Stability of uniformly bounded switched systems and observability

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Table of contents

Uniformly bounded linear switched systems

- Finite collection $B_1, B_2, \dots, B_p$ of $d \times d$ matrices.
- They share a weak quadratic Lyapunov function P, i.e. $B_i^T P + PB_i \leq 0$ for $i = 1, \dots, p$.
- We can assume $P = Id$, so that:

$$B_i^T + B_i \leq 0 \qquad \text{for} \quad i = 1, \dots, p$$

The linear switched system

$$\frac{d}{dt} X = B_{u(t)} X \qquad X \in \mathbb{R}^d, \qquad u(t) \in \{1, 2, \dots, p\}$$

is stable.

Switching laws

- A switching law, or input, is a piecewise constant and right-continuous function from $[0, +\infty)$ to $\{1, \ldots, p\}$.
- For such a switching law u, the trajectory from X is denoted by $\Phi_u(t)X$.
- The ω-limit set, for a given initial point X, is:

$$\Omega_u(X) = \bigcap_{T \geq 0} \overline{\{\Phi_u(t)X;\ t \geq T\}}$$

Two loci

- $\mathcal{K}_i = \{X \in \mathbb{R}^d;\ \ X^T(B_i^T + B_i)X = 0\}$
- $\mathcal{V}_i = \{X \in \mathbb{R}^d;\ \left\| e^{tB_i}X \right\| = \|X\| \ \ \forall t \geq 0\}$
  It is the largest B_i-invariant subspace of $\mathcal{K}_i$.

These loci were previously defined (See Serres, Vivalda, Riedinger, IEEE 2011)

Let u be a switching law:

- For any $X \in \mathbb{R}^d$ the ω-limit set $\Omega_u(X)$ is contained $\bigcup_{i=1}^{p} \mathcal{K}_i$.
- For certain classes of inputs (non-chaotic inputs) $\Omega_u(X)$ is contained $\bigcup_{i=1}^{p} \mathcal{V}_i$.

Pairs of Hurwitz matrices

The linear switched system

$$\dot{X} = B_{u(t)}X \qquad\qquad X \in \mathbb{R}^d$$

is defined by a pair of **Hurwitz** matrices $B_0, B_1 \in \mathcal{M}(d;\mathbb{R})$ assumed to satisfy

$$B_i^T + B_i \leq 0 \qquad i = 0, 1.$$

Problem

Find (necessary and) sufficient conditions for the switched system to be GUAS.

Asymptotic stability

The switched system being linear is **GUAS** *(***G***lobally* **U***niformly* **A***symptotically* **S***table) if and only if for every switching law u the system is globally asymptotically stable, that is*

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It was proved in [B.J. SIAM 2011] that the switched system is GUAS as soon as

$$\mathcal{K} = \mathcal{K}_0 \bigcap \mathcal{K}_1 = \{0\}$$

But this condition is not necessary. It is possible to build GUAS systems for which $\dim \mathcal{K} = d - 1$ regardless of d.

Hurwitz matrices and observability

Theorem (Characterization of Hurwitz matrices)

B is a $d \times d$-matrix s.t. $B^T + B \leq 0$ and $\mathcal{K} = \ker(B^T + B)$. According to the orthogonal decomposition $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$, B writes

$$B = \begin{pmatrix} A & -C^T \\ C & D \end{pmatrix} \tag{1}$$

with $A^T + A = 0$ and $D^T + D < 0$.
Then B is Hurwitz if and only if the pair (C, A) is observable.

Example

Assume B in the previous form, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and C nonzero.

Then B is Hurwitz for any D (that satisfies $D^T + D < 0$).

Sketch of the proof

Consider the linear system:

$$(\Sigma) = \begin{cases} \dot{x} &= Ax \qquad\qquad x \in \mathcal{K} \\ y &= Cx \qquad\qquad y \in \mathcal{K}^{\perp} \end{cases}$$

If (Σ) is not observable, then there exists $x \in \mathcal{K}$, $x \neq 0$, such that $Ce^{tA}x = 0$ for all $t \in \mathbb{R}$.

$$\text{Since} \quad B = \begin{pmatrix} A & -C^T \\ C & D \end{pmatrix} \quad \text{we get} \quad e^{tB}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} e^{tA}x \\ 0 \end{pmatrix}$$

This does not tend to 0 and B is not Hurwitz.
Conversely if B is not Hurwitz, its limit trajectories lie in $\mathcal{K}$ and verifie $Ce^{tA}x = 0$.

Convexification

For $\lambda \in [0,1]$ we consider the matrix $B_\lambda = (1-\lambda)B_0 + \lambda B_1$.

Fundamental space $\mathcal{K} = \mathcal{K}_0 \cap \mathcal{K}_1$.

Lemma

For all $\lambda \in (0,1)$, $\mathcal{K}_\lambda = \ker(B_\lambda^T + B_\lambda) = \mathcal{K}$.
According to the orthogonal decomposition $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$, B_λ writes

$$B_\lambda = \begin{pmatrix} A_\lambda & -C_\lambda^T \\ C_\lambda & D_\lambda \end{pmatrix},$$

with $A_\lambda^T + A_\lambda = 0$ for $\lambda \in [0,1]$,
and $D_\lambda^T + D_\lambda < 0$ for $\lambda \in (0,1)$.

The associated bilinear system

We consider the bilinear controlled and observed system:

$$(\Sigma) = \begin{cases} \dot{x} & = A_\lambda x \\ y & = C_\lambda x \end{cases}$$

where $\lambda \in [0, 1]$, $x \in \mathcal{K}$, and $y \in \mathcal{K}^\perp$.

Definition

The system (Σ) *is said to be* **uniformly observable on** $[0, +\infty[$ *if for any measurable input* $t \longmapsto \lambda(t)$ *from* $[0, +\infty[$ *into* $[0, 1]$*, the output distinguish any two different initial states, that is*

$$\forall x_1 \neq x_2 \in \mathcal{K} \quad m\{t \geq 0;\ C_{\lambda(t)} x_1(t) \neq C_{\lambda(t)} x_2(t)\} > 0,$$

where m *stands for the Lebesgue measure on* $\mathbb{R}$*, and* $x_i(t)$ *for the solution of* $\dot{x} = A_{\lambda(t)} x$ *starting from* x_i*, for* $i = 1, 2$*.*

Main result

Theorem

The linear switched system

$$\dot{X} = B_{u(t)}X \qquad \textit{where} \quad B_i = \begin{pmatrix} A_i & -C_i^T \\ C_i & D_i \end{pmatrix}$$

is GUAS if and only if the bilinear system

$$(\Sigma) = \begin{cases} \dot{x} & = A_\lambda x \\ y & = C_\lambda x \end{cases}$$

is uniformly observable on $[0, +\infty[$.

Sketch of the proof

If the system is not GUAS, we obtain a limit trajectory ψ

$$\psi(t) = \ell + \int_0^t B_{\lambda(s)}\psi(s)\ ds$$

where $t \longmapsto \lambda(t)$ is a measurable function from $[0, +\infty$ into $[0, 1]$.
It is obtained using a weak-$*$ limit of $t \longmapsto B_{u(t)}$ on some sequence of intervals $[t_k, +\infty[$.
We show that $\psi(t)$ is in $\mathcal{K}$ and writes $\psi(t) = (\phi(t), 0)$ according to the decomposition $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$.
Its derivative $\frac{d}{dt}\psi(t) = B(t)\psi(t) = \begin{pmatrix} A_{\lambda(t)}\phi(t) \\ C_{\lambda(t)}\phi(t) \end{pmatrix}$ is also in $\mathcal{K}$, so that

$$C_{\lambda(t)}\phi(t) = 0 \quad \text{for almost every} \quad t \in [0, +\infty[.$$

Observability of the bilinear system

For $\lambda \in [0,1]$, $x \in \mathcal{K}$, and $y \in \mathcal{K}^\perp$

$$(\Sigma) = \left\{ \begin{array}{ll} \dot{x} & = A_\lambda x = (1-\lambda)A_0 + \lambda A_1 \\ y & = C_\lambda x = (1-\lambda)C_0 + \lambda C_1 \end{array} \right.$$

- The trajectories are contained in spheres, because the A_i's are skew-symmetric.

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- The trajectories are contained in spheres, because the A_i's are skew-symmetric.
- A trajectory $x(t)$ on $I = [0, T]$ or $I = [0, +\infty[$ that is contained in $\mathcal{S}^{k-1} = \{x \in \mathcal{K};\ \|x\| = 1\}$ and satisfies

$$C_{\lambda(t)}x(t) = 0 \qquad \text{for almost every } t \in I$$

is a **NTZO trajectory** (**N**on **T**rivial **Z**ero **O**utput) or a bad trajectory.

GUAS Systems with $\dim(\mathcal{K}) \leq 2$

An obvious necessary condition

(Σ) should be observable for every constant input, that is the pair (C_λ, A_λ) should be observable for every $\lambda \in [0, 1]$.

Under this condition no bad trajectory can be constant.

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Proposition 1

If $\dim \mathcal{K} \leq 2$ then (Σ) is uniformly observable on $[0, +\infty[$ if and only if the pair (C_λ, A_λ) is observable for every $\lambda \in [0, 1]$.

Conjecture and counter-example

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The switched system is GUAS if and only if the pair (C_λ, A_λ) is observable for every $\lambda \in [0, 1]$.

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Counter-example (Paolo Mason)

The bad locus

The condition

$$\exists\ \lambda \in [0,1] \quad \text{such that} \quad C_\lambda x = (1-\lambda)C_0 x + \lambda C_1 x = 0$$

holds in **the bad locus** F characterized by

$$\exists \lambda \in [0,1] \text{ s.t. } C_\lambda x = 0 \Longleftrightarrow C_0 x \wedge C_1 x = 0 \text{ and } \langle C_0 x, C_1 x \rangle \leq 0$$

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In $F_0 = F \setminus (\ker C_0 \bigcap \ker C_1)$, the bad input λ is an analytic function of x:

$$\lambda(x) = \frac{\langle C_0 x - C_1 x, C_0 x\rangle}{\|C_0 x - C_1 x\|^2}.$$

Further sufficient conditions

G is the set of points $x \in \mathcal{S}^{k-1} \cap F$ for which there exists $\lambda \in [0, 1]$ such that

$$C_0 A_\lambda x \wedge C_1 x + C_0 x \wedge C_1 A_\lambda x = 0$$

Proposition 2

If the pair (C_λ, A_λ) is observable for every $\lambda \in [0, 1]$ and the set G is discrete then (Σ) is uniformly observable on $[0, T]$ for all $T > 0$. It is in particular true if $\ker C_\lambda = \{0\}$ for $\lambda \in [0, 1]$.

Summarizing Theorem

Theorem

The switched system is GUAS as soon as the pair (C_λ, A_λ) *is observable for every* $\lambda \in [0,1]$, *and one of the following conditions holds:*

1. *the set* G *is discrete;*
2. $\dim \mathcal{K} \leq 2$.

In particular the switched system is GUAS if $\ker C_\lambda = \{0\}$ *for* $\lambda \in [0,1]$.

General examples

1. A is a skew-symmetric $k \times k$ matrix, C is $k' \times k$ matrix, the pair (C, A) is observable.
   Then for any matrices D_0 and D_1 such that $D_i^T + D_i < 0$ the system $\{B_0, B_1\}$ is GUAS, where:

$$B_0 = \begin{pmatrix} A & -C^T \\ C & D_0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} A & -C^T \\ C & D_1 \end{pmatrix}$$

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$$B_0 = \begin{pmatrix} A & -C^T \\ C & D_0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} A & -C^T \\ C & D_1 \end{pmatrix}$$

2. Case where $A_0 = A_1 = 0$.

$$B_0 = \begin{pmatrix} 0 & -C_0^T \\ C_0 & D_0 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & -C_1^T \\ C_1 & D_1 \end{pmatrix} \quad \text{with} \quad D_i^T + D_i < 0.$$

   It is GUAS if and only if C_λ is one-to-one for all $\lambda \in [0, 1]$.

$\dim \mathcal{K} = d - 1$

The skew-symmetric $2q \times 2q$ matrix A has q blocks

$$\begin{pmatrix} 0 & -a_j \\ a_j & 0 \end{pmatrix}$$

on the diagonal and vanishes elsewhere.
Assume $(a_1, \dots, a_q)$ to be rationally independant. Then the orbit of $\dot{x} = Ax$ for **any non zero** initial state $(x_1^0, \dots, x_{2q}^0)$ is dense in the torus

$$x_{2j-1}^2 + x_{2j}^2 = (x_{2j-1}^0)^2 + (x_{2j}^0)^2 = T_j^2 \qquad j = 1, \dots, q$$

where at least one T_j does not vanish.

$\dim \mathcal{K} = d - 1$, end

Therefore this orbit meets the subset of the orthant $\{x_i \geq 0;\ i = 1, \ldots 2q\}$ where $x_{2j-1} > 0$ and $x_{2j} > 0$ for at least one j.
For $C_0 = \begin{pmatrix} 1 & 0 & \ldots & 1 & 0 \end{pmatrix}$ and $C_1 = \begin{pmatrix} 0 & 1 & \ldots & 0 & 1 \end{pmatrix}$ we have $(C_0 x)(C_1 x) > 0$ in this subset.
Every non zero orbit goes out of F.
The bilinear system defined by $A_0 = A_1 = A$, C_0 and C_1 is uniformly observable on $[0, +\infty)$.
The switched system defined by the matrices

$$B_0 = \begin{pmatrix} A & -C_0^T \\ C_0 & -d_0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} A & -C_1^T \\ C_1 & -d_1 \end{pmatrix}$$

is GUAS for any choice of positive numbers d_0 and d_1.

A second conjecture

Consider the assertions

1. The switched system is GUAS and has a non strict quadratic Lyapunov function.
2. The switched system has a strict quadratic Lyapunov function.

It is known that a GUAS linear system has a strict polynomial Lyapunov function (Mason-Boscain-Chitour 2006), but no quadratic one in general.

Can we get a better result under the additional condition that the switched system has a non strict quadratic Lyapunov function?

In other words does $1 \Longrightarrow 2$?

A second Counter-Example

Counter-Example (P. Mason)

$$B_0 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \; B_1 = \begin{pmatrix} -1 & -3-2\sqrt{2} \\ 3-2\sqrt{2} & -1 \end{pmatrix}.$$

The matrix

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 3+2\sqrt{2} \end{pmatrix}$$

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- The system defined by $\{B_0, B_1\}$ is GUAS (BBM, IJC 2009).
- This system admits no strict quadratic Lyapunov function.

Extension of the results

Let $\{B_1, \dots, B_p\}$ be a family of Hurwitz matrices
for which the identity is common weak Lyapunov matrix

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(P_3) Property P_2 holds and for each 3-uple $i, j, k \in \{1, \dots, p\}$ the observed system on $\mathcal{K}_i \bigcap \mathcal{K}_j \bigcap \mathcal{K}_k$ is observable on $[0, +\infty)$.

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(P_p) Properties P_2 to P_{p-1} hold and the observed system on $\bigcap_{i=1}^{p} \mathcal{K}_i$ is observable on $[0, +\infty)$.

Thank you for your attention