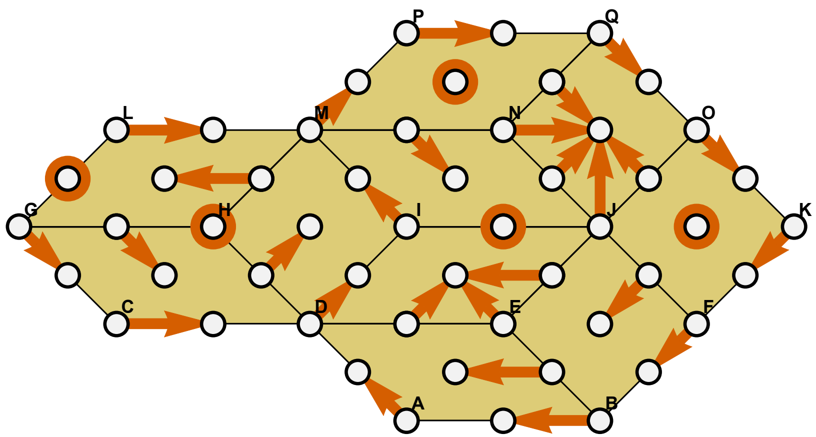
Workshop on switching dynamics & verification
Paris, 29th January 2016

Morse-Conley theory for combinatorial vector fields



Marian Mrozek

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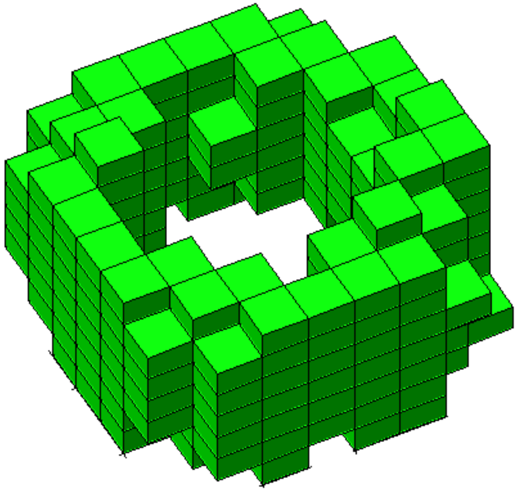
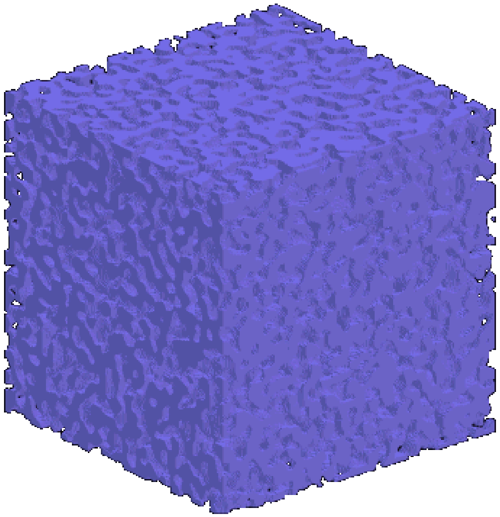
Background

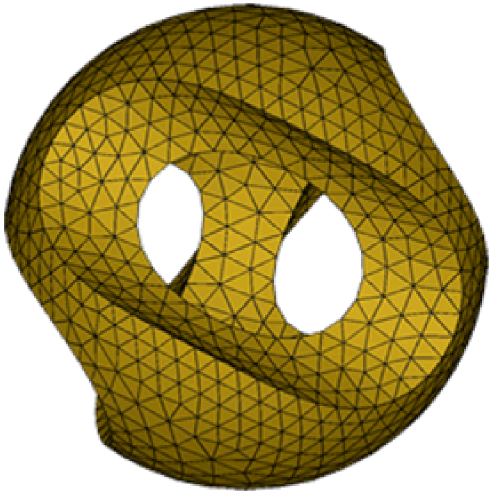
- Topological dynamics
- Topological tools: Lefschetz fixed point theorem, fixed point index, Ważewski criterion, Conley index, Conley-Morse theory
- computer assisted proofs based on topological invariants

Topology 3

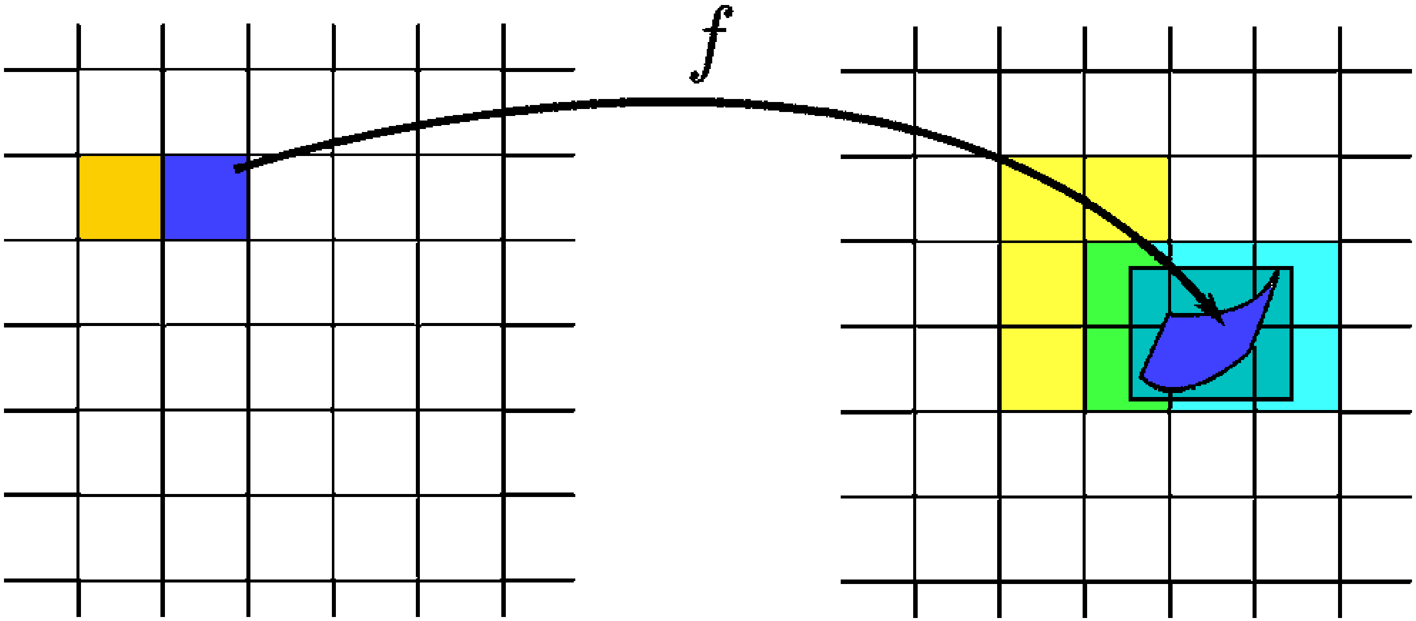
Computational Topology



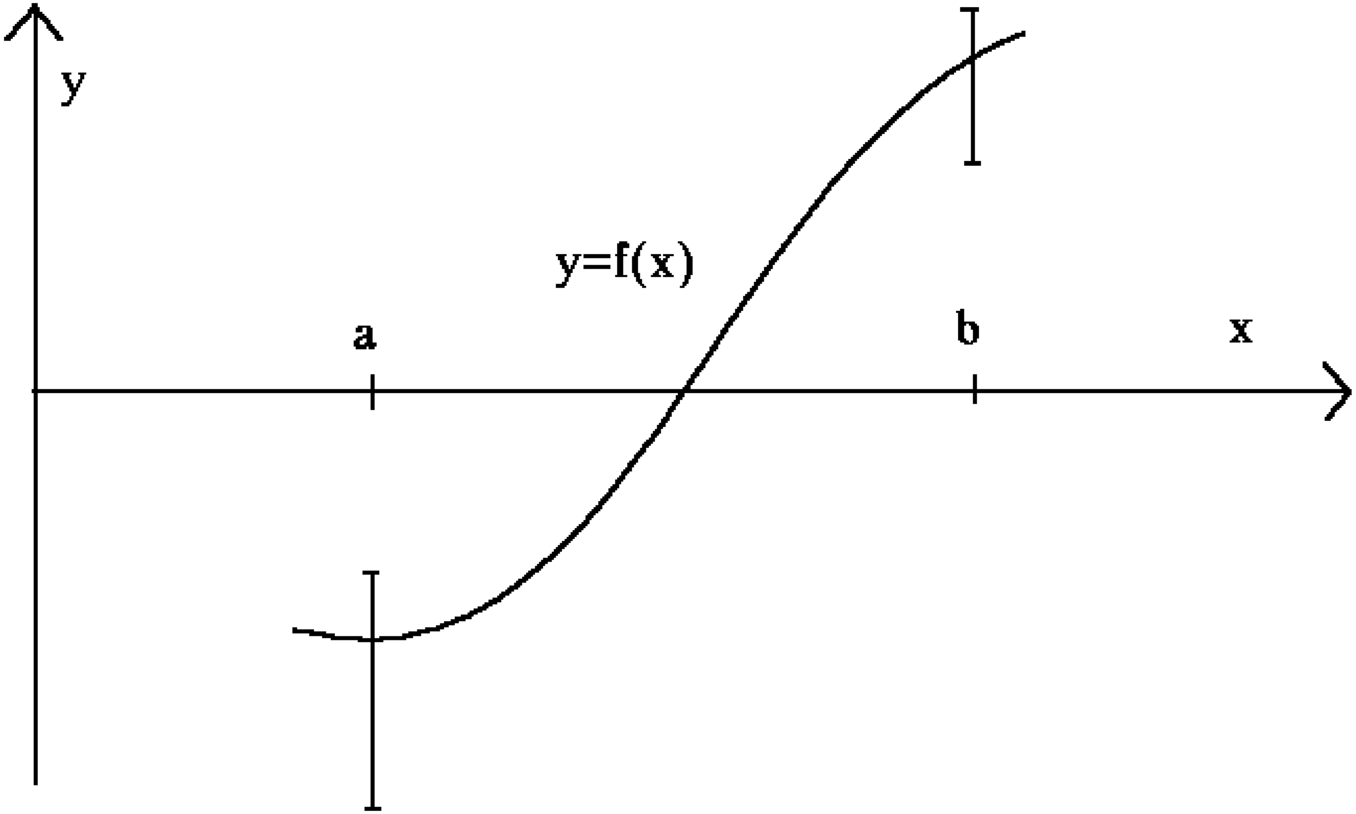




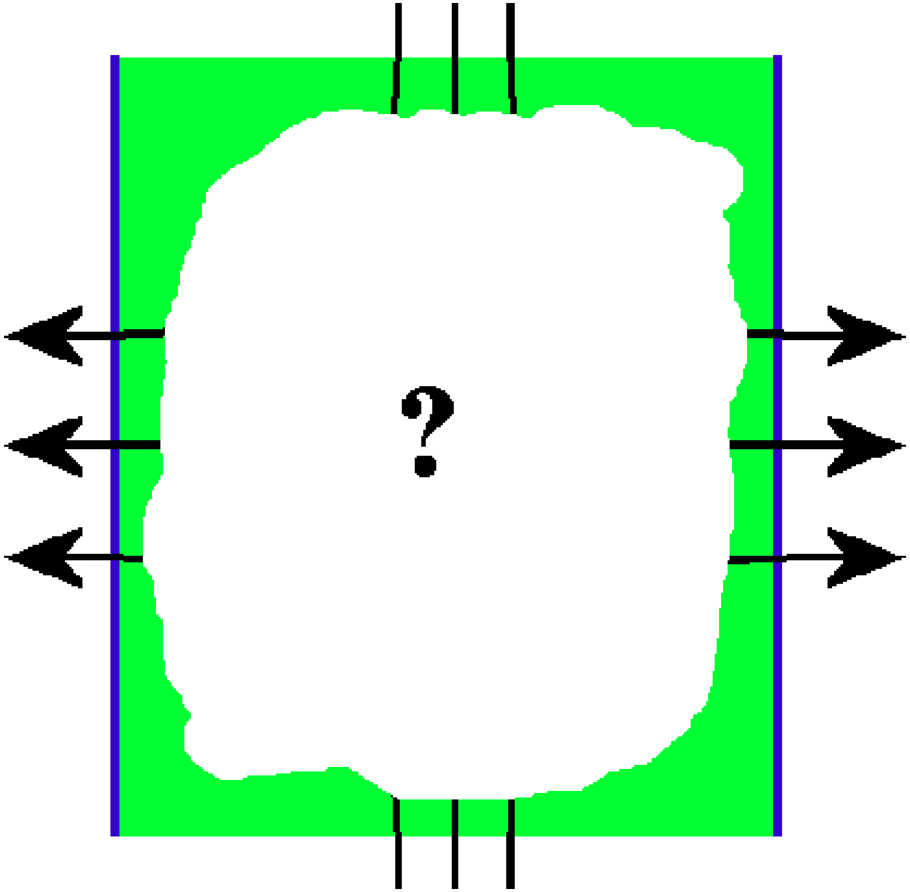
Rigorous numerics of dynamical systems



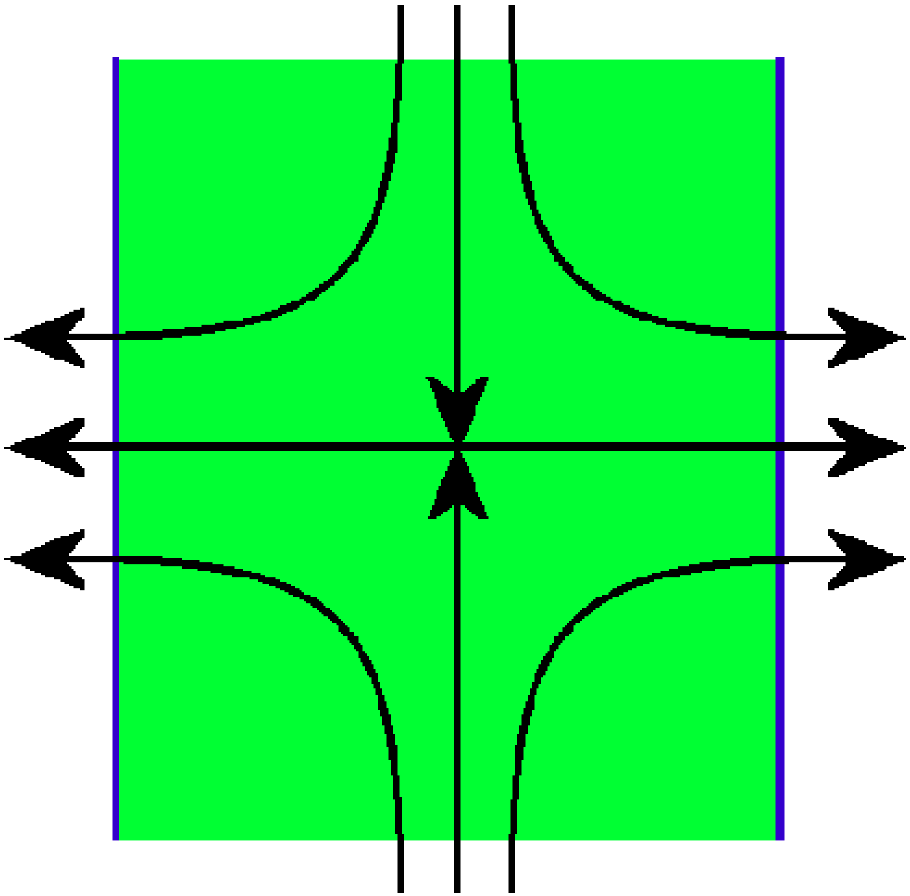
Topological existence criterion



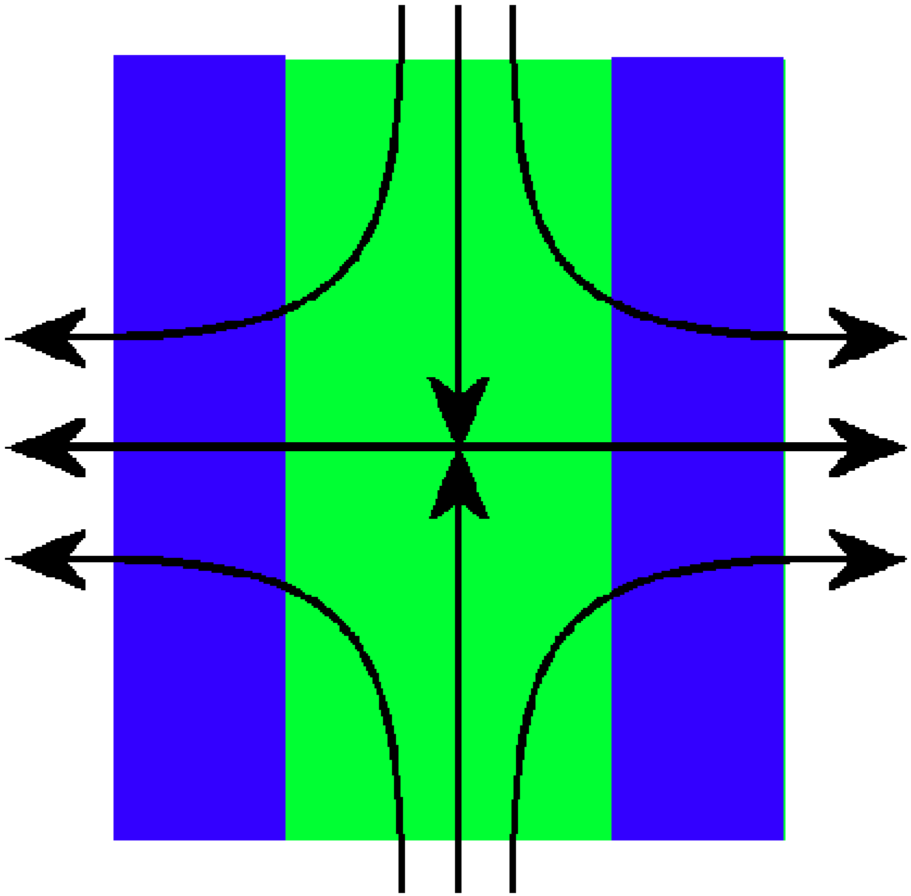
Ważewski Theorem



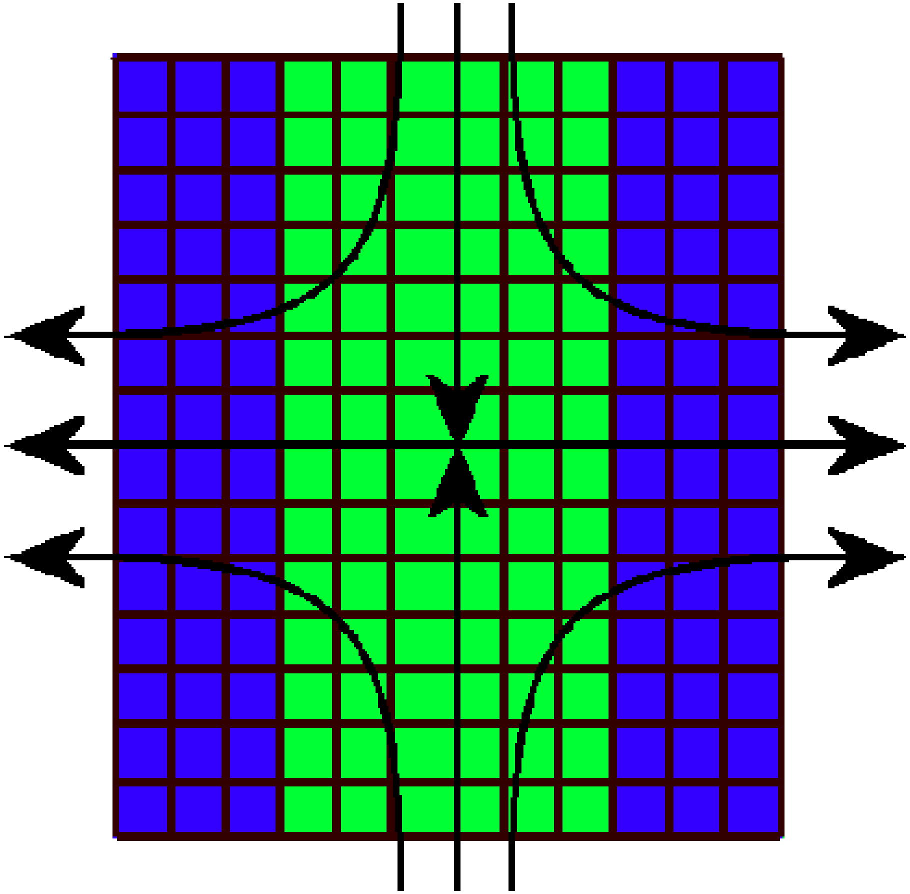
Ważewski Theorem

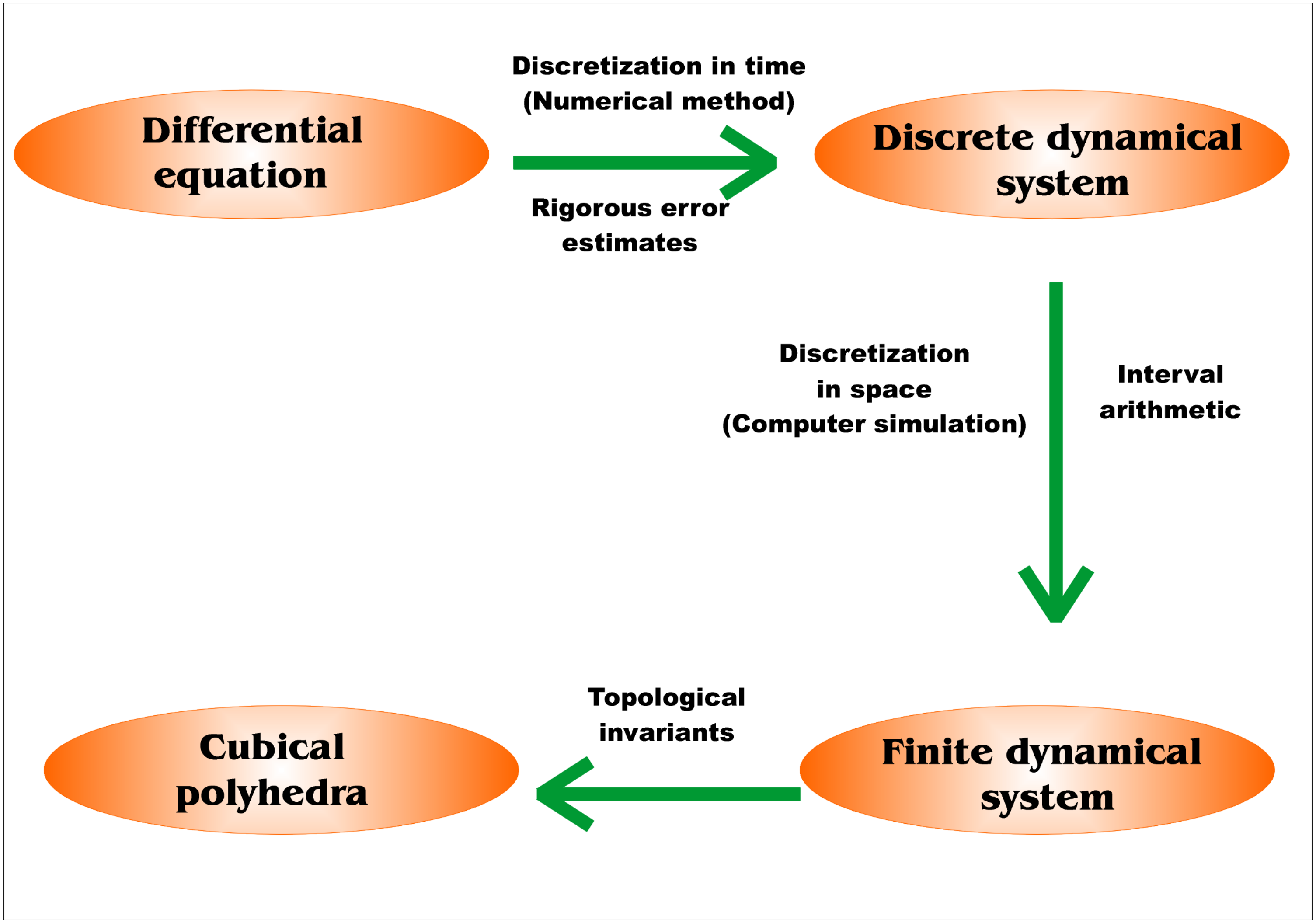


Index pair and Conley index



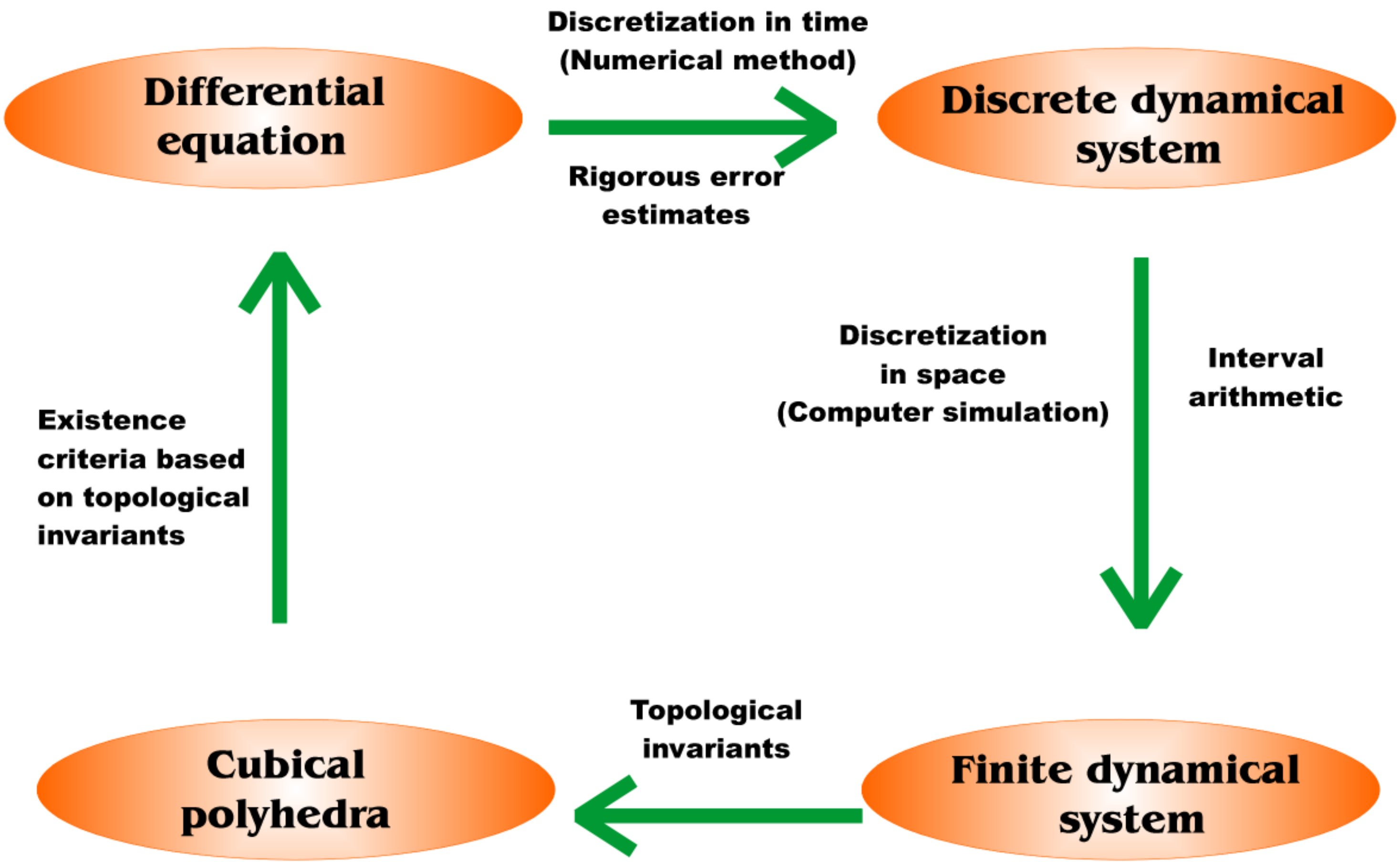
Index pair and Conley index




Differential equation
Discretization in time
(Numerical method)
Rigorous error estimates
Discrete dynamical system
Discretization in space
(Computer simulation)
Interval arithmetic
Finite dynamical system
Topological invariants
Cubical polyhedra

Existence results based on topological invariants

- bounded trajectories
- stationary trajectories
- periodic trajectories
- heteroclinic connections
- chaotic invariant sets
- semiconjugacies onto model dynamics


Differential equation
Discretization in time
(Numerical method)
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Discretization in space
(Computer simulation)
Interval arithmetic
Existence criteria based on topological invariants
Topological invariants
Cubical polyhedra
Finite dynamical system

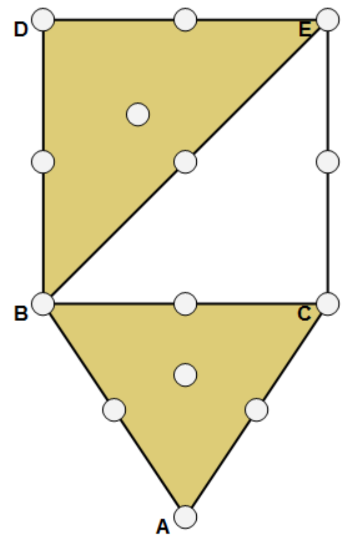
Goal

- Combinatorization of topological dynamics
- Applications to sampled dynamics

Outline

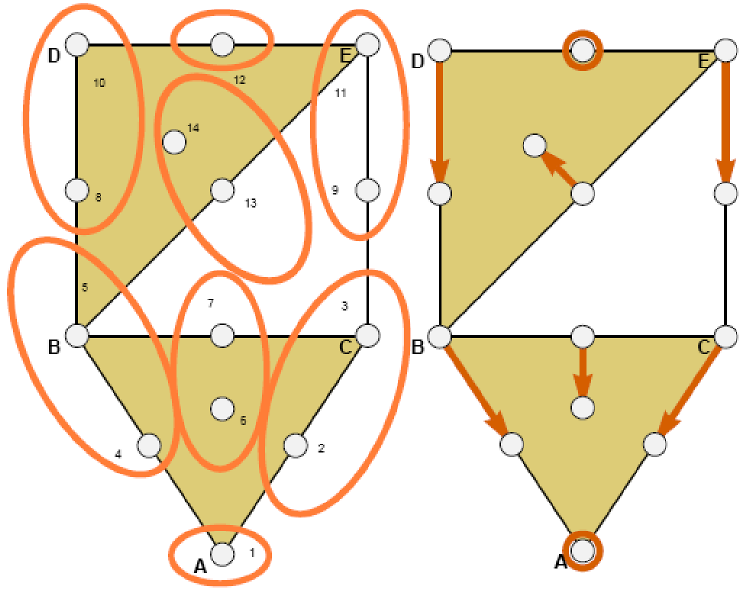
- Review of the combinatorial Morse theory by Forman
- Limitations of the Forman theory
- Combinatorial multivector fields
- Isolated invariant sets, Conley index and Morse inequalities
- Examples
- Relation to classical dynamics (joint with T. Kaczynski and Th. Wanner)

Morse-Forman theory



- $\mathcal{K}$ - the collection of cells of a finite, regular, CW complex X.
- Facet relation: $\tau \prec \sigma \;\Leftrightarrow\; \tau$ is a facet of σ
- Facet digraph: $(\mathcal{K}^2, \{\, (\sigma, \tau) \mid \tau \prec \sigma \,\})$
- $\operatorname{bd} \sigma := \{\, \tau \mid \tau \prec \sigma \,\}$.
- $\operatorname{cbd} \sigma := \{\, \rho \mid \sigma \prec \rho \,\}$.

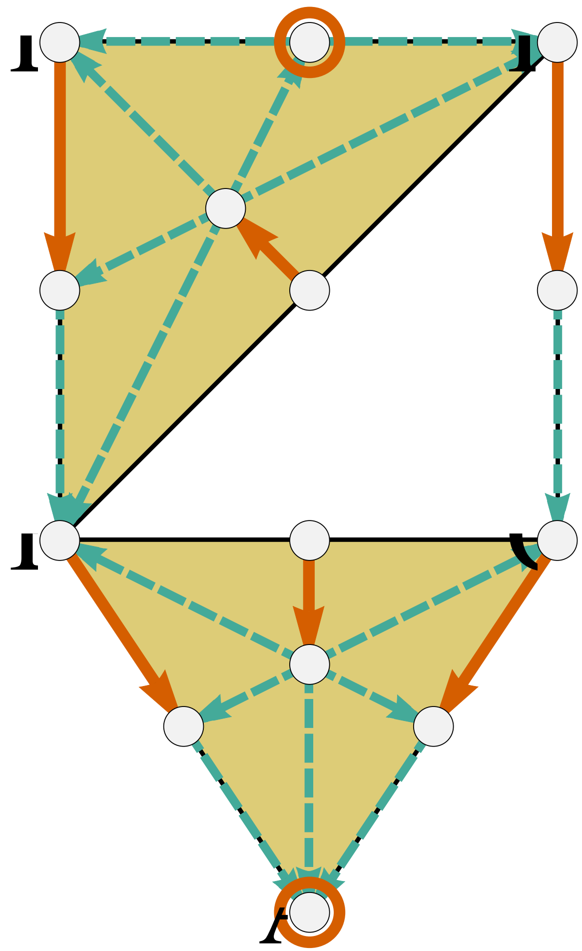
Discrete vector fields



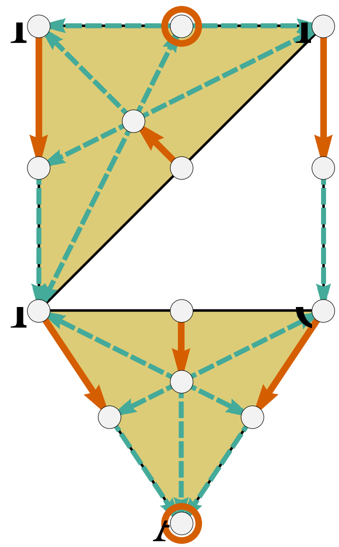
Definition.

- A discrete vector field $\mathcal{V}$ on $\mathcal{K}$ is a partition of $\mathcal{K}$ into doubletons and sigletons such that for each doubleton $\{\tau, \sigma\} \in \mathcal{V}$ either $\tau \prec \sigma$ or $\tau \succ \sigma$.
- The V-digraph of $\mathcal{K}$ is the facet digraph of $\mathcal{K}$ with the direction reversed on the elements of $\mathcal{V}$.

Paths/solutions



Morse Homology



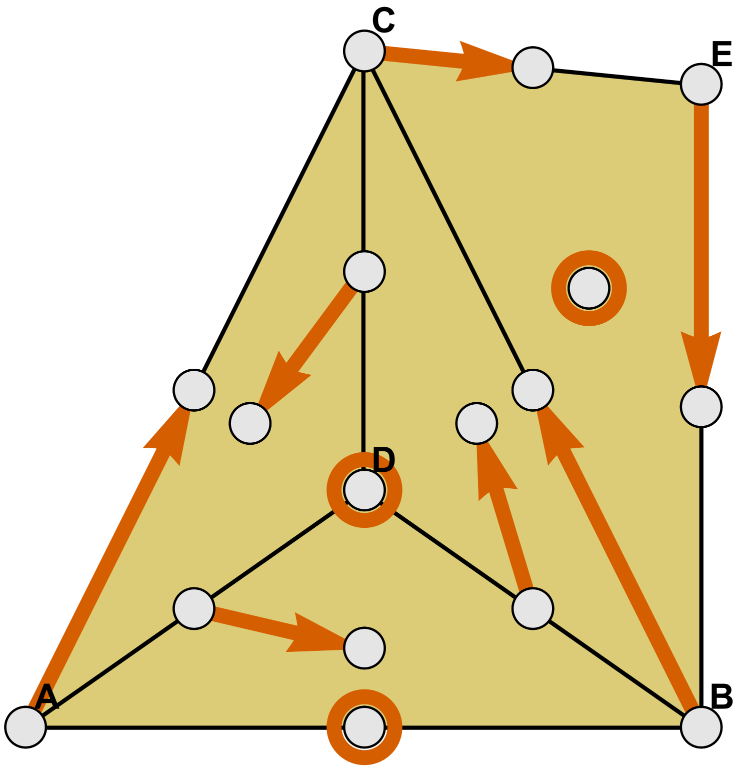
- Morse complex $(M, \Delta) := (M_q(\mathcal{K}, V), \Delta_q(\mathcal{K}, V))_{q\in\mathbb{Z}}$ of a gradient vector field V on $\mathcal{K}$:

$$
\begin{aligned}
M_q &:= \{\, \text{critical cells of dimension } q \,\} \\
\langle \Delta_q \sigma, \tau \rangle &:= \sum_{\alpha \in P_V^a(\sigma,\tau)} w(\alpha).
\end{aligned}
$$

Theorem. (Forman, 1995)

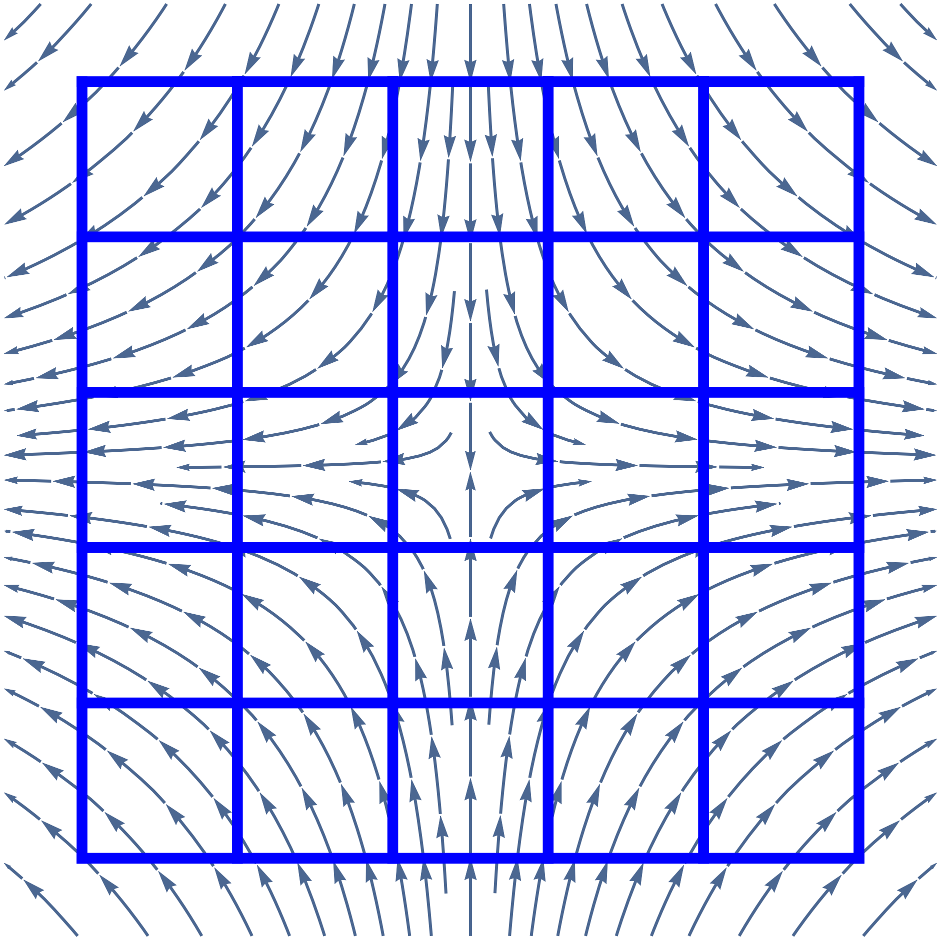
$$H_*(\mathcal{K}) \cong H_*(M, \Delta).$$

Goals



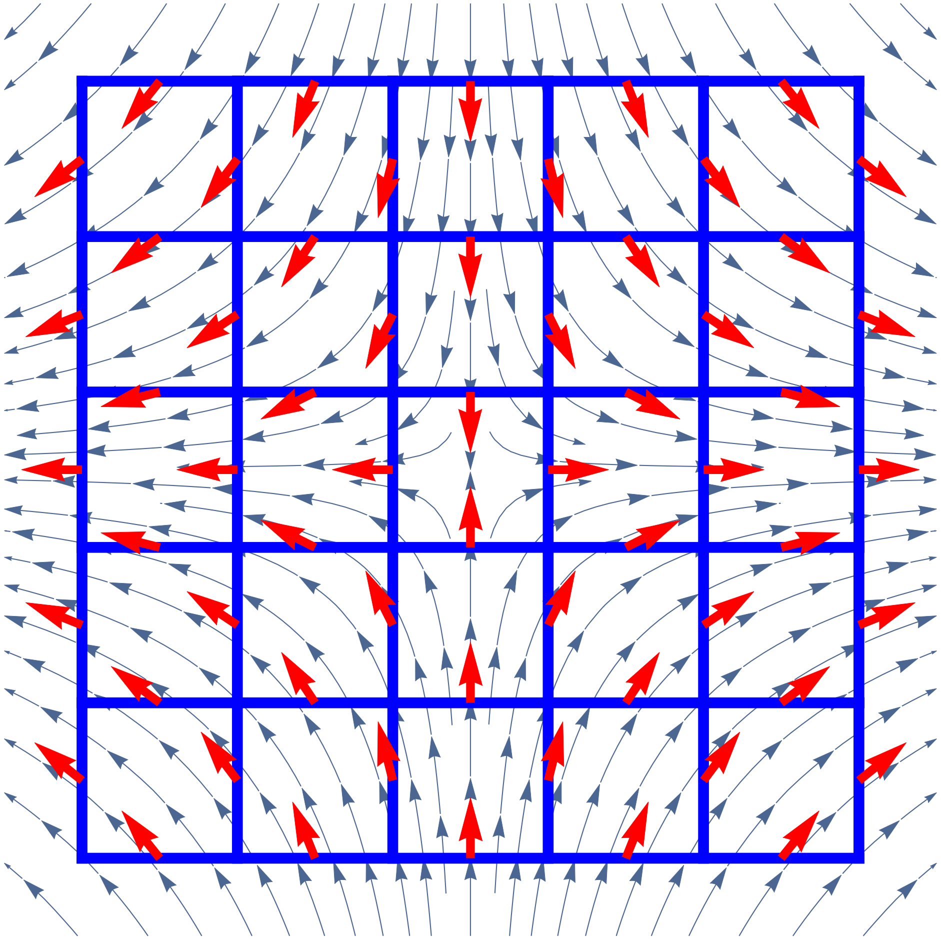
1) Bring Forman's combinatorial vector fields into the framework of classical topological dynamics.

Goals



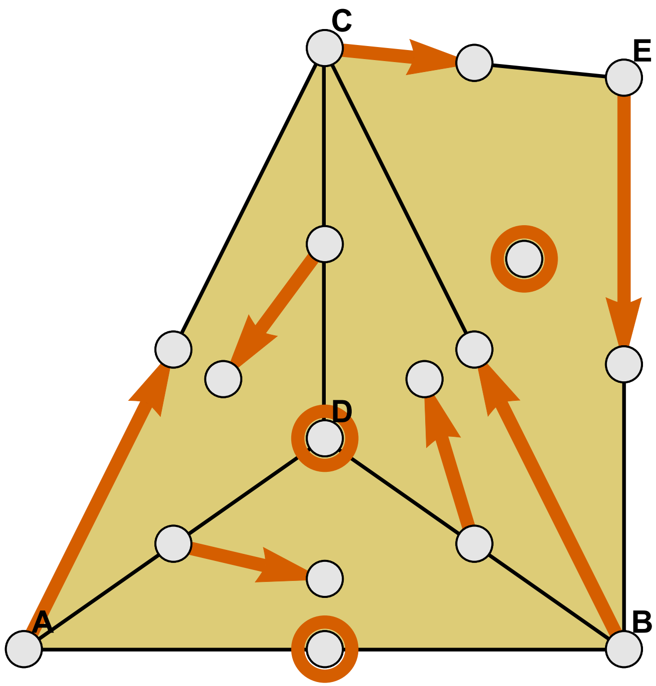
2) Extend the theory to combinatorial multivector fields.

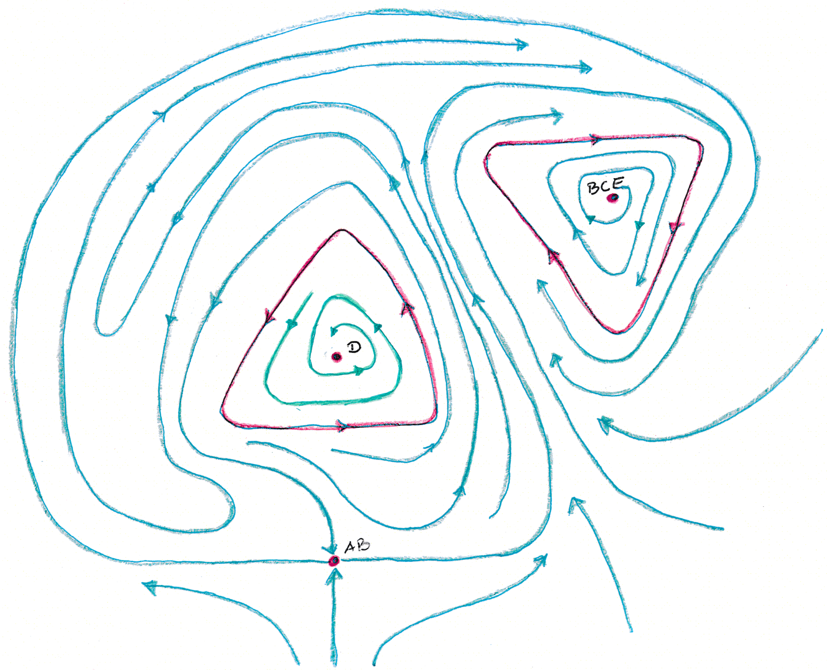
Goals



2) Extend the theory to combinatorial multivector fields.

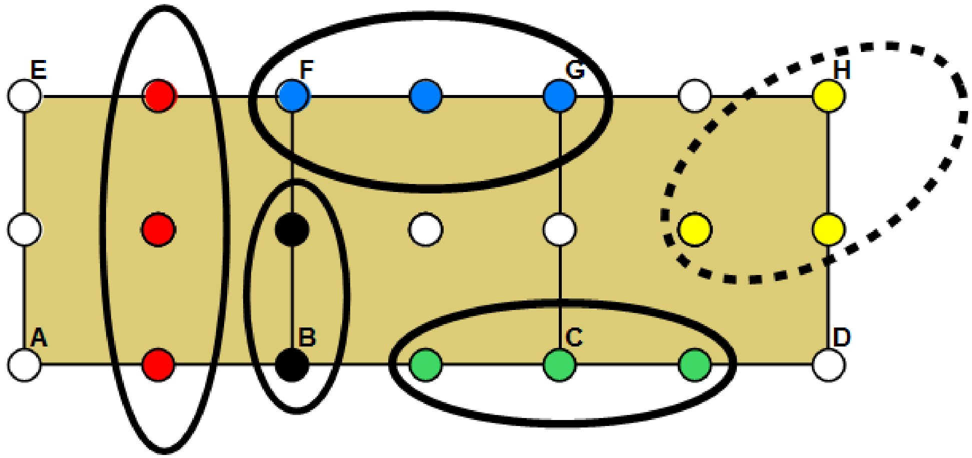
Goals





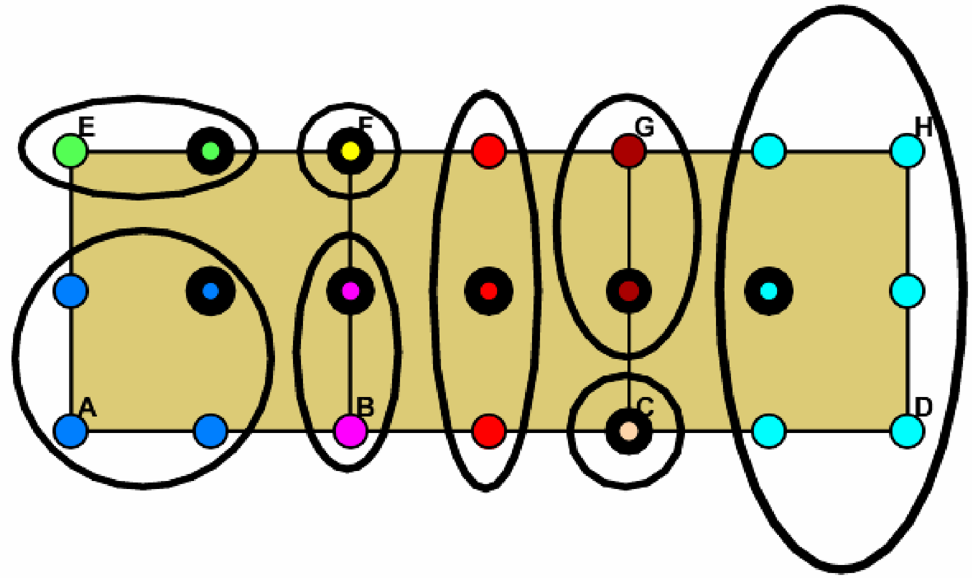
3) Construct bridges between the combinatorial dynamics on the family of cells of CW complexes and continuous dynamics on the topological space of the complex.

Alexandrov topology on $\mathcal{K}$



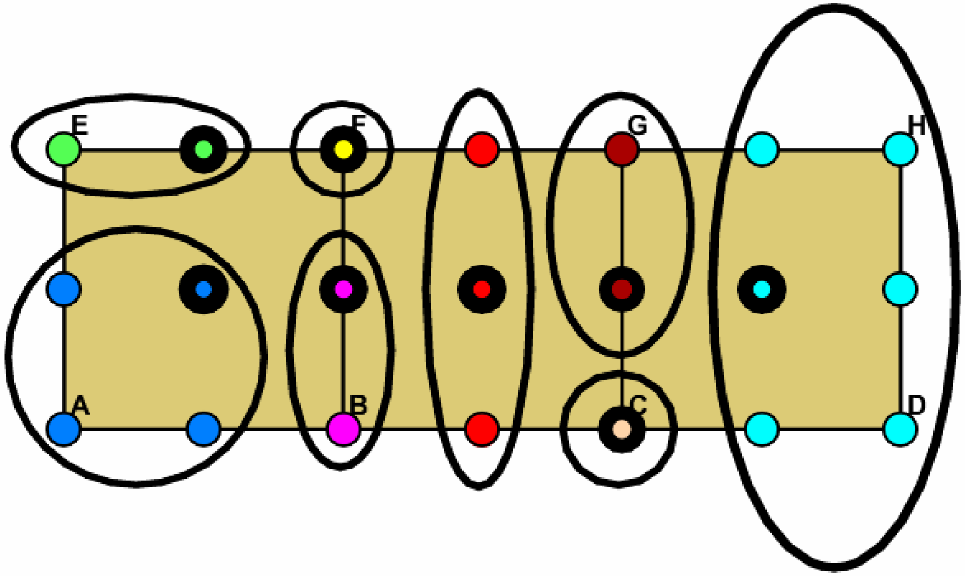
- $\mathcal{A} \subset \mathcal{K}$ is open (closed) iff $\bigcup \mathcal{A}$ is open (closed) in $X = \bigcup \mathcal{K}$.
- $\operatorname{es} \mathcal{A} := \operatorname{cl} \mathcal{A} \setminus \mathcal{A}$ - estuary of $\mathcal{A}$
- $\mathcal{A}$ is proper if $\operatorname{es} \mathcal{A}$ is closed.
- if $\mathcal{A}$ is closed, then $\bigcup \mathcal{A}$ is a subcomplex of the CW complex X.
- A proper $\mathcal{A} \subset \mathcal{K}$ is a zero space if $H(\operatorname{cl} \mathcal{A}, \operatorname{es} \mathcal{A}) = 0$.

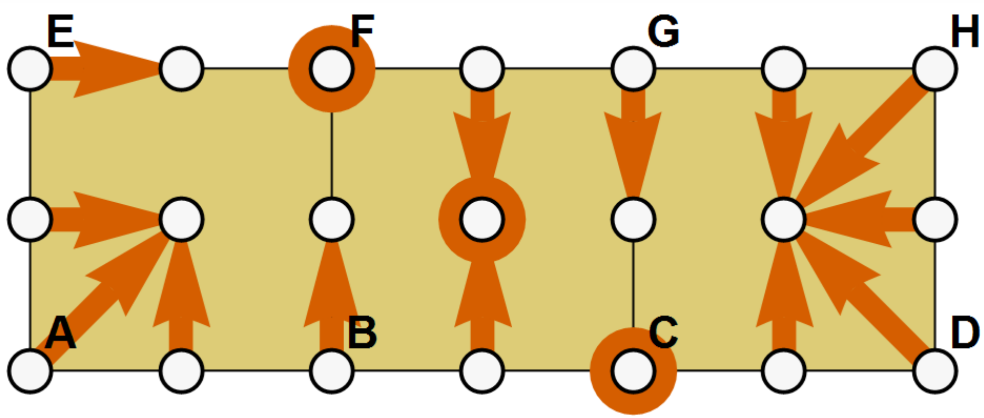
Combinatorial multivector fields



- A multivector is a proper $V \subset \mathcal{K}$ with a unique maximal element.
- A multivector field is a partition $\mathcal{V}$ of $\mathcal{K}$ into multivectors.
- V is regular if V is a zero space. Otherwise it is critical.

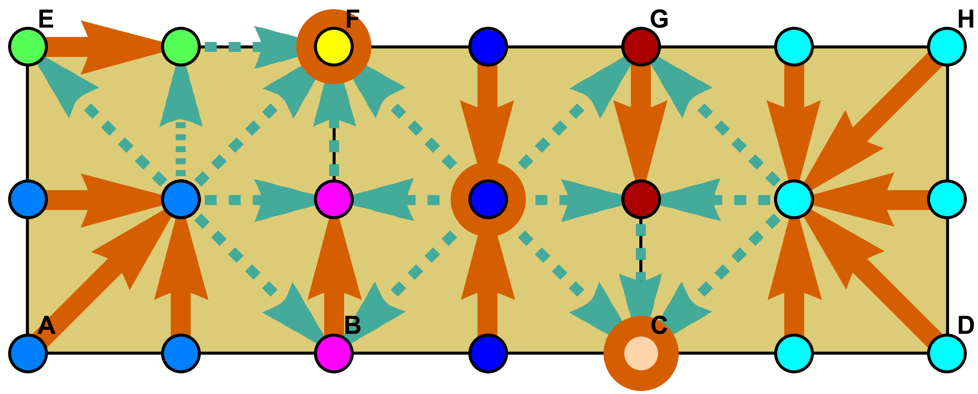
Combinatorial multivector fields





- A multivector is a proper $V \subset \mathcal{K}$ with a unique maximal element V^*.
- A multivector field is a partition $\mathcal{V}$ of $\mathcal{K}$ into multivectors.
- V is regular if V is a zero space. Otherwise it is critical.

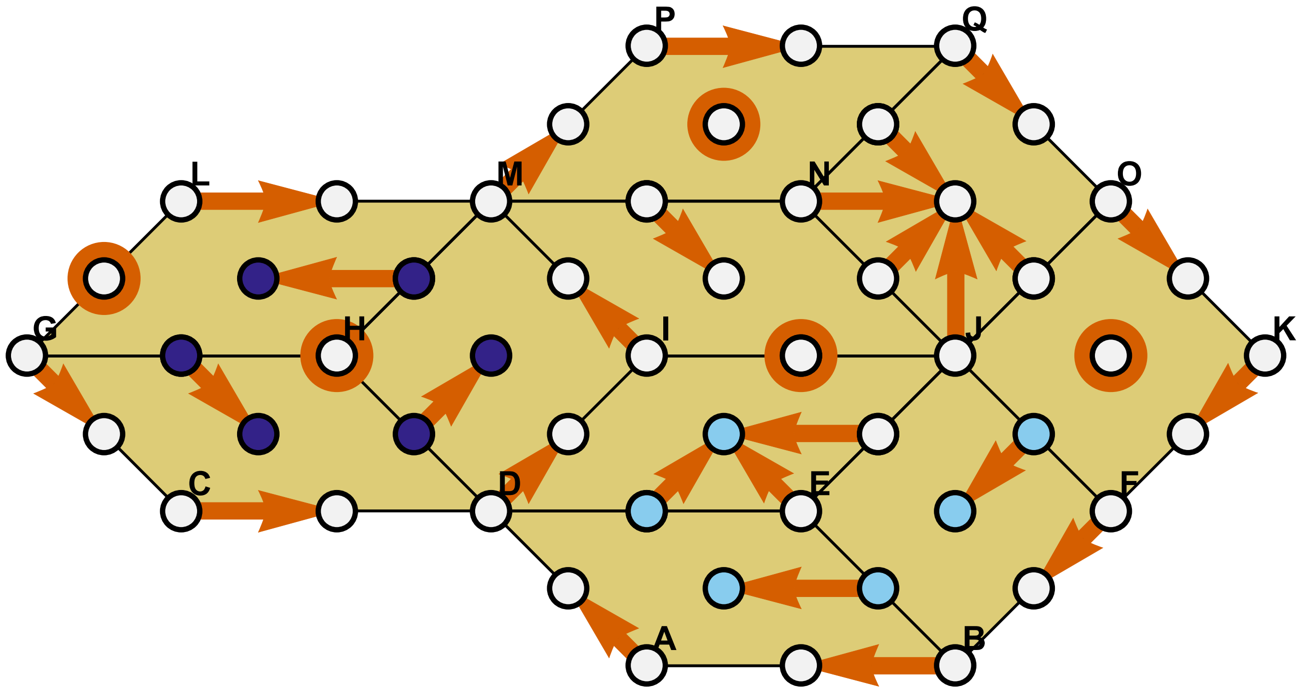
$\mathcal{V}$-digraph



- Vertices: cells in $\mathcal{K}$
- Arrows:
  - explicit: given by $\mathcal{V}$
  - implicit: from each maximal cell of a multivector to all its faces not in the multivector
  - loops: at each maximal cell of a critical multivector

The multivalued map $\Pi_{\mathcal{V}} : \mathcal{K} \rightrightarrows \mathcal{K}$ assigns to σ all targets of edges originating from σ.

Solutions and paths



- A partial map $\gamma : \mathbb{Z} \multimap \mathcal{K}$ is a solution of $\mathcal{V}$ if it is a walk in the $\mathcal{V}$-digraph, that is:

$$\gamma(i+1) \in \Pi_{\mathcal{V}}(\gamma(i)) \text{ for } i, i+1 \in \operatorname{dom} \gamma.$$

Isolated invariant sets

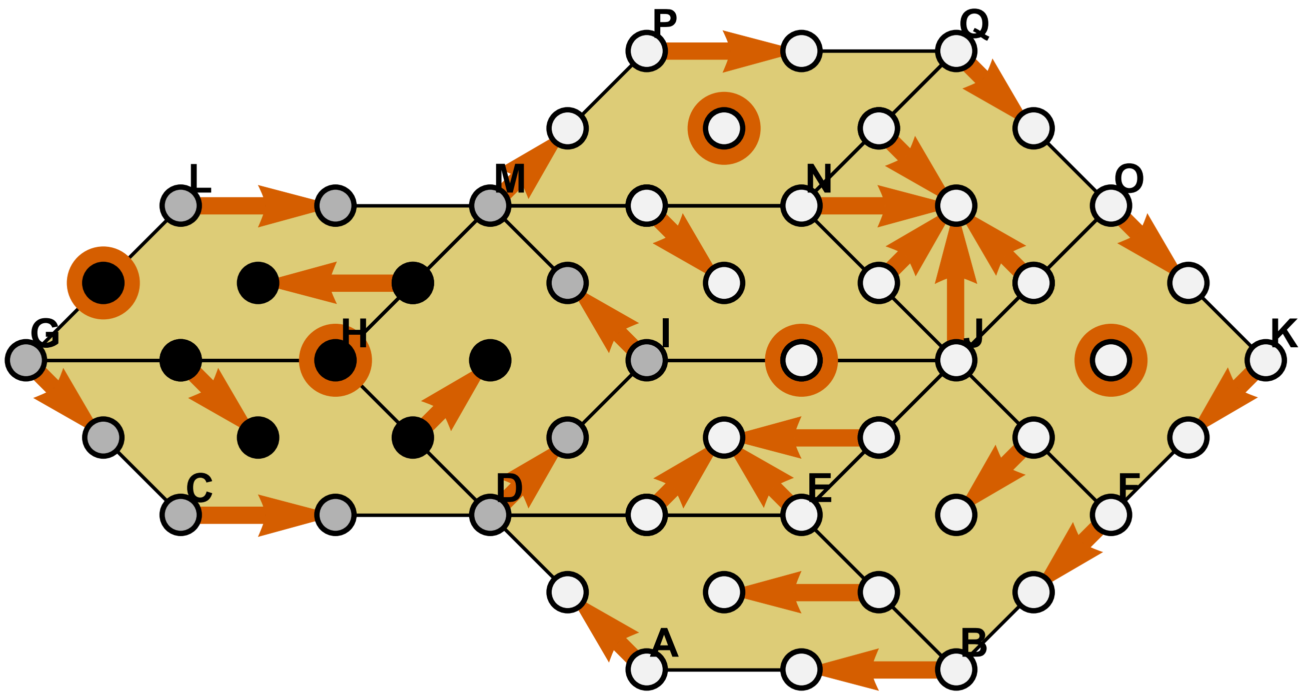
- V^* - maximal cell in $V \in \mathcal{V}$

$$\mathrm{Sol}(x, A) := \{\, \varrho : \mathbb{Z} \to A \text{ a solution s.t. } \varrho(0) = x \,\}.$$
$$\mathrm{Inv}\, A := \bigcup \{\, V \in \mathcal{V} \mid V \subset A \text{ and } \mathrm{Sol}(V^\star, A) \neq \varnothing \,\}$$

Let $S \subset \mathcal{K}$.

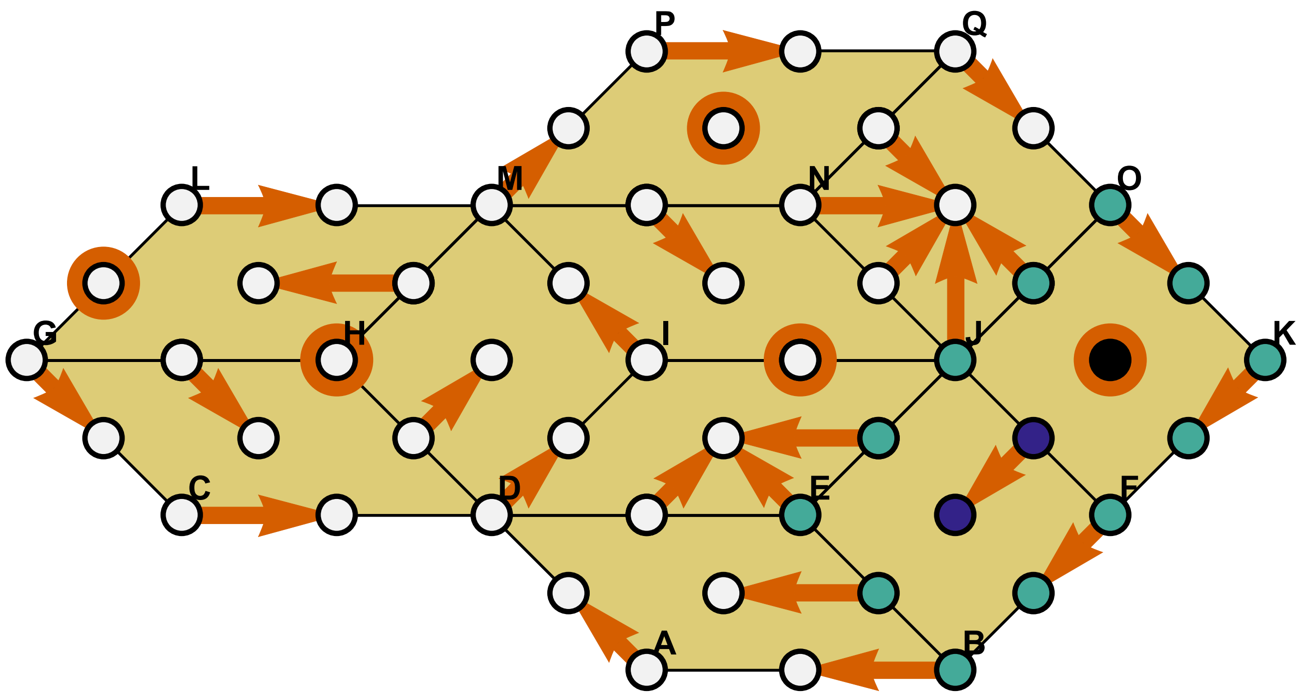
Definition. S is $\mathcal{V}$-invariant if $\mathrm{Inv}\, S = S$.

Definition.
- A solution $\gamma : \mathbb{Z} \to \mathrm{cl}\, S$ is an internal tangency to S if for some $n_1 < n_2 < n_3$ we have $\gamma(n_1), \gamma(n_3) \in S$ but $\gamma(n_2) \notin S$.
- S is an isolated invariant set if it is invariant and admits no internal tangencies.



Theorem. Let $S \subset X$ be invariant. Then, S is an isolated invariant set if and only if S is proper.

Index pairs



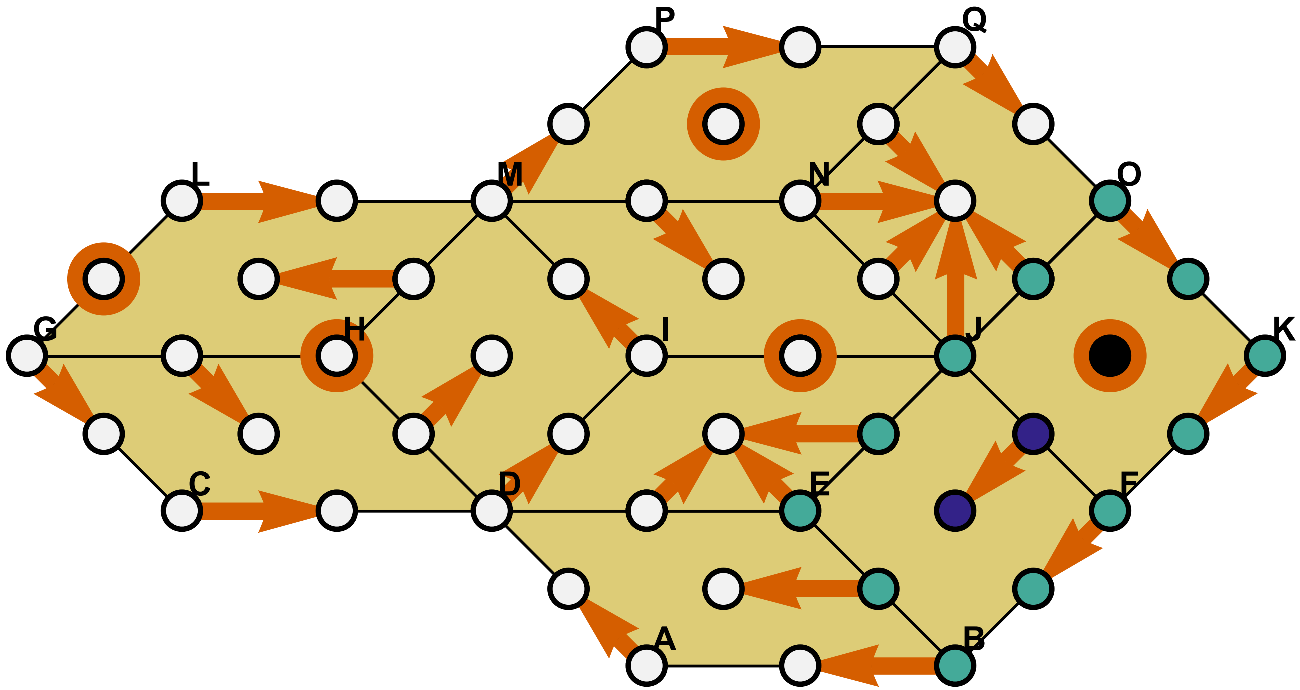
Definition. A pair $P = (P_1, P_2)$ of closed subsets of $\mathcal{X}$ is an index pair for $\mathcal{S}$ iff

(i) $x \in P_2,\ y \in P_1 \cap \Pi_{\mathcal{V}}(x) \ \Rightarrow\ y \in P_2$,

(ii) $x \in P_1,\ \Pi_{\mathcal{V}}(x) \setminus P_1 \neq \varnothing \ \Rightarrow\ x \in P_2$,

(iii) $\mathcal{S} = \operatorname{Inv}(P_1 \setminus P_2)$.

Conley index



Theorem.

- For every S an isolated invariant set $(\operatorname{cl} S, \operatorname{es} S)$ is an index pair for S.
- If P and Q are index pairs for S, then $H(P_1, P_2)$ and $H(Q_1, Q_2)$ are isomorphic .

Definition. The Conley index of S is the homology

$$H(P_1, P_2)$$

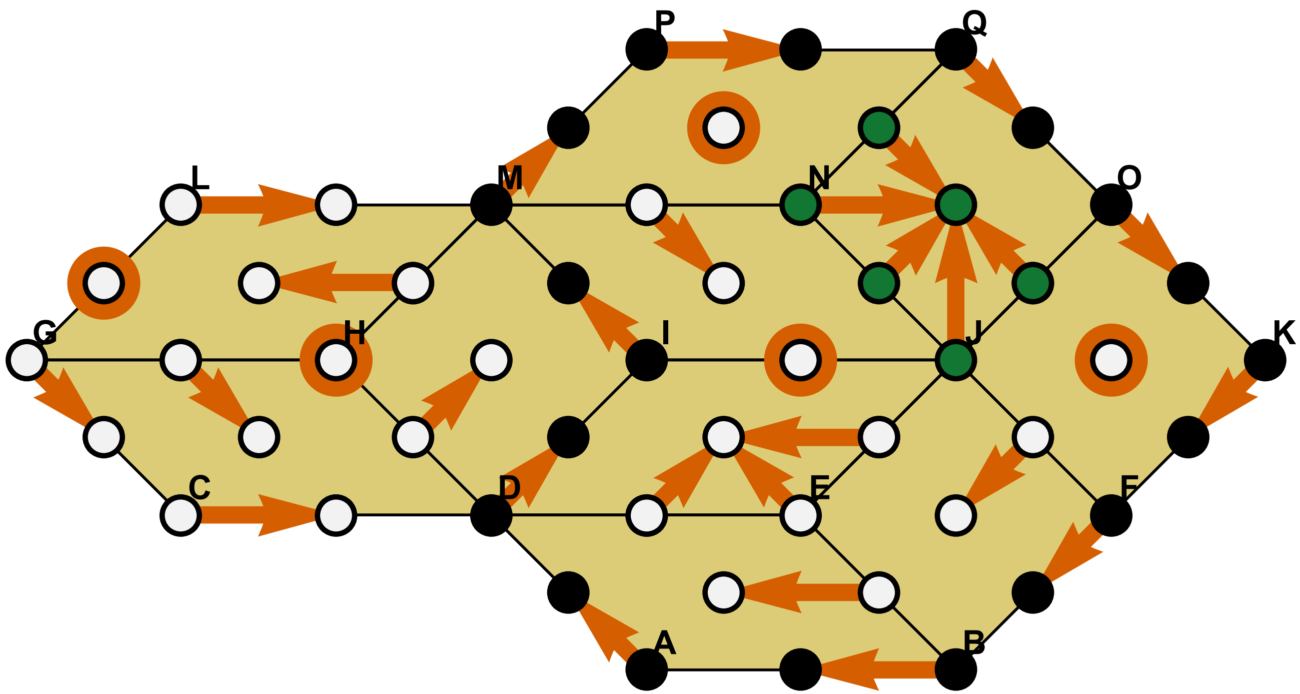
for any index pair P of S.

The Conley polynomial of S is

$$p_S(t) := \sum_{i=0}^{\infty} \beta_i(S) t^i,$$

where $\beta_i(S) := \text{rank } H_i(P_1, P_2)$.

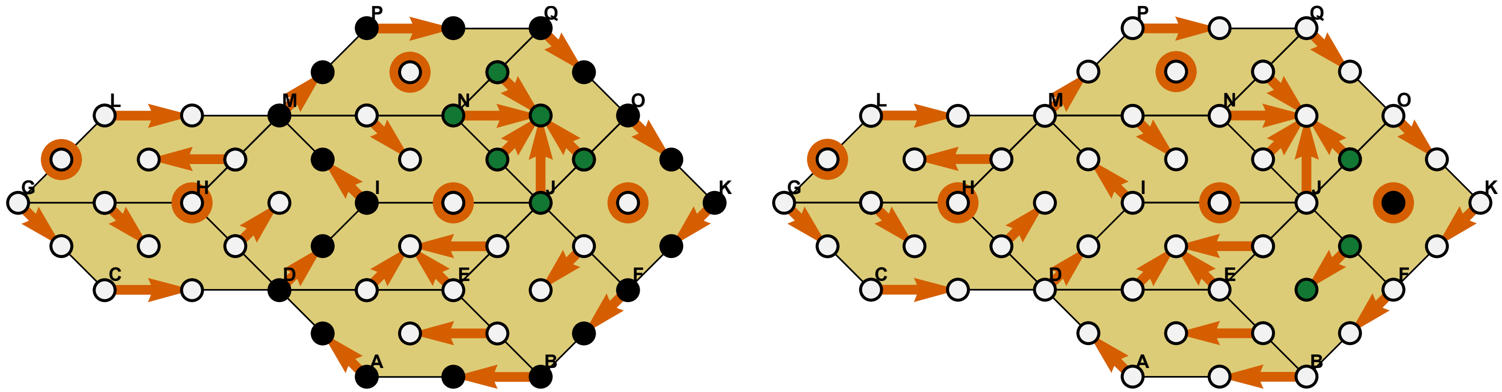
Attractors and repellers



Let $S \subset \mathcal{K}$ be isolated invariant.

- $N \subset S$ is a trapping region (backward trapping region) if for every solution $\gamma : \mathbb{Z}^+ \to S$ ($\gamma : \mathbb{Z}^- \to S$) condition $\gamma(0) \in N$ implies $\operatorname{im} \gamma \subset N$.
- A is an attractor (repeller) in S if iff there is a (backward) trapping region N such that $A = \operatorname{Inv} N$.

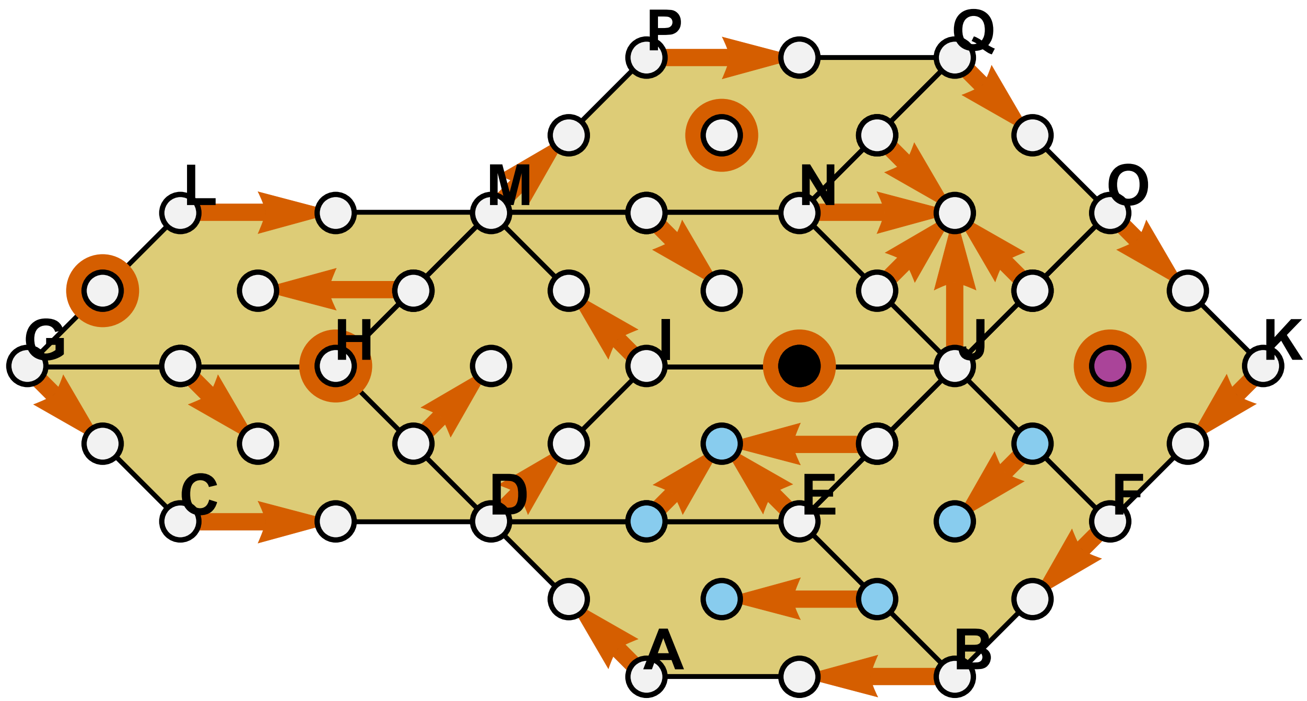
Attractors and repellers



Theorem. The following conditions are equivalent:
(i) A is an attractor,
(ii) A is isolated invariant and closed in S.

Theorem. The following conditions are equivalent:
(i) R is a repeller,
(ii) R is isolated invariant and open in S.

α and ω limit sets

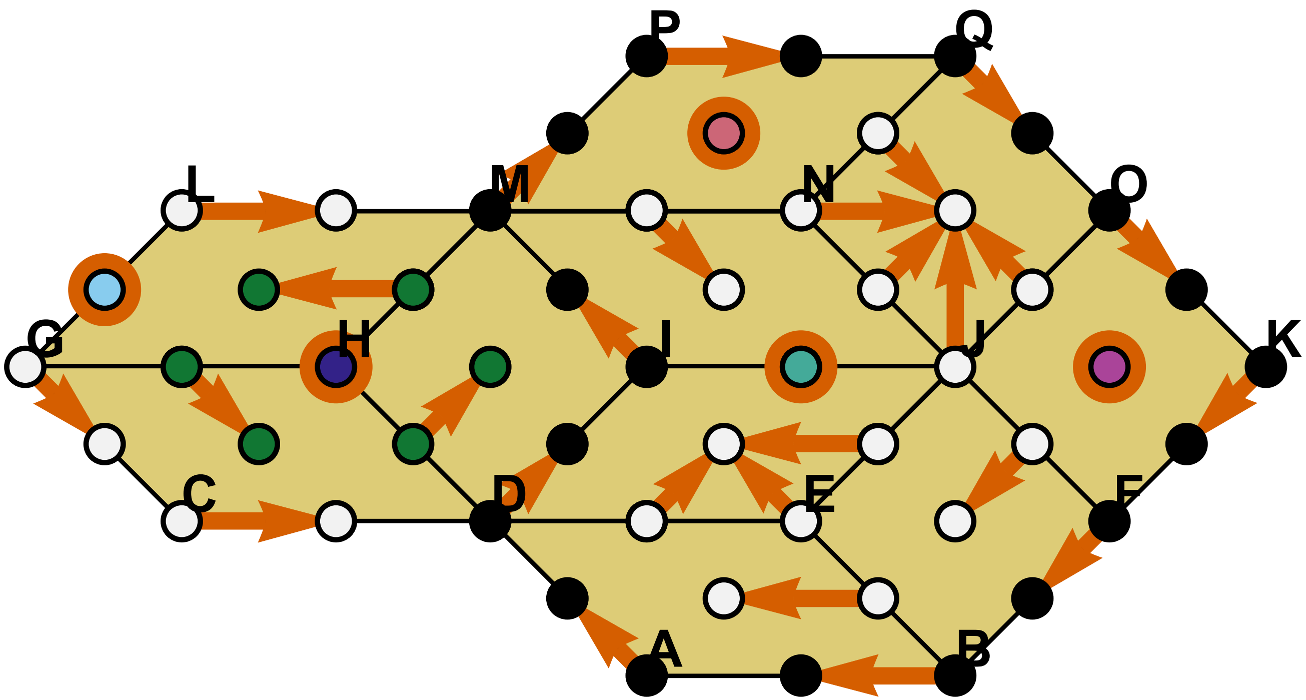


$\varrho : \mathbb{Z} \to S$ - a full solution. The α and ω limit sets of ϱ are

$$\alpha(\varrho) := \bigcap_{k \leq 0} \operatorname{Inv} \operatorname{im} \sigma^k \varrho_{|\mathbb{Z}^-},$$

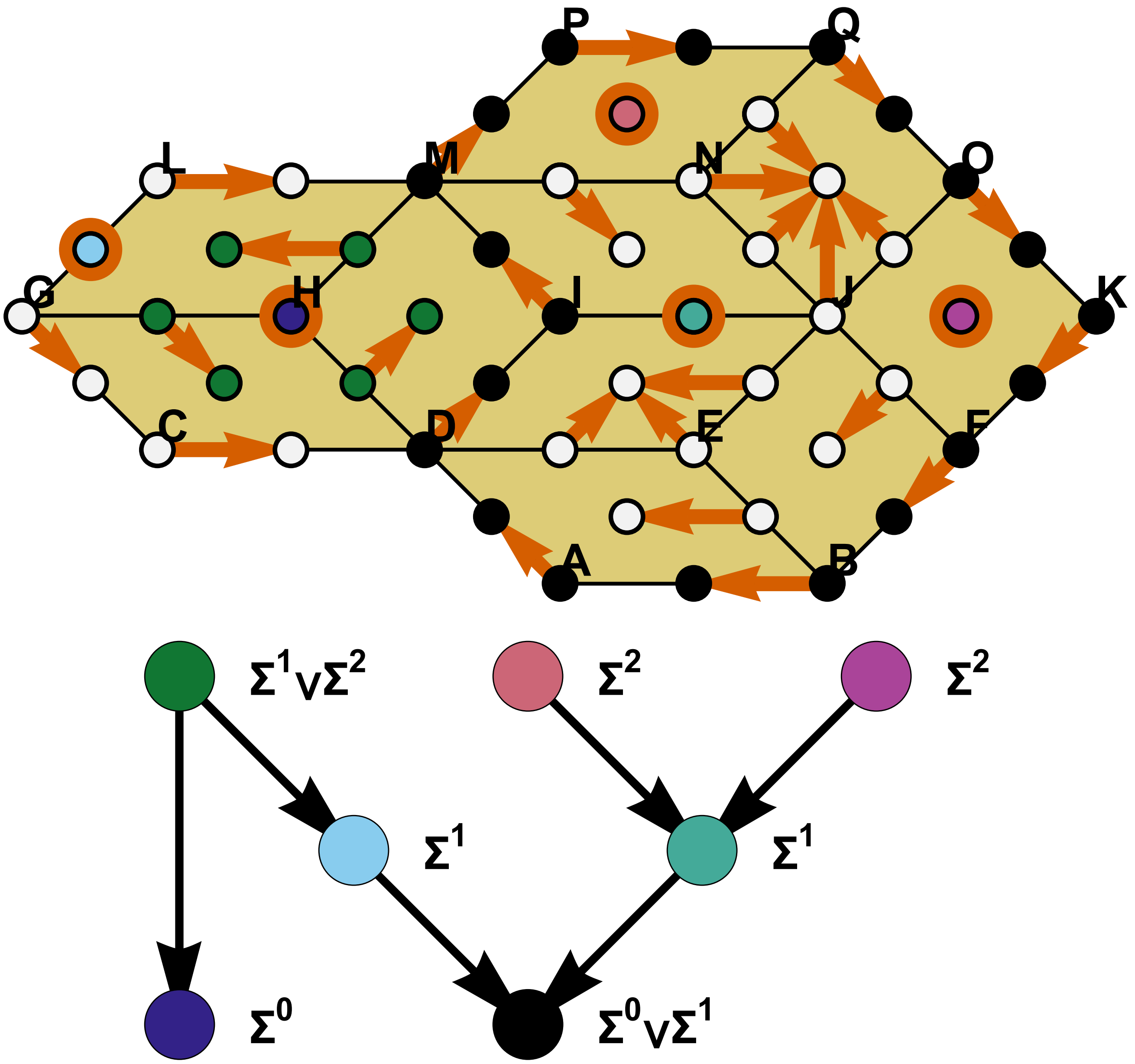
$$\omega(\varrho) := \bigcap_{k \geq 0} \operatorname{Inv} \operatorname{im} \sigma^k \varrho_{|\mathbb{Z}^+}.$$

Morse decompositions



Definition. The collection $M = \{ M_p \mid p \in P \}$ is a Morse decomposition of S if M is a family of mutually disjoint isolated invariant subsets of S and for every solution ϱ either $\operatorname{im} \varrho \subset M_p$ for some $p \in P$ or there exists $p, p' \in P$ such that $p < p'$, $\alpha(\varrho) \subset M_{p'}$, $\omega(\varrho) \subset M_p$.

Morse-Conley graph

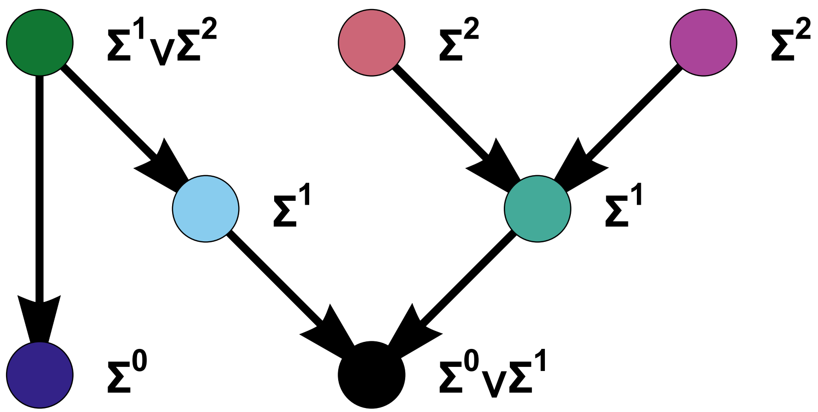


Morse inequalities

Theorem. Given a Morse decomposition $M = \{ M_\iota \mid \iota \in P \}$ of an isolated invariant set S we have

$$\sum_{\iota \in P} p_{M_\iota}(t) = p_S(t) + (1+t)q(t)$$

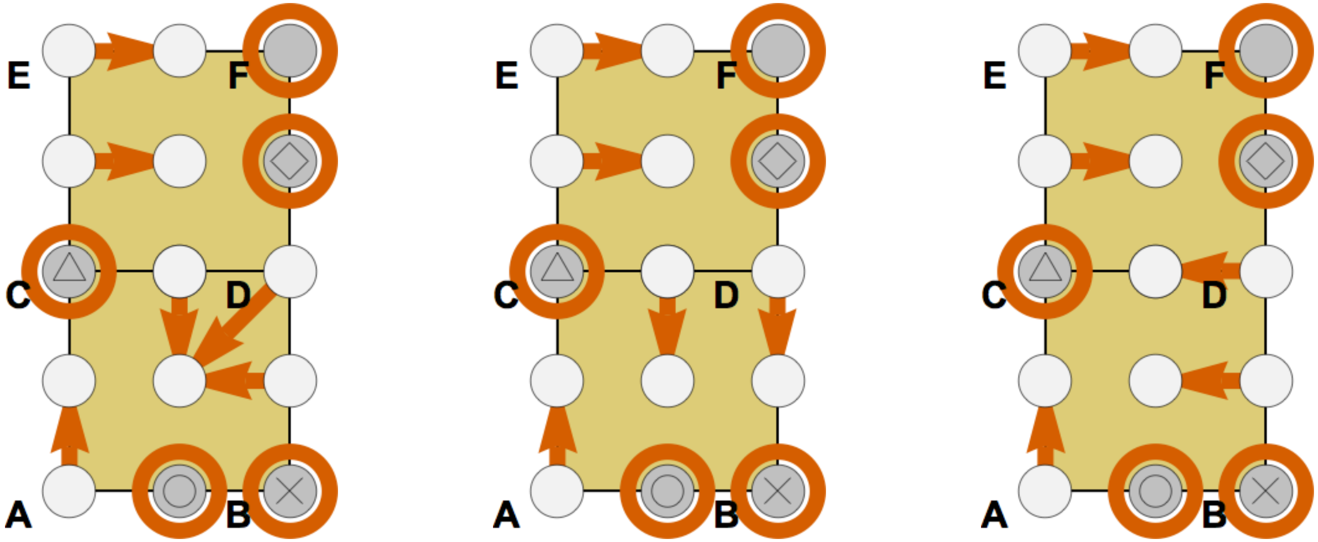
for some non-negative polynomial q.



$$\begin{aligned} p_1(t) &= 1 \\ p_2(t) &= 1+t \\ p_3(t) &= t \\ p_4(t) &= t \\ p_5(t) &= t+t^2 \\ p_6(t) &= t^2 \\ p_7(t) &= t^2 \end{aligned}$$

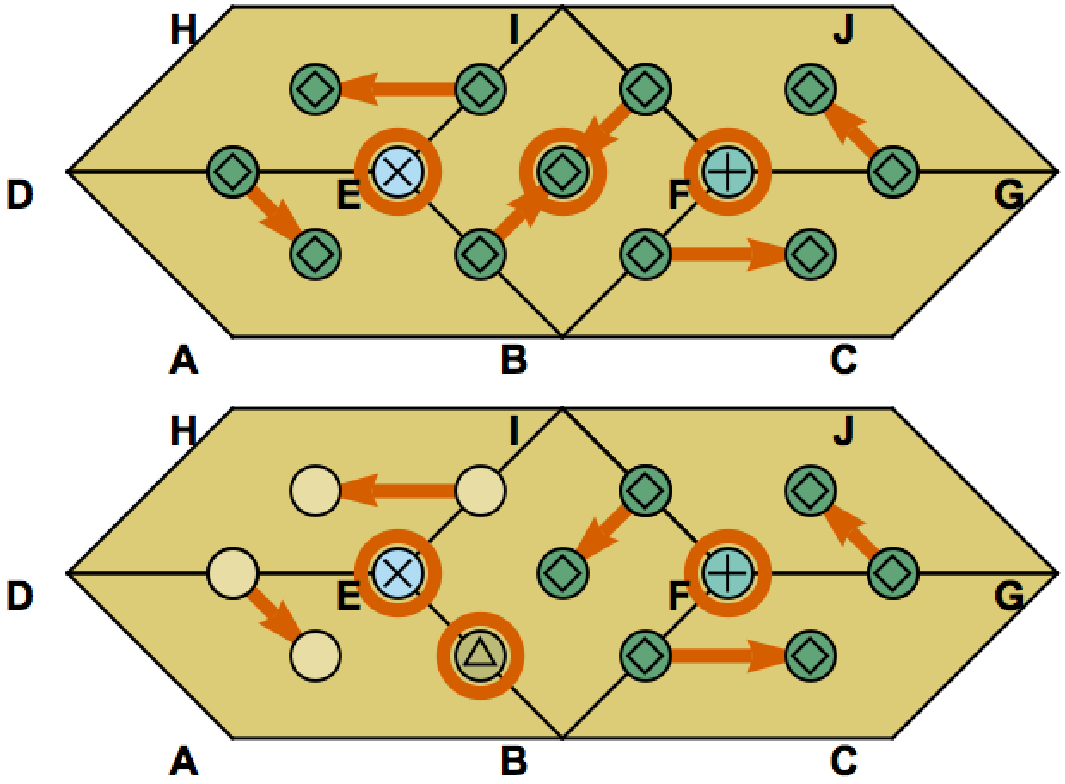
$$\sum_{\iota \in P} p_{M_\iota}(t) = 2 + 4t + 3t^2 = 1 + (1+t)(1+3t) = p_S(t) + (1+t)q(t)$$

Refinements.

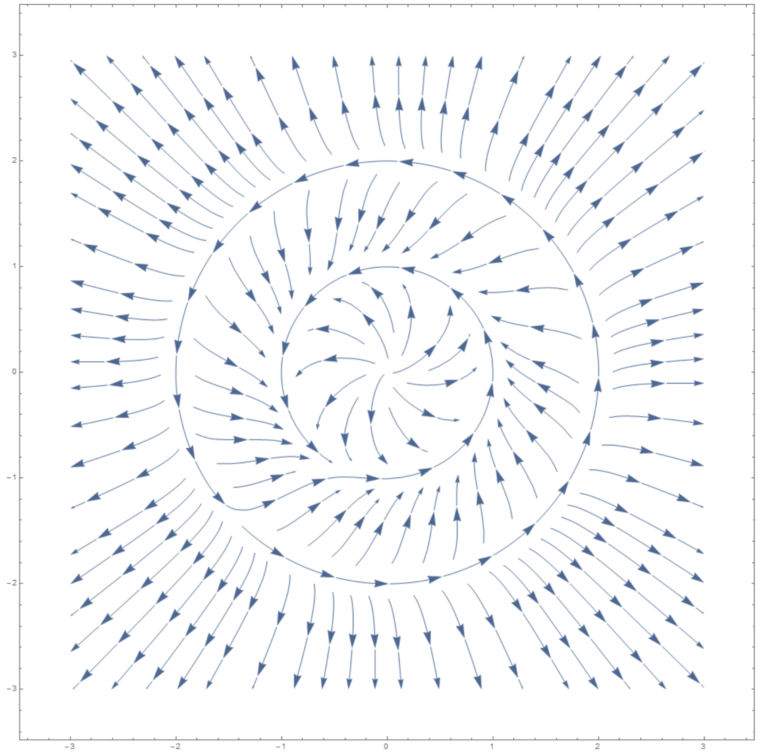


A multivector field $\mathcal{W}$ is a refinement of $\mathcal{V}$ if each multivector in $\mathcal{V}$ is $\mathcal{W}$-compatible.

Refinements.

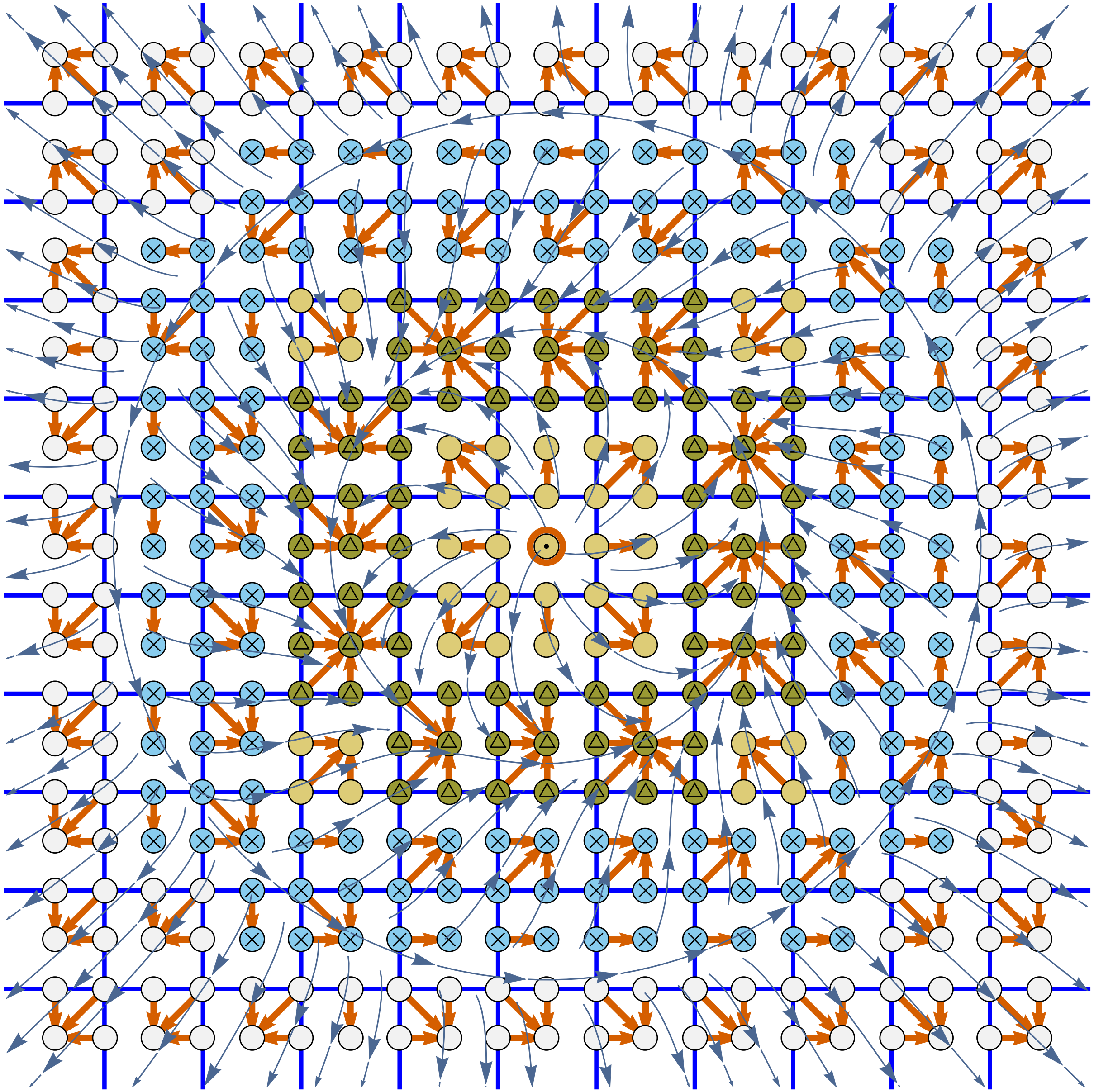


Modelling a differential equation.

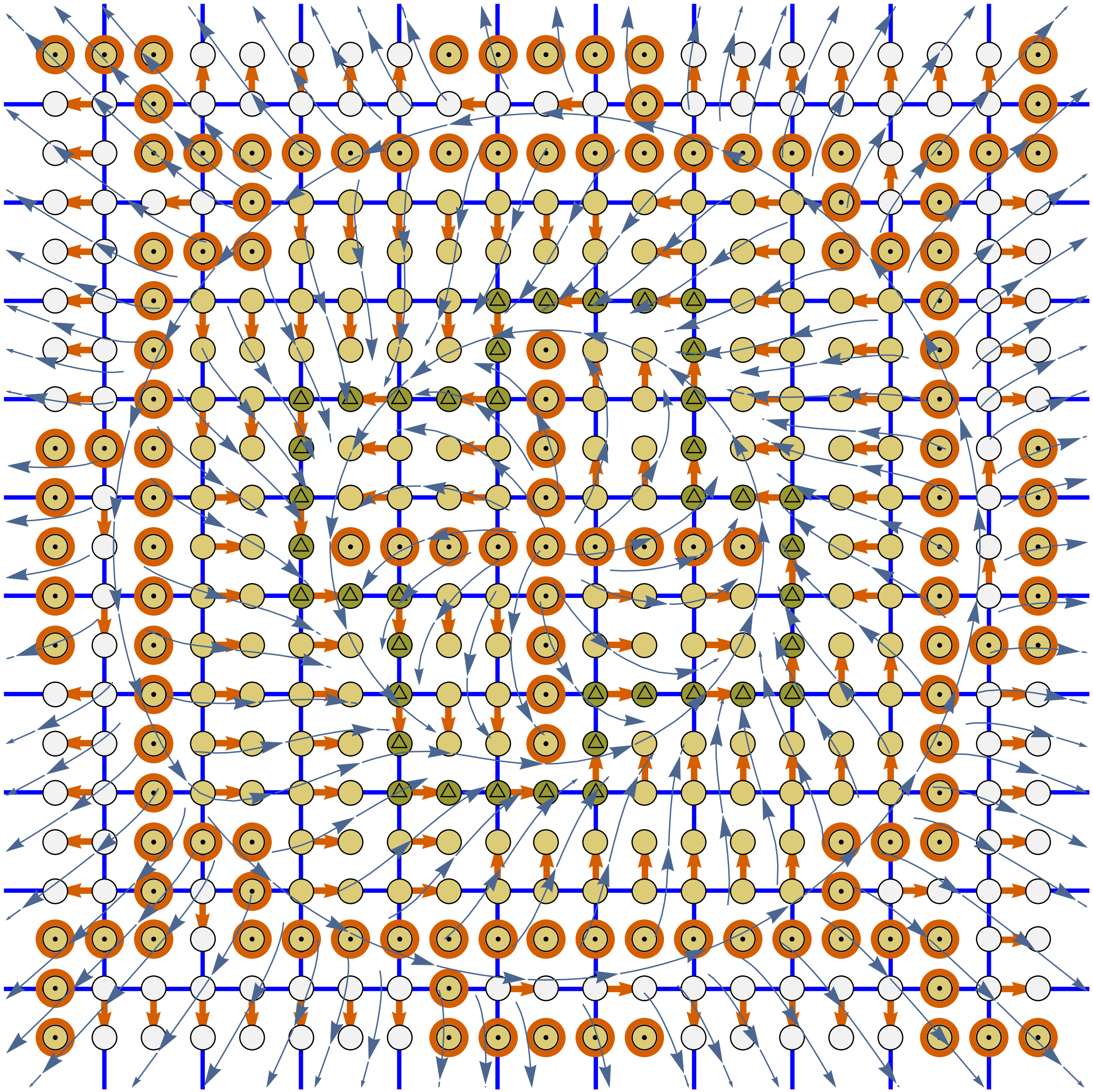


$$
\begin{aligned}
\dot{x}_1 &= -x_2 + x_1(x_1^2 + x_2^2 - 4)(x_1^2 + x_2^2 - 1) \\
\dot{x}_2 &= x_1 + x_2(x_1^2 + x_2^2 - 4)(x_1^2 + x_2^2 - 1)
\end{aligned}
$$

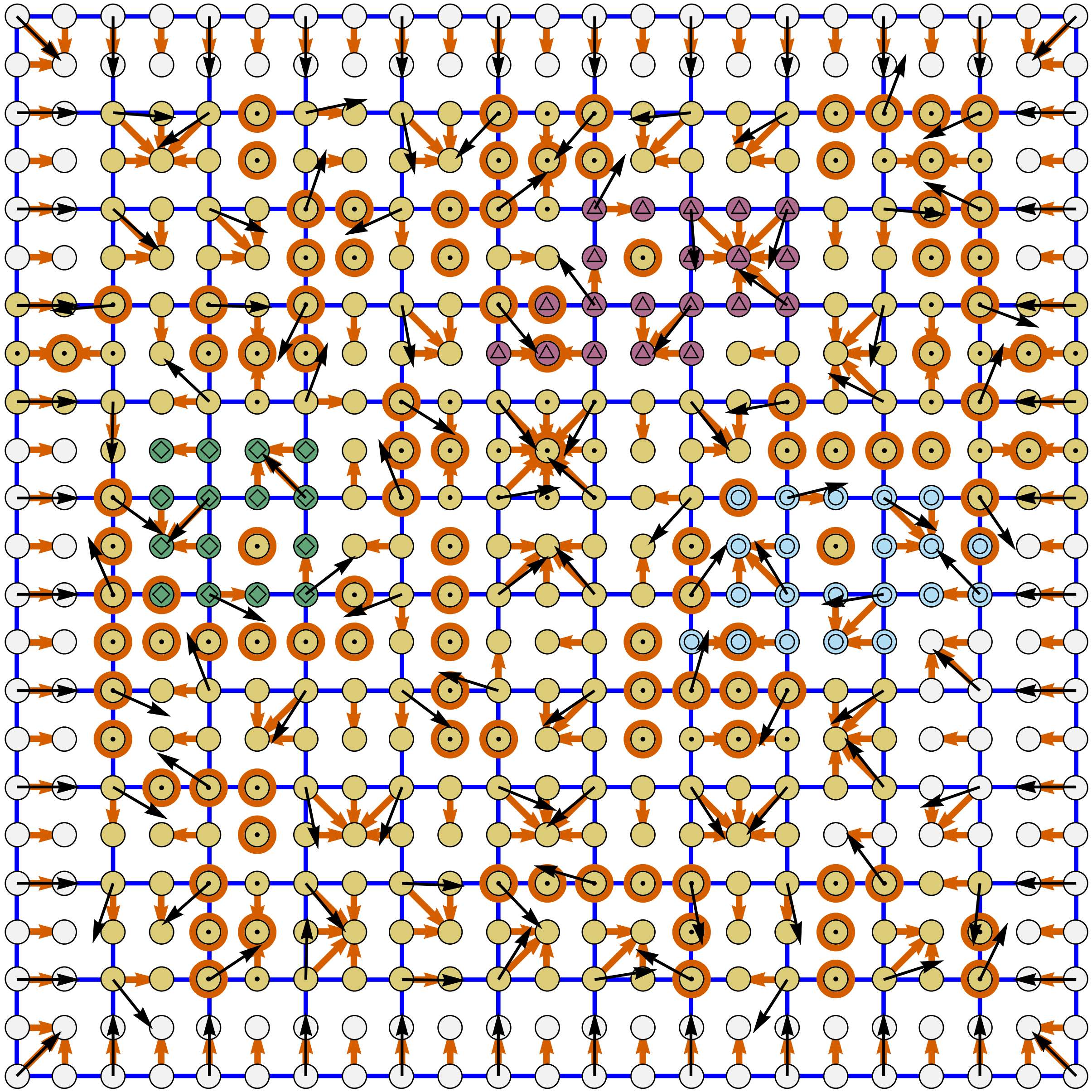
Modelling a differential equation - cmvf.



Modelling a differential equation - cvf.



Modelling a randomly selected vectors.



Relation to classical theory

$\mathcal{X}$ - the collection of cells of a CW complex $X = \bigcup \mathcal{X}$.

Conjecture. Given a Morse decomposition

$$\mathcal{M} = \{ \mathcal{M}_p \mid p \in P \}$$

of $\mathcal{X}$, there exists a flow φ on X and a Morse decomposition $M = \{ M_p \mid p \in P \}$ of φ such that for any interval I in P the Conley indexes of $\mathcal{M}(I)$ and $M(I)$ coincide.

Theorem. (T. Kaczynski, MM, Th. Wanner)
Assume $\mathcal{X}$ is the collection of cells of a simplicial complex $X = \bigcup \mathcal{X}$. Given a Morse decomposition $\mathcal{M} = \{ \mathcal{M}_p \mid p \in P \}$ of $\mathcal{X}$, there exists an usc, acyclic valued, homotopic to identity, multivalued map $F : \mathcal{K} \rightrightarrows \mathcal{K}$ and a Morse decomposition $M = \{ M_p \mid p \in P \}$ of the induced multivalued dynamical system such that for any interval I in P the Conley indexes of $\mathcal{M}(I)$ and $M(I)$ coincide.

Conclusions and future work

- Forman theory generalizes to combinatorial multivector fields.
- Combinatorial multivector fields capture more dynamical features than vector fields do.
- The theory, both for vector and multivector fields, may be extended towards an analogue of Morse-Conley theory.
- It resembles in many, but not all aspects the classical theory.
- It provides a very concise description of dynamics.

Current and future work:

- applications to the analysis of sampled dynamics
- formal ties between classical and combinatorial theory
- efficient algorithms for concise approximation of classical dynamics
- continuation results
- connection matrix theory
- time-discrete dynamical systems

- R. Forman, Morse Theory for Cell Complexes, *Advances in Mathematics* (**1998**).
- R. Forman, Combinatorial vector fields and dynamical systems, *Math. Z.* (**1998**).
- T. Kaczynski, M. Mrozek, and Th. Wanner, Towards a Formal Tie Between Combinatorial and Classical Vector Field Dynamics, *IMA Preprint Series #2443* (**2014**).
- M. Mrozek, Conley-Morse-Forman theory for combinatorial multivector fields on Lefschetz complexes, *preprint arXiv:1506.00018v1 [math.DS]* (**2015**).