# A method to approximate Lyapunov exponents and most unstable trajectories of switching systems. 

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Inspired by joint works with Vladimir Yu. Protasov (Moscow State University) and Marino Zennaro (Universitá di Trieste).

## Outline of the talk

(1) A class of switched linear systems

- Upper and lower Lyapunov exponent. Stability concepts
(2) Discretization: joint spectral radius
- Generalization of the spectral radius of a matrix
- Polytope norms and related algorithms
(3) Approximating the upper Lyapunov exponent
- A bilateral convergent estimate
(4) Approximating most unstable trajectories
(5) Summary and Outlook

A class of switched linear dynamical systems For a given finite (or compact) set of $k \times k$ matrices

$$
\mathcal{C}=\left\{C_{i}\right\}_{i \in \mathcal{I}} \quad(\mathcal{I} \text { set of indeces })
$$

and $u:(0,+\infty) \rightarrow \mathcal{I}, u \in \mathcal{U}$ (set of measurable switching functions), consider the linear dynamical system (for $x \in \mathbb{C}^{k}$ )
(S) $\quad\left\{\begin{array}{l}\dot{x}(t)=C(u(t)) x(t), \quad C(i)=C_{i} \text { for } i \in \mathcal{I}, \\ x(0)=x_{0} \in \mathbb{C}^{k}\end{array}\right.$

The switching function $u(t)$ jumps among the values of $\mathcal{I}$.
We mostly consider here the finite illustrative case $\mathcal{I}=\{0,1\}$, that is $\mathcal{C}=\left\{C_{0}, C_{1}\right\}$.
Example of $u(t)$ :


Lyapunov exponents, stability and stabilizability
The upper Lyapunov exponent $\sigma(\mathcal{C})$ is the infimum of the numbers $\alpha$ such that, for some constant $L>0$,

$$
\|x(t)\| \leq L e^{\alpha t} \quad \forall t \geq 0
$$

for any $u \in \mathcal{U}$ and initial value $x_{0}$ in (S).
If $\sigma(\mathcal{C})<0$, then the system is uniformly asymptotically stable

The lower Lyapunov exponent $\widetilde{\sigma}(\mathcal{C})$ is the infimum of the numbers $\beta$ for which there exists a switching function $\tilde{u} \in \mathcal{U}$ such that, for some constant $M>0$, the corresponding trajectory of (S) satisfies, $\forall x_{0}$,

$$
\|x(t)\| \leq M e^{\beta t} \quad \forall t \geq 0
$$

If $\widetilde{\sigma}(\mathcal{C})<0$, then the system is stabilizable.

Extremal norms and their approximations
Definition. A norm $\|\cdot\|$ is called extremal if for every trajectory of (S) it holds $\|x(t)\| \leq e^{\sigma(\mathcal{C}) t}\|x(0)\|, t \geq 0$. If equality holds for all $t$ and for all $x(0),\|\cdot\|$ is called a Barabanov norm.

## Theorem (Opoitsev '77, Barabanov '88)

An irreducible set of operators possesses an extremal Barabanov norm.

## Polytope approximate extremal norms.

Advantages: (i) can reach arbitrary accuracy; (ii) very efficient for sets of matrices whose exponential has an invariant cone (e.g. Metzler matrices and the non-negative orthant).

Drawbacks: computationally expensive in the general case.
Common Quadratic Lyapunov Functions alias ellipsoid norms. Advantages: computationally efficient till dimension $k \approx 25$.
Drawbacks: (i) not arbitrarily accurate; (ii) costly if $k>25$.

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## Discretization

Assumption: $\mathcal{C}=\left\{C_{i}\right\}_{i \in \mathcal{I}}, \mathcal{I}=\{0, \ldots, m\}$ is finite.
(1) Restrict the switching function $u(t)$ to $\mathcal{U}_{\Delta t}$, the space of piecewise constant functions on $\left\{t_{j}\right\}_{j \geq 0}, t_{j}=j \Delta t$ (that is $\left.\left.u\right|_{\left(t_{j-1}, t_{j}\right]}=i_{j} \in \mathcal{I}\right)$.
(2) Let $\mathcal{A}_{\Delta t}=\left\{A_{i}\right\}_{i \in \mathcal{I}}$, with $A_{i}=\mathrm{e}^{\Delta t} C_{i}$.

We have the discrete switched system (with $x_{n}:=x\left(t_{n}\right)$ )

$$
x_{n+1}=A_{i_{n}} x_{n} \quad \text { with } i_{n} \in \mathcal{I}, \quad n \geq 0 .
$$

(3) Lower bound to $\sigma(\mathcal{C})$. Compute the Lyapunov exponent of the discretized problem, which is obtained restricting the set of switching functions as in (1). This is related to the computation of the joint spectral radius ${ }^{1}$ of the matrix family $\mathcal{A}_{\Delta t}$.

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## Discrete time switched linear systems

We are lead to consider discrete switched systems of the form

$$
\begin{equation*}
x_{n+1}=A_{i_{n}} x_{n}=A_{i_{n}} \cdot \ldots \cdot A_{i_{1}} \cdot A_{i_{0}} x_{0}, \quad n=0,1,2, \ldots \tag{DSS}
\end{equation*}
$$

where $x_{0} \in \mathbb{C}^{k}$ and $A_{i_{j}} \in \mathbb{C}^{k, k}$ is an element of

$$
\mathcal{A}=\left\{A_{i}\right\}_{i \in \mathcal{I}} \quad(\mathcal{I} \text { set of indices })
$$

Product semigroup: $\Sigma(\mathcal{A})=\bigcup_{n \geq 1} \Sigma_{n}(\mathcal{A})$, where

$$
\Sigma_{n}(\mathcal{A})=\left\{A_{j_{n}} \cdot \ldots \cdot A_{j_{1}} \mid\left(j_{1}, \ldots, j_{n}\right) \in \mathcal{I} \times \mathcal{I} \times \ldots \times \mathcal{I}\right\}
$$

Goal. Computing the highest rate of growth of trajectories of (DSS) (or equivalently of sequences in $\Sigma(\mathcal{A})$ ). The problem is not trivial...

Generalizations of the spectral radius of a matrix
(1) Joint spectral radius (Rota \& Strang '60): for $\mathcal{A}$ bounded

$$
\widehat{\rho}(\mathcal{A})=\limsup _{n \rightarrow \infty} \widehat{\rho}_{n}(\mathcal{A})^{1 / n} \text { with } \widehat{\rho}_{n}(\mathcal{A})=\sup _{P \in \Sigma_{n}(\mathcal{A})}\|P\|
$$

(2) Generalized spectral radius (Daubechies \& Lagarias '92):

$$
\bar{\rho}(\mathcal{A})=\limsup _{n \rightarrow \infty} \bar{\rho}_{n}(\mathcal{A})^{1 / n} \quad \text { with } \bar{\rho}_{n}(\mathcal{A})=\sup _{P \in \Sigma_{n}(\mathcal{A})} \rho(P)
$$

General result (Berger \& Wang '92):

$$
\widehat{\rho}(\mathcal{A})=\bar{\rho}(\mathcal{A}) \quad=: \rho(\mathcal{A}) .
$$

$\rho(\cdot)$ is a positively homogeneous function $(\rho(c \mathcal{A})=c \rho(\mathcal{A}))$. Note that for a single matrix (1) and (2) reduce to the spectral radius.

## A third generalization: extremal norms

(3) Common spectral radius (Elsner '95):

$$
\rho(\mathcal{A})=\inf _{\| \| \in \mathcal{N}}\|\mathcal{A}\|, \quad\|\mathcal{A}\|=\sup _{A \in \mathcal{A}}\|A\|
$$

with $\mathcal{N}$ set of operator norms. If the inf is a $\min \mathcal{A}$ is non-defective,

$$
\|\cdot\|_{\star} \longrightarrow \min _{\|\cdot\| \in \mathcal{N}}\|\mathcal{A}\| \text { is said extremal norm for } \mathcal{A} \text {. }
$$

Useful estimate (Daubechies \& Lagarias '92).

$$
\rho(P)^{1 / n} \leq \rho(\mathcal{A}) \leq\|\mathcal{A}\| \quad \text { for any } P \in \Sigma_{n}(\mathcal{A})
$$

This suggests the natural scaling $\mathcal{A}^{*}=\mathcal{A} / \rho(P)^{1 / n}$ s.t. $\rho\left(\mathcal{A}^{*}\right) \geq 1$.

$$
\text { If } \rho(P)^{1 / n}=\rho(\mathcal{A}), P \text { is called spectrum maximizing product (s.m.p.). }
$$

Illustrative example: a discrete switched system Consider $x_{n+1}=A_{i_{n}} x_{n}, \quad i_{n} \in\{0,1\}, \quad n \geq 0$, with

$$
A_{0}=\beta\left(\begin{array}{rrr}
-1 & 1 & -1 \\
-1 & -1 & 1 \\
0 & 1 & 1
\end{array}\right), \quad A_{1}=\beta\left(\begin{array}{rrr}
-1 & 1 & -1 \\
-1 & -1 & 0 \\
1 & 1 & 1
\end{array}\right)
$$

with $\beta=0.559$. Note: $\rho\left(A_{0}\right)<1, \rho\left(A_{1}\right)<1$.
Question. Is the solution stable (bounded) for any sequence?
Answer. Yes. Maximal growth is obtained for the periodic sequence $\{001001011010010010100100101\}^{k}$.
This corresponds to the iterated application of the s.m.p. of degree 27

$$
P=\left(A_{0}^{2} A_{1}\right)^{2} A_{0} A_{1}^{2} A_{0} A_{1}\left(\left(A_{0}^{2} A_{1}\right)^{2} A_{0} A_{1}\right)^{2} \quad \text { s.t. } \quad \rho(P)=1 .
$$

Illustrative example: a switched system (ctd.)
Daubechies and Lagarias estimate provides

$$
1=\rho(P)^{1 / 27} \leq \rho(\mathcal{A}) \leq\|\mathcal{A}\|
$$

We prove stability determining an optimal norm s.t. $\|\mathcal{A}\|_{\text {opt }}=1$


The norm $\|\cdot\|_{\text {opt }}$ is s.t.
$\left\|A_{0}\right\|_{\text {opt }}=\left\|A_{1}\right\|_{\text {opt }}=1$.
This implies $\|Q\|_{\text {opt }} \leq 1$ for any product $Q$ of $A_{0}, A_{1}$

A goal of next slides is to explain how to get $\|\cdot\|_{\text {opt }}$

## Difficulties

The computation of the j. .s.r. is a challenging problem.
It is known (Blondel \& Tsitsiklis, Math. Contr. '97) that there is no algorithm able to approximate (with an a priori accuracy) the joint spectral radius in polynomial time.

Finiteness conjecture (Lagarias \& Wang '95).
It stated that every finite family has an s.m.p. (i.e. a product $P$ of degree $n$ such that $\rho(P)^{1 / n}=\rho(\mathcal{A})$ ). Alas it has been disproved by Bousch \& Mairesse, J. AMS '02 and Blondel et al.,SIMAX '03).
Therefore it may not be possible to find a finite product $P$ which gives the highest rate of growth in the product semigroup.

Our goal.
For families with the finiteness property we aim to compute (in a finite way) the j.s.r. by means of an extremal norm. How to proceed?

## Extremal norms and trajectories

The set of trajectories. Let $\mathcal{A}^{*}$ s.t. $\rho\left(\mathcal{A}^{*}\right) \geq 1$ (natural scaling).
Given an initial vector $x \neq 0$ we consider the set

$$
\mathcal{T}\left[\mathcal{A}^{*}, x\right]:=\{x\} \cup\left\{P x \mid P \in \Sigma\left(\mathcal{A}^{*}\right)\right\}
$$

Theorem (e.g. G., Wirth \& Zennaro '05)
Assume that for a given $x \in \mathbb{C}^{k}$, the set $\mathcal{T}\left[\mathcal{A}^{*}, x\right]$ satisfies
(1) $\operatorname{span}\left(\mathcal{T}\left[\mathcal{A}^{*}, x\right]\right)=\mathbb{C}^{k}$;
(2) $\mathcal{T}\left[\mathcal{A}^{*}, x\right]$ is bounded.

Then $\rho\left(\mathcal{A}^{*}\right)=1$ and the set
$\mathcal{S}=\operatorname{absco}\left(\overline{\mathcal{T}\left[\mathcal{A}^{*}, x\right]}\right) \quad$ (absolutely convex hull)
is the unit ball of an extremal norm for $\mathcal{A}^{*},\left\|\mathcal{A}^{*}\right\|_{\mathcal{S}}=1$.

## Illustration of the theorem: a special case

Trajectory and extremal norm. Often $\mathcal{S}$ is a polytope...


Assume $\rho\left(\mathcal{A}^{*}\right) \geq 1$.
The theorem suggests how
to check if $\rho\left(\mathcal{A}^{*}\right)=1$.
Compute recursively $\mathcal{T}\left[\mathcal{A}^{*}, x\right]$
$\mathcal{T}^{(0)}=\{x\}$
$\mathcal{T}^{(\ell+1)}=\mathcal{A}^{*} \mathcal{T}^{(\ell)}, \quad \ell \geq 0$
until absco $\left(\mathcal{T}^{(\ell)}\right)$ invariant.

## Example 1

Consider the family $\mathcal{A}=\left\{A_{0}, A_{1}\right\}$

$$
A_{0}=\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right), \quad A_{1}=\left(\begin{array}{rr}
9 / 10 & 0 \\
9 / 10 & 9 / 10
\end{array}\right)
$$

After sampling the semigroup (e.g by branch-and-bound Gripenberg algorithm) guess $P=A_{0} A_{1} \in \Sigma_{2}(\mathcal{A})$ is an s.m.p.

Lower bound: $\vartheta:=\rho(P)^{1 / 2} \leq \rho(\mathcal{A})$. Then define the scaled family

$$
\mathcal{A}^{*}=\left\{A_{0}^{*}, A_{1}^{*}\right\}=\left\{A_{0} / \vartheta, A_{1} / \vartheta\right\} ; \quad \rho\left(\mathcal{A}^{*}\right) \geq 1 .
$$

Starting vector: choose the unique leading eigenvector of $P$,

$$
x=\left(\begin{array}{ll}
(1+\sqrt{5}) / 2 & 1
\end{array}\right)^{\mathrm{T}} \text {. }
$$

## Computing the trajectory: step 1



New vertices are drawn in red, old vertices as black points.

## Computing the trajectory: step 2



New vertices are drawn in red, old vertices as black points. Internal points of the trajectory are in white.

## Computing the trajectory: step 3



Since span $\left(\mathcal{P}^{(3)}\right)=\mathbb{R}^{2}, \mathcal{P}=\mathcal{P}^{(3)}$ is a real invariant polytope giving an extremal norm.

## The polytope extremal norm



The success of the algorithm implies $\rho\left(\mathcal{A}^{*}\right)=1$ and thus $\rho(\mathcal{A})=\vartheta$.

Definition: A bounded set $\mathcal{P} \subset \mathbb{C}^{k}$ is a balanced complex polytope (b.c.p.) if there exists a finite set $\mathcal{V}$ such that

$$
\mathcal{P}=\operatorname{absco}(\mathcal{V}) \quad \text { and } \quad \operatorname{span}(\mathcal{V})=\mathbb{C}^{k}
$$

Complex polytope norm: is a norm whose unit ball is a b.c.p.

## Some general finiteness results

## Definition

Let $\rho\left(\mathcal{A}^{*}\right)=1$. A spectrum maximizing product $P^{*}$ is said dominant for the family $\mathcal{A}^{*}$ if exists $q<1$ such that $\rho\left(Q^{*}\right) \leq q$ for any product $Q^{*}$ that is not a power of $P^{*}$ nor a power of its cyclic permutations.

Theorem (G. \& Protasov '13, see also G., Wirth \& Zennaro '05) Let $\mathcal{A}^{*}$ be finite and irreducible. Assume $\rho\left(\mathcal{A}^{*}\right)=1$ and

- $x$ is the unique leading eigenvector of a dominant s.m.p. then $\mathcal{A}^{*}$ has a complex polytope extremal norm with unit ball

$$
\operatorname{absco}\left(\left\{x, P_{1}^{*} x, \ldots, P_{s}^{*} x\right\}\right) \quad \text { with } P_{1}^{*}, \ldots, P_{s}^{*} \in \Sigma\left(\mathcal{A}^{*}\right)
$$

Algorithms by G., Wirth \& Zennaro '05 and G. \& Protasov '13.

## Usefulness of Example 1

Let $\mathcal{C}=\left\{C_{0}, C_{1}\right\}$ related to the considered $A_{0}, A_{1}$ by
$C_{0}=\log \left(A_{0}\right)=\log \left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right), C_{1}=\log \left(A_{1}\right)=\log \left(\begin{array}{rr}9 / 10 & 0 \\ 9 / 10 & 9 / 10\end{array}\right)$.
As seen the s.m.p. is $P=A_{0} A_{1}$ corresponding to the function

$$
u_{\Delta t}^{\star} \in \mathcal{U}_{\Delta t}(\Delta t=1):{ }^{1} \varlimsup_{0} \prod_{0} \underbrace{}_{2}
$$

Lower bound for $\sigma(\mathcal{C})$ : $\quad$ Since $\rho(\mathcal{A})=\rho\left(A_{0} A_{1}\right)^{\frac{1}{2}} \approx 1.53$, we get $\sigma(\mathcal{C}) \geq \log (1.53 .)=$.0.428 .., the exponent associated to computed extremal discrete trajectory with constant switching time $\Delta t=1$.

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## Back to ODEs. Lower bound computation

(1) Fix $\Delta t$ and compute $\mathcal{A}_{\Delta t}=\left\{\mathrm{e}^{C_{i} \Delta t}\right\}_{i \in \mathcal{I}}$.
(2) Compute the j.s.r. $\rho\left(\mathcal{A}_{\Delta t}\right)$ by detecting an s.m.p. $P$ of length $n$ and computing a polytope extremal norm for $\mathcal{A}_{\Delta t}$
(3) Lower bound for the upper exponent of (S).

The discrete trajectory growing faster is given by $P^{k},(k \geq 1)$.
Thus the associated exponent provides the lower bound

$$
\beta:=\frac{1}{\Delta t} \log \left(\rho\left(\mathcal{A}_{\Delta t}\right)\right) \leq \sigma(\mathcal{C})
$$

Shifting property: let $\mathcal{C}^{*}=\mathcal{C}-\beta \mathbf{I}$ then $\sigma\left(\mathcal{C}^{*}\right)=\sigma(\mathcal{C})-\beta$.
This means that $\sigma(C) \geq \beta \quad \Longleftrightarrow \quad \sigma\left(\mathcal{C}^{*}\right) \geq 0$. Discretizing $\mathcal{C}$ we get $\mathcal{A}_{\Delta t}$ while discretizing $\mathcal{C}^{*}$ we get $\mathcal{A}_{\Delta t}^{*}$.

## Example 2

Let $\mathcal{C}=\left\{C_{0}, C_{1}\right\}$ with

$$
C_{0}=\left(\begin{array}{rr}
0.346 \ldots & 0.785 \ldots \\
-0.785 \ldots & 0.346 \ldots
\end{array}\right) \quad C_{1}=\left(\begin{array}{rr}
0.604 \ldots & 1.209 \ldots \\
-1.209 \ldots & -0.604 \ldots
\end{array}\right)
$$

For $\Delta t=1$ we set $\mathcal{A}_{\Delta t}=\left\{A_{0}, A_{1}\right\}=\left\{e^{C_{0}}, e^{C_{1}}\right\}$ with

$$
A_{0}=\left(\begin{array}{rr}
1 & 1 \\
-1 & 1
\end{array}\right), \quad A_{1}=\left(\begin{array}{rr}
1 & 1 \\
-1 & 0
\end{array}\right) .
$$

The product $P=A_{0}^{2} A_{1} A_{0}^{3} A_{1}$ is an s.m.p. $\left(\rho(\mathcal{A})=\rho(P)^{1 / 7}\right)$ and determines the lower bound $\sigma(\mathcal{C}) \geq \beta=0.3734 \ldots$

Equivalently the family $\mathcal{A}_{\Delta t}^{*}=\mathcal{A}_{\Delta t} / \rho(P)^{1 / 7}$ has spectral radius 1 and

$$
\sigma\left(\mathcal{C}^{*}\right)=\sigma(\mathcal{C}-\beta \mathbf{I}) \geq 0 .
$$

The extremal norm, i.e. an invariant set for $\mathcal{A}_{\Delta t}^{*}$


Figure: In red the vectors $A_{0}^{*} v$ and in blue $A_{1}^{*} v$, for all vertices $v$ of $\mathcal{P}_{\Delta t}$.

## To recap...

Setting $\mathcal{C}^{*}=\mathcal{C}-\beta \mathbf{I}$ and then discretizing we get $\mathcal{A}_{\Delta t}^{*}$.

We interpret the computation as:


$$
\mathcal{A}_{\Delta t}^{*}=\mathrm{e}^{\mathcal{C}^{*} \Delta t} \text { has spectral radius } 1 \text { and invariant polytope } \mathcal{P}_{\Delta t} .
$$

Consequently $\sigma\left(\mathcal{C}^{*}\right) \geq 0$ (either marginally stable or unstable).

## Computing an upper bound for the Lyapunov exponent

$\left(S^{*}\right) \quad \begin{cases}\dot{x}(t)=C^{*}(u(t)) x(t), \quad C^{*}(u(t)) \in \mathcal{C}^{*}=\left\{C_{i}^{*}\right\}_{i \in \mathcal{I}}, \\ x(0)=x_{0}\end{cases}$
Observation. Given a polytope $\mathcal{P}$, if all vectorfields $C_{i}^{*} v$, for any vertex $v$ of $\mathcal{P}$ are oriented inside $\mathcal{P}$, then $\mathcal{P}$ is positively invariant for $\left(S^{*}\right)$ and all solutions of $\left(S^{*}\right)$ are bounded. Hence $\sigma\left(\mathcal{C}^{*}\right) \leq 0$.
Remark. If this is not true, we can always find $\gamma>0$ such that $\forall i$ the modified vectorfield $\left(C_{i}^{*}-\gamma \mathbf{l}\right) v$ is oriented inside $\mathcal{P}$ for any $v$.

## Main idea.

Starting from $\mathcal{C}^{*}$ which is such that $\sigma\left(\mathcal{C}^{*}\right) \geq 0$ and using the computed extremal polytope $\mathcal{P}_{\Delta t}$, we obtain the bilateral estimate

$$
0 \leq \sigma\left(\mathcal{C}^{*}\right) \leq \gamma
$$

## Example 2: the optimal shift



Figure: In red $\left(C_{0}^{*}-\gamma I\right) v$, in blue $\left(C_{1}^{*}-\gamma I\right) v$ for all vertices $v$ of $\mathcal{P}_{\Delta t}(\gamma \approx 0.4)$.

## Summary of the approximations and main result

(LB) For a given $\Delta t$ we compute the lower bound $\beta$ by the joint spectral radius of the discretized system.
(UB) By means of the obtained polytope $\mathcal{P}_{\Delta t}$ we compute the optimal shift $\gamma$. This provides the bilateral estimate

$$
\beta \leq \sigma(\mathcal{C}) \leq \alpha, \quad \alpha=\beta+\gamma .
$$

## Theorem (convergence, G., Laglia \& Protasov '16)

For every compact irreducible family $\mathcal{A}$ of matrices, there exists a constant $K>0$ such that

$$
\alpha-\beta \leq K \Delta t, \quad \Delta t>0,
$$

with $\alpha, \beta$ (depending on $\Delta t$ ) the computed upper and lower bounds.
An interpretation of the Theorem is that the polytope $\mathcal{P}_{\Delta t}$ converges to an extremal norm of the system (that usually is not a polytope).

## Example 3 (derived from Teichner and Margaliot)

 Let $\mathcal{C}=\left\{C_{0}, C_{1}\right\}$ with $C_{0}, C_{1} \in \mathbb{R}^{3,3}$ :$$
C_{0}=\left(\begin{array}{rrr}
-2 & 0 & 0 \\
10 & -2 & 0 \\
0 & 0 & -11
\end{array}\right) \quad \text { and } \quad C_{1}=\left(\begin{array}{rrr}
-11 & 0 & 10 \\
0 & -11 & 0 \\
0 & 10 & -2
\end{array}\right) .
$$

| $\Delta t$ | $\beta$ | $\alpha$ | $\gamma$ | s.m.p. $P$ | $\# V$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $1 / 32$ | -0.0442 | 0.2548 | 0.299 | $A_{0}^{16} A_{1}^{9}$ | 34 |
| $1 / 128$ | -0.0427 | 0.0425 | 0.085 | $A_{0}^{62} A_{1}^{37}$ | 165 |
| $1 / 256$ | -0.0426 | -0.0006 | 0.042 | $A_{0}^{125} A_{1}^{35}$ | 587 |
| $1 / 512$ | -0.0426 | -0.0175 | 0.025 | $A_{0}^{249} A_{1}^{149}$ | 2228 |

As $\Delta t$ decreases, the computations are longer and longer and it is necessary to use $\Delta t=1 / 256$ to prove uniform stability. On the other hand no semidefinite matrix $M$ can be found s.t. $C_{i}^{\mathrm{T}} M+M C_{i} \preceq 0$ for $i=0,1$. Hence the CQLF approach is not effective.

## Example 4 (structured, high dimension)

Let $\mathcal{C}=\left\{C_{0}, C_{1}\right\}$ with $C_{0}, C_{1} \in \mathbb{R}^{100,100}$ with elements in $\{-1,0,1\}$ with non-negative exponentials (Metzler matrices).

Table: Approximation of the upper Lyapunov exponent

| $\Delta t$ | $\beta$ | $\alpha$ | $\gamma$ | s.m.p. $P$ | $\# V$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $1 / 32$ | 48.556 | 49.244 | 0.688 | $A_{0}^{2} A_{1}^{2}$ | 5 |
| $1 / 64$ | 48.669 | 49.058 | 0.389 | $A_{0} A_{1} A_{0} A_{1}^{2}$ | 10 |
| $1 / 128$ | 48.727 | 48.973 | 0.246 | $A_{0} A_{1}$ | 38 |
| $1 / 256$ | 48.737 | 48.881 | 0.144 | $A_{0} A_{1}$ | 434 |

The number of vertices to obtain an accuracy $\approx 10^{-1}$ is quite small.

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## Duality

Given a b.c.p. $\mathcal{P}=\operatorname{absco}(\mathcal{V}), V=\left\{v_{1}, \ldots, v_{n}\right\}$, we define

$$
\mathcal{P}^{\text {ad }}=\operatorname{adj}(\mathcal{X})=\left\{y \in \mathbb{C}^{k}| |\left\langle y, v_{i}\right\rangle \mid \leq 1, i=1, \ldots, n\right\} .
$$

## Theorem (G. \& Zennaro '15, also Plischke \& Wirth '08)

Let $\mathcal{P}$ be a balanced complex polytope defining an extremal norm $\|\cdot\|_{\mathcal{P}}$ for $\mathcal{A}$ and assume that every vertex $v_{\ell}, \ell=1, \ldots, n$, of the polytope $\mathcal{P}$ has been generated in such a way that

$$
v_{\ell}=A_{i_{\ell}}^{*} v_{j_{\ell}} \quad \text { for some } j_{\ell} \in\{1, \ldots, n\} \text { and } i_{\ell} \in\{1, \ldots, m\} .
$$

Then the norm $\|\cdot\|_{\mathcal{P}_{\text {ad }}}$ is a Barabanov norm for the adjoint set $\mathcal{A}^{\text {ad }}$.
Note: $\ln \mathbb{R}^{k}$ the geometry of symmetric and adjoint symmetric polytopes is the same; this fact is not inherited by b.c.p's in $\mathbb{C}^{k}$.

## The general methodology

Given a family $\mathcal{A}=\left\{A_{0}, \ldots, A_{m}\right\}$ we apply the polytope algorithm to the adjoint family $\mathcal{A}^{\text {ad }}=\left\{A_{0}^{\text {ad }}, \ldots, A_{m}^{\text {ad }}\right\}$ and, if the algorithm ends successfully, we get an extremal polytope norm $\|\cdot\|_{\mathcal{P}}$.

The norm $\|\cdot\|_{\mathcal{P}}$ satisfies the hypothesis of the previous Theorem. Thus, since $\left(\mathcal{A}^{\text {ad }}\right)^{\text {ad }}=\mathcal{A}$, the dual norm is a Barabanov norm for $\mathcal{A}$.

Most unstable switching law for the discrete system
Starting from $x(0)$ on the boundary of the unit ball of $\|\cdot\|_{\mathcal{P a d}^{\text {ad }}}$, we can find a switching law $\sigma$ (MUSL) such that the whole trajectory $\{x(n)\}_{n \geq 0}$ lies on the boundary $\partial \mathcal{P}^{\text {ad }}$, i.e. a most unstable trajectory.

## Example

Let $\mathcal{A}=\left\{A_{0}, A_{1}\right\}$ where

$$
A_{0}=\left(\begin{array}{rr}
1 & 1 \\
-1 & 0
\end{array}\right), \quad A_{1}=\left(\begin{array}{rr}
0 & -1 \\
1 & 0
\end{array}\right) .
$$



Figure: Extremal norm (left) for the family $\mathcal{A}^{\text {ad }}$ and Barabanov norm for $\mathcal{A}$ (right)

## Approximate most unstable trajectories of the ODE

 By previous theory, for small $\Delta t$ a most unstable trajectory of $\mathcal{A}=\left\{\mathrm{e}^{\Delta t C_{0}}, \ldots, \mathrm{e}^{\Delta t C_{m}}\right\}$ is "close" to a most unstable trajectory of the time-continuous switched system (with matrices $C_{0}, \ldots, C_{m}$ ).

## Outline of the talk

(1) A class of switched linear systems

- Upper and lower Lyapunov exponent. Stability concepts
(2) Discretization: joint spectral radius
- Generalization of the spectral radius of a matrix
- Polytope norms and related algorithms

3 Approximating the upper Lyapunov exponent

- A bilateral convergent estimate

4 Approximating most unstable trajectories
(5) Summary and Outlook

## A glance to the lower Lyapunov exponent

In order to compute the lower Lyapunov exponent, again we discretize using piecewise constant switching functions and proceed similarly.
Bounds to the lower Lyapunov exponent are obtained now by computing the lower spectral radius (I.s.r.) of the family $\mathcal{A}_{\Delta t}$.

## Definition (Gurvits, 1995)

The lower spectral radius (I.s.r.) of a matrix family $\mathcal{A}$ is the minimal rate of growth in the product semigroup and is given by

$$
\tilde{\rho}(\mathcal{A})=\inf _{n \geq 1} \tilde{\rho}_{n}, \quad \text { where } \tilde{\rho}_{n}=\inf _{P \in \Sigma_{n}(\mathcal{A})} \rho(P)^{1 / n} .
$$

In general, the function $\tilde{\rho}(\mathcal{A})$ is not continuous. But is continuous on set of families $\mathcal{A}$ with an invariant cone $K$ (for example the set of non negative matrix families). Here norms are replaced by antinorms...

## Summary and outlook

- A framework for the approximation of Lyapunov exponents of switched systems.
- Linear convergence in the dwell time (discretization stepsize).
- Application to control theory, uniform stability and stabilizability of switched control systems.
- Software (Matlab) developed by the author.
- Robustness analysis, effect of perturbations and varying parameters.
- Acceleration issues, exploiting the structure.
- No constraints included, extension to Markovian switched dynamical systems, where switching is not arbitrary.


[^0]:    ${ }^{1}$ e.g. R. Jungers: The joint spectral radius. Theory and applications, 2009.

