilon \in M^1(\mathbb{R})$ such that $W_1(\rho^\varepsilon, \bar\rho) \leq \varepsilon$ and*

$$E(\rho^\varepsilon) < E(\bar\rho), \tag{22}$$

*where $E$ is the energy defined by* (8)*.*

The following proposition provides a criterion for a sum of Dirac masses to be a stationary state of (9).

**Proposition 3.2** (Stationary condition for sums of Diracs)**.** *Assume $V$, $W$ satisfy the Assumptions 1 and 2A or 2B. Then, for a given integer $n \in \mathbb{N}$, a non-negative measure $\bar\rho$ as given in* (11) *or, equivalently, $\bar u$ as given in* (12) *is a stationary state of eq.* (9) *or* (10)*, respectively, if and only if the following condition holds (with $W'(0) = 0$):*

$$\sum_{j=1}^{n} W'(\bar u_j - \bar u_i)\, \bar\rho_j = V'(\bar u_i), \qquad i = 1, \dots, n\,. \tag{23}$$

*Proof.* The proof can be found in [FR]. ☐

## 3.2 Proof of Theorem 1.1

*Proof of Theorem 1.1.* We consider $u = \bar u + v \in L^\infty([0,1])$ a non-decreasing perturbation of $\bar u$.

*Step 1: After a finite time $T > 0$ the pseudoinverse $u$ consists of $n$ steps functions, i.e. $\rho$ is a sum of $n$ Dirac masses.*

We recall the formula (12) of $\bar u$. For $t \geq 0$ choose $i \in 1, \dots, n$ such that $v(t,\cdot)$ is not constant on $I_i$ and consider $z' < z'' \in I_i$ such that $v(t,z') < v(t,z'')$. Then, we compute using Assumption 2B (and $\operatorname{sign}(0) = 0$)

$$\begin{aligned}
\partial_t(v(z'') - v(z')) &= \left[\int_{\{\xi\in[0,1];\, u(\xi)\neq u(z'')\}} W'(u(\xi) - u(z''))\, d\xi - V'(u(z''))\right] \\
&\quad - \left[\int_{\{\xi\in[0,1];\, u(\xi)\neq u(z')\}} W'(u(\xi) - u(z'))\, d\xi - V'(u(z'))\right] \\
&= W'(0^+)\int [\operatorname{sign}(u(\xi) - u(z'')) - \operatorname{sign}(u(\xi) - u(z'))]\, d\xi \\
&\quad + \int \big[\tilde W'(u(\xi) - u(z'')) - \tilde W'(u(\xi) - u(z'))\big] d\xi - V'(u(z'')) + V'(u(z'))
\end{aligned}$$

We recall that $u$ is non-decreasing and compute the first term as

$$\int \operatorname{sign}(u(\xi)-u(z))\,d\xi = -|\{\xi\in[0,1];\, u(\xi)<u(z)\}| + |\{\xi\in[0,1];\, u(\xi)>u(z)\}|,$$

and estimates the second and the third term by the mean value theorem after defining $\Delta(t) := \sup\{|v(t,z_1)-v(t,z_2)|;\, i\in\{1,\dots,n\},\, z_1,\, z_2\in I_i\}$. Thus, as long as $\Delta(t)>0$,

$$\begin{aligned}
\partial_t\left(v(z')-v(z'')\right) = O(\Delta)& \\
&+ W'(0^+)\Big(|\{\xi\in[0,1];\, u(\xi)>u(z'')\}| + |\{\xi\in[0,1];\, u(\xi)<u(z')\}|\Big)\\
&- W'(0^+)\Big(|\{\xi\in[0,1];\, u(\xi)<u(z'')\}| + |\{\xi\in[0,1];\, u(\xi)>u(z')\}|\Big)\\
&= O(\Delta) + W'(0^+)\Big(1-|\{\xi\in[0,1];\, u(z')\le u(\xi)\le u(z'')\}|\Big)\\
&- W'(0^+)\Big(1+|\{\xi\in[0,1];\, u(z')< u(\xi)< u(z'')\}|\Big)\\
&\le O(\Delta) - W'(0^+)|\{\xi\in[0,1];\, u(z')\le u(\xi)\le u(z'')\}|\\
&\le O(\Delta) - W'(0^+)|z''-z'|,
\end{aligned}$$

where $|O(\Delta)| \le \Big(\|\tilde W''\|_\infty + \|\tilde V''\|_\infty\Big)\,|\Delta|$. As the above estimate holds for any $z'<z''\in I_i$ such that $v(t,z')<v(t,z'')$, we get in particular :

$$\frac{d}{dt}\Delta(t) \le O(\Delta) - W'(0^+)\min_{i=1,\dots,n}|I_i|.$$

Moreover, $\Delta(0)$ is small as the initial data $\rho_{in}$ are close to $\bar\rho$ in $W_\infty$ :

$$\Delta(0) \le 2\|u_{in}-\bar u\|_\infty = 2W_\infty(\rho_{in},\bar\rho) \le \varepsilon.$$

Thus, for $\varepsilon>0$ small enough,

$$|O(\Delta(0))| \le (\|\tilde W''\|_\infty + \|\tilde V''\|_\infty)\,\varepsilon \le \frac{1}{2}W'(0^+)\min_{i=1,\dots,n}|I_i|,$$

and there exist a time $T\in(0, 2\,\varepsilon/(W'(0^+)\min_{i=1,\dots,n}|I_i|)]$ such that $\Delta(T)=0$, that is $v(T,z)=\sum_{i=1}^n v_i(T)\mathbb{1}_{I_i}(z)$ and for all times after we have

$$\forall t\ge T,\quad \forall z\in[0,1]:\quad v(t,z)=\sum_{i=1}^n v_i(t)\mathbb{1}_{I_i}(z). \tag{24}$$

Finally, $\|(v_i(T))\|$ is small as $\varepsilon>0$ is small :

$$\begin{aligned}
\|v_i(T)\| &\le \|u_{in}-\bar u\|_\infty + T\|\partial_t v\|_\infty\\
&\le \varepsilon + \frac{2\varepsilon}{W'(0^+)\min_{i=1,\dots,n}|I_i|}\left(\|W'\|_{\infty([-2C,2C])} + \|V'\|_{\infty([-C,C])}\right) \le O(\varepsilon),
\end{aligned} \tag{25}$$

where $C$ is the uniform bound on the support supp $\rho$ as stated in Prop 2.1.

Note that $u(T,z) = \sum_{i=1}^n (\bar{u}_i + v_i(T)) \mathbb{1}_{I_i}(z)$ denotes a shift perturbation of $\bar{u}$ as defined in (13). Hence, it remains to show that the linear stability condition with respect to shifts, i.e. (**SS1**) or (**SS2**) if $V = 0$ implies the convergence of $u$ to $\bar{u}$.

*Step 2: We show that $v_i(t) \to 0$ for $t \geq T$.*

Since $v(t,z) = v_i(t)$ is constant on $I_i$ for $t \geq T$, we have for $z \in I_i$ :

$$\begin{aligned}
\frac{d}{dt} v_i = \partial_t v(t,z) &= \int_{[0,1]/I_i} W'(\bar{u}(\xi) + v(\xi) - \bar{u}(z) - v(z))\, d\xi - V'(\bar{u}(z) + v(z)) \\
&= \sum_{j \neq i} \bar{\rho}_j W'(\bar{u}_j + v_j(t) - \bar{u}_i - v_i(t)) - V'(\bar{u}_i + v_i(t)),
\end{aligned}$$

since $|I_j| = \bar{\rho}_j$. If we multiply this equation with $\bar{\rho}_i$ we get the ODE system :

$$\frac{d}{dt}(\bar{\rho}_i v_i) = \sum_{j \neq i} \bar{\rho}_i \bar{\rho}_j W'(\bar{u}_j + v_j(t) - \bar{u}_i - v_i(t)) - \bar{\rho}_i V'(\bar{u}_i + v_i(t)) \quad \text{for } i = 1, \ldots, n. \tag{26}$$

Obviously $v_i = 0$ is a stationary state of (26) since $\bar{u}$ is a stationary state of (3). To check the stability we linearise the equation (26) around $v_i = 0$. By recalling that $V \in C^{2,\alpha}(\mathbb{R})$ and $W = W'(0^+)|x| + \tilde{W} \in C^{2,\alpha}(\mathbb{R} \setminus \{0\})$ for a $\alpha > 0$, we calculate :

$$\begin{aligned}
\frac{d}{dt}(\bar{\rho}_i v_i) &= \sum_{j \neq i} \bar{\rho}_i \bar{\rho}_j \left[ W'\left(\bar{u}_j - \bar{u}_i\right) + W''\left(\bar{u}_j - \bar{u}_i\right)\left(v_j(t) - v_i(t)\right) + O(|v_j(t) - v_i(t)|^{1+\alpha}) \right] \\
&\qquad - \bar{\rho}_i V'(\bar{u}_i) - \bar{\rho}_i V''(\bar{u}_i)\, v_i(t) + O(|v_i(t)|^{1+\alpha}) \\
&= \sum_{j \neq i} \Big( \bar{\rho}_i W''\left(\bar{u}_j - \bar{u}_i\right) \bar{\rho}_j v_j(t) \Big) - \Big( \sum_{j \neq i} \bar{\rho}_j W''\left(\bar{u}_j - \bar{u}_i\right) \Big) \bar{\rho}_i v_i(t) \\
&\qquad - V''(\bar{u}_i) \bar{\rho}_i v_i(t) + O\left( \|(\bar{\rho}_j v_j)(t)\|^{1+\alpha} \right),
\end{aligned}$$

and thus,

$$\frac{d}{dt}(\bar{\rho}_i v_i) = -M\,(\bar{\rho}_i v_i) + O\left( \|(\bar{\rho}_j v_j)(t)\|^{1+\alpha} \right), \tag{27}$$

where $M$ is the matrix defined in (14).

In the following we distinguish the cases **(SS1)** and **(SS2)** if $V = 0$ :

*We show Theorem 1.1 in the case where* **(SS1)** *is satisfied.*

For $t \geq T$ we have $v(t,z) = \sum_{i=1}^n v_i(t) \mathbb{1}_{I_i}(z)$ and the $v_i(t)$ satisfy (27). Thus, the solution $(\bar{\rho}_i v_i)$ is given by

$$(\bar{\rho}_i v_i) = e^{-(t-T)M}\,(\bar{\rho}_i v_i)(T) + \int_T^t e^{-(t-s)M}\, O(\|(\bar{\rho}_i v_i)(s)\|_\infty^{1+\alpha})\, ds.$$

Then, as **(SS1)** is satisfied, Lemma 6.1 applied to $e^{-(t-T)M}$ and $e^{-(t-s)M}$ yields

$$\begin{aligned}\|(\bar{\rho}_i v_i)\| \leq \|(\bar{\rho}_i v_i)(T)\|(1+(t-T)^{n-1})\, e^{-(t-T)\nu} \\ + C\int_T^t \|(\bar{\rho}_i v_i)(s)\|^{1+\alpha}(1+(t-s)^{n-1})\, e^{-(t-s)\nu}\, ds.\end{aligned}$$

Moreover the estimate (25) implies that $\|(\bar{\rho}_i v_i)(T)\|$ can be made sufficiently small to apply the second part of Lemma 6.2, which yields for $\varepsilon > 0$ small enough

$$\|(\bar{\rho}_i v_i)(t)\| \leq C(1+t^{n-1})\, e^{-\nu t}.$$

*We show Theorem 1.1 in the case where* **(SS2)** *is satisfied.*

Since $V = 0$ the centre of mass of $\rho(t,\cdot)$ is conserved by (2) we have $0 = \int_0^1 v(t,z)\, dz = \int_0^1 u_{in}(z) - \bar{u}(z)\, dz$ and

$$\forall t \geq T, \quad (\bar{\rho}_i v_i) \in H = \left\{ (w_i)_{i=1,\dots,n};\, \sum_{i=1}^n w_i = 0 \right\}.$$

Let $(\tilde{e}_1, \dots, \tilde{e}_{n-1})$ be a basis of $H$ with $\tilde{e}_n := \sum_{i=1}^n e_i$ and $(e_i)$ denoting the canonical base of $\mathbb{R}^n$. Then, in the basis $(\tilde{e}_i)$ of $\mathbb{R}^n$, the vector $(\bar{\rho}_i v_i)(t)$ writes as $\sum_{i=1}^n (\bar{\rho}_i v_i)(t)\, e_i = \sum_{i=1}^{n-1} w_i(t)\, \tilde{e}_i$. In particular $w_n(t) = 0$ for all $t \geq 0$ and $(w_1(t), \dots, w_{n-1}(t))$ is given by

$$(w_i)(t) = e^{-(t-T)M|_H}\, (w_i)(T) + \int_T^t e^{-(t-s)M|_H}\, O(\|(w_i)(s)\|_\infty^{1+\alpha})\, ds,$$

where $M|_H$ is the matrix $M$ restricted to $H$ (and expressed in the basis $(\tilde{e}_i)$ of $H$). Since **(SS2)** is satisfied, Lemma 6.1 applies to $e^{-(t-T)M|_H}$ and $e^{-(t-s)M|_H}$ and yields

$$\begin{aligned}\|\,(w_i)\,\| \leq \|(w_i)(T)\|(1+(t-T)^{n-2})\, e^{-(t-T)\nu} \\ + C\int_T^t \|(w_i)(s)\|^{1+\alpha}(1+(t-s)^{n-2})\, e^{-(t-s)\nu}\, ds.\end{aligned}$$

and similar as above, since $\|(\bar{\rho}_i v_i)(T)\|$ can be made sufficiently small by (25), we can apply the second part of Lemma 6.2, to get for $\varepsilon > 0$ small enough

$$\|(\bar{\rho}_i v_i)(t)\| \leq C(1+t^{n-2})\, e^{-\nu t},$$

which concludes the proof of Theorem 1.1. $\square$

# 4 Singular repulsive interaction potentials

## 4.1 Large time behaviour for singular repulsive $W$

The following result (which is a generalisation of a result of [Rao1]) shows that $\rho(t,\cdot)$ converges, up to extraction of a subsequence, to a stationary state of (2). Proposition 4.1 implies in particular the existence of a stationary state $\bar{\rho}$ of (2) for any $V$, $W$ satisfying Assumptions 1 and 2C.

**Proposition 4.1** (Convergence of subsequences to a stationary state)**.** *Let $\rho_{in}$, $V$ and $W$ satisfy the Assumptions 1 and 2C. Let $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap Lip_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$ be the solution of* (9) *with initial data $\rho_{in}$ given by Proposition 2.1.*

*Then, for any sequence $t_k \to \infty$, there exist a subsequence still denoted by $t_k$ such that:*

$$W_1\left(\rho(t_k,\cdot),\bar{\rho}\right) = \|u(t_k,\cdot)-\bar{u}\|_1 \to 0 \quad \text{as } k\to\infty, \tag{28}$$

*where $W_1$ denotes the $1-$Wasserstein distance and $\bar{\rho}$ is a steady-state of* (9)*.*

**Remark 4.1.** *Notice that the limit $\bar{\rho}$ in* (28) *is not necessarily unique as it may depend both on the sequence $t_k$ and on the subsequence.*

*Proof of Proposition 4.1.* Proposition 4.1 has been proven in the case of regular interaction potentials in [Rao1]. Here, we will show how this proof extends to the case where Assumption 2C is satisfied.

The pseudo-inverse $u(t,\cdot)$ of $\rho(t,\cdot)$ is an increasing function and is uniformly bounded thanks to Proposition 2.1. Any sequence $u(t_k,\cdot)$ is thus uniformly bounded in $BV([0,1])$. Therefore, there exists a subsequence, still denoted by $u(t_k,\cdot)$, which converges in $L^1$ to a limit denoted by $\bar{u}$:

$$\|u(t_k,\cdot)-\bar{u}\|_{L^1} \to 0.$$

It remains to prove that $\bar{u}$ is a stationary state of (3). In order to do so we shall use the estimate (18) from Proposition 4.1, which writes in the pseudo-inverse variables as :

$$\frac{dE}{dt}(t_k) = -\int_0^1 \left(\int W'(u(t_k,z)-u(t_k,\xi))\,d\xi + V'(u(t_k,z))\right)^2 dz.$$

Next, we define $\bar{F} := -\int_0^1 \left(\int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi + V'(\bar{u}(z))\right)^2 dz$ and it follows (see [Rao1] for the details)

$$\bar{F} - \frac{dE}{dt} \le C\Big\|\int_{u(z)\neq u(\xi)} W'(u(z)-u(\xi))\,d\xi - \int_{\bar{u}(z)\neq\bar{u}(\xi)} W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi\Big\|_{L^1} + C\|u-\bar{u}\|_{L^1}.$$

and by (7) and $\mathrm{sign}(0) = 0$,

$$\bar{F} - \frac{dE}{dt} \leq C \Big\| \int_{u(z)\neq u(\xi)} \mathrm{sign}\,(u(z) - u(\xi)) + \tilde{W}'(u(z) - u(\xi))\, d\xi \\ - \int_{\bar{u}(z)\neq \bar{u}(\xi)} \mathrm{sign}\,(\bar{u}(z) - \bar{u}(\xi)) + \tilde{W}'(\bar{u}(z) - \bar{u}(\xi))\, d\xi \Big\|_{L^1} + C\|u - \bar{u}\|_{L^1}.$$

As Assumption 2C is satisfied it follows from Proposition 2.1 that $\rho$, $\bar{\rho} \in L^\infty(\mathbb{R})$ and, hence, that $u$, $\bar{u}$ are strictly increasing, which yields

$$\mathrm{sign}\,(u(z) - u(\xi)) = \mathrm{sign}\,(z - \xi) = \mathrm{sign}\,(\bar{u}(z) - \bar{u}(\xi))\,,$$

and moreover

$$\bar{F} - \frac{dE}{dt} \leq C\|\tilde{W}''\|_{L^\infty(-2C,2C)}\, \|u - \bar{u}\|_{L^1} + C\|u - \bar{u}\|_{L^1}.$$

Finally,

$$\bar{F} \leq \frac{dE}{dt}(t_k) + C\|u(t_k,\cdot) - \bar{u}\|_{L^1} \to 0 \quad \text{as} \quad k \to \infty.$$

and $\bar{F} = 0$. Thus,

$$\mathrm{supp}\ \bar{\rho} \subset \left\{ x \in \mathbb{R};\ \int W'(x-y)\bar{\rho}(y)\, dy + V'(x) = 0 \right\},$$

and $\bar{\rho}$ is a stationary state of (2). This shows Proposition 4.1. □

## 4.2 Proof of Theorem 1.2

*Proof of Theorem 1.2.* By Proposition 4.1 there exists a stationary state $\bar{\rho}$ of (2). In particular, if $V = 0$, there exist such a stationary state with the same centre of mass as $\rho_{in}$ (since in this case the equation is invariant w.r.t. translation in space).

Let $u(t,\cdot)$ be the pseudo-inverse of $\rho(t,\cdot)$. By the assumptions of the theorem eq. (3) writes as :

$$\partial_t u(z) = \int_0^1 [W'(0^+)\,\mathrm{sign}(u(\xi) - u(z)) + \tilde{W}'(u(\xi) - u(z))]\, d\xi - V'(u(z)).$$

Moreover, Proposition 2.1 yields $\rho$, $\bar{\rho} \in L^\infty(\mathbb{R})$ and $u$, $\bar{u}$ are strictly increasing. Thus we calculate using $\int_0^1 \mathrm{sign}(u(\xi) - u(z))\, d\xi = \int_0^1 \mathrm{sign}(\xi - z)\, d\xi = 1 - 2z$ :

$$\partial_t u(z) = W'(0^+)(1 - 2z) + \int_0^1 \tilde{W}'[(\bar{u}(\xi) - \bar{u}(z)) + (v(\xi) - v(z))]\, d\xi - V'(u(z)).$$

Using the mean value theorem, there exist functions $\theta_1(z,\xi), \theta_2(z) \in [-2C, 2C]$ for $z$, $\xi \in [0,1]$ (where $C$ is the uniform bound on the support as stated in Proposition 2.1) such that :

$$
\begin{aligned}
\partial_t v(z) = {} & W'(0^+)(1-2z) + \int_0^1 \tilde{W}'(\bar{u}(\xi) - \bar{u}(z))\, d\xi - V'(\bar{u}(z)) \\
& + \int_0^1 \tilde{W}''(\theta_1(z,\xi))(v(\xi) - v(z))\, d\xi - V''(\theta_2(z))v(z) \\
= {} & - \Big( \int_0^1 \tilde{W}''(\theta_1(z,\xi))\, d\xi + V''(\theta_2(z)) \Big) v(z) + \int_0^1 \tilde{W}''(\theta_1(z,\xi)) v(\xi)\, d\xi,
\end{aligned}
$$

since $\bar{u}$ is a stationary state of (3).

In the following we shall distinguish the two cases :

*Case where $V'' > C > 0$ and $W|_{(0,\infty)}$ is convex.*

Consider $z^* \in [0,1]$ such that $v(t,z^*) = \|v(t,\cdot)\|_\infty$ and assume w.l.o.g. that $v(t,z^*) > 0$. Since $W$ is convex on $(0,\infty)$ we have $\tilde{W}'' \geq 0$ and

$$
\begin{aligned}
\partial_t v(z^*) \leq & - \Big( \int_0^1 \tilde{W}''(\theta_1(z^*,\xi))\, d\xi \Big) v(z^*) + \|v\|_\infty \int_0^1 \tilde{W}''(\theta_1(z,\xi))\, d\xi - V''(\bar{u}(z))\, v(z^*) \\
\leq & - V''(\bar{u}(z))\, v(z^*),
\end{aligned}
$$

by the definition of $z^*$. Thus,

$$
\frac{d}{dt} \|v(t,\cdot)\|_\infty \leq - \left( \inf_{x \in \mathrm{supp}(\rho)} V''(x) \right) \|v\|_\infty
$$

holds for all $t \geq 0$, which yields

$$
\|v(t,\cdot)\|_\infty \leq \|v(0,\cdot)\|_\infty \, e^{-\left(\inf_{x\in \mathrm{supp}(\rho)} V''(x)\right) t}.
$$

This proves the Theorem 1.2 in this case.

*Case where $V = 0$ and $W''|_{(0,\infty)} > C > 0$.*

Consider again $z^* \in [0,1]$ such that $v(t,z^*) = \|v(t,\cdot)\|_\infty$ and assume w.l.o.g. that $v(t,z^*) > 0$. Since $V = 0$ the centre of mass of $\rho(t,\cdot)$ is conserved (see (4)) and $\int_{\mathbb{R}} x\, \rho(t,x)\, dx = \int_{\mathbb{R}} x\, \rho_{in}(x)\, dx = \int_{\mathbb{R}} x\, \bar{\rho}(x)\, dx$ and $\int_0^1 v(\xi)\, d\xi = 0$ holds for all $t \geq 0$. Moreover, $\tilde{W}''(\bar{u}(\xi) - \bar{u}(z)) - \inf_{x\in \mathrm{supp}(\rho)} \tilde{W}''(x) \geq 0$. Thus,

$$
\begin{aligned}
\partial_t v(z^*) = & \int_0^1 \Big( \tilde{W}''(\bar{u}(\xi) - \bar{u}(z^*)) - \inf_{\mathrm{supp}(\rho)} \tilde{W}''(x) \Big) v(\xi)\, d\xi - \int_0^1 \tilde{W}''(\bar{u}(\xi) - \bar{u}(z^*))\, d\xi\, v(z^*) \\
\leq & \|v\|_\infty \int_0^1 \Big( \tilde{W}''(\bar{u}(\xi) - \bar{u}(z^*)) - \inf_{\mathrm{supp}(\rho)} \tilde{W}''(x) \Big)\, d\xi - \int_0^1 \tilde{W}''(\bar{u}(\xi) - \bar{u}(z^*))\, d\xi\, v(z^*) \\
\leq & - \Big( \inf_{\mathrm{supp}(\rho)} \tilde{W}''(x) \Big) \|v\|_\infty,
\end{aligned}
$$

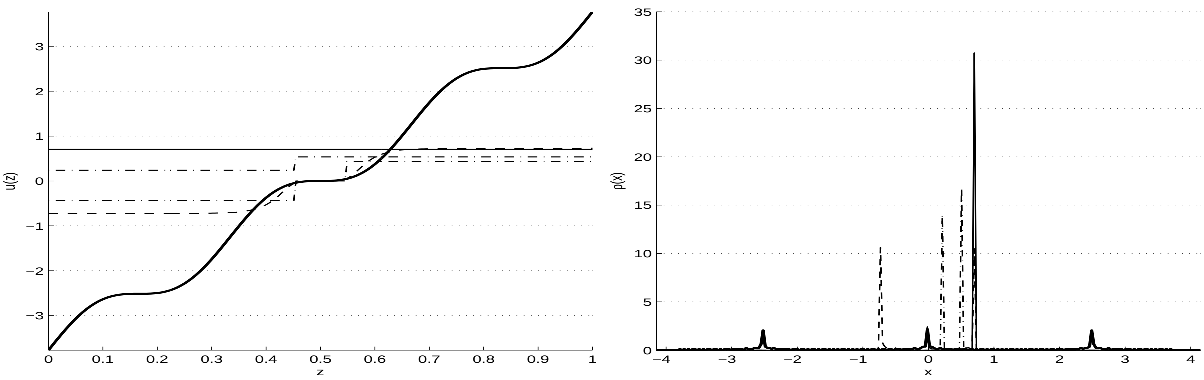


Figure 1: Convergence towards an approximative single Dirac stationary state for the ($\varepsilon$-smoothed) attractive interaction potential $W(x) = |x|_\varepsilon$ with $\varepsilon = 0.03$ and the external potential $V(x) = x^4 - x^2$. The left image plots $u(t, z)$ at time $t = 0$ (initial data, bold line), $t = 0.5$ (dashed line), $t = 5, 20$ (dash-dotted lines) and $t = 30$ (stable stationary state, solid line). The right image plots the measure $\rho(t, x)$ at the times $t = 0$ (bold lines), $t = 0.5$ (dashed line), $t = 20$ (dash-dotted line) and $t = 30$ (solid line).

by the definition of $z^*$. Then $\frac{d}{dt}\|v(t,\cdot)\|_\infty \leq -\big(\inf_{x\in\mathrm{supp}(\rho)} \tilde{W}''(x)\big)\|v\|_\infty$ holds for all $t \geq 0$ and

$$\|v(t,\cdot)\|_\infty \leq \|v(0,\cdot)\|_\infty \, e^{-\big(\inf_{x\in\mathrm{supp}(\rho)} \tilde{W}''(x)\big)t},$$

proves the Theorem 1.2 in this case where $V = 0$. □

# 5 Numerical examples

We perform numerical simulations using both an explicit and an implicit Euler scheme for the pseudo-inverse equation (3). Note that approximating $u(z)$ on $z \in [0, 1]$ by piecewise constant step functions on an equidistant grid with $n + 1$ grid points (we have used $n = 256$) is equivalent to a particle method for equation (2), where a measure $\rho(x)$ is approximated by a sum of $n$ Diracs with mass $\frac{1}{n}$. As expected, the implicit Euler scheme remains stable for singular repulsive interaction potentials, for which the explicit Euler scheme fails.

The numerics are implemented and plotted in Matlab. In order to depict a measure $\rho(x)$, we represent each Dirac mass by a triangle centred at the position $u_i$ with basis-length 1/90 and with area equivalent to the mass of the represented Dirac.

In a first example we consider the confining external potential $V(x) = x^4 - x^2$ and the attractive interaction potential $W(x) = |x|_\varepsilon$, which is a piecewise $C^2$-approximation of the singular attractive potential $W(x) = |x|$ with $W''(x) = 1/\varepsilon$ for $x \in (-\varepsilon, \varepsilon)$. Figure 1 shows convergence of initially three smoothed Dirac masses $u_{in}(z) = 0.4\,(\sin(6\pi z) + 6\pi z) - C$

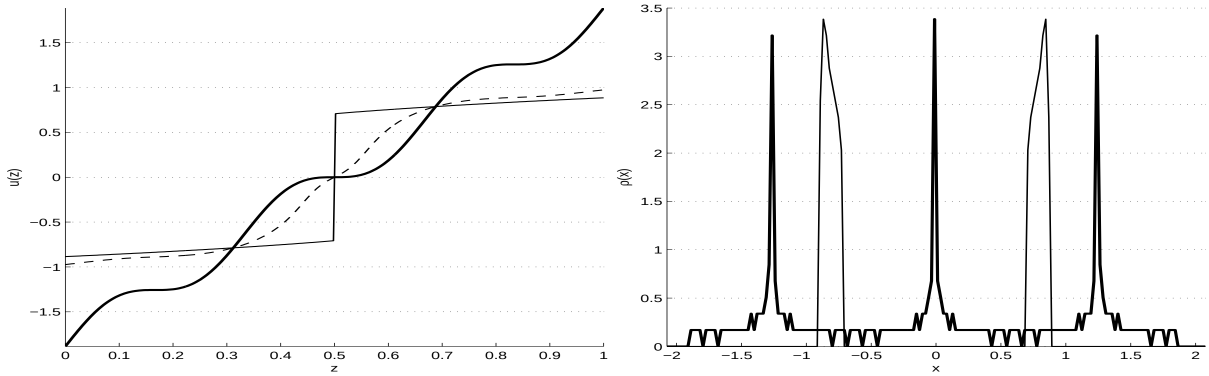


Figure 2: Convergence towards a stationary state consisting of two separated continuous parts for the interaction potential $W(x) = -|x|$ and the external potential $V(x) = x^4 - x^2$. The left image plots $u(t, z)$ at time $t = 0$ (initial data, bold line), $t = 0.5$ (dashed line) and $t = 10$ (stable stationary state, solid line). The right image plots the density $\rho(t, x)$ at the times $t = 0$ (bold lines) and $t = 10$ (solid line).

with $C$ such that $\int_0^1 u_{in}(z)\,dz = 0$ towards a $\varepsilon$-smoothed single Dirac stationary state. In a first phase the solution converges quickly towards three $\varepsilon$-smoothed instable Dirac masses (see $t = 5$). This first phase corresponds to *Step 1.* in the proof of Theorem 1.1.

In a second slow phase, the numerical scheme is able to resolve the instability of the $\varepsilon$-smoothed Diracs. In fact, this is the reason why we take $W(x) = |x|_\varepsilon$ and not $W(x) = |x|$ in this example in the first place. Thus, Figure 1 shows in the following how the three Diracs collapse to two Diracs (see $t = 20$) and finally to a single Dirac stationary state ($t = 30$). This corresponds to *Step 2.* in the proof of Theorem 1.1. Note that since in this example the external potential $V$ is locally repulsive stationary states can consist of more than a single Dirac mass.

In Figure 2 we show convergence to the stationary state of (3) for the singular repulsive interaction potential $W(x) = -|x|$ and the confining external potential $V(x) = x^4 - x^2$ subject to the initial data $u_{in}(z) = 0.2\,(\sin(6\pi z) + 6\pi z) - C$ with $C$ such that $\int_0^1 u_{in}(z)\,dz = 0$. Note that $\bar{\rho}$ consists of two continuous parts separated by the effect of the external potential being locally repulsive.

Finally, the Figures 3 and 4 consider strongly singular repulsive interaction potentials. Figure 3 shows how the solution of (2) with the symmetric (in the $\rho$ picture) initial data $u_{in}(z) = 0.2\,(\sin(8\pi z) + 8\pi z) - C$ with $C$ such that $\int_0^1 u_{in}(z)\,dz = 0$ (bold line) converges to a smoothed Dirac stationary state for the singular repulsive double-well potential $W(x) = x^2 - |x|^\alpha$ with $\alpha = 0.05$. Figure 4 shows the same for $W(x) = x^2 - |x|^\alpha$ with $\alpha = 0.001$.

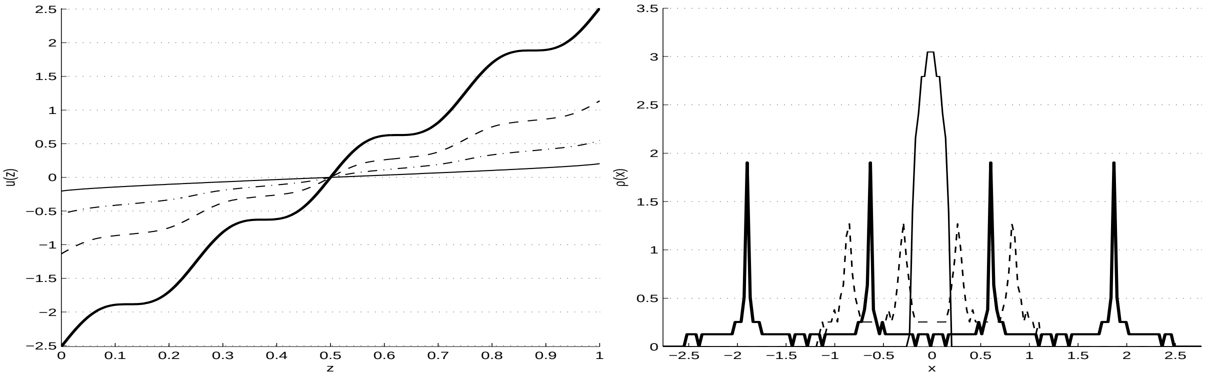


Figure 3: Convergence towards a smoothed Dirac stationary state for the doublewell potential $W(x) = x^2 - |x|^\alpha$ with $\alpha = 0.05$. The left image plots $u(t, z)$ at time $t = 0$ (initial data, bold line), $t = 0.5$ (dashed line), $t = 1$ (dash-dotted line), and $t = 10$ (stable stationary state, solid line). The right image plots the density $\rho(t, x)$ at the times $t = 0$ (bold lines), $t = 0.5$ (dashed line) and $t = 10$ (solid line).

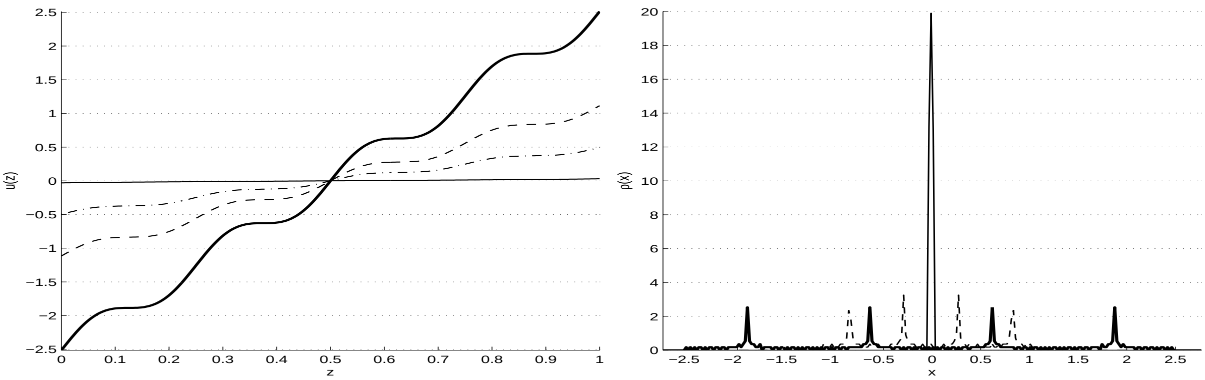


Figure 4: Convergence towards a smoothed Dirac stationary state for the doublewell potential $W(x) = x^2 - |x|^\alpha$ with $\alpha = 0.001$. The left image plots $u(t, z)$ at time $t = 0$ (initial data, bold line), $t = 0.5$ (dashed line), $t = 1$ (dash-dotted line), and $t = 10$ (stable stationary state, solid line). The right image plots the density $\rho(t, x)$ at the times $t = 0$ (bold lines), $t = 0.5$ (dashed line) and $t = 10$ (solid line).

# 6 Appendix

The proof of the following Lemma is classical but shall be recalled for the sake of the reader.

**Lemma 6.1.** *If a matrix $M \in M_n(\mathbb{C})$ satisfies $\sigma(M) \subset \{z \in \mathbb{C} : \mathcal{R}e(z) > \eta\}$, then, for any induced matrix-norm $\|\cdot\|$ of $\mathbb{R}^{n\times n}$, there exist a constant $C > 0$ such that for $t \geq 0$*

$$\left\|e^{-M\,t}\right\| \leq C(1 + t^{n-1})\, e^{-\eta t}.$$

*Proof of Lemma 6.1.* The Dunford decomposition theorem implies that there exists $D \in M_n(\mathbb{C})$ diagonalisable (i.e. $P^{-1}DP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ for a $P \in GL_n(\mathbb{C})$) and $N$ nilpotent (i.e. $N^n = 0$) with $M = D + N$ such that $D$ and $N$ commute, i.e. $D\,N = N\,D$. Thus,

$$\begin{aligned} e^{-t\,M} = e^{-t\,D} e^{-t\,N} &= P\Big(\sum_{k=0}^{\infty} \frac{t^k}{k!}(-P^{-1}DP)^k\Big)P^{-1}\Big(\sum_{k=0}^{n-1} \frac{1}{k!}(-tN)^k\Big) \\ &= P\,\mathrm{diag}(e^{-t\,\lambda_1}, \dots, e^{-t\,\lambda_n})P^{-1}\Big(\sum_{k=0}^{n-1} \frac{1}{k!}(-tN)^k\Big) \end{aligned}$$

and we can then estimate $\left\|e^{-tM}\right\|$ as follows :

$$\left\|e^{-t\,M}\right\| \leq \|P\| \left(C \max_{i=1,\dots,n}\{e^{-t\,\lambda_i}\}\right) \|P^{-1}\| \left(\sum_{k=0}^{n-1} \frac{1}{k!} t^k \|N\|^k\right) \leq C(1 + t^{n-1})\, e^{-\eta t},$$

for constants $C$ and with $\eta = \min_{i=1,\dots,n} \mathcal{R}e(\lambda_i)$. ☐

Next, we prove the following Gronwall-type lemma:

**Lemma 6.2.** *Let $u \in Lip([T, \infty), \mathbb{R}_+)$ satisfies for all $t \geq T$:*

1. *If*
$$u(t) \leq \alpha(t) + \gamma(t) \int_0^t \beta(s)u(s)\, ds,$$
*then*
$$u(t) \leq \alpha(t) + \gamma(t) \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(\sigma)\gamma(\sigma)\, d\sigma\right) ds.$$

2. *If $u(T)$ is small enough and*
$$u(t) \leq C\,u(T)(1 + t^k)\, e^{-(t-T)\kappa} + C \int_T^t u(s)^{1+\alpha}(1 + (t-s)^k)\, e^{-(t-s)\kappa}\, ds,$$
*then there exist $C > 0$ such that:*
$$u(t) \leq C(1 + t^k)\, e^{-t\kappa}.$$

*Proof. To show 1.* we define:

$$v(t) := \exp\left(-\int_0^t \beta(s)\gamma(s)\,ds\right)\int_0^t \beta(s)u(s)\,ds. \tag{29}$$

with $v(0) = 0$ and

$$\begin{aligned}
v'(t) &= \beta(t)\left(u(t) - \gamma(t)\int_0^t \beta(s)u(s)\,ds\right)\exp\left(-\int_0^t \beta(s)\gamma(s)\,ds\right) \\
&\le \beta(t)\alpha(t)\exp\left(-\int_0^t \beta(s)\gamma(s)\,ds\right).
\end{aligned}$$

Then,

$$v(t) \le \int_0^t \alpha(s)\beta(s)\exp\left(-\int_0^s \beta(\sigma)\gamma(\sigma)\,d\sigma\right)\,ds. \tag{30}$$

Using the definition (29) of $v$ and (30), we can estimate $u$:

$$\begin{aligned}
u(t) &\le \alpha(t) + \gamma(t)\int_0^t \beta(s)u(s)\,ds \le \alpha(t) + \gamma(t)v(t)\exp\left(\int_0^t \beta(s)\gamma(s)\,ds\right) \\
&\le \alpha(t) + \gamma(t)\exp\left(\int_0^t \beta(s)\gamma(s)\,ds\right)\int_0^t \alpha(s)\beta(s)\exp\left(-\int_0^s \beta(\sigma)\gamma(\sigma)\,d\sigma\right)\,ds \\
&\le \alpha(t) + \gamma(t)\int_0^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(\sigma)\gamma(\sigma)\,d\sigma\right)\,ds.
\end{aligned}$$

*To show 2.* we define $y_1(t) := u(t+T)e^{t\kappa}$. Then,

$$\begin{aligned}
y_1(t) &\le C\,u(T)(1+t^k) + C\int_0^t y_1(s)^{1+\alpha}e^{-\kappa\alpha s}(1+(t-s)^k)\,ds \\
&\le C\,u(T)(1+t^k) + C(1+t^k)\int_0^t y_1(s)^{1+\alpha}e^{-\kappa\alpha s}\,ds
\end{aligned}$$

Let $M > 0$. As long as $y_1(t)^\alpha e^{-\frac{\kappa}{2}t} \le M$, we have

$$y_1(t) \le C\,u(T)(1+t^k) + CM(1+t^k)\int_0^t y_1(s)e^{-\frac{\kappa\alpha}{2}s}\,ds.$$

We can then apply part 1 of this lemma to $u = y_1$ and get :

$$\begin{aligned}
y_1(t) &\le C\,u(T)(1+t^k) \\
&\quad + CM(1+t^k)\int_0^t C\,u(T)(1+s^k)e^{-\frac{\kappa\alpha}{2}s}\exp\left(\int_s^t e^{-\frac{\kappa\alpha}{2}\sigma}CM(1+\sigma^k)\,d\sigma\right)\,ds \\
&\le C\,u(T)(1+t^k)Me^{CM},
\end{aligned}$$

and if $u(T)$ is small enough, then $y_1(t)^\alpha e^{-\frac{\kappa}{2}t} \le M$ holds at all times, which implies $u(t) \le C(1+t^k)\,e^{-\kappa t}$ and the Lemma 6.2. $\square$

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