# Positivity and Monotonicity in Switched Systems: A Miscellany

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# Talk Outline - 3 Problems

- D-stability for switched positive systems.
- Stability Vs persistence for switched epidemiological models:
    - stability of the disease free equilibrium;
    - persistence and periodic orbits.
- Monotonicity and continuity for state-dependent switching.

# Notation

- For $A \in \mathbb{R}^{n \times n}$: $\rho(A)$ denotes its spectral radius; $\mu(A)$ denotes the spectral abscissa

$$\mu(A) = \max\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)\}.$$

- $A$ is *Metzler* if $a_{ij} \geq 0$ for $i \neq j$.
- For a finite set $\mathcal{M} \subset \mathbb{R}^{n \times n}$ $\mathrm{conv}(\mathcal{M})$ denotes its convex hull.
- $A \in \mathbb{R}^{n \times n}$ nonnegative or Metzler is irreducible if the associated digraph is strongly connected.

# Background

- The LTI system
$$\dot{x} = Ax \qquad (1)$$
is positive if $x_0 \geq 0$ implies $x(t, x_0) \geq 0$ for all $t \geq 0$.
- It is well known that (1) is positive if and only if $A$ is Metzler.

**Theorem**

*Let $A \in \mathbb{R}^{n \times n}$ be Metzler. The following are equivalent:*

1. *$A$ is Hurwitz ($\mu(A) < 0$);*
2. *there exists some $v \gg 0$ with $Av \ll 0$;*
3. *D-Stability: $DA$ is Hurwitz for all diagonal matrices $D \in \mathbb{R}^{n \times n}$ with positive diagonal entries.*

# Cooperative Systems

- $D \subseteq \mathbb{R}^n$ open, connected; $f : D \to \mathbb{R}^n$ $C^1$ is *cooperative* if $\frac{\partial f}{\partial x}(a)$ is Metzler for every $a \in D$.
- Assume that $D$ is an invariant set for

$$\dot{x}(t) = f(x(t)). \tag{2}$$

- Well known that if $f$ is cooperative then (2) is monotone/order-preserving:

$$x_0 \leq y_0 \Rightarrow x(t, x_0) \leq x(t, y_0)$$

for all $t \geq 0$.

# Cooperative and Monotone Systems

- Converse of this is true also if state space is locally convex.
- More generally, conditions for monotonicity are so-called Kamke-Müller conditions:

$$x \le y, x_i = y_i \Rightarrow f_i(x) \le f_i(y).$$

- When this holds, $x_0 \le y_0 \Rightarrow x(t, x_0) \le x(t, y_0)$ but also $x_0 \ll y_0$ implies $x(t, x_0) \ll x(t, y_0)$ for all $t \ge 0$.

# D-Stability for Switched Linear Systems

## The Problem

Given a set of Metzler matrices $\mathcal{M} := \{A_1, \ldots, A_m\} \subseteq \mathbb{R}^{n \times n}$, the switched system

$$\dot{x}(t) = A_{\sigma(t)} x(t) \ \sigma : [0, \infty) \to \{1, \ldots, m\} \tag{3}$$

is D-stable if

$$\dot{x}(t) = D_{\sigma(t)} A_{\sigma(t)} x(t) \tag{4}$$

is globally asymptotically stable for all diagonal matrices $D_1, \ldots, D_m$ with positive diagonal entries.

**M, Bokharaie, Shorten, 2009**

1. If there exists $v \gg 0$ in $\mathbb{R}^n$ with $A_i v \ll 0$ for $1 \leq i \leq m$, then (4) is D-stable.
2. If (4) is D-stable, then there exists some non-zero $v \geq 0$ with $A_i v \leq 0$ for $1 \leq i \leq m$.

In general, there is a gap between these two conditions. Consider

$$A_1 = \begin{pmatrix} -2 & 1 \\ 2 & -2 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}.$$

# D-Stability for Switched Linear Systems

It is possible to close this gap if our system matrices are irreducible.

**Bokharaie, M, Wirth, 2010**

If each $A_i$ is irreducible then (4) is D-stable if and only if there exists some $v \gg 0$ with $A_i v < 0$ for $1 \leq i \leq m$.

- Combine $D_i A_i v < 0$ with irreducibility to show that any solution starting at $v$ decreases in every component initially.
- This combined with monotonicity properties of positive LTI systems allows us to show that $x(t, v, \sigma) \to 0$ as $t \to \infty$ for any switching signal $\sigma$.
- Another application of monotonicity allows us to conclude that solutions corresponding to all initial conditions tend to zero asymptotically.

# D-Stability for Switched Linear Systems

These results can be used to characterise D-stability for systems with commuting matrices.

**Bokharaie, M, Wirth, 2010**

If $A_i A_j = A_j A_i$ for all $i, j$, then (4) is D-stable if and only if $A_i$ is Hurwitz for $1 \leq i \leq n$.

- This follows easily as it is straightforward to show that there must exist some $v \gg 0$ with $A_i v \ll 0$ for $1 \leq i \leq n$.
- This result and the original sufficient condition for D-stability extends to nonlinear cooperative vector fields.

# SIS model for structured population

A compartmental SIS model for structured populations was analysed in [Fall, Iggidr, Sallet and Tewa, 2007].

- Population divided into $n$ groups; each group divided into *susceptibles* ($S_i$) and *infectives* ($I_i$).
- $N_i$ - total population of group $i$.
- $\mu_i$ - birth rate and death (non-disease related) rate of group $i$.
- $\beta_{ij}$ - infectious rate for contacts between group $j$ and $i$.
- $\gamma_i$ - recovery rate for group $i$.

# SIS model for structured population

- This leads to the time-invariant SIS model:

$$\begin{aligned}\dot{S}_i(t) &= \mu_i N_i - \mu_i S_i - \sum_{j=1}^{n} \beta_{ij} \frac{S_i(t) I_j(t)}{N_i} + \gamma_i I_i(t) \\ \dot{I}_i(t) &= \sum_{j=1}^{n} \beta_{ij} \frac{S_i(t) I_j(t)}{N_i} - (\gamma_i + \mu_i) I_i(t).\end{aligned}$$

- Clearly, $N_i$ is constant for each group.

# SIS model for structured population

- Let $x_i(t) = \frac{I_i(t)}{N_i}$ denote the proportion of group $i$ infected at time $t$;
- $\hat{\beta}_{ij} = \frac{\beta ij N_j}{N_i}$, $\alpha_i = \gamma_i + \mu_i$. We can write the system as:

$$\dot{x}_i(t) = (1 - x_i(t)) \sum_{j=1}^{n} \hat{\beta}_{ij} x_j(t) - \alpha_i x_i(t),$$

with $\alpha_i > 0$, $\beta_{ij} \geq 0$.

# SIS model - compact description

- This basic model can be written in the compact form:

$$\dot{x} = [-D + B - \mathrm{diag}\,(x)B]x, \tag{5}$$

- $D = \mathrm{diag}\,(\alpha_i)$ and $B = (\hat{\beta}_{ij})$.
- $\Sigma_n := \{x \in \mathbb{R}^n_+ : x_i \leq 1, i = 1, \ldots, n\}$ is invariant and the origin is an equilibrium - *disease-free equilibrium*.

# Stability of Disease-Free Equilibrium (DFE)

Let $R_0 = \rho(D^{-1}B)$. This plays the role of the basic reproduction number and acts as a threshold parameter for the model.

**Fall et al, 2007**

Consider the system (5). Assume that the matrix $B$ is irreducible. The DFE at the origin is globally asymptotically stable if and only if $R_0 \leq 1$.

Not difficult to see that

$$R_0 \leq 1 \Leftrightarrow \mu(-D + B) \leq 0.$$

# Endemic Equilibria

**Fall et al, 2007**

Consider the system (5) and assume that $B$ is irreducible. There exists a unique endemic equilibrium $\bar{x}$ in $\mathrm{int}\,(\mathbb{R}^n_+)$ if and only if $R_0 > 1$. Moreover, in this case, $\bar{x}$ is asymptotically stable with region of attraction $\Sigma_n \setminus \{0\}$.

As above, the condition $R_0 > 1$ is equivalent to $\mu(-D + B) > 0$.

# Switched Model

We consider a switched version of this model to handle uncertainty and time-variation.

- $D_1, \ldots, D_m$ diagonal, $B_1, \ldots, B_m$ nonnegative in $\mathbb{R}^{n \times n}$.
- $$\dot{x} = (-D_{\sigma(t)} + B_{\sigma(t)} - \mathrm{diag}\,(x) B_{\sigma(t)})x. \tag{6}$$
- $\sigma : [0, \infty) \to \{1, \ldots, m\}$ measurable switching signal.

# Linearised System

The linearisation of this system is

$$\dot{x} = (-D_{\sigma(t)} + B_{\sigma(t)})x \tag{7}$$

with the associated set of system matrices

$$\mathcal{M} = \{-D_1 + B_1, \ldots, -D_m + B_m\}.$$

1. System matrices are all Metzler.
2. A natural generalisation of the condition $R_0 \leq 1$ is to consider the *joint Lyapunov exponent* of $\mathcal{M}$.

# Joint Lyapunov Exponent

- For each switching signal $\sigma$, the evolution operator is given by the solution of the matrix differential equation:

$$\dot{\Phi}_\sigma(t) = A_{\sigma(t)}\Phi_\sigma(t),\ \Phi(0) = I.$$

- For each $t$, $\mathcal{H}_t$ denotes the set of all time evolution operators for time $t$.
- We then define the operator semigroup

$$\mathcal{H} := \cup_{t \geq 0}\mathcal{H}_t.$$

$(\mathcal{H}_0 = \{I\})$

# Joint Lyapunov Exponent

- The growth rate at time $t$ is given by

$$\rho_t(\mathcal{M}) := \sup_{\sigma} \frac{1}{t} \log \|\Phi_\sigma(t)\|.$$

- The joint Lyapunov exponent (JLE) is then given by

$$\rho(\mathcal{M}) = \lim_{t\to\infty} \rho_t(\mathcal{M}).$$

- The JLE can be thought of a generalisation of the spectral abscissa to othe switched system.

# Stability of the DFE for Switched SIS Model

Assume $\mathrm{conv}(\mathcal{M})$ contains an irreducible matrix.

**Ait-Rami, Bokharaie, M, Wirth, 2014**

The DFE of (6) is uniformly globally asymptotically stable if $\rho(\mathcal{M}) \leq 0$.

- Proving the result for $\rho(\mathcal{M}) < 0$ is straightforward using monotonicity techniqes.
- The existence of extremal norms plays a key role in the (far) subtler case $\rho(\mathcal{M}) = 0$.

# Persistence

Let a mapping $\eta : \Sigma_n \to \mathbb{R}_+$ be given.

## Strong Persistence

A semiflow $\phi : \Sigma_n \times \mathbb{R}_+ \to \Sigma_n$ is strongly persistent if

$$\liminf_{t\to\infty} \eta(\phi(t,x)) > 0 \ \forall x, \eta(x) > 0.$$

## Uniform Strong Persistence

A semiflow $\phi : \Sigma_n \times \mathbb{R}_+ \to \Sigma_n$ is uniformly strongly persistent if there is some $\epsilon > 0$ such that:

$$\liminf_{t\to\infty} \eta(\phi(t,x)) > \epsilon \ \forall x, \eta(x) > 0.$$

# Endemic Behaviour - Persistence

**Ait-Rami, Bokharaie, M, Wirth, 2014**

Consider the switched SIS model (6). Assume that there exists some $R \in \mathrm{conv}(\mathcal{M})$ with $\mu(R) > 0$. Then there exists a switching signal $\sigma$ such that for all $x^0 > 0$, $1 \leq i \leq n$

$$\liminf_{t \to \infty} x_i(t, x^0, \sigma) > 0.$$

Under the hypotheses of the theorem, there is a switching signal for which the resulting semiflow is strongly persistent in every population group.

# Outline of Proof

- We take $R = \kappa_1(D_1 + B_1) + \ldots + \kappa_m(D_m + B_m)$ and consider the autonomous SIS system

$$\dot{x}(t) = \hat{f}(x) = (\hat{D} + \hat{B})x - \mathrm{diag}(x)\hat{B}x \qquad (8)$$

  with $\hat{D} = \kappa_1 D_1 + \cdots + \kappa_m D_m$, $\hat{B} = \kappa_1 B_1 + \cdots + \kappa_m B_m$ .
- This has an endemic equilibrium $\hat{x}$ which is asymptotically stable with region of attraction $\mathbb{R}^n_+ \setminus \{0\}$.
- Moreover, there is some vector $v \gg 0$ such that the solution $\phi(t, v)$ of (8) is monotonically increasing.

# Outline of Proof

- For any $T > 0$, we can define a periodic switching signal $\sigma$ as follows.

$$\sigma(t) = 1 \text{ for } 0 \le t < \kappa_1 T$$

$$\sigma(t) = i \text{ for } (\sum_{j=1}^{i-1} \kappa_j)T \le t < (\sum_{j=1}^{i} \kappa_j)T$$

  for $2 \le i \le m$. Finally, $\sigma(t+T) = \sigma(t)$ for all $t \ge 0$.
- Using techniques from averaging theory for ODEs, we can then approximate the solution of the switched system with that of the autonomous system possessing an endemic equilibrium.
- Methods from monotone systems and differential inequalities allow us to conclude the result.

# Endemic Behaviour - Periodic Orbits

In fact, under the same hypotheses, we can establish the existence of a periodic orbit.

**Ait-Rami, Bokharaie, M, Wirth, 2014**

Consider the switched SIS model (6). Assume that there exists some $R \in \mathrm{conv}(\mathcal{M})$ with $\mu(R) > 0$. Then there exists a switching signal $\sigma$ and some $x^0 \gg 0$ such that the orbit

$$x(t, x^0, \sigma)$$

is periodic.

# Outline of Proof

- Use same averaged system and switching signal as in the previous result to define:

$$S_1(x^0) := \int_0^1 \hat{f}(\phi(s, x^0))ds, \; S_2(x^0) := \int_0^1 f_{\sigma(s)}(x(s, x^0, \sigma))ds.$$

- From the properties of $\hat{f}$, we can find a neighbourhood $\Omega$ in $\mathrm{int}(\Sigma_N)$ and $\hat{x} \in \Omega$ such that:
    1. $S_1(\hat{x}) = 0$;
    2. $S_1(z) \neq 0$ for all $z \in \mathrm{bd}(\Omega)$.

# Outline of Proof

- We next apply an approximation theorem from averaging theory for ODEs to conclude:

$$\max_{z \in \bar{\Omega}} \|S_1(z) - S_2(z)\|_\infty < \min_{z \in \mathrm{bd}(\Omega)} \|S_1(z)\|_\infty$$

  provided we choose the period $T > 0$ appropriately.
- Using a result from Degree Theory for nonlinear maps, this implies that $S_1$ and $S_2$ have the same number of zeros in $\Omega$.
- The zero of $S_2$ corresponds to a periodic orbit of the switched system.

# Summary of the Situation

- $\mathcal{M}$ irreducible and $\rho(\mathcal{M}) \leq 0$ means that DFE is GAS - Disease dies out.
- Each $B_i$ irreducible and $\mu(R) > 0$ for some $R \in \mathrm{conv}(\mathcal{M})$ means that there is a strongly persistent switching signal, and an endemic periodic orbit.
- If $\mu(R) > 0$ for some $R \in \mathrm{conv}(\mathcal{M})$, then $\rho(\mathcal{M}) > 0$.
- In general there is a gap between these two conditions - (Fainshil, Margaliot, Chigansky, 2009).

# JLE and Convex Hull

- For $\mathcal{M} \subseteq \mathbb{R}^{2\times 2}$ consisting of Metzler matrices:

$$\rho(\mathcal{M}) > 0$$

  implies the existence of some $R \in \mathrm{conv}(\mathcal{M})$ with $\mu(R) > 0$ (Gurvits, Shorten, M, 2007).
- This means that for a population consisting of two groups if each $B_i$ is irreducible:
  1. $\rho(\mathcal{M}) \leq 0$ implies the DFE is GAS;
  2. $\rho(\mathcal{M}) > 0$ implies that there is a persistent switching signal and a periodic orbit.

# Some Natural Extensions

- Does our condition for strong persistence imply *uniform* strong persistence? If not, what extra conditions are required?
- Does the same condition imply persistence in the case where $\mathrm{conv}(\mathcal{M})$ is irreducible?
- Does $\rho(\mathcal{M}) > 0$ imply that there is a persistent switching signal in general?
- Is the periodic orbit attractive?
- Can the analysis be extended to more complex epidemiological and rumour-spreading models?

# Monotonicity and Piecewise Systems

- $\mathcal{D}$ a region in $\mathbb{R}^n$, $\phi : \mathcal{D} \to \mathbb{R}$ a $C^2$ function.
- $D_f = \{x \in \mathcal{D} : \phi(x) < 0\}$ ,$D_g = \{x \in \mathcal{D} : \phi(x) > 0\}$.
- $f$ and $g$ are $C^1$ vector fields defined on neighbourhoods of $D_f$ and $D_g$ respectively.

**Key Question**

When is

$$\dot{x}(t) = \begin{cases} f(x) & \text{if } x \in D_f \\ g(x) & \text{if } x \in \overline{D_g}. \end{cases} \tag{9}$$

*monotone*?

# Monotonicity and Piecewise Systems

- $S := \{x \in \mathcal{D} : \phi(x) = 0\}$.

### Locally Monotone

For all $x_0, y_0$ in $\mathcal{D} \backslash S$ with $x_0 \leq y_0$ $x(t, x_0) \leq x(t, y_0)$ for $t \in [0, \delta]$ for some $\delta > 0$.

### Monotone

For all $x_0$ in $\mathcal{D}$ there exists a unique solution and $x_0 \leq y_0$ implies $x(t, x_0) \leq x(t, y_0)$ for all $t$ for which they are defined.

# Monotonicity and Piecewise Systems

- For local monotonicity, $f$ and $g$ must satisfy the Kamke Muller conditions in $U_f$, $U_g$.
- For $a \in S$, $I_0(a)$ denote those $i$ in $\{1, \ldots, n\}$ such that there are indices $j_1, j_2$ distinct to $i$ with $(\nabla\phi(a))_{j_1} < 0$, $(\nabla\phi(a))_{j_2} > 0$.

**O'Donoghue, M, Middleton, 2012**

If $I_0(a) = \{1, \ldots, n\}$, the following are equivalent:

1. (9) is locally monotone;
2. $f(a) = g(a)$ for all $a \in S$;
3. (9) is monotone.

# Some References

More details can be found in:


- Extremal Norms for Positive Linear Inclusions, O. Mason and F. Wirth, Linear Algebra and its Applications, 2014.
- Stability Criteria for SIS Epidemiological Models under Switching Policies, M. Ait-Rami, V. Bokharaie, O. Mason and F. Wirth, Discrete and Continuous Dynamical Systems, 2014.
- On the D-stability of linear and nonlinear positive switched systems, V. Bokharaie, O. Mason and F. Wirth, Proceedings of the 19th MTNS, 2010
- On the Kamke-Muller conditions, monotonicity and continuity for bi-modal, piecewise smooth systems. Y. O'Donoghue, O. Mason and R. Middleton, Systems and Control Letters, 2012.

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- Robert Shorten;
- Leonid Gurvits.

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