Eigenspace structure of fuzzy \((\text{max}, t)\)-matrices

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Introduction
Extremal algebras
Extremal algebras

Schedual algebra \((\mathbb{R}_{max}, \oplus, \otimes)\)

where \(\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}\)

\[ a \oplus b = \max(a, b) \]
\[ a \otimes b = a + b \]

"\(\oplus\)" is the usual addition in \(\mathbb{R}\) and \((-\infty) + b = -\infty\) for all \(b \in \mathbb{R}_{max}\)

Optimization algebra \((\mathbb{R}_{min}, \oplus, \otimes)\)

where \(\mathbb{R}_{min} = \mathbb{R} \cup \{\infty\}\)

\[ a \oplus b = \min(a, b) \]
\[ a \otimes b = a + b \]
Extremal algebras

max-min algebra \((\mathcal{B}, \oplus, \otimes)\)
Linearly ordered set \(\mathcal{B}\)
Operations: \(\oplus = \text{maximum}, \otimes = \text{minimum}\)

min-max algebra \((\mathcal{B}, \oplus, \otimes)\)
Linearly ordered set \(\mathcal{B}\)
Operations: \(\oplus = \text{minimum}, \otimes = \text{maximum}\)

\(\mathcal{B}(n)\) vectors of dimension \(n\)
\(\mathcal{B}(m, n)\) matrices of dimension \(m \times n\)
Operations \(\oplus, \otimes\) extended to vectors and matrices in a formal way
Eigenvectors and permutations

Given permutations $\varphi, \psi \in P_n$

matrix $A_{\varphi,\psi}$ has rows permuted by $\varphi$, columns by $\psi$

vector $x_\varphi$ is permuted by $\varphi$

Theorem 1 [1]

Let $A \in \mathcal{B}(n,n)$, $x \in \mathcal{B}(n)$, $\varphi \in P_n$. Then $x \in \mathcal{F}(A)$ if and only if $x_\varphi \in \mathcal{F}(A_{\varphi,\varphi})$
Triangular norms and fuzzy algebras
Max-t fuzzy algebras

Max-t fuzzy algebra \((\mathcal{I}, \oplus, \otimes)\)
Linearly ordered unit interval \(\mathcal{I} = \langle 0, 1 \rangle\)
Triangular norm \(t\)

Operations: \(\oplus = \text{maximum}, \otimes = t\)

\(\mathcal{I}(n)\) vectors of dimension \(n\)
\(\mathcal{I}(m, n)\) matrices of dimension \(m \times n\)

Operations \(\oplus, \otimes\) extended to vectors and matrices in a formal way
Triangular norms

- Gödel norm \( G(x, y) = \min(x, y) \)
- Product norm \( \text{prod}(x, y) = (x \cdot y) \)
- Łukasiewicz norm \( L(x, y) = \max(x + y - 1, 0) \)
- Drastic norm \( \text{drast}(x, y) = \begin{cases} 
\min(x, y) & \text{if } \max(x, y) = 1 \\
0 & \text{if } \max(x, y) < 1 
\end{cases} \)
Eigenvectors in max-t fuzzy algebras

Eigenspace of $A \in \mathcal{I}(n, n)$
$$\mathcal{F}(A) := \{ x \in \mathcal{I}(n); A \otimes x = x \}$$

Increasing eigenspace of $A \in \mathcal{I}(n, n)$
$$\mathcal{F}^{\leq}(A) := \{ x \in \mathcal{F}(A); (\forall i, j)[i \leq j \Rightarrow x_i \leq x_j] \}$$

Strictly increasing eigenspace of $A \in \mathcal{I}(n, n)$
$$\mathcal{F}^{<}(A) := \{ x \in \mathcal{F}(A); (\forall i, j)[i < j \Rightarrow x_i < x_j] \}$$

Similar notation: $\mathcal{I}^{\leq}(n)$, $\mathcal{I}^{<}(n)$
Non-strictly increasing eigenvectors

For a given matrix $A \in I(3, 3)$

$$I^<(D_{12}, 3) = \{ x \in I(3); x_1 = x_2 < x_3 \}$$
$$I^<(D_{23}, 3) = \{ x \in I(3); x_1 < x_2 = x_3 \}$$

and

$$F^<(D_{12}, A) = \{ x \in I^<(D_{12}, 3); A \otimes x = x \}$$
$$F^<(D_{23}, A) = \{ x \in I^<(D_{23}, 3); A \otimes x = x \}$$
Monotone Eigenspace Structure
Eigenproblem in max-prod fuzzy algebra
Max-prod eigenvectors

**Max-prod fuzzy algebra** \((\mathcal{I}, \oplus, \otimes_p)\)

Linear order on unit interval \(\mathcal{I} = \langle 0, 1 \rangle\)

For \(x, y \in \mathcal{I}(n)\) we have \(x \oplus y = \max(x, y)\), \(x \otimes_p y = \text{prod}(x, y)\)

\[
(x \oplus y)_i = \max(x_i, y_i)
\]

\[
\text{prod}(x, y) = (x \cdot y)
\]

for every \(i \in \mathbb{N}\)
Max-prod eigenvectors

For every vector $x \in I(n)$ we define the quotient vector $q = q(x) \in I(n)$ by putting

$$q_i = \begin{cases} 
  x_i / x_{i+1} & \text{if } i \in N \setminus \{n\}, x_{i+1} \neq 0 \\
  1 & \text{if } i \in N \setminus \{n\}, x_{i+1} = 0 \\
  x_n & \text{if } i = n
\end{cases}$$

**Theorem 2** [5]

Let $x \in I(n)$, $n > 1$. Then $x \in I^<(n)$ if and only if the vector $q = q(x)$ fulfills the following inequalities

$$0 \leq q_1 < 1$$
$$0 < q_i < 1 \quad \text{for every} \quad i \in N \setminus \{1, n\}$$
$$0 < q_n \leq 1.$$
Max-prod eigenvectors

**Theorem 3** [5]
Let $A \in \mathcal{I}(n,n)$. If $\mathcal{F}^<(A) \neq \emptyset$ then the following conditions are satisfied

(P1) $a_{ij} < 1$ for all $i, j \in \mathbb{N}, i < j$
(P2) $a_{nn} = 1$.

**Theorem 4** [5]
Let $A \in \mathcal{I}(n,n)$, then there is a non-zero constant eigenvector $x \in \mathcal{F}^\leq(A)$ if and only if

(P3) $\max \{a_{ij}; j \in \mathbb{N}\} = 1$ for every $i \in \mathbb{N}$, then
\[ \mathcal{F}^\leq(A) = \{x = (c, c, \ldots, c); c \in \mathcal{I}\}. \]
If $A$ does not satisfy the condition (P3), then
\[ \mathcal{F}^\leq(A) = \{x = (0, 0, \ldots, 0)\}. \]
Eigenspace of a three-dimensional max-prod matrix
Eigenvectors of a three-dimensional max-prod matrix

Similarly as in previous section, we define for any $x \in \mathcal{I}(3)$ the quotient vector $q = q(x) = (q_1, q_2, q_3) \in \mathcal{I}(3)$ where

$q_1 = x_1/x_2, \quad q_2 = x_2/x_3 \quad and \quad q_3 = x_3$

We put

$q_1 = 1$ if $x_2 = 0$, and $q_2 = 1$ if $x_3 = 0$

In view of Theorem 2, $x \in \mathcal{I}^<(3)$ holds if and only if

(P4) $0 \leq q_1 < 1, \quad 0 < q_2 < 1, \quad 0 < q_3 \leq 1$
Eigenvectors of a three-dimensional max-prod matrix

\textbf{Theorem 5} \textsuperscript{[5]}

Let $A \in \mathcal{I}(3, 3)$. Then $\mathcal{F}(A) \neq \emptyset$ if and only if the following conditions are satisfied

(P5) $a_{12} < 1$, $a_{13} < 1$, $a_{23} < 1$,

(P6) $a_{22} = 1$ or $a_{13} < a_{23}$,

(P7) $a_{33} = 1$. 
Eigenspace of a three-dimensional max-prod matrix

**Theorem 6 [5]**

Let $A \in \mathcal{I}(3, 3)$ fulfills (P5)-(P7) of Theorem 5. Then $\mathcal{F}^<(A)$ consists of all vectors $(x_1, x_2, x_3) = (q_1 q_2 q_3, q_2 q_3, q_3) \in \mathcal{I}(3)$ with $q_1, q_2, q_3$ satisfying (p4) and the following conditions

(P8) if $a_{11} = 1, a_{22} = 1$, then
\[
\max(a_{12}, a_{13}) \leq q_1, \max(a_{23}, a_{13}) \leq q_2, \max(a_{12} a_{23}, a_{13}) \leq q_1 q_2
\]

(P9) if $a_{11} = 1, a_{22} < 1$, then
\[
\max(a_{12}, a_{13}/a_{23}) \leq q_1, a_{23} = q_2
\]

(P10) if $a_{11} < 1, a_{22} = 1$, then
\[
\max(a_{12}, a_{13}/q_2) = q_1, a_{23} \leq q_2
\]

(P11) if $a_{11} < 1, a_{22} < 1$, then
\[
\max(a_{12}, a_{13}/a_{23}) = q_1, a_{23} = q_2.
\]
Monotone eigenspace of a three-dimensional max-prod matrix
Non-strictly increasing eigenvectors

Remark

Let \( x \in \mathcal{I}(3) \). It is easy to see that \( x \in \mathcal{I}^< (D_{12}, 3) \) holds if and only if the vector \( q = q(x) \) fulfills the inequalities

\[
(P12) \quad q_1 = 1, \quad 0 < q_2 < 1, \quad 0 < q_3 \leq 1.
\]

Similarly, \( x \in \mathcal{I}^< (D_{23}, 3) \) is equivalent to the inequalities

\[
(P13) \quad 0 \leq q_1 < 1, \quad q_2 = 1, \quad 0 < q_3 \leq 1.
\]

Conditions (P12) and (P13) are analogous to condition (P4) in the previous section.
Non-strictly increasing eigenvectors

**Theorem 7** [5]
Let $A \in \mathcal{I}(3, 3)$. Then $\mathcal{F}^{<}(D_{12}, A) \neq \emptyset$ if and only if

1. $(P14)$ $a_{13} < 1$, $a_{23} < 1$,
2. $(P15)$ $\max(a_{11}, a_{12}) = 1$ or $a_{13} \geq a_{23}$,
3. $(P16)$ $\max(a_{21}, a_{22}) = 1$ or $a_{13} \leq a_{23}$,
4. $(P17)$ $a_{33} = 1$.

**Theorem 8** [5]
Let $A \in \mathcal{I}(3, 3)$. Then $\mathcal{F}^{<}(D_{23}, A) \neq \emptyset$ if and only if

1. $(P18)$ $a_{12} < 1$, $a_{13} < 1$,
2. $(P19)$ $\max(a_{22}, a_{23}) = 1$,
3. $(P20)$ $\max(a_{32}, a_{33}) = 1$. 
Non-strictly increasing eigenvectors

**Theorem 9** \([5]\)

Let \(A \in \mathcal{I}(3, 3)\), let the conditions (P14)-(P17) of Theorem 7 be satisfied. Then \(\mathcal{F}^<(D_{12}, A)\) consists exactly of all vectors 
\((x_1, x_2, x_3) = (q_1 q_2 q_3, q_2 q_3, q_3) \in \mathcal{I}(3, 3)\) with \(q_1, q_2, q_3\) satisfying the inequalities (P12) and the following conditions

(P21) if \(\max(a_{11}, a_{12}) = 1, \max(a_{21}, a_{22}) = 1\), then \(\max(a_{13}, a_{23}) \leq q_2\),

(P22) if \(\max(a_{11}, a_{12}) < 1, \max(a_{21}, a_{22}) = 1\), then \(a_{13} = q_2\),

(P23) if \(\max(a_{11}, a_{12}) = 1, \max(a_{21}, a_{22}) < 1\), then \(a_{23} = q_2\),

(P24) if \(\max(a_{11}, a_{12}) < 1, \max(a_{21}, a_{22}) < 1\), then \(a_{13} = a_{23} = q_2\).
Non-strictly increasing eigenvectors

**Theorem 10 [5]**

Let $A \in \mathcal{I}(3, 3)$, let the conditions (P18)-(P20) of Theorem 8 be satisfied. Then $\mathcal{F}^<(D_{23}, A)$ consists exactly of all vectors $(x_1, x_2, x_3) = (q_1 q_2 q_3, q_2 q_3, q_3) \in \mathcal{I}(3, 3)$ with $q_1, q_2, q_3$ satisfying the inequalities (P13) and the following conditions

(P25) if $a_{11} = 1$, then $\max(a_{12}, a_{13}) \leq q_1$,

(P26) if $a_{11} < 1$, then $\max(a_{12}, a_{13}) = q_1$. 
Eigenproblem in max-Ł fuzzy algebra
Max-$\mathcal{I}$ eigenvectors

Max-$\mathcal{I}$ fuzzy algebra $(\mathcal{I}, \oplus, \otimes)$

Linear order on unit interval $\mathcal{I} = \langle 0, 1 \rangle$

For $x, y \in \mathcal{I}(n)$ we have

$$(x \oplus y)_i = \max(x_i, y_i)$$

$$(x \otimes_t y)_i = \begin{cases} x_i + y_i - 1 & \text{if } \min(x_i + y_i - 1, 0) = 0 \\ 0 & \text{otherwise} \end{cases}$$

for every $i \in \mathbb{N}$. 
Max-Ł eigenvectors

**Theorem 11 [9]**

Let $A \in \mathcal{I}(n, n)$ and $x \in \mathcal{I}^<(n)$. Then $x \in \mathcal{F}^<(A)$ if and only if for every $i \in N$ the following hold

(L1) $a_{ij} \leq 1 + x_i - x_j$, for every $j \in N, j \geq i$
(L2) if $i = 1$ then $x_1 = 0$ or $a_{1j} = 1 + x_1 - x_j$ for some $j \in N$
(L3) $a_{ij} = 1 + x_i - x_j$ for some $j \in N, j \geq i > 1$

**Theorem 12 [9]**

Let $A \in \mathcal{I}(n, n)$. If $\mathcal{F}^<(A) \neq \emptyset$ then the following conditions are satisfied

(L4) $a_{ij} < 1$ for all $i, j \in N, i < j$
(L5) $a_{nn} = 1$
Theorem 13 [9]

Let $A \in \mathcal{I}(n, n)$, then there is a non-zero constant eigenvector $x \in \mathcal{F}^\equiv(A)$ if and only if

\[(L6) \quad \max\{a_{ij}; j \in N\} = 1 \quad \text{for every } i \in N, \text{ then}\]

$\mathcal{F}^\equiv(A) = \{x = (c, c, \ldots, c); \ c \in \mathcal{I}\}.$

If $A$ does not satisfy the condition $(L6)$, then

$\mathcal{F}^\equiv(A) = \{x = (0, 0, \ldots, 0)\}.$
Eigenspace of a three-dimensional max-$\ell$ matrix
Eigenvectors of a three-dimensional max-Ł matrix

**Theorem 14** [9]

Let $A \in I(3, 3)$. Then $\mathcal{F}^< (A) \neq \emptyset$ if and only if the following conditions are satisfied

(L7) $a_{12} < 1$, $a_{13} < 1$, $a_{23} < 1$

(L8) $a_{22} = 1$ or $a_{13} < a_{23}$

(L9) $a_{33} = 1$

**Theorem 15** [9]

Let $A \in I(3, 3)$ satisfies the conditions (L7), (L8) and (L9). Then $\mathcal{F}^< (A)$ is the union of two disjoint sets $\mathcal{F}^<_0 (A)$ and $\mathcal{F}^<_1 (A)$. 
Eigenvectors of a three-dimensional max-$\Lambda$ matrix

$\mathcal{F}_0^<(A)$ consists exactly of all vectors $x = (x_1, x_2, x_3) \in \mathcal{I}^<(3)$ with $x_1 = 0$ and satisfying conditions

(L10) if $a_{22} < 1$, then

$0 < x_2 \leq \min(1 - a_{12}, a_{23} - a_{13}), x_2 < 1 - a_{13}, x_3 = x_2 + 1 - a_{23}$

(L11) if $a_{22} = 1$, then

$0 < x_2 < \min(1 - a_{12}, 1 - a_{13}), x_2 < x_3 \leq \min(1 - a_{13}, x_2 + 1 - a_{23})$
Eigenspace of a three-dimensional max-$\ell$ matrix

$\mathcal{F}_1^<(A)$ consists exactly of all vectors $x = (x_1, x_2, x_3) \in I^<(3)$ with $x_1 > 0$ and satisfying conditions

(L12) if $a_{11} = 1$, $a_{22} = 1$, then

$x_1 < 1$, $x_1 < x_2 \leq x_1 + 1 - a_{12}$, $x_3 \leq \min(x_1 + 1 - a_{13}, x_2 + 1 - a_{23}, 1)$

(L13) if $a_{11} = 1$, $a_{22} < 1$, then

$x_1 < a_{23}$, $x_2 \leq \min(a_{23}, x_1 + 1 - a_{12}, x_1 + (a_{23} - a_{13}))$, $x_3 = x_2 + 1 - a_{23}$
Eigenspace of a three-dimensional max-Ł matrix

(L14) if \( a_{11} < 1, a_{22} = 1 \), then
\[
x_1 \leq a_{13}, \quad x_1 + (a_{23} - a_{13}) \leq x_2 < x_1 + 1 - a_{12}, \quad x_3 = x_1 + 1 - a_{13}
\]
or \( x_1 < a_{12}, x_2 = x_1 + 1 - a_{12}, x_3 \leq \min(x_1 + 2 - (a_{12} + a_{23}), x_1 + 1 - a_{13}, 1) \)

(L15) if \( a_{11} < 1, a_{22} < 1 \), then we have two possibilities
\[
\text{if } a_{12} - a_{13} + a_{23} \leq 1 \text{ then } \\
x_1 \leq a_{13}, \quad x_2 = x_1 + (a_{23} - a_{13}), \quad x_3 = x_1 + 1 - a_{13}
\]
\[
\text{if } a_{12} - a_{13} + a_{23} \geq 1 \text{ then } \\
x_1 \leq a_{12} + a_{23} - 1, \quad x_2 = x_1 + 1 - a_{12}, \quad x_3 = x_1 + 2 - (a_{12} + a_{23})
\]
Monotone eigenspace of a three-dimensional max-$\land$ matrix
Non-strictly increasing eigenvectors

**Theorem 16** [9]
Let \( A \in \mathcal{I}(3, 3) \). Then \( \mathcal{F}^{<}(D_{12}, A) \neq \emptyset \) iff the following hold

(L16) \( a_{13} < 1 \), \( a_{23} < 1 \)

(L17) \( \max(a_{11}, a_{12}) = 1 \) or \( a_{13} \geq a_{23} \)

(L18) \( \max(a_{21}, a_{22}) = 1 \) or \( a_{13} \leq a_{23} \)

(L19) \( a_{33} = 1 \)

**Theorem 17** [9]
Let \( A \in \mathcal{I}(3, 3) \). Then \( \mathcal{F}^{<}(D_{23}, A) \neq \emptyset \) iff the following hold

(L20) \( a_{12} < 1 \), \( a_{13} < 1 \)

(L21) \( \max(a_{22}, a_{23}) = 1 \)

(L22) \( \max(a_{32}, a_{33}) = 1 \)
Non-strictly increasing eigenvectors

**Theorem 18 [9]**

Let $A \in \mathcal{I}(3, 3)$ and let the conditions (L16)-(L19) of Theorem 16 be satisfied. Then $\mathcal{F}<(D_{12}, A)$ consists of exactly all vectors $(x_1, x_2, x_3) \in \mathcal{I}(3)$ such that $x_1 = x_2 < x_3$ and satisfying the following

(L23) if $\max(a_{11}, a_{12}) = 1$, $\max(a_{21}, a_{22}) = 1$, then $1 - \max(a_{13}, a_{23}) \geq x_3 - x_1$,

(L24) if $\max(a_{11}, a_{12}) < 1$, $\max(a_{21}, a_{22}) = 1$, then $1 - a_{13} = x_3 - x_1$,

(L25) if $\max(a_{11}, a_{12}) = 1$, $\max(a_{21}, a_{22}) < 1$, then $1 - a_{23} = x_3 - x_1$,

(L26) if $\max(a_{11}, a_{12}) < 1$, $\max(a_{21}, a_{22}) < 1$, then $1 - a_{13} = 1 - a_{23} = x_3 - x_1$. 

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Non-strictly increasing eigenvectors

**Theorem 19** [9]

Let $A \in \mathcal{I}(3,3)$ and let the conditions (L20)-(L22) of Theorem 17 be satisfied. Then $\mathcal{F}^{<}(D_{23}, A)$ consists of exactly all vectors $(x_1, x_2, x_3) \in \mathcal{I}(3)$ such that $x_1 < x_2 = x_3$ and satisfying the following

(L27) if $a_{11} = 1$, then $1 - \max(a_{12}, a_{13}) \geq x_2 - x_1$,

(L28) if $a_{11} < 1$, then $1 - \max(a_{12}, a_{13}) = x_2 - x_1$. 

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Eigenproblem in max-drast fuzzy algebra
Max-drast eigenvectors

**Max-drast fuzzy algebra** \((\mathcal{I}, \oplus, \otimes_d)\)

Linear order on unit interval \(\mathcal{I} = \langle 0, 1 \rangle\)

For \(x, y \in \mathcal{I}(n)\) we have \(x \oplus y = \max(x, y)\), \(x \otimes_d y = \text{drast}(x, y)\)

\[
(x \oplus y)_i = \max(x_i, y_i)
\]

\[
(x \otimes_d y)_i = \begin{cases} 
\min(x_i, y_i) & \text{if } \max(x_i, y_i) = 1 \\
0 & \text{if } \max(x_i, y_i) < 1 
\end{cases}
\]

for every \(i \in \mathbb{N}\)
Max-dраст eigenvectors

**Theorem 20**[6]

Let $A \in \mathcal{I}(n, n)$. Then $\mathcal{F}^<(A) \neq \emptyset$ if and only if

(D1) $a_{ij} < 1$ for all $i, j \in N, i < j$

(D2) $a_{in} > 0$ for all $i \in N \setminus \{1\}$ with $a_{ii} < 1$

(D3) $a_{kn} < a_{in}$ for all $i, k \in N, k < i$ with $a_{ii} < 1$

(D4) $a_{nn} = 1$
Max-drast eigenvectors

**Theorem 21 [6]**

Let $A \in \mathcal{I}(n, n)$ fulfills the conditions (D1) - (D4) from Theorem 20, let $x \in \mathcal{I}^<(n)$. Then $x \in \mathcal{F}^<(A)$ if and only if

(D5) $x_i = a_{in}$ for all $i \in N$ with $a_{ii} < 1$

(D6) if $x_n = 1$, then $x_i \geq a_{in}$ for all $i \in N \setminus \{n\}$ with $a_{ii} = 1$

(D7) $x_n = 1$ if $a_{11} < 1$, $0 < a_{1n}$

(D8) $x_n = 1$ if $a_{ii} < 1$ for some $i \in N \setminus \{1\}$
Max-dраст eigenvectors

**Theorem 22 [6]**

Let $A \in \mathcal{I}(n, n)$, then there is a non-zero constant eigenvector $x \in \mathcal{F}^= (A)$ if and only if

$$(D9) \quad \max \{a_{ij}; j \in N\} = 1 \text{ for every } i \in N,$$

then

$$\mathcal{F}^= (A) = \{x = (c, c, \ldots, c); c \in \mathcal{I}\}.$$

If $A$ does not satisfy the condition $(D9)$, then

$$\mathcal{F}^= (A) = \{x = (0, 0, \ldots, 0)\}.$$
Monotone eigenspace of a max-drast matrix
Non-strictly increasing eigenvectors

**Theorem 23** [6]

Let $A \in \mathcal{I}(n, n)$. Then $\mathcal{F}<(D, A) \neq \emptyset$ if and only if the following conditions are satisfied

\begin{align*}
(D10) & \quad a_{ij} < 1 \text{ for all } i, j \in N, \ D[i] < D[j] \\
(D11) & \quad 0 < \max_{k \in D[n]} a_{ik} \text{ for all } i \in N \setminus D[1] \text{ with } \max_{j \in D[i]} a_{ij} < 1 \\
(D12) & \quad \max_{k \in D[n]} a_{hk} < \max_{k \in D[n]} a_{ik} \text{ for all } i, h \in N, \ D[h] < D[i]
\end{align*}

with $\max_{j \in D[i]} a_{ij} < 1$
Non-strictly increasing eigenvectors

(D13) \( \max_{k \in D[n]} a_{hk} = \max_{k \in D[n]} a_{ik} \) for all \( i, h \in N, D[h] = D[i] \)

with \( \max_{j \in D[h]} a_{hj} < 1, \max_{j \in D[i]} a_{ij} < 1 \)

(D14) \( \max_{k \in D[n]} a_{hk} \leq \max_{k \in D[n]} a_{ik} \) for all \( i, h \in N, D[h] = D[i] \)

with \( \max_{j \in D[h]} a_{hj} = 1, \max_{j \in D[i]} a_{ij} < 1 \)

(D15) \( \max_{k \in D[n]} a_{ik} = 1 \) for all \( i \in N, D[i] = D[n] \).

**Theorem 24 [6]**
Let matrix \( A \in \mathcal{I}(n, n) \) fulfill conditions (D10)-(D15). Then \( \mathcal{F}^<(D, A) \) consists exactly of all vectors \( x \in \mathcal{I}^<(D, n) \) fulfilling
Non-strictly increasing eigenvectors

\[(D16) \text{ if } \max_{j \in D[i]} a_{ij} = 1 \text{ for every } i \in N, \text{ then either } x_n < 1, \text{ or}\]
\[x_n = 1 \text{ and } \max_{h \in S, D[h] \leq D[i]} \max_{k \in D[n]} a_{hk} \leq x_i \text{ for every } i \in N\]

\[(D17) \text{ if } \max_{j \in D[i]} a_{ij} < 1 \text{ for some } i \in N, \text{ then } x_n = 1 \text{ and for every } h \in N \]
\[x_h = x_i = \max_{k \in D[n]} a_{ik}, \text{ if } \max_{j \in D[i]} a_{ij} < 1 \text{ for some } i \in D[h]\]
\[\max_{i \in S, D[i] \leq D[h]} \max_{k \in D[n]} a_{ik} \leq x_h, \text{ if } \max_{j \in D[i]} a_{ij} = 1 \text{ for every } i \in D[h]\]
Algorithms and Programs
Computing eigenspace in max-prod algebra
Algorithm max-prod

Step 1: consider a matrix $A$ in max-prod algebra and go to Step 2;

Step 2: if $A$ satisfies each of the conditions of Theorem 3 then go to Step 3;
else write(’No strictly increasing eigenvectors’) and go to Step 7;

Step 3: if $a_{11} = 1$ then write(’Strictly increasing eigenvectors: $a_{12}x_2 \leq x_1 < x_2, 0 < x_2 \leq 1’) and go to Step 4;
else write(’Strictly increasing eigenvectors: $a_{12}x_2 = x_1, 0 < x_2 \leq 1’) and go to Step 7;

Step 4: for strictly decreasing eigenvectors, consider a permutation $\varphi = (1\ 2)$ on rows and columns of $A$ and write(’$A_{\varphi\varphi’}) and go to Step 5;
Algorithm max-prod

Step 5: repeat Step 2 to Step 3 for $A_{\varphi\varphi}$ and go to Step 6;

Step 6: permute the results of Step 5 by an inverse permutation $\varphi^{-1}$ write('Strictly decreasing eigenvectors for $A$ are exactly the increasing eigenvectors of $A_{\varphi\varphi}$') and go to Step 7;

Step 7: if $A$ satisfies the condition of Theorem 4 then write('Constant eigenvectors: $x = (c, c)$ with $0 \leq c \leq 1$') and go to Step 8;
else write('Only constant eigenvector: $x = (0, 0)$') and go to Step 8;

Step 8: STOP.
Example 1

\[ B = \begin{pmatrix} 1 & 0.3 \\ 0.2 & 1 \end{pmatrix}, \quad B_{\varphi \varphi} = \begin{pmatrix} 1 & 0.2 \\ 0.3 & 1 \end{pmatrix} \]

The eigenspace \( \mathcal{F}(B) \) consists of

- strictly **increasing** eigenvectors: \( 0.3 x_2 \leq x_1 < x_2, \ 0 < x_2 \leq 1 \)
- strictly **decreasing** eigenvectors: \( 0.2 x_1 \leq x_2 < x_1, \ 0 < x_1 \leq 1 \)
- constant eigenvectors: \( x = (c, c) \) with \( 0 \leq c \leq 1 \)
Computing eigenspace in max-$\ell$ algebra
Algorithm max-Ł

**Step 1:** consider a matrix $A$ in max-Ł algebra and go to **Step 2**;

**Step 2:** if $A$ satisfies each of the conditions of Theorem 12 then go to **Step 3**;
else write(’No strictly increasing eigenvectors’) and go to **Step 7**;

**Step 3:** if $a_{11} = 1$ then write(’Strictly increasing eigenvectors: $0 \leq x_1 < 1, x_1 < x_2 \leq \min(1, 1 + x_1 - a_{12})$’) and go to **Step 4**;
else write(’strictly increasing eigenvectors: $(0 = x_1 < x_2 \leq 1 - a_{12})$ or $(0 < x_1 \leq a_{12}, x_2 = 1 + x_1 - a_{12})$’) and go to **Step 7**;

**Step 4:** for strictly decreasing eigenvectors, consider a permutation $\varphi = (12)$ on rows and columns of $A$ and write(’$A_{\varphi\varphi}$’) and go to **Step 5**;
Algorithm \text{max-Ł}

\begin{enumerate}
\item[Step 5:] repeat \textit{Step 2} and \textit{Step 3} for $A_{\varphi \varphi}$ and go to \textit{Step 6};
\item[Step 6:] permute the results of \textit{Step 6} by an inverse permutation $\varphi^{-1}$ write(‘Decreasing eigenvectors for $A$ are exactly the increasing eigenvectors of $A_{\varphi \varphi}$’) and go to \textit{Step 7};
\item[Step 7:] if $A$ satisfies the condition of Theorem 13 then write(‘Constant eigenvectors: $x = (c, c)$ with $0 \leq c \leq 1$’) and go to \textit{Step 8};
\item[Step 8:] STOP.
\end{enumerate}
Example 2

\[ B = \begin{pmatrix} 1 & 0.3 \\ 0.2 & 1 \end{pmatrix}, \quad B_{\varphi\varphi} = \begin{pmatrix} 1 & 0.2 \\ 0.3 & 1 \end{pmatrix} \]

The eigenspace \( \mathcal{F}(B) \) consists of

- strictly increasing eigenvectors: \( 0 \leq x_1 < x_2 \leq \min(1, x_1 + 0.7) \leq 1 \)
- strictly decreasing eigenvectors: \( 0 \leq x_2 < x_1 \leq \min(1, x_2 + 0.8) \leq 1 \)
- constant eigenvectors: \( x = (c, c) \) with \( 0 \leq c \leq 1 \)
Computing eigenspace in max-drast algebra
Algorithm max-drast

Step 1: consider a matrix $A$ in max-drast algebra and go to Step 2;

Step 2: if $A$ satisfies each of the conditions of Theorem 20 then go to Step 3; else write('No strictly increasing eigenvectors') and go to Step 4;

Step 3: if $a_{11} < 1$ then

if $a_{12} > 0$ then
write('Exactly one strictly increasing eigenvector: $x = (a_{12}, 1)$') and go to Step 8;
else write('Strictly increasing eigenvectors: $0 = x_1 < x_2 \leq 1$') and go to Step 8;
end;
else write('Strictly increasing eigenvectors: $a_{12} \leq x_1 < x_2 \leq 1$') and go to Step 8;
end;
Algorithm max-drast

Step 4: if $A$ satisfies the condition of Theorem 22 then write('Constant eigenvectors: $x = (c, c)$ with $0 \leq c \leq 1$') and go to Step 5; else write('Only constant eigenvector: $x = (0, 0)$') and go to Step 5;

Step 5: for strictly decreasing eigenvectors, consider a permutation $\varphi = (12)$ on rows and columns of $A$ and write('$A_{\varphi\varphi}$') and go to Step 6;

Step 6: repeat Step 2 and Step 3 for $A_{\varphi\varphi}$;

Step 7: permute the results of Step 6 by an inverse permutation $\varphi^{-1}$ write('Decreasing eigenvectors for $A$ are exactly increasing eigenvectors of $A_{\varphi\varphi}$');

Step 8: STOP.
Example 3

\[ B = \begin{pmatrix} 1 & 0.3 \\ 0.2 & 1 \end{pmatrix}, \quad B_{\varphi \varphi} = \begin{pmatrix} 1 & 0.2 \\ 0.3 & 1 \end{pmatrix}. \]

The eigenspace \( \mathcal{F}(B) \) consists of
- strictly increasing eigenvectors: \( 0 \leq x_1 < x_2 < 1 \),
- strictly increasing eigenvectors: \( 0.3 \leq x_1 < x_2 = 1 \),
- strictly decreasing eigenvectors: \( 0 \leq x_2 < x_1 < 1 \),
- strictly decreasing eigenvectors: \( 0.2 \leq x_2 < x_1 = 1 \),
- constant eigenvectors \( x = (c, c) \) with \( 0 \leq c \leq 1 \).
Conclusions
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- Investigated the eigenproblem in max-prod, max-Łukasiewicz, max-drast fuzzy algebras.
- Necessary and sufficient conditions for the existence of constant, strictly increasing and non-strictly increasing eigenvector.
- Description of the complete structure of eigenspace for three-dimensional max-prod and max-Łukasiewicz matrices.
- Description of the complete structure of eigenspace for a max-drast matrix.
- Presented algorithms for computing the eigenspace of two-dimensional max-prod, max-Łukasiewicz and max-drast matrices.
Thank you for attention!