On Piecewise Deterministic Markov Processes

Michel Benaïm (Neuchâtel)

IHP, January 28, 2016

Talk based on recent works with

- **Stephane Le Borgne (Rennes), Florent Malrieu (Tours) & Pierre-André Zitt (Marne la Vallée)**

(Annales de l'IHP, 2015).

- **Fritz Colonius & Ralph Lettau (Augsburg)**

(Work in Progress).

- **Claude Lobry (Nice)**

(Arxiv preprint Oct. 2015).

Introduction, Goals and Examples

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Here we restrict attention to the following specific class:

$\bullet E = \{1, \ldots, m\}$,

$\bullet F^1, \ldots, F^m$ smooth bounded vector fields on $\mathbb{R}^d$,

$\bullet$ $(\Phi_t^1), \ldots, (\Phi_t^m)$ induced flows

$\bullet$ $M \subset \mathbb{R}^d$ = compact positively invariant set under each Φ^i,

$\bullet$ For $x \in M$, $(Q_{ij}(x))$ = Markov transition matrix over E (irreducible aperiodic and continuous in x) .

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– Repeat

End

This makes

(Z_t) a continuous time Markov process, and

$(\tilde{Z}_n) = (Z_{T_n})$ a discrete time Markov chain,

$$T_n = U_1 + \ldots + U_n$$

are the jump times.

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Feller :

$$P_t f(z) = \mathbb{E}_z(f(Z_t)), \; \tilde{P} f(z) = \mathbb{E}_z(f(\tilde{Z}_1))$$

map $C_b(M)$ into itself.

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- Uniqueness (of invariant measure),
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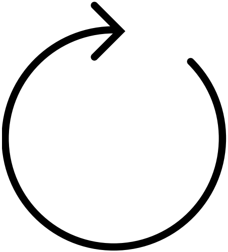

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Figure: designed by freepick

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Both Z and $(\tilde{Z})$ are uniquely ergodic (unique invariant measure is the uniform distribution μ on S^1)

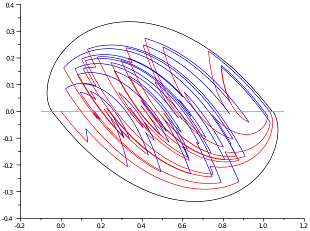
$$Law(Z_t) = \delta_{\Phi_t(x)}, \ Law(\tilde{Z}_n) = (x + \Gamma(n)) \mod 1 \rightarrow \mu.$$

Another (less trivial) example

$E = \{1,2\}, M \subset \mathbb{R}^2$

$$F^1(x) = Ax, F^2(x) = A(x - e)$$

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$



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A non trivial example (from Benaim and Lobry, 15)

$E = \{1, 2\}, M \subset \mathbb{R}_+ \times \mathbb{R}_+.$

F^1, F^2 two Lotka-Volterra vector fields

$$F^i(x, y) = \begin{cases} \alpha_i x(1 - a_i x - b_i y) \\ \beta_i y(1 - c_i x - d_i y) \end{cases}$$

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favorable to the same species x

$$\alpha_i, \beta_i, a_i, b_i > 0,$$

$$a_i < c_i \text{ and } b_i < d_i.$$

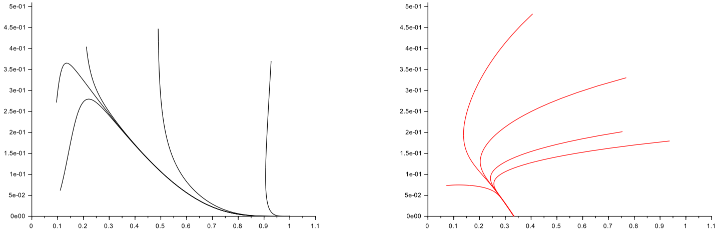

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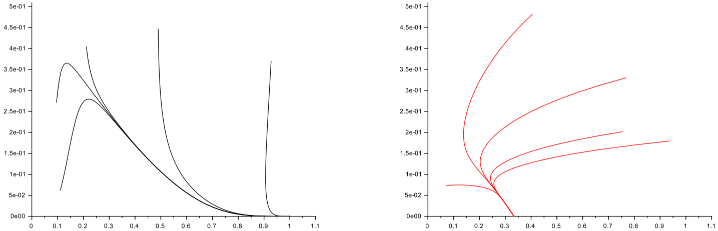
Figure: Phase portraits of F^1 and F^2



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$$Q(x) = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}.$$

Different values of p, λ lead to different behaviors $\hookrightarrow$

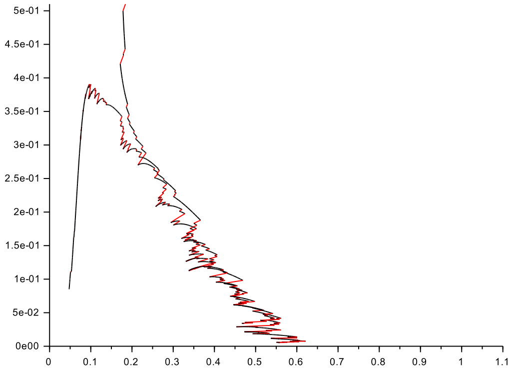


Figure: extinction of 2

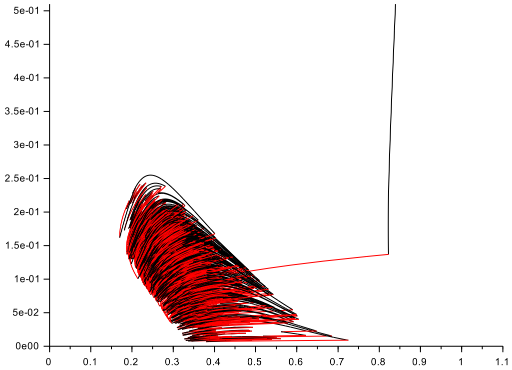
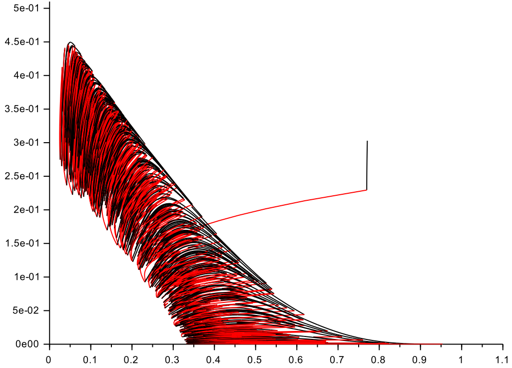


Figure: Persistence



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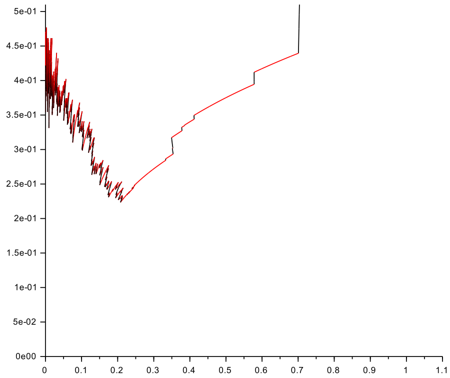

Figure: Extinction of 1

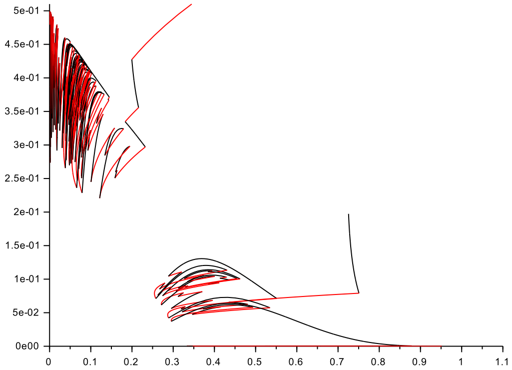

Figure: Extinction of 1 or 2

Some Maths

A support Theorem

$$co(F)(x) := \mathrm{conv}(F^1(x), \ldots, F^m(x)),$$

• S^x the set of (absolutely continuous) maps $\eta : \mathbb{R}_+ \mapsto \mathbb{R}^d$ solutions to

$$\dot{\eta} \in co(F)(\eta), \eta(0) = x$$

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Theorem (Benaim, Leborgne, Malrieu, Zitt, 15)

If $X_0 = x$ then, the support of the law of (X_t) equals S^x.

Invariant Measures

A probability measure on $M \times E$ is called *invariant* for (Z_t) whenever

$$Law(Z_0) = \mu \Rightarrow Law(Z_t) = \mu$$

$$\Leftrightarrow \int \mathsf{P}_z(Z_t \in \cdot)\mu(dz) = \mu(\cdot)$$

$$\Leftrightarrow \int P_t f(z)\mu(dz) = \int f \mu(dz).$$

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Similar definition for $(\tilde{Z}_n)$
Ergodic measure = extremal invariant

- $\mathcal{P}_{inv}, (\mathcal{P}_{erg}) :=$ invariant (ergodic) measures for (Z_t)
- $\tilde{\mathcal{P}}_{inv}, (\tilde{\mathcal{P}}_{erg}) :=$ invariant (ergodic) measures for (Z_t)

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Proposition (Correspondence for invariant measures, BLMZ, 15)

(i) *$\tilde{\mathcal{P}}_{inv}, (\tilde{\mathcal{P}}_{erg})$ and $\tilde{\mathcal{P}}_{inv}, (\tilde{\mathcal{P}}_{erg})$ are homeomorphic*

(ii) *This homeomorphism preserves the support*

(iii) *If μ is ergodic it is either absolutely continuous or singular*

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Open problem Nothing is known in general (in the a-c case) on the regularity of the density **except in dimension** 1 :
(Y. Bakhtin, T. Hurt and J- Mattingly, Nonlinearity, 2015)

Accessible Set

- (Ψ_t) the *set valued semi flow* induced by $\dot{\eta} \in co(F)(\eta)$:

$$\Psi_t(x) = \{\eta(t) : \ \eta \in S^x\}.$$

Omega limit set:

$$\omega_{\Psi}(x) = \bigcap_{t \geq 0} \overline{\{\Psi_s(x), s \geq t\}}$$

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Γ is a compact possibly empty set

Proposition (Properties of the accessible set, BLMZ, 15)

(i) $\Gamma = \omega_\Psi(p)$ *for all* $p \in \Gamma$

(ii) Γ *is compact connected invariant* $(\forall t \geq 0 \Psi_t(\Gamma) = \Gamma)$.

(iii) *Either* Γ *has empty interior or its interior is dense in* Γ.

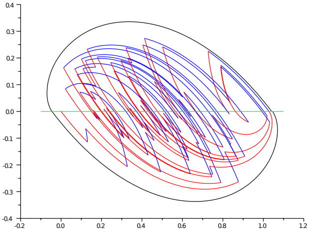

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Figure: Example of accessible set

Proposition (Accessible set and invariant measures, BLMZ, 15)

(i) *If* $\Gamma \neq \emptyset$ *then* $\Gamma \times E \subset \mathbf{supp}(\mu)$ *for all* $\mu \in \mathcal{P}_{inv}$ *and there exists* $\mu \in \mathcal{P}_{inv}$ *such that* $\mathbf{supp}(\mu) = \Gamma \times E$.

(ii) *If* Γ *has non empty interior, then* $\Gamma \times E = \mathbf{supp}(\mu)$ *for all* $\mu \in \mathcal{P}_{inv}$.

(iii) *Suppose that there is a* **unique invariant measure** *for* Z *(or* $\tilde{Z}$*)* π, *then* $\mathbf{supp}(\pi) = \Gamma \times E$.

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When Γ has empty interior, inclusion $\Gamma \times E \subset \mathbf{supp}(\mu)$ can be strict !

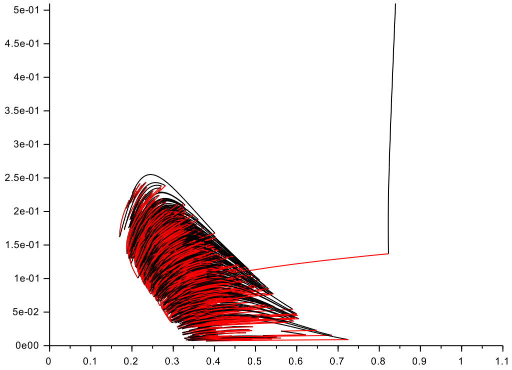

Figure: $\Gamma \subset \mathbb{R}_+ \times \{0\}$ $\Gamma \times E \neq \mathbf{supp}(\mu)$

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When Γ has non empty interior, there may be several invariant measures ! (Furstenberg's type example)

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If $\Gamma = \emptyset$ some of the previous result still hold by considering *Invariant Control Sets*

•$\{F^1, \ldots, F^m\}$ is said *locally accessible* in M if for all $x \in M$ and $T > 0$

$$O_T^+(x) = \{\Psi_t(x) : 0 \leq t \leq T\}$$

and

$$O_T^-(x) = \{y \in M : x \in \Psi_t(y) \text{ for some } 0 \leq t \leq T\}$$

have non empty interior

Proposition (Benaim, Colonius & Lettau, 16)

Suppose $\{F^1, \ldots, F^m\}$ *locally accessible in* M.

(i) *There are finitely many invariant control sets* $\Gamma_1, \ldots, \Gamma_l$

(ii) *They coincide with minimal compact invariant sets for* Ψ *and have non empty interiors*

(iii) *For every* $\mu \in \mathcal{P}_{inv}$ $\mathbf{supp}(\mu) \subset \bigcup_{i=1}^{l} \Gamma_i \times E$.

(iv) *For every* $\mu \in \mathcal{P}_{erg}$ *there is some* i *for which* $\mathbf{supp}(\mu) = \Gamma_i \times E$.

Uniqueness and Convergence

The Bracket conditions

Weak Bracket at $x \in M$:

The Lie algebra generated by $\{F^i, i = 1, \ldots, m\}$ has full rank at x

Strong Bracket at $x \in M$:

$$G_0 = \{F^i - F^j \ : i, j = 1, \ldots m\} \ G_{k+1} = G_k \cup \{[F^i, V] \ : V \in G_k\}$$

has full rank at x for some k.

Convergence in discrete time

Theorem (BLMZ, 15)

Suppose $\exists x \in \Gamma$ *at which the* **weak bracket** *condition holds. Then* $\tilde{Z}$ *admits a unique invariant probability* $\tilde{\pi}$*, absolutely continuous with respect to the Lebesgue measure* $\lambda_{M \times E}$ *on* $M \times E$*; and*

$$\|\mathbf{P}(\tilde{Z}_n \in \cdot) - \tilde{\pi}\| \leq c\rho^n$$

for some $c > 1$ *and* $\rho \in (0, 1)$*.*
Here $\|\cdot\| =$ *total variation norm.*

Corollary

Suppose $\exists x \in \Gamma$ at which the weak bracket condition holds. Then Z admits a unique invariant probability π, absolutely continuous with respect to the Lebesgue measure $\lambda_{M \times E}$ on $M \times E$;

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- Weak Bracket $\not\Rightarrow$ convergence in law of (Z_t) !

$$M = S^1 = \mathbb{R}/\mathbb{Z}, \Phi_t(x) = (x + t) \mod 1.$$

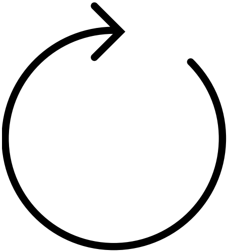

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Figure: designed by freepick

$$Z_t = \Phi_t(x), \; \tilde{Z}_n \sim x + \Gamma(n)) \mod 1 \text{ converges in distribution}$$

Convergence in continuous time

Theorem (BLMZ, 15)

Suppose $\exists p \in \Gamma$ *at which the* **strong bracket** *condition holds. Let* π *be the invariant probability of* Z. *Then*

$$\|\mathbf{P}(Z_t \in \cdot) - \pi\| \leq Ce^{-\kappa t}$$

for some $C > 1$ *and* $\kappa > 0$

What if $\Gamma = \emptyset$?

Proposition (Benaim, Colonius & Lettau, 16)

Suppose the **weak** *bracket condition holds for all $x \in M$.*

(i) *There are finitely many compact invariant control sets $\Gamma_1, \ldots, \Gamma_l$*

(ii) $\tilde{\mathcal{P}}_{erg} = \{\tilde{\pi}^1, \ldots, \tilde{\pi}^l\}$ *and* **supp**$(\tilde{\pi}^i) = \Gamma_i \times E$.

(iii) *On Γ_i the law of $\tilde{Z}$ converges exponentially fast toward $\tilde{\pi}^i$.*

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(ii) $\mathcal{P}_{erg} = \{\pi^1, \ldots, \pi^l\}$ *and* $\mathbf{supp}(\pi^i) = \Gamma_i \times E$.

(iii) *On* Γ_i *the law of* Z *converges exponentially fast toward* π^i.

Time to Switch to Lunch...