Path-complete Lyapunov techniques

And applications

Raphaël Jungers (UCL, Belgium)

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Outline

- Joint spectral characteristics
- Path-complete methods for switching systems stability
- Applications:
  - WCNs and packet dropouts
  - Switching delays
- Conclusion and perspectives

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- Joint spectral characteristics
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Switching systems

$$\mathbf{x}_{t+1} = \begin{matrix} \mathbf{A}_0\, \mathbf{x}_t \\ \mathbf{A}_1\, \mathbf{x}_t \end{matrix}$$

Point-to-point Given x_0 and x_*, is there a product (say, $A_0\ A_0\ A_1\ A_0\ \dots\ A_1$) for which $x_*=A_0\ A_0\ A_1\ A_0\ \dots\ A_1\ x_0$?

Mortality Is there a product that gives the zero matrix?

Boundedness Is the set of all products $\{A_0, A_1, A_0A_0, A_0A_1, \dots\}$ bounded?

Global convergence to the origin Do all products of the type $A_0\ A_0\ A_1\ A_0\ \dots\ A_1$ converge to zero?

The joint spectral characteristics

$$\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i \in \Sigma} ||A_1 A_2 \dots A_t||\right]^{1/t}$$

The joint spectral radius

1 $||A_0||$

2 $||A_0 A_0||$

$||A_0 A_0 A_0||$ 3 …

$||A_0 A_0 A_1||$ 1 …

3 $||A_0 A_1||$

$||A_0 A_1 A_0||$ 5

$||A_0 A_1 A_1||$ 2

2 $||A_1||$

4 $||A_1 A_0||$

$||A_1 A_0 A_0||$ 6

$||A_1 A_0 A_1||$ 2

4 $||A_1 A_1||$

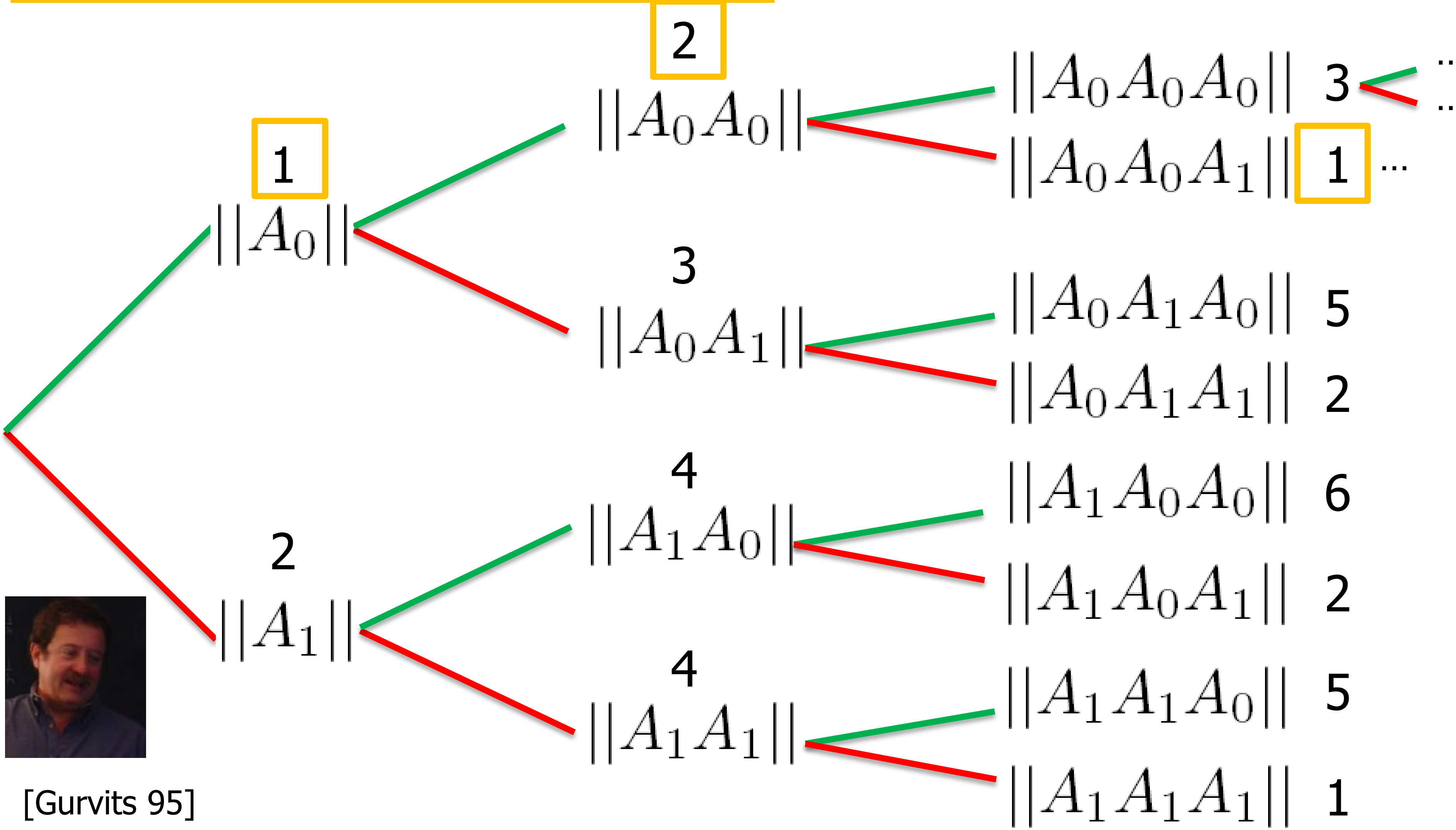
$||A_1 A_1 A_0||$ 5

$||A_1 A_1 A_1||$ 1

The joint spectral characteristics

$$\check{\rho}(\Sigma) = \lim_{t \to \infty} \left[\min_{A_i \in \Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}$$

The joint spectral subradius



[Gurvits 95]

The joint spectral characteristics

$$\boxed{\rho_p(\Sigma) = \lim_{t\to\infty} \left[m^{-t} \sum_{A_i \in \Sigma^t} \|A_1 A_2 \ldots A_t\|^p \right]^{1/(pt)}}$$

The p-radius

(m is the number of matrices in Σ)

1 $||A_0||$

2 $||A_1||$

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3 $||A_0A_1||$

4 $||A_1A_0||$

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$\boxed{||(1,2)||_p}$ $\boxed{||(2,3,4,4)||_p}$ $\boxed{||(3,1,5,2,6,2,5,1)||_p}$

[Protasov 97]

The joint spectral characteristics

$$\bar{\rho}(\Sigma) = \lim_{t\to\infty} \left[\prod_{A_i \in \Sigma^t} \|A_1 A_2 \dots A_t\| \right]^{1/(tm^t)}$$

The Lyapunov Exponent

(m is the number of matrices in Σ)

1 $||A_0||$

2 $||A_0A_0||$

$||A_0A_0A_0||$ 3 …

…

$||A_0A_0A_1||$ 1 …

3 $||A_0A_1||$

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$||A_1A_1A_0||$ 5

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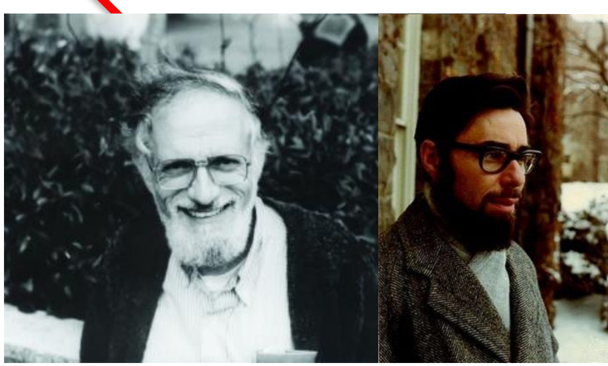


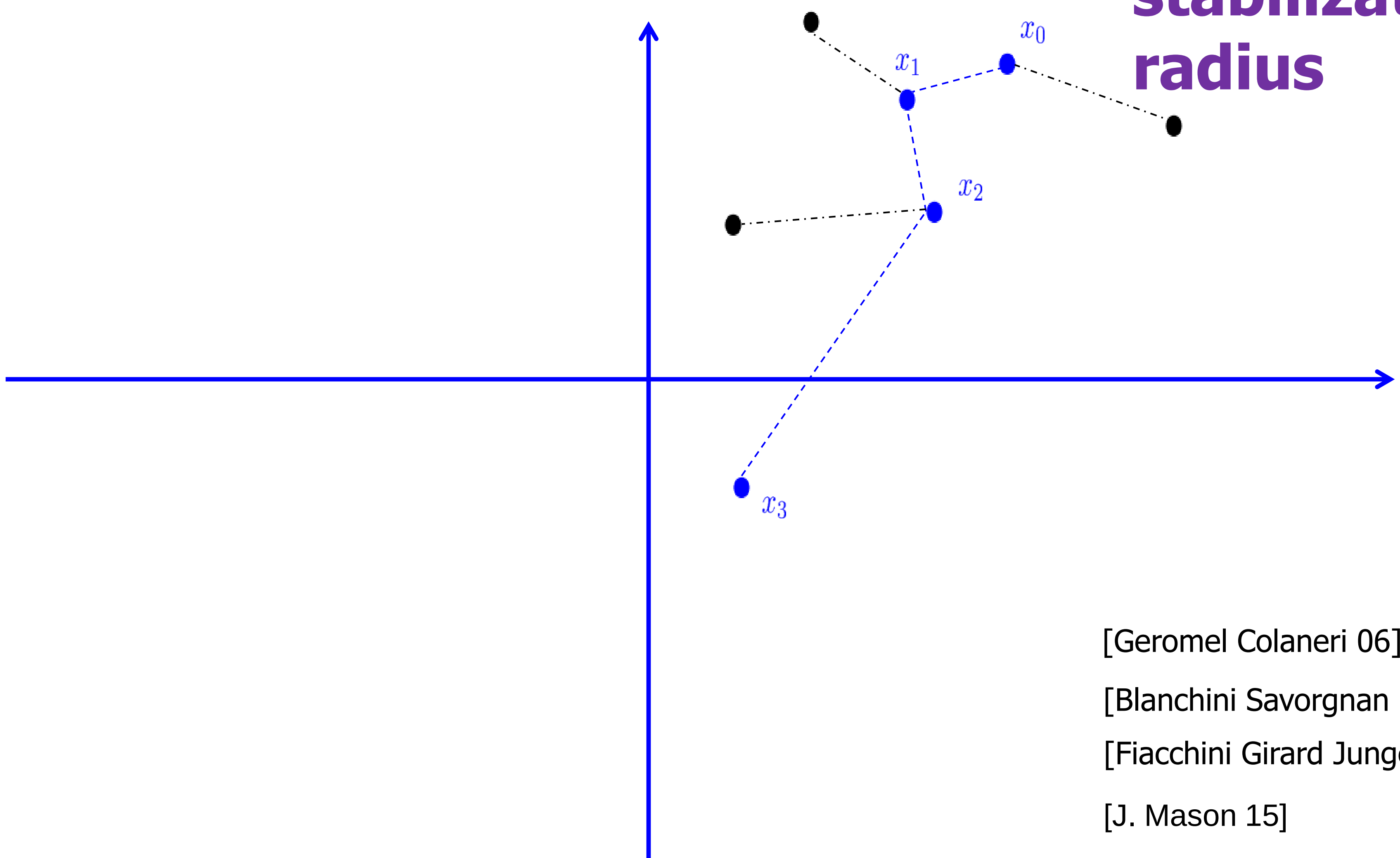
[Furstenberg Kesten, 1960]

The joint spectral characteristics

$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists \sigma(0), \sigma(1), \ldots\ldots, \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t |x|, \forall t \geq 0\}$

$\tilde{\rho}(\Sigma) = \sup_{x \in \mathbb{R}^n} \tilde{\rho}_x(\Sigma)$

The feedback stabilization radius



[Geromel Colaneri 06]

[Blanchini Savorgnan 08]

[Fiacchini Girard Jungers 15]

[J. Mason 15]

The joint spectral characteristics

$$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists \sigma(0), \sigma(1), \ldots.., \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t |x|, \forall t \geq 0\}$$

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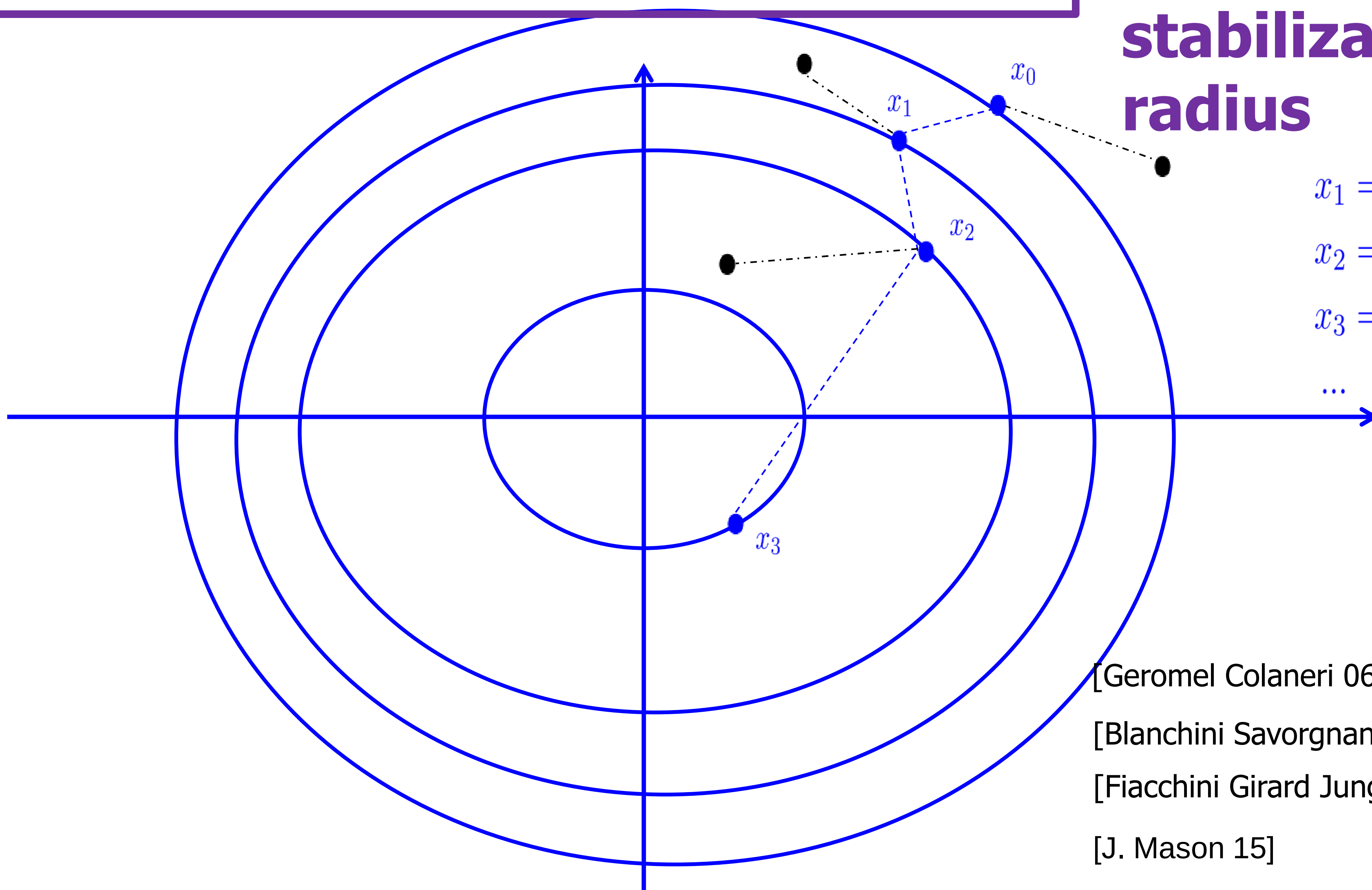

$x_1 = A_{\sigma(0)} x_0$

$x_2 = A_{\sigma(1)} x_1$

$x_3 = A_{\sigma(2)} x_2$

...

[Geromel Colaneri 06]

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The joint spectral characteristics

$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists \sigma(0), \sigma(1), \ldots\ldots, \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t |x|, \forall t \geq 0\}$

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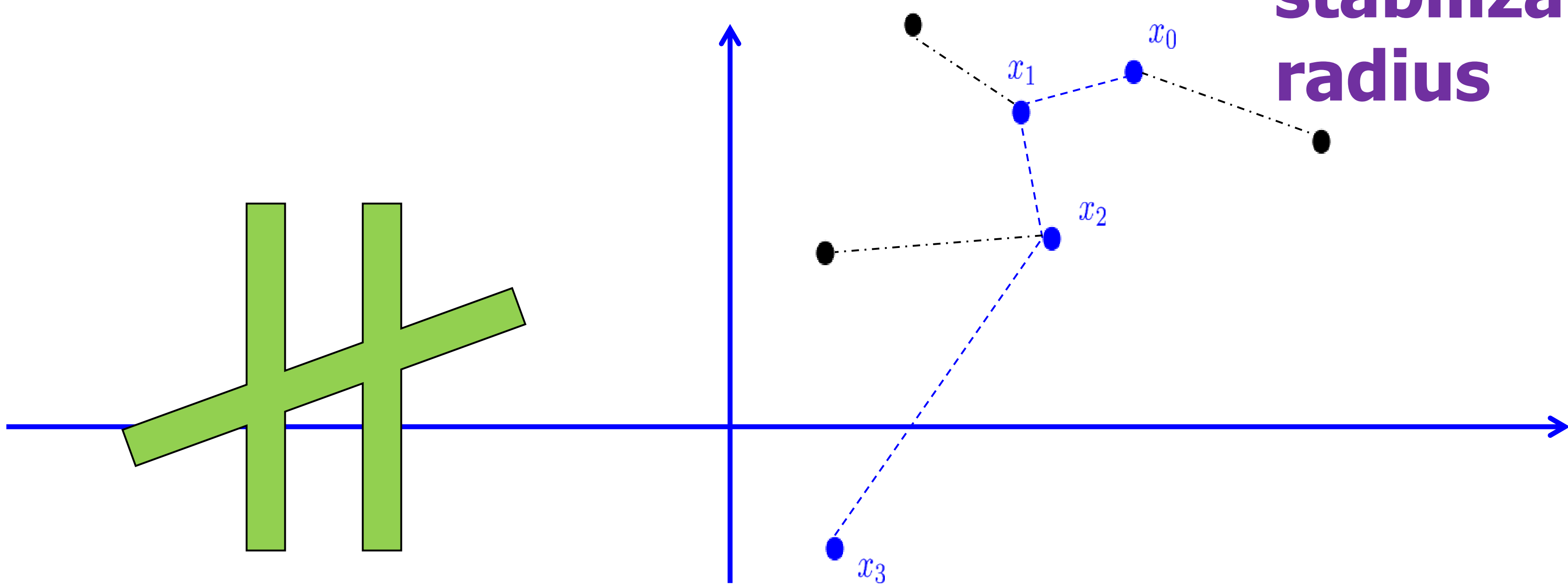
The feedback stabilization radius



Alternative definition: suppose you can observe x(t) at every step, and apply the switching you want, as a function of the x(t)

[Geromel Colaneri 06]

[Blanchini Savorgnan 08]

[Fiacchini Girard Jungers 15]

[J. Mason 15]

The joint spectral characteristics

$$\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i\in\Sigma} ||A_1 A_2 \dots A_t||\right]^{1/t}$$

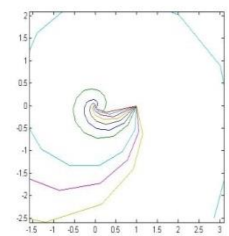
The joint spectral radius addresses the **stability** problem

$$\check{\rho}(\Sigma) = \lim_{t\to\infty} \left[\min_{A_i\in\Sigma} ||A_1 A_2 \dots A_t||\right]^{1/t}$$

The joint spectral subradius addresses the **stabilizability** problem

$$\rho_p(\Sigma) = \lim_{t\to\infty} \left[m^{-t} \sum_{A_i\in\Sigma^t} ||A_1 A_2 \dots A_t||^p \right]^{1/(pt)}$$

The p-radius addresses the… **p-weak stability**



[J. Protasov 10]

$$\bar{\rho}(\Sigma) = \lim_{t\to\infty} \left[\prod_{A_i\in\Sigma^t} ||A_1 A_2 \dots A_t|| \right]^{1/(tm^t)}$$

The Lyapunov exponent addresses the **stability with probability one**

(Cfr. Oseledets Theorem)

$$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists\sigma(0), \sigma(1), \dots\dots, \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t|x|, \forall t \geq 0\}$$

$$\tilde{\rho}(\Sigma) = \sup_{x\in\mathbb{R}^n} \tilde{\rho}_x(\Sigma)$$

The feedback stabilization radius addresses the **feedback stabilizability**

[J. Mason 16]

[Fiacchini Girard Jungers 15]

The joint spectral characteristics: Mission Impossible?

Theorem Computing or approximating ρ is NP-hard

Theorem The problem $\rho>1$ is algorithmically undecidable

Conjecture The problem $\rho<1$ is algorithmically undecidable

Theorem Even the question « $|\check{\rho} - r| \leq a + b\check{\rho}$?» is algorithmically undecidable for all (nontrivial) a and b

Theorem The same is true for the Lyapunov exponent

Theorem The p-radius is NP-hard to approximate

Theorem The feedback stabilization radius is turing-uncomputable

See [Blondel Tsitsiklis 97,
Blondel Tsitsiklis 00,
J. Protasov 09
J. Mason 15]

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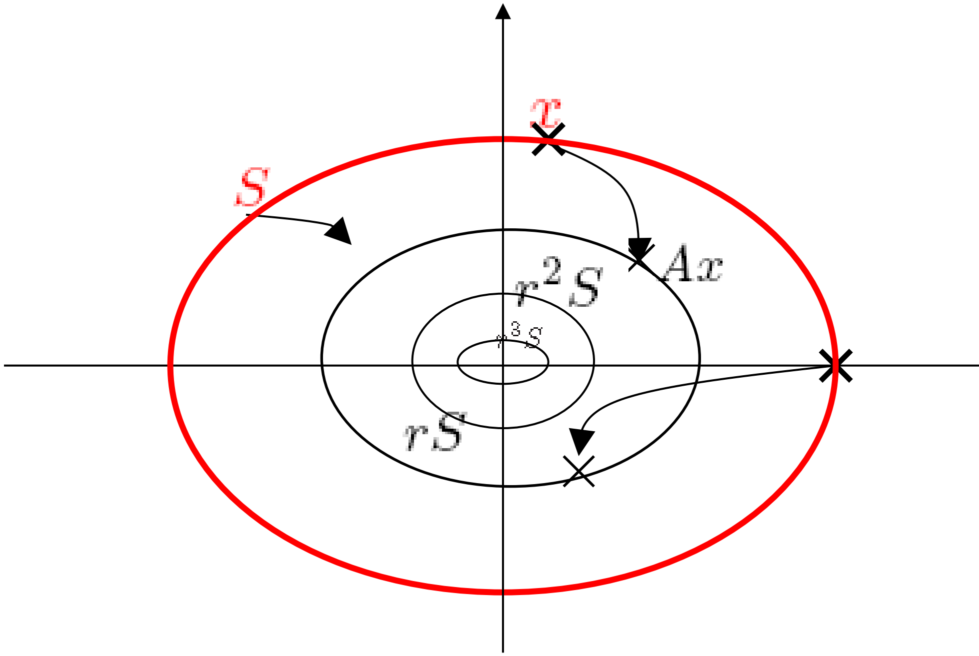
LMI methods

- The CQLF method

$$
\begin{array}{lll}
\inf_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A^T P A & \preceq & r^2 P, \quad \forall A \in \Sigma \\
P & \succeq & 0.
\end{array}
$$

$$\Leftrightarrow \frac{|Ax|_P}{|x|_P} \leq r$$

$$\rho \leq r$$



SDP methods

- Theorem For all $\epsilon > 0$ there exists a norm such that

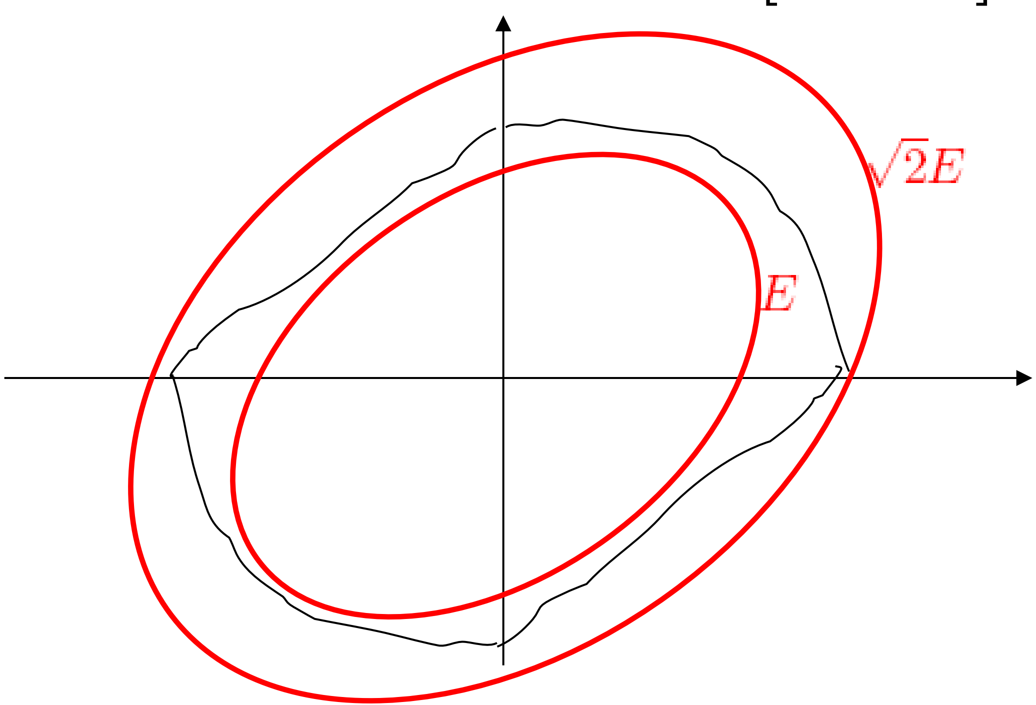
$$\forall A \in \Sigma, \forall x, |Ax| \leq (\rho + \epsilon)|x|$$

[Rota Strang, 60]

- John's ellipsoid Theorem: Let K be a compact convex set with nonempty interior symmetric about the origin. Then there is an ellipsoid E such that $E \subset K \subset \sqrt{n}E$

[John 1948]

•So we can approximate the unit ball of an extremal norm with an ellipsoid

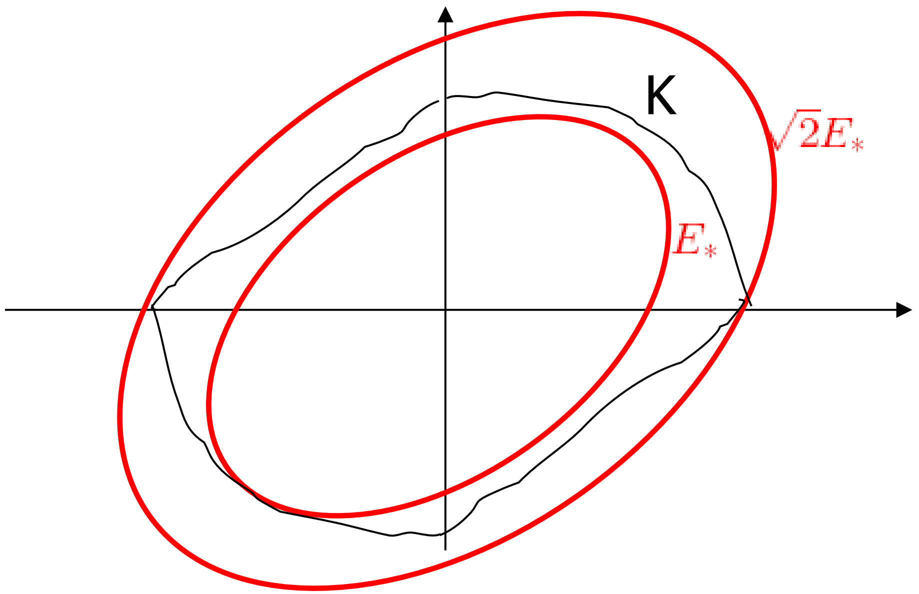


SDP methods

- Theorem The best ellipsoidal norm $||\cdot||_{E_*}$ approximates the joint spectral radius up to a factor $\sqrt{n}$ [Ando Shih 98]

$$\rho \leq \max ||A||_{E_*} \leq \sqrt{n}\rho$$

$$\frac{1}{\sqrt{n}}\rho* \leq \rho \leq \rho*$$



$$\frac{1}{\sqrt[2d]{n}}\rho* \leq \rho \leq \rho*$$

$\rho < 1/n^{\frac{1}{2d}} \Rightarrow$ There exists a Lyap. function of degree d

One can improve this method by lifting techniques [Nesterov Blondel 05] [Parrilo Jadbabaie 08]

Algorithm that approximates the joint spectral radius of arbitrary sets of m (nXn)-matrices up to an arbitrary accuracy ϵ in $\mathcal{O}(n^{m\frac{1}{\epsilon}})$ operations

PTAS

Yet another LMI method

- A strange semidefinite program

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P & \succeq & 0.
\end{array}
$$

$$\rho \leq r$$

[Goebel, Hu, Teel 06]

- But also… [Daafouz Bernussou 01]

[Bliman Ferrari-Trecate 03]

[Lee and Dullerud 06] …

Yet another LMI method

- An even stranger program:

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P A_1 & \preceq & r^2 P, \\
(A_2 A_1)^T P (A_2 A_1) & \preceq & r^4 P, \\
(A_2^2)^T P (A_2^2) & \preceq & r^4 P, \\
P & \succeq & 0.
\end{array}
$$

$$\rho \leq r$$

[Ahmadi, J., Parrilo, Roozbehani10]

Yet another LMI method

- Questions:
  - Can we characterize all the LMIs that work, in a unified framework?
  - Which LMIs are better than others?
  - How to prove that an LMI works?
  - Can we provide converse Lyapunov theorems for more methods?

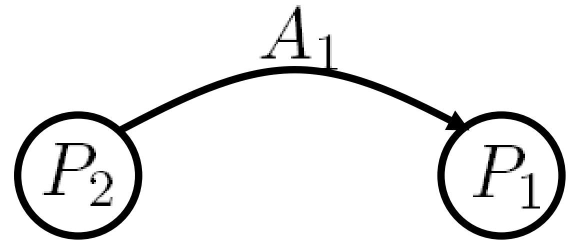
$$\frac{1}{\sqrt[2d]{n}}\rho* \leq \rho \leq \rho*$$

$\rho < 1/n^{\frac{1}{2d}} \Rightarrow$ There exists a Lyap. function of degree d

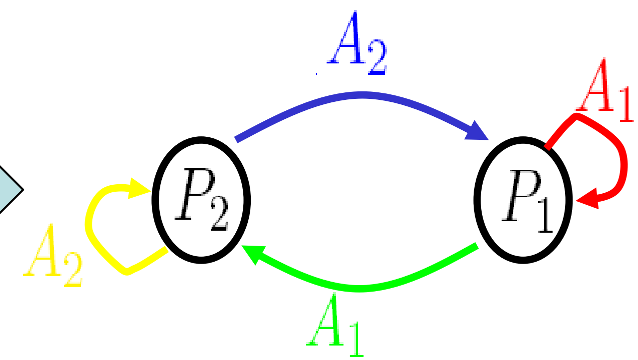
From an LMI to an automaton

- Automata representation Given a set of LMIs, construct an automaton like this:

$$A_1^T P_1 A_1 \preceq r^2 P_2$$



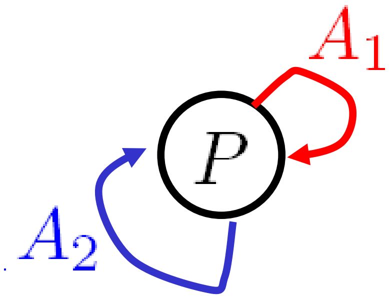
$$\begin{array}{lll} \min_{r \in \mathbb{R}^+} & & r \\ \text{s.t.} & & \\ A_1^T P_1 A_1 & \preceq & r^2 P_1, \\ A_2^T P_1 A_2 & \preceq & r^2 P_2, \\ A_1^T P_2 A_1 & \preceq & r^2 P_1, \\ A_2^T P_2 A_2 & \preceq & r^2 P_2, \\ P_i & \succeq & 0. \end{array}$$

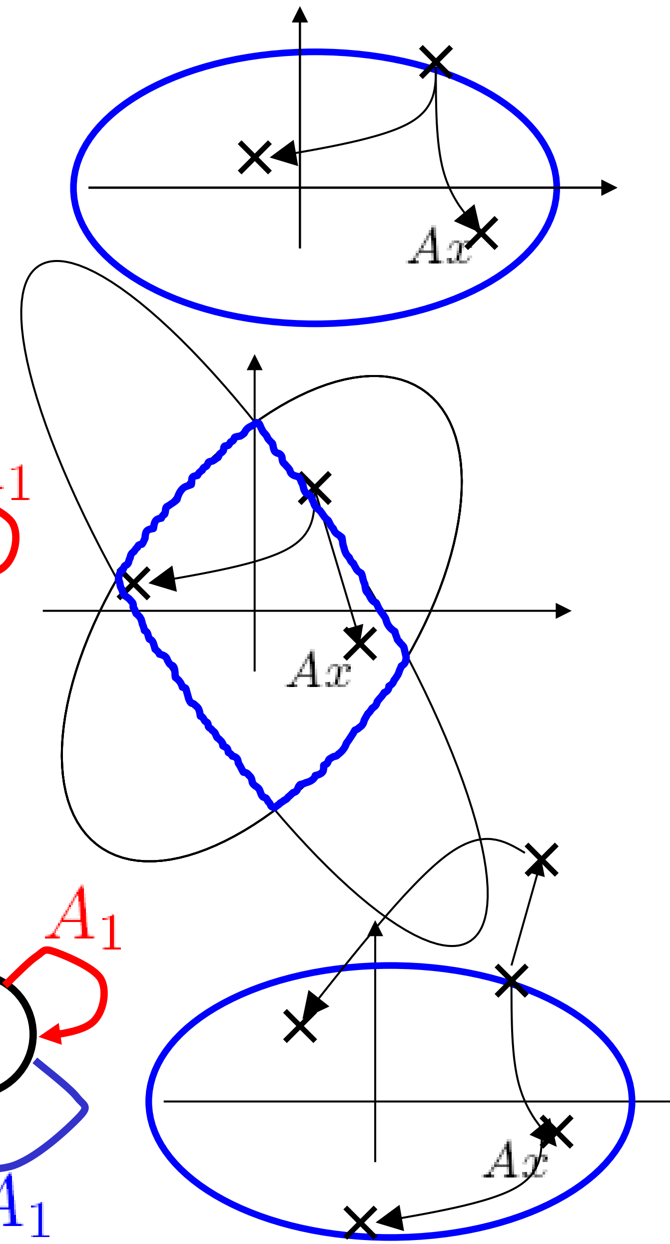


- Definition A labeled graph (with label set A) is path-complete if for any word on the alphabet A, there exists a path in the graph that generates the corresponding word.
- Theorem If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability. [Ahmadi J. Parrilo Roozbehani 14]

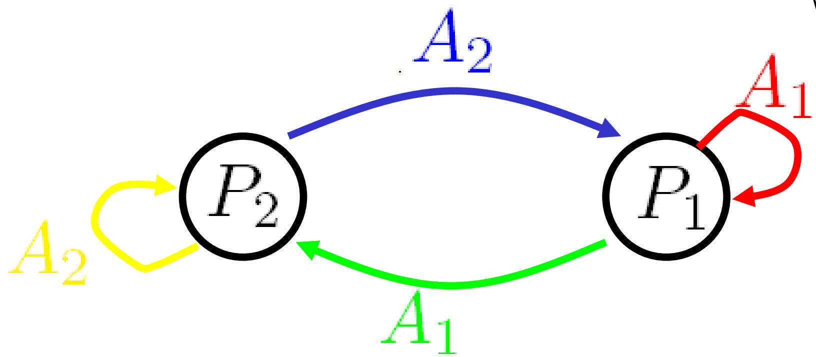
Some examples

- Examples:
  - CQLF





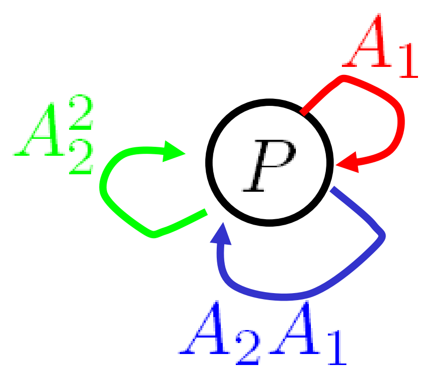
  - Example 1



This type of graph gives a max-of-quadratics Lyapunov function (i.e. intersection of ellipsoids)

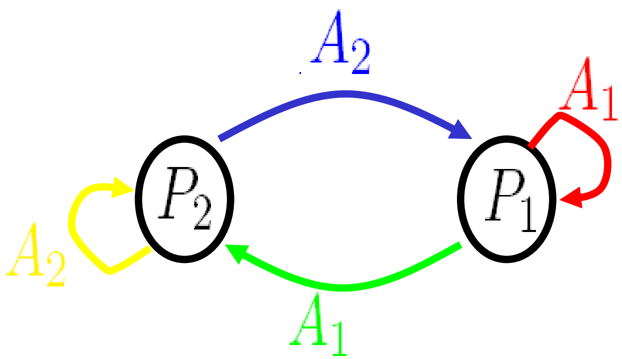
  - Example 2

This type of graph gives a common Lyapunov function for a generating set of words



An obvious question: are there other valid criteria?

- Theorem



$$
\begin{array}{lcl}
\min_{r\in\mathbb{R}+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
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A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

Path complete ??? Sufficient condition for stability

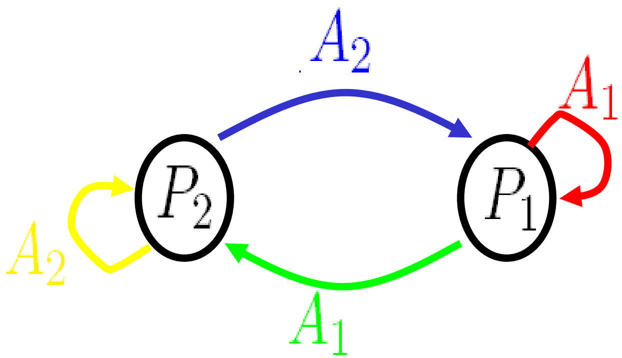
If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

- Are all valid sets of equations coming from path-complete graphs?
- …or are there even more valid LMI criteria?

Are there other valid criteria?

- Theorem Non path-complete sets of LMIs are not sufficient for stability.

[J. Ahmadi Parrilo Roozbehani 15]



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$$

Path complete ??? Sufficient condition for stability

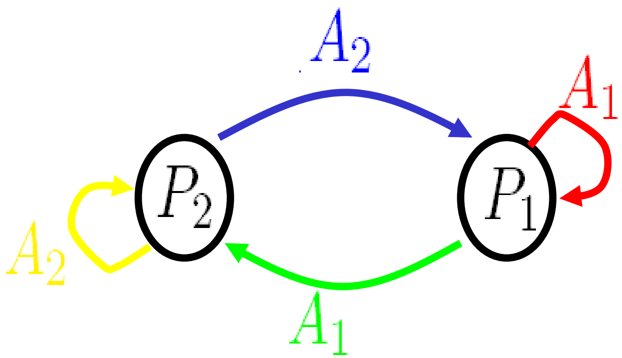
- Corollary

  It is PSPACE complete to recognize sets of equations that are a sufficient condition for stability

- These results are not limited to LMIs, but apply to other families of conic inequalities

So what now?

After all, what are all these results useful for?



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Optimize on optimization problems!

This framework is generalizable to harder problems

- Constrained switching systems
- Controller design for switching systems
- Automatically optimized abstractions of cyber-physical systems
- …

So what now?

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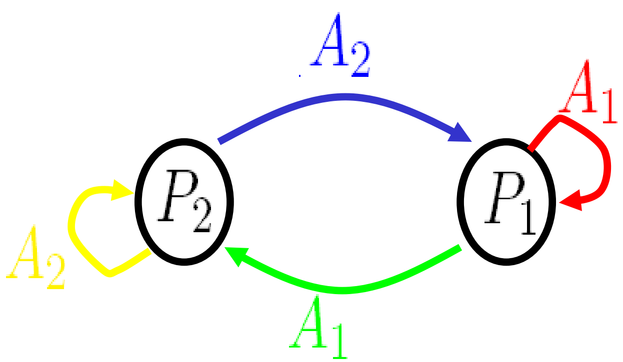

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Optimize on optimization problems!

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- **Constrained switching systems**
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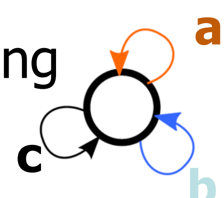
Constrained switching sequences

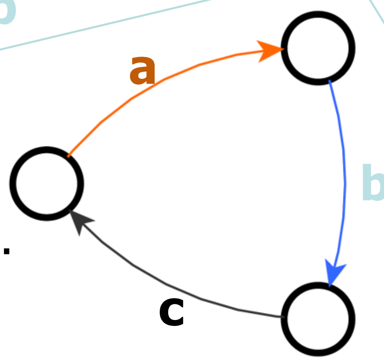
Switching sequences on regular languages

$G(V,E)$ Directed & Labeled $\quad e=(v_i,v_j,k)\in E \quad k\in\{1,\cdots,N\}$

$\sigma(1),\sigma(2),\cdots$ **admissible** if $\exists p=\{(v_i,v_j,\sigma(1)),\ (v_j,v_\ell,\sigma(2)),\ \cdots\}$

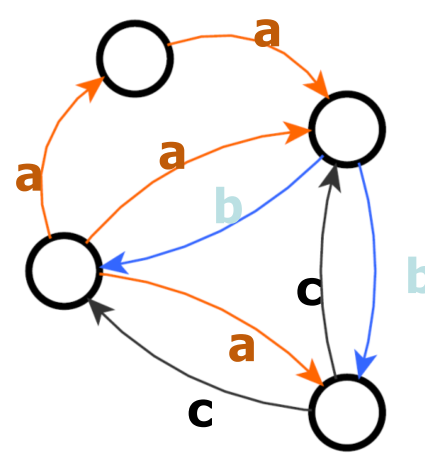
everything





... abcabcabc ...

~~... ac ...~~



~~... bb ...~~

~~... cc ...~~

... aab

...

Constrained switching sequences

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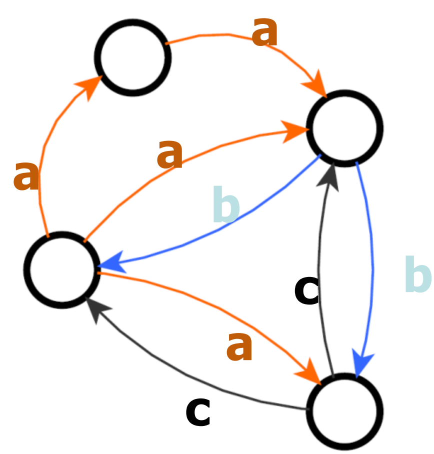


Stability

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} A_{\sigma(t-1)}\cdot\ldots\cdot A_{\sigma(0)}x_0 = 0$$
$$\forall x_0\in\mathbb{R}^n,\ \forall\,\sigma(0),\sigma(1),\cdots\in G$$

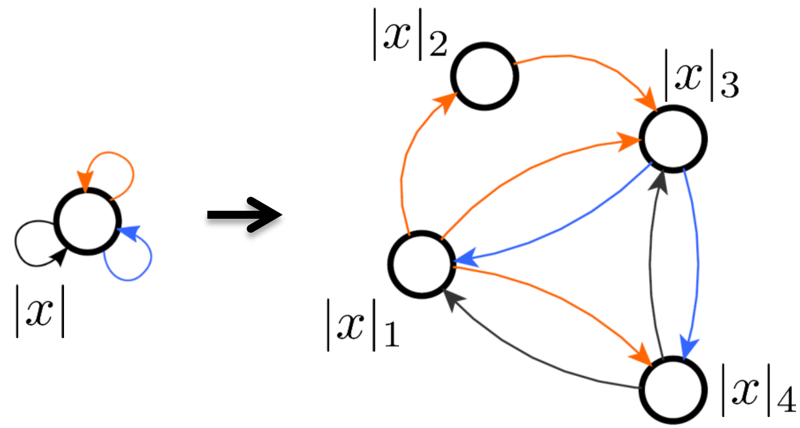
Constrained switching and multinorms

- CJSR as an infimum over **sets** of norms

$$\rho(G(V,E),\, M) = \inf_{|\cdot|_1,\cdots,|\cdot|_{|V|}} \gamma$$

$$|A_k x|_j \leq \gamma |x|_i,$$

$$\forall (v_i, v_j, k) \in E,\ \forall x \in \mathbb{R}^n$$



Theorem:

$\rho(G,\, M) < n^{-1/2T} \Rightarrow S(G_T, M^T)$ admits a Quadratic Multinorm

Corollary: One can again develop a PTAS based on Path-complete methods

[Philippe, Essick, Dullerud, J. 2014]

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Applications of Wireless Control Networks

Industrial automation



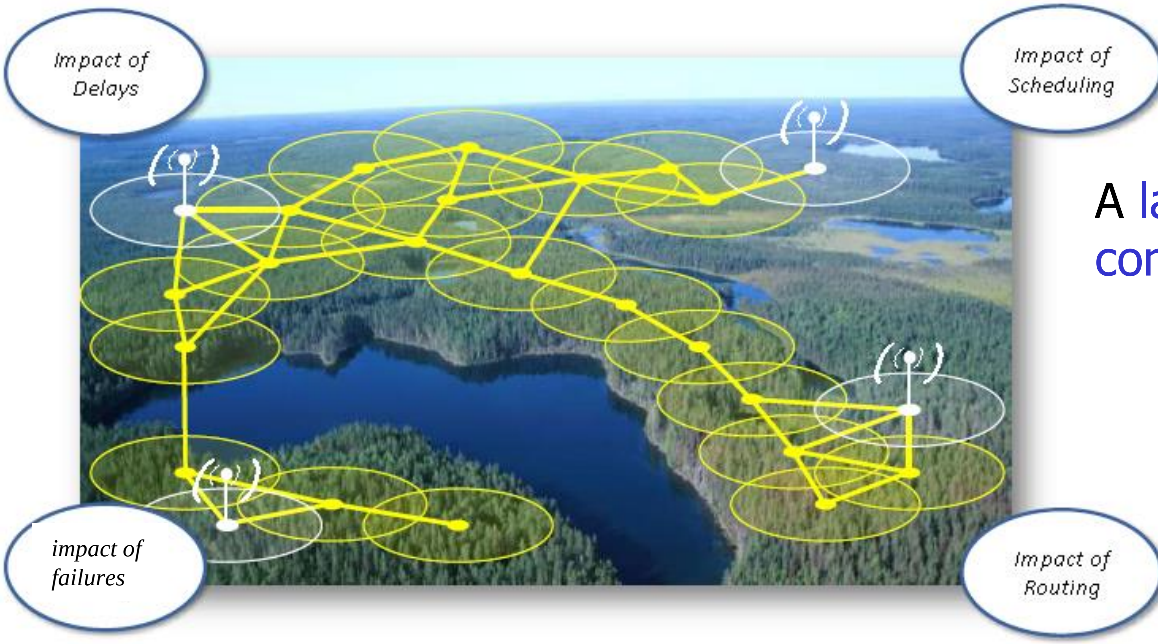
Physical Security and Control

Supply Chain and Asset Management



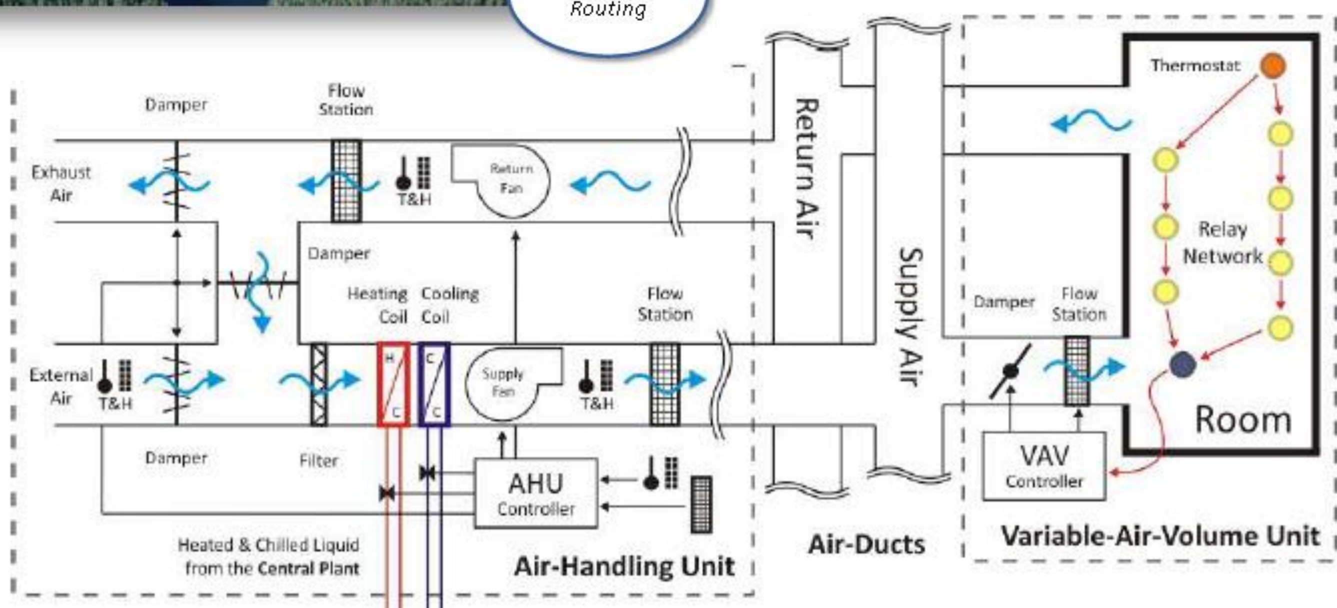
Environmental Monitoring, Disaster Recovery and Preventive Conservation

Wireless control networks



A large scale decentralized control network

A green building



[Ramanathan Rosales-Hain 00]
[alur D'Innocenzo Johansson Pappas Weiss 10]
[Mazo Tabuada 10]
[Zhu Yuan Song Han Başar 12]

Motivation

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 55, NO. 8, AUGUST 2010 1781

Networked Control Systems With Communication Constraints: Tradeoffs Between Transmission Intervals, Delays and Performance

W. P. Maurice H. Heemels, *Member, IEEE*, Andrew R. Teel, *Fellow, IEEE*, Nathan van de Wouw, *Member, IEEE*, and Dragan Nešić, *Fellow, IEEE*

Roughly speaking, the network-induced imperfections and constraints can be categorized in five types:

(i) Quantization errors in the signals transmitted over the network due to the finite word length of the packets;
(ii) Packet dropouts caused by the unreliability of the network;
(iii) Variable sampling/transmission intervals;
(iv) Variable communication delays;
(v) Communication constraints caused by the sharing of the network by multiple nodes and the fact that only one node is allowed to transmit its packet per transmission.

Previous work

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[Jungers D'Innocenzo Di Benedetto, TAC 2015]

Today

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Networked Control Systems With Communication Constraints: Tradeoffs Between Transmission Intervals, Delays and Performance

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[Jungers Kundu Heemels, 2016]

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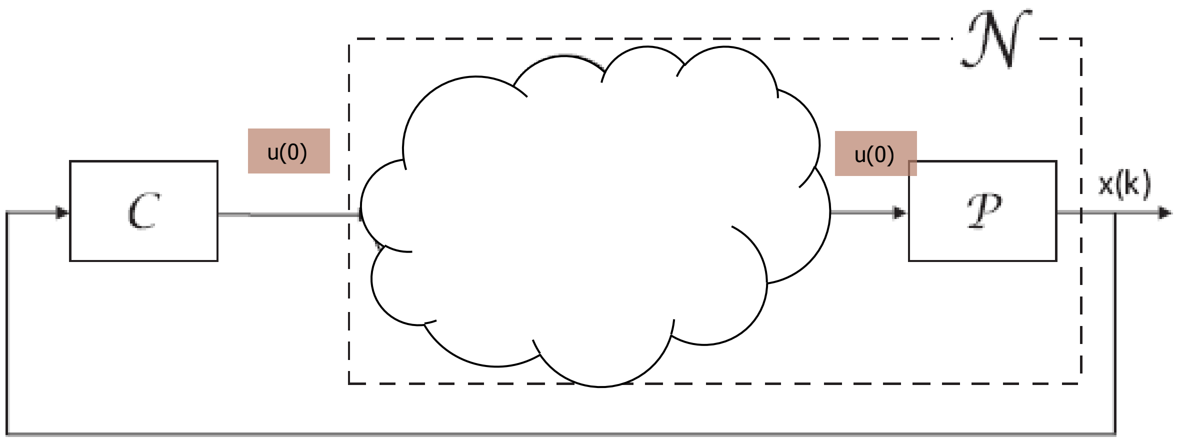
Controllability with packet dropouts

The delay is constant, but some packets are dropped

$$x(1) = Ax(0) + Bu(0)$$

$\boxed{\sigma = 1001\ldots}$

$\sigma(0) = 1$



A data loss signal determines the packet dropouts $\sigma(t) =$ 1 or 0

…this is a switching system!

$$x(t+1) = \begin{cases} Ax(t) + bu(t), & \text{if } \sigma(t) = 1, \\ Ax(t), & \text{if } \sigma(t) = 0 \end{cases}$$

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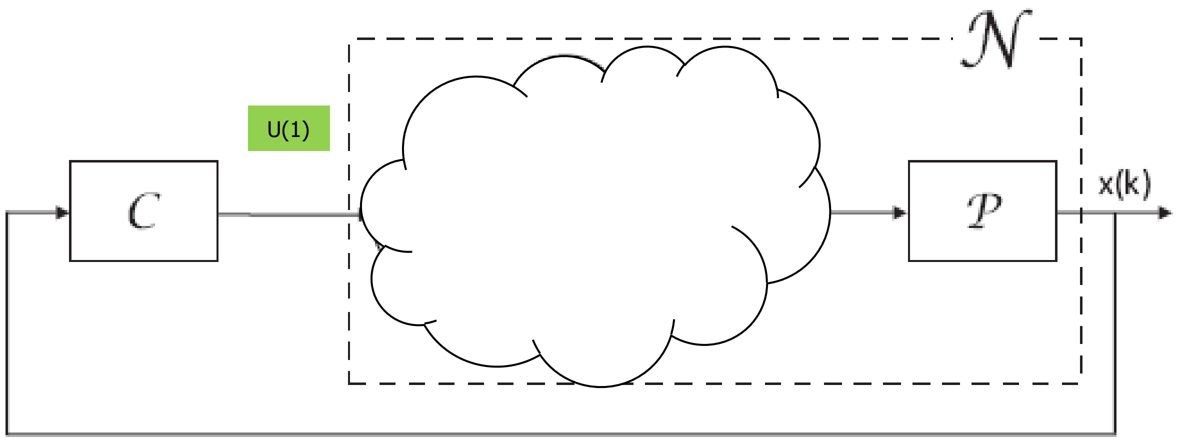

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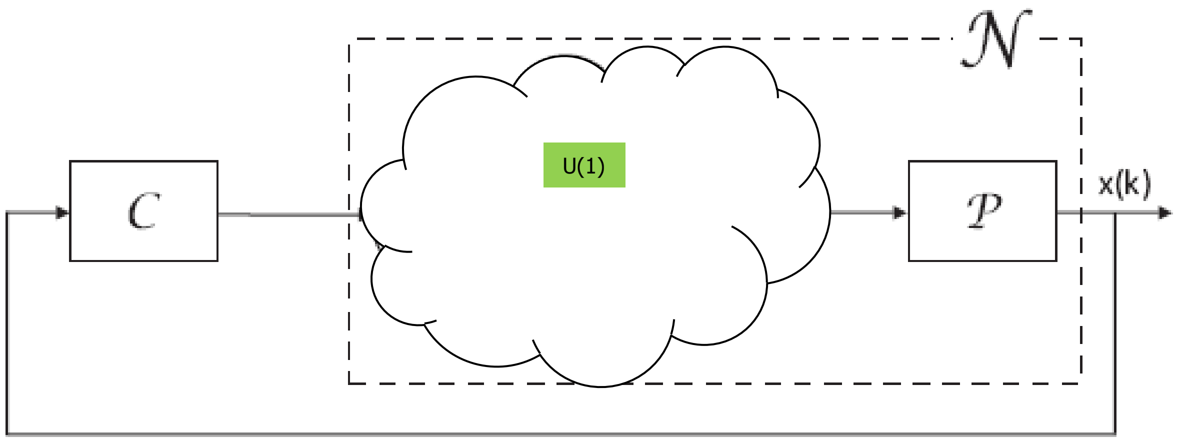

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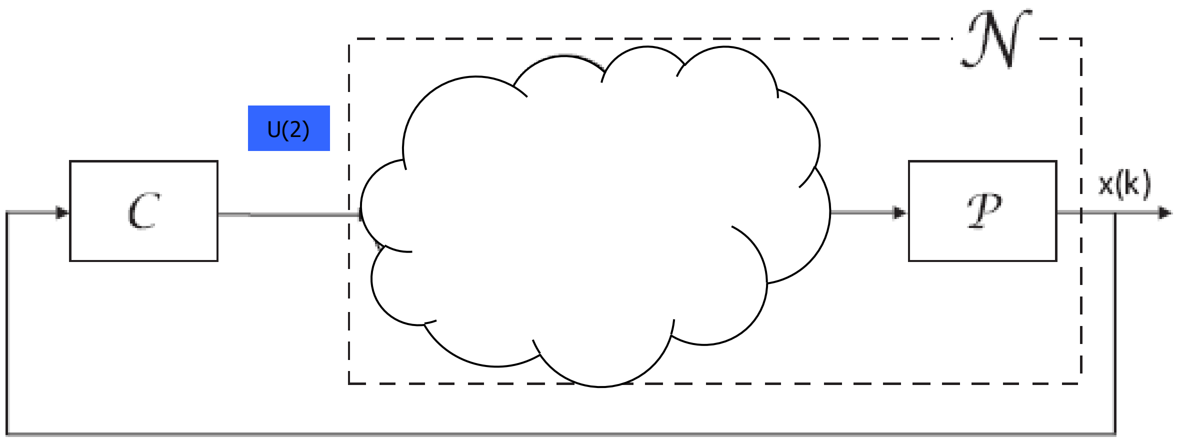

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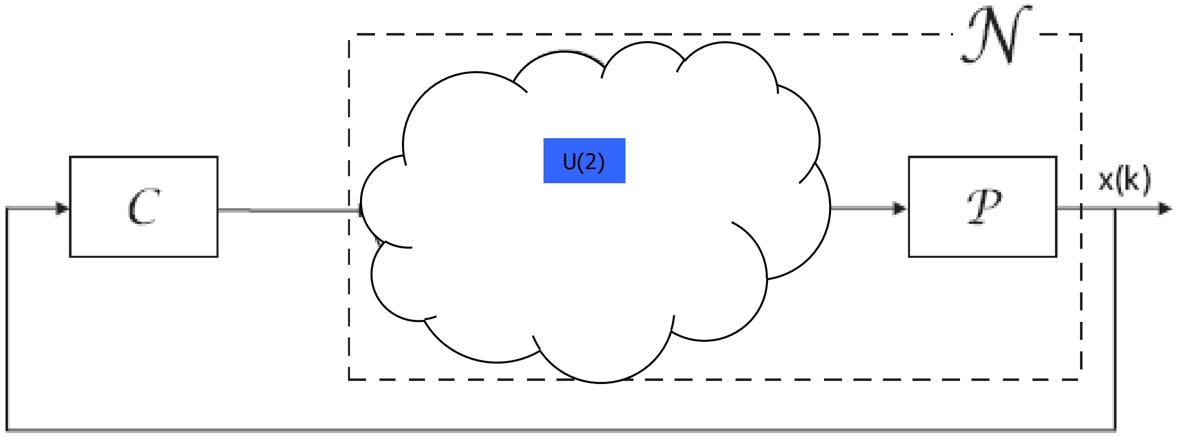

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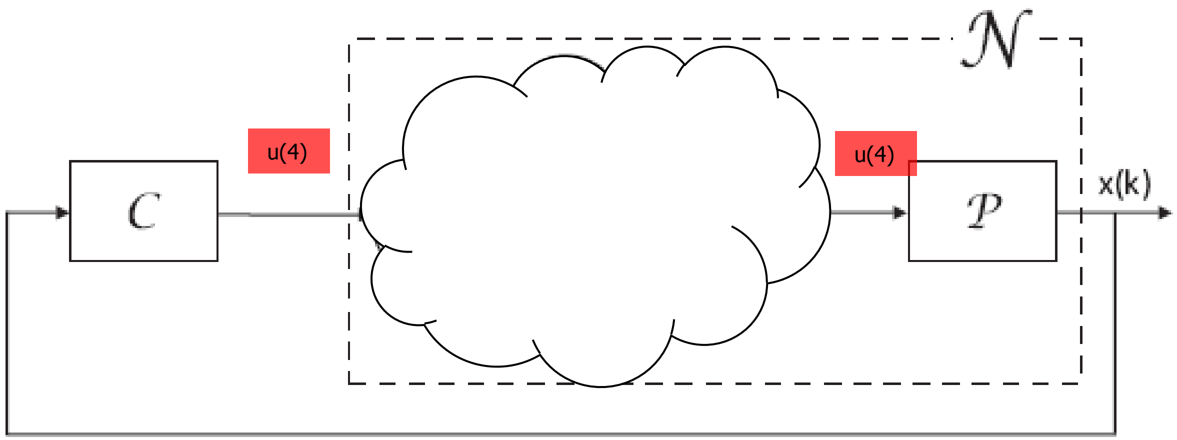

A data loss signal determines the packet dropouts $\sigma(t) =$ 1 or 0

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$$x(t+1) = \begin{cases} Ax(t) + bu(t), & \text{if } \sigma(t) = 1, \\ Ax(t), & \text{if } \sigma(t) = 0 \end{cases}$$

The switching signal

We are interested in the controllability of such a system

$$\boxed{\sigma = 1001\ldots}$$

$$\sigma(0) = 1 \qquad \sigma(1) = 0 \qquad \sigma(2) = 0$$

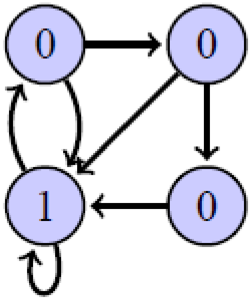
$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ x(2) &= A^2x(0) + ABu(0) \\ x(3) &= A^3x(0) + A^2Bu(0) \\ x(4) &= A^4x(0) + A^3Bu(0) + Bu(3) \end{aligned}$$

Of course we need an assumption on the switching signal

The switching signal is constrained by an automaton

Example:

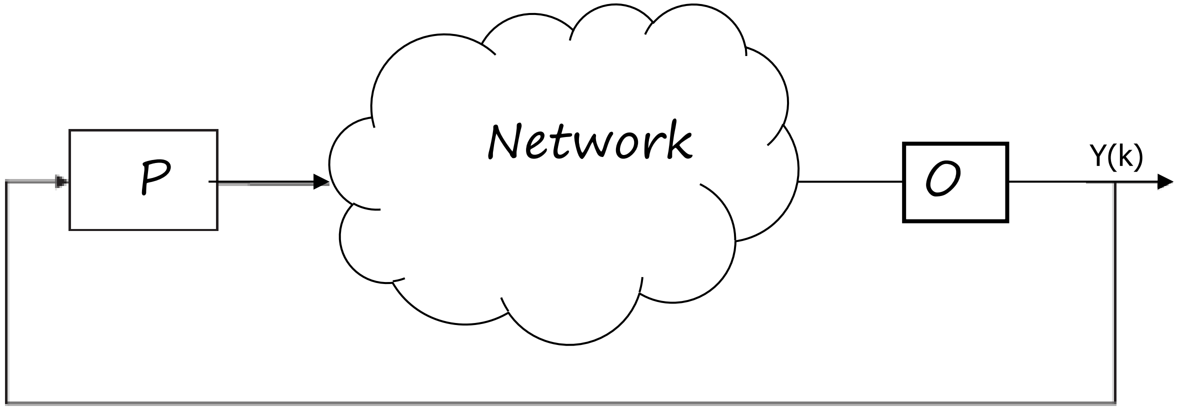
Bounded number of consecutive dropouts (here, 3)



The controllability problem: For any starting point x(0), and any target x*, does there exist, for any switching signal, a control signal u(.) and a time T such that x(T)=x* ?

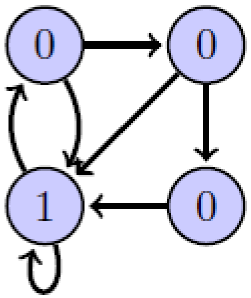
The dual observability problem

Observability under intermittent outputs is algebraically equivalent (and perhaps more meaningful)



$$x(t+1) = Ax(t),$$
$$y(t) = \sigma(t)Cx(t)$$

Controllability with Packet Dropouts



We are given a pair (A,b) and an automaton

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$$x(1) = Ax(0) + Bu(0)$$
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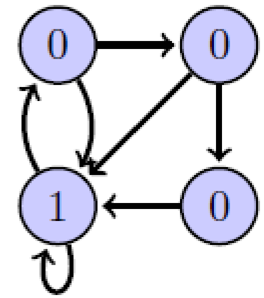
$$\boxed{\sigma = 1001\ldots}$$

The controllability problem: for any starting point x(0), and any target x*, does there exist, for any switching signal, a control signal u(.) and a time T such that x(T)=x* ?

Theorem: Deciding controllability of switching systems is undecidable in general (consequence of [Blondel Tsitsiklis, 97])

Controllability with Packet Dropouts



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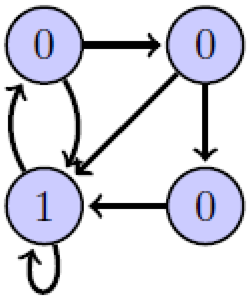
Baabali & Egerstedt's framework (2005)

$X(t+1)=Ax + B_i\, u(t)$ → Here, the switching is on the input matrix B_i

Theorem [Baabali Egerstedt 2005]: There exists some l such that : If for all l<L, the pairs (A^l, B_i) are controllable, then the system is controllable

- Only a sufficient condition
- The set of pairs to check can be huge (more than exponential)

Controllability with Packet Dropouts



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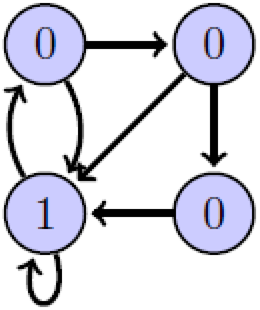
$$\boxed{\sigma = 1001\dots}$$

The controllability problem: for any starting point x(0), and any target x*, does there exist, for any switching signal, a control signal u(.) and a time T such that x(T)=x* ?

Proposition: The system is controllable iff the generalized controllability matrix

$$C_\sigma(t) = [A^{(t-1)}b\sigma(0)|A^{(t-2)}b\sigma(1)|\dots|Ab\sigma(t-2)|b\sigma(t-1)]$$

is bound to become full rank at some time t



Our algorithm

Thus, we have a purely algebraic problem: is it possible to find a path in the automaton such that the controllability matrix is never full rank?

$$C_\sigma(t) = [A^{(t-1)}b\sigma(0)|A^{(t-2)}b\sigma(1)|\ldots|Ab\sigma(t-2)|b\sigma(t-1)]$$

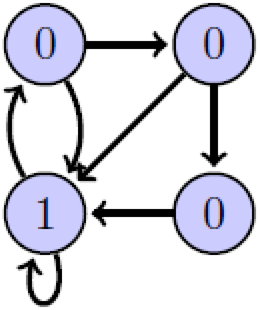
➔ Theorem: Given a matrix A and two vectors b,c, the set of paths such that

$$C_\sigma(t)$$

is never full rank is either empty, or contains a cycle in the automaton.

From this, we obtain an algorithm to decide controllability:

Semi-algorithm 1: For every cycle of the automaton, check if it leads to an infinite uncontrollable signal
Semi-algorithm 2: For every finite path, check whether it leads to a controllable signal (i.e. a full rank controllability matrix).



Proof of our theorem

Theorem ([Skolem 34]): Given a matrix A and two vectors b,c, the set of values n such that

$$c^{\top} A^n b = 0$$

is eventually periodic.

We managed to rewrite our controllability conditions in terms of a linear iteration

➔ Theorem: Given a matrix A and two vectors b,c, the set of paths such that

$$C_{\sigma}(t)$$

is never full rank is either empty, or contains a cycle in the automaton.

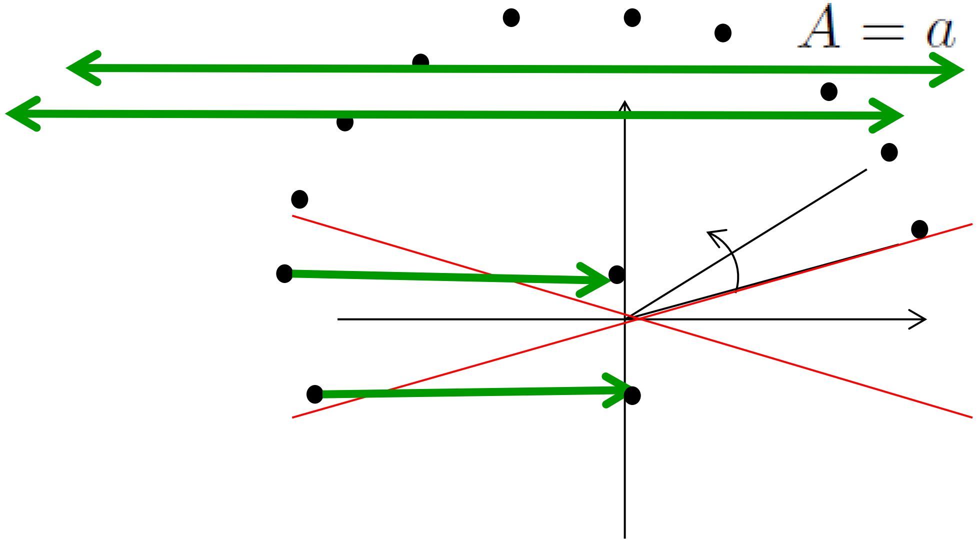
Now, how to optimally chose the control signal, if one does not know the switching signal in advance?

Outline

- Joint spectral characteristics
- Path-complete methods for switching systems stability
- Applications:
    - WCNs and packet dropouts
    - Switching delays
- Conclusion and perspectives

LTIs with switched delays Example

The controller design problem: a 2D system with two possible delays



$$A = a\begin{pmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(alpha) \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$D = \{0, 1\}$$

- **Theorem:** For the above system, there exist values of the parameters such that no linear controller can stabilize the system, but a nonlinear bang-bang controller does the job. [J. D'Innocenzo Di Benedetto 2014]

That is, a linear controller is not always sufficient

Outline

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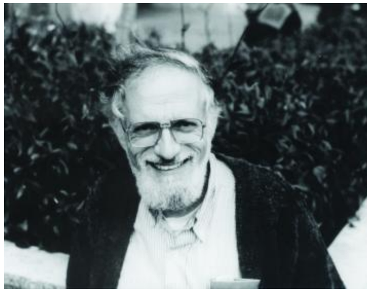
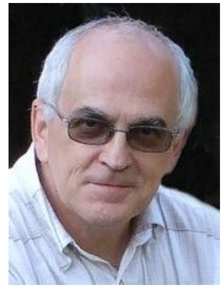
Conclusion: a perspective on switching systems




[Furstenberg Kesten, 1960]

[Gurvits, 1995]



[Kozyakin, 1990]


[Rota, Strang, 1960]


[Blondel Tsitsiklis, 98+]

(sensor) networks

Wireless control

Bisimulation design

consensus problems

Social/big data control

…

60s 70s

Mathematical properties

90s

TCS inspired
Negative
Complexity results

2000s

Lyapunov/LMI
Techniques
(S-procedure)

now

CPS applic.
Ad hoc
techniques

Thanks!

Questions?

Ads

The JSR Toolbox:
http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox
[Van Keerberghen, Hendrickx, J. HSCC 2014]
The CSS toolbox, 2015

Several open positions:
raphael.jungers@uclouvain.be

References:
http://perso.uclouvain.be/raphael.jungers/

Joint work with

A.A. Ahmadi (Princeton), M-D di Benedetto (l'Aquila), V. Blondel (UCLouvain), J. Hendrickx (UCLouvain) A. D'innocenzo (l'Aquila), M. Heemels (TU/e), A. Kundu (TU/e), P. Parrilo (MIT), M. Philiippe (UCLouvain), V. Protasov (Moscow), M. Roozbehani (MIT),…

EECI Course,
L'Aquila, April 4-8