Dubins’ Problem on Surfaces.
I. Nonnegative Curvature

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ABSTRACT. Let $M$ be a complete, connected, two-dimensional Riemannian manifold. Consider the following question: Given any $(p_1, v_1)$ and $(p_2, v_2)$ in $TM$, is it possible to connect $p_1$ to $p_2$ by a curve $\gamma$ in $M$ with arbitrary small geodesic curvature such that, for $i = 1, 2$, $\dot{\gamma}$ is equal to $v_i$ at $p_i$? In this article, we bring a positive answer to the question if $M$ verifies one of the following three conditions: (a) $M$ is compact, (b) $M$ is asymptotically flat, and (c) $M$ has bounded nonnegative curvature outside a compact subset.

1. Introduction

Let $(M, m)$ be a connected, oriented, complete Riemannian manifold and $N = UM$ its unit tangent bundle. Points of $N$ are pairs $(p, v)$, where $p \in M$ and $v \in T_p M$, $m(v, v) = 1$. Given $\varepsilon > 0$, Dubins’ problem consists of finding, for every $(p_1, v_1), (p_2, v_2) \in N$, a curve $\gamma : [0, T] \to M, T \geq 0$, parameterized by arc-length such that $\gamma(0) = p_1$, $\dot{\gamma}(0) = v_1$, $\gamma(T) = p_2$, $\dot{\gamma}(T) = v_2$, with geodesic curvature bounded by $\varepsilon$ and $T$ as small as possible (depending on $(p_1, v_1), (p_2, v_2)$). When the dimension of $M$ is equal to two, Dubins’ problem can be formulated as the time optimal control problem for the following control system,

$$ (D_\varepsilon) : \quad \dot{q} = f(q) + ug(q), \quad q \in N, \quad u \in [-\varepsilon, \varepsilon], $$

where $f$ is the geodesic spray on $N$ (i.e., $f$ is the infinitesimal generator of the geodesic flow on $M$), $g$ is the smooth vector field generating the fiberwise rotation with angular velocity equal to one and the admissible controls are measurable functions $u : J \to [-\varepsilon, \varepsilon]$, where $J$ is an interval of $\mathbb{R}$. The trajectories of $(D_\varepsilon)$ are absolutely continuous curves $\gamma = \gamma_{u,q}(\cdot)$, with $\gamma$ the solution of $(D_\varepsilon)$ starting at $q$ and associated with the admissible control $u$. A trajectory $\gamma : [0, T] \to N$ of $(D_\varepsilon)$ is said to be time optimal if, for every trajectory $\gamma' : [0, T'] \to N$ of $(D_\varepsilon)$ such that $\gamma'(0) = \gamma(0)$ and $\gamma'(T') = \gamma(T)$, we have $T \leq T'$.

Note that, in the statement of Dubins’ problem, the existence of a curve $\gamma$ of minimal length is not guaranteed. In the language of control theory, a controllability issue should be solved in

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order to tackle the time optimal problem. Recall that \((D_\varepsilon)\) is completely controllable (CC) if, for every \(q_1, q_2 \in N\), \(q_2\) is reachable from \(q_1\), i.e., there exists a trajectory of \((D_\varepsilon)\) steering \(q_1\) to \(q_2\). For \(q \in N\), let \(A_q \subset N\) be the set of points of \(N\) reachable from \(q\).

If \(M\) is the Euclidean plane, the dynamics defined by \((D_\varepsilon)\) represents, in the robotics literature (cf. [18]), the motion of a unicycle (or a rolling penny) and the projections of the trajectories of \((D_\varepsilon)\) on the plane are planar curves parameterized by arc-length with curvature bounded by \(\varepsilon\). It is easy to see that \((D_\varepsilon)\) is completely controllable for every \(\varepsilon > 0\) and, given any pair \((p_1, v_1), (p_2, v_2) \in U\mathbb{R}^2\), there exists a time optimal trajectory of \((D_\varepsilon)\) connecting \((p_1, v_1)\) and \((p_2, v_2)\). In 1957, Dubins [8] determined the global structure of time optimal trajectories of \((D_\varepsilon)\) in the case where \(M\) is the Euclidean plane: He showed that such trajectories are concatenations of at most three pieces made of circles of radius \(\frac{1}{2}\) or straight lines. Further restrictions on the length of the arcs of an optimal concatenation have been proved by Sussmann and Tang [24].

Dubins’-like problems have been proposed by considering more general manifolds \(M\). For instance, the case where \(M\) is a two-dimensional manifold of constant Gaussian curvature was investigated in [2, 5, 10, 15] and the case where \(M = \mathbb{S}^n, n \geq 3\) was studied in [16, 17, 23]. Another line of generalization consists of considering the distributional version of \((D_\varepsilon)\). For simplicity, suppose \(M\) to be two-dimensional. The distributional dynamics can be represented by the two-input control system \((DD_\varepsilon) : \dot{q} = uf(q) + vg(q)\) with \(|u|, |v| \leq \varepsilon\) (cf. [1]). The controllability issue is trivial since it can be solved infinitesimally: Let \(h = [f, g]\), where \([\ldots]\) denotes the Lie bracket; then the distribution \((f, g)\) is strongly bracket generating, i.e., for every \(q \in N\), the triple \((f(q), g(q), h(q))\) spans \(T_q N\).

In this article, we follow the first path of generalization, i.e., we assume that \(M\) is a two-dimensional connected Riemannian manifold, oriented and complete (with possibly nonconstant curvature). Our aim is to find geometric or topological conditions on \(M\), such that, for every \(\varepsilon > 0\), \((D_\varepsilon)\) is completely controllable. We refer to that property as the unrestricted complete controllability (UCC) for Dubins’ problem (we still use the word “Dubins” although we will not consider any optimal control problem). Geometrically, the (UCC) property can be stated as follows: For every \((p_1, v_1), (p_2, v_2) \in N\), there exists a curve \(\gamma\) connecting \(p_1\) to \(p_2\) with prescribed initial and final directions \(v_1\) and \(v_2\) and with arbitrary small geodesic curvature.

To establish (CC) of \((D_\varepsilon)\), \(\varepsilon > 0\), we use a standard reduction (cf., for instance, [10]): We will show that \((D_0)\) is completely controllable if and only if \((D_\varepsilon)\) is weakly symmetric, i.e., for every \(q = (p, v) \in N\), \(q^- = (p, -v) \in A_q\). If, for instance, \(M = \mathbb{R}^2\), then a control strategy which shows that \((D_\varepsilon)\) is weakly symmetric can be given by \(u\) so that the resulting trajectory is a teardrop of size \(\frac{1}{\varepsilon}\). For \(R > 0\) a teardrop of size \(R\) is described in Figure 1.

Let \(\Phi : \tilde{M} \to M\) be a Riemannian covering and \((\tilde{D}_\varepsilon)\) be Dubins’ problem on \(\tilde{M}\); then \(\Phi|_{\tilde{N}}\) maps trajectories of \((\tilde{D}_\varepsilon)\) onto trajectories of \((D_\varepsilon)\). Therefore, if the (CC) property holds for \((\tilde{D}_\varepsilon)\), then it also holds for \((D_\varepsilon)\). Equivalently, if \((\tilde{D}_\varepsilon)\) is weakly symmetric, then \((D_\varepsilon)\) is. For instance, if \(M\) is flat, then a controllability strategy for \(M\) is obtained by projecting the one of the Euclidean plane, seen as the universal covering of \(M\). This simple idea of applying strategies which are valid on a Riemannian covering of the manifold (not necessarily a universal covering) will be repeatedly exploited in the article.

The first condition ensuring (UCC) which we obtain is purely topological: If \(M\) is compact, then, by means of a Poisson stability argument, (UCC) turns out to hold for Dubins’ problem. Thus, we are let to the case where \(M\) is noncompact.

The geometric quantity which plays a crucial role in the characterization of controllable Dubins’ problems is the Gaussian curvature of \(M\), denoted by \(K\). The curvature appears quite soon in the study of the Lie bracket configuration defined by \((D_\varepsilon)\) and, therefore, of local controllability
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FIGURE 1 The teardrop trajectory of size $R$.

issues: Indeed, for every $q \in N$,

$$[f, [f, g]](q) = -K(\pi(q))g(q),$$

where $\pi : N \to M$ denotes the bundle projection. The role of $K$ is reinforcedly suggested by the following fact: If $M$ is the Poincaré half-plane ($K \equiv -1$), then $(D_\varepsilon)$ is completely controllable if and only if $\varepsilon > 1$ (cf. [5, 15]). Roughly speaking, this happens because the negativeness of $K$ not only prevents the geodesics to have conjugate points but actually is an obstacle for the controlled turning action to overcome the spreading of the geodesics. When $|u| \leq \varepsilon \leq 1$, the steering effect of $u \, g$ is not strong enough and (CC) fails to hold.

It is therefore natural to formulate necessary conditions for (UCC) in terms of the Gaussian curvature $K$. For instance, an extension of the case $K \equiv 0$ is given by the situation in which $M$ is asymptotically flat, i.e., $K$ tends to zero at infinity. Under this hypothesis, we are able to prove the (UCC) property: The control strategy is based on the possibility of tracking a teardrop loop in a covering domain over a piece of $M$ at infinity. We will see that a suitable covering manifold can be globally described by a single appropriate geodesic chart.

Bearing in mind the previous example of noncontrollability, it is reasonable to study first the situation where the curvature is nonnegative. The negative curvature case is the subject of subsequent article [22], where we provide a complete characterization of two-dimensional nonpositively curved manifolds $M$, with either uniformly negative or bounded curvature, that satisfy property (UCC). Such characterization involves the limit set of $M$ as well as an integral decay condition of $K$ at infinity.

In the nonnegative curvature case, the one treated here, no local spreading effect due to the drift term has to be compensated. A result by Cohn-Vossen (cf. [6]) implies that, if $K \geq 0$ and $K$ is not identically equal to 0, then $M$ is diffeomorphic to a plane and, more importantly for the
controllability issue,

$$\int_M K \, dA \leq 2\pi,$$  \hspace{1cm} (1.2)

where $dA$ is the surface element in $M$. (It is actually true that, in any dimension, a complete noncompact Riemannian manifold whose sectional curvatures are all everywhere nonnegative and with one point where they all are positive is diffeomorphic to an Euclidean space, as proved by Perelman in [19].) As a consequence of (1.2), for any fixed radius $R > 0$, the total curvature on the disk centered at $p$ with radius $R$ tends to zero as $p$ tends to infinity. The same is true under the relaxed hypothesis that $K$ is nonnegative outside a compact subset of $M$. The integral decay of $K$ to zero can be interpreted as a kind of asymptotic flatness condition and it suggests that $(D_\epsilon)$ should be completely controllable for every $\epsilon > 0$. We are able to confirm that intuition under the additional assumption that $K$ is bounded over $M$, i.e., $K_\infty = \sup_M K$ is finite. The existence of a control strategy which allows to track a teardrop loop at infinity is much more delicate to prove than in the asymptotically flat case. The key step is the identification of a simply connected covering domain on which the teardrop strategy can be applied. The covering domain $D$ is not described anymore by one single chart, as in the asymptotic flat case, but by gluing rectangular strips, each of them obtained by one regular geodesic chart. There are $O(\frac{1}{\epsilon})$ such strips and each of them has width proportional to $\frac{1}{\sqrt{K_\infty}}$. Then, using these strips, one is able to mesh $D$ by geodesic quadrilaterals $P_{j,k}$ with edges of length proportional to $\frac{1}{\sqrt{K_\infty}}$. The tracking operation is now decomposed in $O(\frac{1}{\epsilon})$ steps: We design a discrete approximation of the teardrop, by fixing a sequence of $O(\frac{1}{\epsilon})$ points on the edges of the polygons $P_{j,k}$ and by associating with each of them a corresponding direction. After that, we solve the problem of connecting pairs of subsequent points of the approximating sequence by an admissible trajectory, being tangent to the associated directions. Each elementary problem of this kind can be formulated in a single coordinate strip. Intuitively, its solution is based on the topological description of small time attainable sets for nondegenerate single-input control-affine three-dimensional systems, due to Lobry [13]: The set of points which are reachable from $q_0 \in N$ in small time, is given by the region enclosed by two surfaces, obtained as union of all small-time bang-bang trajectories from $q_0$ with one switch. What we do in practice, is to estimate the coordinate expression of such surfaces and to check whether they enclose the final state of the elementary problem.

The article is organized as follows. In Section 2, we gather the notations used in the article, describe the general construction of local covering domains, establish basic properties for Dubins’ problem and, finally, study the case where $M$ is compact. Section 3 is devoted to the asymptotically flat case, where the Gaussian curvature tends to zero at infinity. In Section 4, the unrestricted complete controllability property is established when $K$ is nonnegative and bounded outside a compact domain.

2. Basic notations and first results

2.1. Differential geometric notions

Let $(M, m)$ be a complete, connected, oriented, two-dimensional Riemannian manifold. Denote by $K$ its Gaussian curvature and by $N$ the unit tangent bundle $UM$. Let $\pi : N \rightarrow M$ be the canonical bundle projection of $N$ onto $M$. We will usually denote by $p$ a point in $M$ and by $q = (p, v)$ one in $N$, where $p = \pi(q)$ and $v \in T_pM$, $m(v, v) = 1$. Given $v \in T_pM$, we write $v^\perp$ for its counterclockwise rotation in $T_pM$ of angle $\pi/2$. For every $q = (p, v) \in N$, we set $q^\perp = (p, v^\perp)$ and $q^- = (p, -v)$. 
Given $p_1, p_2 \in M$, $d(p_1, p_2)$ denotes the geodesic distance between $p_1$ and $p_2$. When no confusion is possible, we simply write $\|p\|$ (respectively, $\|q\|$) to denote the distance $d(p, p_0)$ (respectively, $d(\pi(q), p_0)$) from a fixed point $p_0 \in M$.

Let $f$ be the geodesic spray on $TM$, whose restriction to $N$ (still denoted by $f$) is a well-defined vector field on $N$. Recall that $f$ is characterized by the following property: $p(\cdot)$ is a geodesic on $M$ if and only if $(p(\cdot), \dot{p}(\cdot))$ is an integral curve of $f$. In particular, $f$ satisfies the relation

$$\pi_*(f(q)) = q.$$  
(2.1)

Denote by $g$ the smooth vector field on $N$, whose corresponding flow at time $t$ is the fiberwise rotation of angle $t$. In terms of the covariant derivative on $M$, the integral curves of $f$ and $g$ are solutions, respectively, of the following equations,

$$\begin{cases}
\dot{p} = v, \\
\nabla_v v = 0,
\end{cases} \quad \text{and} \quad \begin{cases}
\dot{p} = 0, \\
\nabla_v v = v^\perp.
\end{cases}$$

We write $e^{tf}$ (respectively, $e^{tg}$) to denote the flow of $f$ (respectively, $g$) at time $t$.

If $x_0$ belongs to a metric space $(X, d_0)$ and $\rho > 0$, then $B_\rho(x_0)$ denotes the open ball of center $x_0$ and radius $\rho$. Given a subset $Y$ of $X$, $\text{Clos}(Y)$ and $\text{Int}(Y)$ are, respectively, the closure and the interior of $Y$.

In the sequel of the article, we will systematically use as local coordinates the geodesic ones, whose definition is recalled below. Its construction has a crucial role in the present exposition, since it allows to define a wide class of local covering domains of $M$.

Given $q \in N$, consider the map

$$\phi_q : \mathbb{R}^2 \rightarrow M, \quad (x, y) \mapsto \pi(e^{yf}e^{xg}f(q)).$$

Fix $R = [x_1, x_2] \times [y_1, y_2] \subset \mathbb{R}^2$ and assume that the origin $(0, 0)$ belongs to $R$. If $\phi_q$ is a local diffeomorphism at every point of $R$, then $R$ can be endowed with the Riemannian structure lifted from $M$, in such a way that $\phi_q$ becomes a local isometry. If this happens, we denote by $R(q)$ the manifold with boundary which is obtained. The segment $[x_1, x_2] \times \{0\}$, which is the support of a geodesic in $R(q)$, is called the base curve of $R(q)$. The Gaussian curvature of $R(q)$ at a point $(x, y)$ is given by $K(\phi_q(x, y))$, and, where no confusion can arise, will be denoted by $K(x, y)$. If $R$ is a neighborhood of $(0, 0)$ and $\phi_q|_R$ is injective, then $\phi_q|_R$ is a geodesic chart on $M$. In the coordinates $(x, y)$, $m$ has the form

$$m(x, y) = B^2(x, y) dx^2 + dy^2,$$

where $B : R \rightarrow \mathbb{R}$ is the solution of the system

$$B(x, 0) \equiv 1, \quad B_y(x, 0) \equiv 0, \quad \text{and} \quad B_{yy} + KB \equiv 0,$$  
(2.2)

in which the index $y$ appearing in $B_y, B_{yy}$ stands for the partial differentiation with respect to $y$. (See, for instance, [11].)

Notice that, for every point $q \in N$ and every small enough rectangular neighborhood $R$ of $(0, 0)$, $\phi_q|_R$ is a geodesic chart on $M$. In general, if $B$ is the solution of (2.2) on $R$, with
Let $K = K \circ \phi_q$, then $R(q)$ is well defined if and only if $B$ is everywhere positive on $R$. In $R(q)$, define the real-valued function $F$ by

$$F(x, y) = \frac{B_y(x, y)}{B(x, y)}. \tag{2.3}$$

The unit bundle $UR(q)$ can be identified with

$$\{(x, y, v_x, v_y) \in \mathbb{R}^4 \mid (x, y) \in R, B^2(x, y)v_x^2 + v_y^2 = 1\}.$$

Equivalently, a set of coordinates in $UR(q)$ is given by $(x, y, \theta) \in \mathbb{R} \times S^1$, with the identification $Bv_x = \cos \theta, v_y = \sin \theta$. \tag{2.4}

**Remark 2.1.** Notice that, for all points $(x, y, \theta)$ in $UR(q)$ such that $y = 0$, the coordinate $\theta$ measures the Riemannian angle between the corresponding unit vector and the base curve of $R(q)$. On the other hand, a unit vector of coordinates $(x, y, 0)$ or $(x, y, \pi)$ is always $\phi_q^*m$-orthogonal to the segment $\{x\} \times [y_1, y_2]$.

More generally, the function which associates with $\theta$ the angle between $(x, y, 0)$ and $(x, y, \theta)$ is a Lipschitz continuous map from $S^1$ into itself, with Lipschitz constant continuously depending on $B(x, y)$.

In geodesic coordinates, $f$ and $g$ are given by

$$f(x, y, \theta) = \left(\frac{\cos \theta}{B(x, y)}, \sin \theta, F(x, y) \cos \theta\right)^T, \quad g(x, y, \theta) = (0, 0, 1)^T. \tag{2.5}$$

The pair of vector fields $(f, g)$ define a contact distribution on $N$, i.e., the triple $(f(q), g(q), [f, g](q))$ spans $T_qN$ for every $q \in N$, where $[\cdot, \cdot]$ stands for the Lie bracket. The Lie-algebraic structure of the contact distribution $\{f, g\}$ is characterized by the relations

$$\text{(i)} \quad [f, g] = h, \quad \text{(ii)} \quad [g, h] = f, \quad \text{(iii)} \quad [h, f] = Kg, \tag{2.6}$$

where $h$, defined by (i), is represented in geodesic coordinates as

$$h(x, y, \theta) = \left(\frac{\sin \theta}{B(x, y)}, -\cos \theta, F(x, y) \sin \theta\right)^T. \tag{2.7}$$

A proof of (2.6) can be obtained, for instance, by using the expressions (2.5) of $f$ and $g$ in geodesic coordinates. Equivalently, (2.6) could have been derived from the structure equations arising from the moving frame approach (see [5]).

A metric $\tilde{m}$ on $N$ can be introduced by requiring that $(f(q), g(q), h(q))$ is an $\tilde{m}$-orthonormal basis of $T_qN$, for every $q \in N$. Such $\tilde{m}$ is usually called the Sasaki metric inherited from $m$ and endows $N$ with a complete Riemannian structure. (See, for instance, [20].)

### 2.2. The control system

Recall that, for every $\varepsilon > 0$, $(D_\varepsilon)$ denotes the control system

$$(D_\varepsilon) : \quad \dot{q} = f(q) + ug(q), \quad q \in N, \quad u \in [-\varepsilon, \varepsilon].$$
By definition, an admissible control is a measurable function $u(\cdot)$, defined on some interval of $\mathbb{R}$, with values in $[−\varepsilon, \varepsilon]$. The solutions of $(D_{\varepsilon})$ corresponding to admissible controls are called admissible trajectories.

It follows from (2.1) and the definition of $g$ that, for every admissible trajectory $q : [0, T] \to N$ of $(D_{\varepsilon})$, $d(\pi(q(0)), \pi(q(T))) \leq T$. Therefore, $M$ being complete, for every admissible control $u : \mathbb{R} \to [−\varepsilon, \varepsilon]$, the non-autonomous vector field $f + u(t)g$ is complete, that is, with any initial condition $q_0 \in N$ we can associate a solution $q(\cdot)$ of $(D_{\varepsilon})$, defined on the whole real line, such that $q(0) = q_0$. In other words, the control system $(D_{\varepsilon})$ is complete.

For every $q \in N$ and $T > 0$, the attainable set from $q$ within time $T$ is the set $A_q^T$ consisting of the endpoints of all admissible trajectories starting from $q$ and defined on a time interval of length smaller than $T$. Similarly, the attainable set from $q$ is the set $A_q$ consisting of the endpoints of all admissible trajectories starting from $q$. The control system $(D_{\varepsilon})$ is called completely controllable if $A_q = N$ for every $q \in N$.

**Definition 2.2.** We say that Dubins’ problem on $M$ has the unrestricted complete controllability (UCC) property if, for every $\varepsilon > 0$, $(D_{\varepsilon})$ is completely controllable.

In local geodesic coordinates, $(D_{\varepsilon})$ can be written as follows,

\begin{align*}
\dot{x} &= \frac{\cos \theta}{B}, \\
\dot{y} &= \sin \theta, \\
\dot{\theta} &= u + F \cos \theta.
\end{align*}

(2.8)

(2.9)

(2.10)

More intrinsically, we can rewrite system (2.8)–(2.10) in the form

\begin{equation}
\begin{cases}
\dot{p} = v, \\
\nabla_v v = uv^\perp,
\end{cases}
\end{equation}

(2.11)

which accounts for a clear geometric interpretation of the unrestricted controllability property: The Dubins problem on $M$ is unrestrictedly completely controllable if and only if, for every $(p_1, v_1), (p_2, v_2) \in N$, for every $\varepsilon > 0$, there exists a curve $p : [T_1, T_2] \to M$ with geodesic curvature smaller than $\varepsilon$ such that $p(T_1) = p_1$, $p(T_2) = \bar{p}_i$, $i = 1, 2$.

The fact that $f$ and $g$ define a contact distribution on $N$ has the important consequence that, for every $0 < t < T$ and $q \in N$, $e^{tf}(q)$ belongs to $\text{Int}(A_q^T)$. This follows, for instance, from the description of small-time attainable sets for single-input nondegenerate three-dimensional control systems given by Lobry in [13].

From the viewpoint of control theory, the property that the distribution defining $(D_{\varepsilon})$ has a contact structure implies that $(D_{\varepsilon})$ is bracket generating, i.e., such that the iterated Lie brackets of $f$ and $g$ span the tangent space to $N$ at every point.

**Remark 2.3.** If $q : [0, T] \to N$ is a trajectory of $(D_{\varepsilon})$ corresponding to some admissible control $u : [0, T] \to [−\varepsilon, \varepsilon]$, then the trajectory $q(T - \cdot)^-$ obtained from $q(\cdot)$ by reflection and time-reversion is itself an admissible trajectory of $(D_{\varepsilon})$ and steers $q(T)^-$ to $q(0)^-$. Its corresponding control function is given by $-u(T - \cdot)$, which is indeed admissible. Therefore, for every $q, q' \in N$, $q'$ belongs to $A_q$ if and only if $q^-$ belongs to $A_{(q')^-}$.

**Remark 2.4.** Assume that, for every $q$ in $N$, $q^- \in A_q$, i.e., that $(D_{\varepsilon})$ is weakly symmetric. Then, due to Remark 2.3, $q' \in A_q$ if and only if $A_q = A_{q'}$. It follows that, for every $q \in N$ and
every $q' \in A_q$,  
\[ q' \in \text{Int} \left( A e^{-tf(q')} \right) = \text{Int} \left( A_q \right) = \text{Int} \left( A_q' \right), \]

where $t > 0$ and the first inclusion follows from the previously quoted Lobry’s result. Therefore, 
\{A_q\}_{q \in N}$ is an open partition of $N$. Since $N$ is connected, then $(D_\epsilon)$ is completely controllable.

Thanks again to Remark 2.3, we obtain the following equivalence: $(D_\epsilon)$ is completely controllable if and only if, for every $q \in N$, there exists $q' \in A_q$ such that $(q')^- \in A_q'$.

**Remark 2.5.** If $M$ is nonorientable, then the vector field $g$ is not well defined. Anyhow, the control problem still makes sense, since locally $g$ can be defined fixing arbitrarily an orientation, and the system is independent of this choice. Formally, Dubins’ problem can be defined as a control-affine system with multiple controls, using a partition of unity on $M$ in order to glue the local definitions of $g$ together. It is known that $M$ admits an oriented Riemannian double covering $\Phi : \tilde{M} \to M$. Since the additional hypothesis under which we get the (UCC) property (cf. Proposition 2.7 and Theorems 3.1, 4.1) are shared by any finite Riemannian covering, then the results of this article extend to nonorientable manifolds.

A sufficient condition for unrestricted complete controllability is the compactness of $M$. This fact is a consequence of a more general result on controllability on compact manifolds of bracket generating systems made of conservative vector fields due to Lobry [14]. Lemma 2.6 gives a stronger formulation of Lobry’s result, adapted to the specific control system $(D_\epsilon)$, which implies also that every attainable set is unbounded when $M$ is open. The proof is a variation on the classical one of Poincaré’s theorem on volume-preserving flows.

**Lemma 2.6.** If $q \in N$ exists such that $A_q$ is relatively compact in $N$, then $M$ is compact and $A_q = N$.

**Proof of Lemma 2.6.** Fix $q \in N$ and assume that $G = \text{Clos}(A_q)$ is compact in $N$. As already remarked, for every $t > 0$ and every $q' \in N$, $e^{tf(q')} \in \text{Int}(A_q^\epsilon)$. Fix $t = 1$. The compactness of $G$ and the continuous dependence of $A_q^\epsilon$ on $q'$ imply that there exists $\rho > 0$ such that, for every $q' \in G$,  
\[ B_\rho(e^t(q')) \subset A_q'. \quad (2.12) \]

We want to prove that $\partial A_q$ is empty. Let, by contradiction, $r \in \partial A_q$. A well-known theorem by Krener [12] states that any attainable set of a bracket generating system is contained in the closure of its interior. Therefore, $V = A_q \cap B_\rho(r)$ has nonempty interior and, in particular, its volume is strictly positive. Since $e^t$ is a volume preserving diffeomorphism of $N$ (see, for instance, [20]) and $A_q$ has finite volume (it is bounded), then $\{e^{nt}(V)\}_{n \in \mathbb{N}}$ cannot be a disjoint family, being $e^{nt}(V) \subset A_q$ for every $n \in \mathbb{N}$. Therefore, there exist $n_1 < n_2$ such that $e^{n_1t}(V) \cap e^{n_2t}(V)$ is not empty. Equivalently, there exists a point in $e^{(n_2-n_1)t}f(V)$ whose image by $e^t$ lies in $V$. Due to (2.12), it follows that $r \in \text{Int}(A_q)$ and the contradiction is reached.

**Proposition 2.7.** Let $M$ be a complete, connected, oriented, two-dimensional Riemannian manifold. Assume, in addition, that $M$ is compact. Then Dubins’ problem is unrestrictedly completely controllable.

For the rest of the article, we deal with the case $M$ noncompact.
3. Asymptotically flat manifolds

Throughout this section, we assume that \( M \) is asymptotically flat, that is,

\[
\lim_{|p| \to \infty} K(p) = 0. \tag{3.1}
\]

For every \( L > 0 \), let \( Q_L = [0, 2L] \times [-L, L] \). According to the notation introduced in Section 2.1, if the map \( \phi_{q_0}, q_0 \in N \), is a local diffeomorphism at every point of \( Q_L \), then \( Q_L(q_0) \) denotes the Riemannian manifold (with boundary) obtained endowing \( Q_L \) with the Riemannian structure lifted from \( M \).

Let us characterize values of \( L \) for which the construction of \( Q_L(q_0) \) can be carried out. Let \( B \) be the solution of (2.2) on \( Q_L \), with \( K = K \circ \phi_{q_0} \). Set

\[
\delta = \max_{Q_L} \left| K \circ \phi_{q_0} \right|. \tag{3.2}
\]

By Sturm-Liouville theory, we can compare \( B \) with the solution of (2.2) corresponding to \( K \) constantly equal to \( \delta \). We obtain that, if \( \sqrt{\delta} |y| \leq \frac{\pi}{2} \), then \( B(x, y) \geq \cos(\sqrt{\delta} y) \geq 0 \) for every \( x \in [0, 2L] \). Thus, if

\[
L < \frac{\pi}{2\sqrt{\delta}}, \tag{3.3}
\]

then \( Q_L(q_0) \) is well defined.

In particular, since \( M \) is asymptotically flat, then, for every \( L > 0 \) and every \( q_0 \) outside a compact subset of \( N \) (depending, in general, on \( L \)), \( Q_L(q_0) \) is well defined.

We stress that no global finiteness property is stated (nor needed) for the projection \( \phi_{q_0} \) from \( Q_L(q_0) \) onto its image. In general, for \( L \) fixed, the cardinality of the set of preimages \( \phi_{q_0}^{-1}(q_0) \) can fail to have a uniform bound when \( q_0 \) varies in \( N \). The situation will be different in Section 4.

Together with \( m \), also the control problem \((D_\varepsilon)\) is lifted from \( N \) to \( U Q_L(q_0) \). Let us stress the trivial, but crucial, property that every admissible trajectory of the lifted control system is projected by \( \phi_{q_0} \) to an admissible trajectory of \((D_\varepsilon)\). In the coordinates \((x, y, \theta)\) of \( U Q_L(q_0) \), the dynamics of the lifted system is described by (2.8)–(2.10). Due to Remark 2.4 and Lemma 2.6, the proof of the complete controllability of \((D_\varepsilon)\) reduces to show that \( q_0 \in A_{q_0} \) if \( \delta \) is small enough. This will be done by designing an admissible trajectory for the lifted control problem on \( U Q_L(q_0) \), steering \((0, 0, 0)\) to \((0, 0, \pi)\).

Fix \( q_0 \in N, L > 0 \) and assume that

\[
\sqrt{\delta} \leq \frac{\pi}{3L}, \tag{3.4}
\]

where \( \delta \) is defined as in (3.2). The Sturm-Liouville theory, together with the well definedness of \( Q_L(q_0) \), implies that

\[
\cos \left( \sqrt{\delta} y \right) \leq B(x, y) \leq \cosh \left( \sqrt{\delta} y \right), \tag{3.5}
\]

and

\[
|F(x, y)| = \left| \frac{B_y(x, y)}{B(x, y)} \right| \leq \sqrt{\delta} \frac{\sinh \left( \sqrt{\delta} |y| \right)}{\cos \left( \sqrt{\delta} y \right)}. \tag{3.6}
\]
for every \((x, y) \in Q_L(q_0)\). An upper bound for \(|F|\) in \(Q_L(q_0)\) is given by \(\sqrt{\delta\sinh(\sqrt{\delta}L)}\). Then, we can assume that
\[
\max_{Q_L(q_0)} |F| \leq \frac{\varepsilon}{2},
\]
by taking
\[
\sqrt{\delta} \leq \frac{\varepsilon}{4 \sinh \left(\frac{\pi}{2}\right)}.
\]

Consider now the control system \((D_{\varepsilon/2})\) on the unit bundle of the Euclidean plane. Let \(\bar{u}(\cdot)\) be the control function corresponding to the trajectory whose projection on \(\mathbb{R}^2\) is a teardrop of size \(2/\varepsilon\) which leaves the origin horizontally and arrives at the origin with the opposite direction (Figure 1, with \(R = 2/\varepsilon\)). Thus, \(\bar{u}(\cdot)\), is piecewise constant, taking alternately the values \(-\varepsilon/2\) and \(\varepsilon/2\).

Denote the coordinates of the teardrop trajectory in \(\mathbb{R}^2 \times S^1\) by \(x(\cdot), y(\cdot)\) and \(\theta(\cdot)\). It follows from straightforward computations that \((x(\cdot), y(\cdot))\) takes values in the rectangle \([-2/\varepsilon, 2/\varepsilon]\) and that the teardrop has length \(14\pi/3\varepsilon\). Fix \(L = 3\varepsilon/\varepsilon\).

The idea is to apply to the lifted system the time-variant feedback control
\[
u(t) = \bar{u}(t) - F(x, y)\cos\theta,
\]
which is admissible, as long as the corresponding trajectory stays in \(U Q_L(q_0)\), since \((3.7)\) holds.

Consider the solution \(\gamma(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))\) of \((2.8)–(2.10)\) corresponding to \(u(\cdot)\), with initial condition \(\gamma(0) = (x_0, 0, 0)\). As long as \((x(t), y(t))\) stays in \(Q_L\), we have \(y(t) = \bar{y}(t)\) and \(\theta(t) = \bar{\theta}(t)\). Therefore,
\[
|x(t) - \bar{x}(t) - x_0| \leq \int_0^t |\cos(\theta(s))| \left| \frac{1}{B(x(t), y(t))} - 1 \right| ds \\
\leq \frac{14\pi}{3\varepsilon} \max_{Q_L(q_0)} \left| \frac{1}{B} - 1 \right|.
\]
It follows from \((3.5)\) that, for every \(\alpha \in \left(0, \frac{\pi}{2}\right)\), if
\[
\sqrt{\delta}L \leq \alpha,
\]
then
\[
\max_{Q_L(q_0)} \left| \frac{1}{B} - 1 \right| \leq \frac{\cosh(\alpha) - 1}{\cos(\alpha)}.
\]

Therefore, it is possible to fix \(\alpha\), independent of \(\varepsilon\), such that, whenever \(\delta\) satisfies \((3.10)\),
\[
\frac{14\pi}{3\varepsilon} \max_{Q_L(q_0)} \left| \frac{1}{B} - 1 \right| \leq \frac{1}{4\varepsilon}.
\]

Assume that \((3.10)\) is satisfied and fix \(x_0 = \frac{1}{4\varepsilon}\). Then \(\gamma(\cdot)\) is defined for the entire time duration of \(\bar{u}(\cdot)\). At its final point, its coordinates are of the type \((x_1, 0, \pi)\). Concatenating \(\gamma\) with two trajectories corresponding to control equal to zero, we obtain an admissible trajectory.
Dubins’ Problem on Surfaces. I. Nonnegative Curvature

for Dubins’ problem lifted to $U_{Q_L(q_0)}$, steering $(0, 0, 0)$ to $(0, 0, \pi)$. We proved the following theorem.

**Theorem 3.1.** Let $M$ be a complete, connected, oriented, two-dimensional Riemannian manifold. Assume, in addition, that $M$ is asymptotically flat. Then, Dubins’ problem is unrestrictedly completely controllable.

Actually, from the nature of the above argument, a stronger result follows.

**Proposition 3.2.** There exists a universal constant $\mu > 0$ such that, if $\limsup_{\|p\| \to \infty} |K(p)| \leq \mu \varepsilon^2$, then $(D_\varepsilon)$ is completely controllable.

Proposition 3.2 can be recovered from the smallness conditions (3.4), (3.8), and (3.10) imposed on $\delta$, where $L$ should be replaced by $3/\varepsilon$ and $\alpha$ can be given explicitly.

4. Manifolds with nonnegative curvature outside a compact set

4.1. Construction of the covering domain

From now on, in addition to the general assumptions on $M$ made in Section 2, we will assume $K$ to be nonnegative outside a compact subset of $M$. Since $\int_M K \, dA$, the total curvature of $M$, is well defined (allowing extended values), and larger than $-\infty$, then it follows from a result by Huber [9], that $M$ is finitely connected. Therefore, Cohn-Vossen theorem [6] applies, i.e.,

$$\int_M K \, dA \leq 2\pi \chi(M),$$

(4.1)

where $\chi(M)$ is the Euler characteristic of $M$.

The main result of the section is the following.

**Theorem 4.1.** Let $M$ be a complete, connected, oriented, two-dimensional Riemannian manifold. Assume, in addition, that the Gaussian curvature $K$ of $M$ is bounded and $\{p \in M | K(p) < 0\}$ is relatively compact in $M$. Then, Dubins’ problem is unrestrictedly completely controllable.

In the sequel, we assume that $K$ is bounded on $M$ and we set

$$K_\infty = \sup_{p \in M} K(p).$$

Without loss of generality, $K_\infty > 0$.

For every $L, d > 0$, let $Q_{L,d} = [-L, L] \times [-d, d]$. As remarked in Section 3, if $d$ is smaller than $\frac{\pi}{2\sqrt{K_\infty}}$, then the local covering domain $Q_{L,d}(q_0)$ is well defined. Fix

$$d = \frac{\pi}{3\sqrt{K_\infty}}.$$  

(4.2)

We claim that, for every $L > 0$,

$$\lim_{\|q\| \to \infty} \int_{Q_{L,d}(q)} K \, dA = 0.$$  

(4.3)
The result follows from general properties of Riemannian surfaces. Recall that the \textit{injectivity radius at a point} \( p \) of \( M \) is defined as the least upper bound of all \( R > 0 \) such that the exponential map

\[
\exp_p : T_p M \rightarrow M,
\]

restricted to the disk \( B_R(0) \), is injective. It is denoted by \( i_p(M) \), while \( i(M) = \inf_{p \in M} i_p(M) \) is called the \textit{injectivity radius} of \( M \). Property (4.3) is proved as soon as we show that \( i(M) > 0 \), since, due to Cohn-Vossen theorem, for every \( R > 0 \),

\[
\lim_{\|p\| \to \infty} \int_{B_R(p)} K \, dA = 0. \quad (4.4)
\]

\textbf{Lemma 4.2.} Let \( M \) be a complete, connected, two-dimensional Riemannian manifold. Assume that \( K \) is bounded on \( M \) and nonnegative outside a compact subset. Then \( i(M) > 0 \).

We were not able to find in the literature an explicit statement of Lemma 4.2, although it happens to be a rather straightforward consequence of known facts. It worth mentioning, anyhow, that, in the special case when \( 0 \leq K \leq K_\infty \), a positive lower bound on the injectivity radius of \( M \) is known. Indeed, a theorem by Sharafutdinov [21, 7] states that \( i(M) \) is larger than or equal to the minimum between \( \pi/\sqrt{K_\infty} \) and the injectivity radius of the Cheeger-Gromoll soul of \( M \), which is a compact submanifold of \( M \). In the nonflat two-dimensional case, the soul is always atomic and its injectivity radius can be set to be \( +\infty \).

\textbf{Proof of Lemma 4.2.} A general result [4, Lemma 5.6], which holds for any complete Riemannian manifold, states that, for every \( p \in M \),

\[
i_p(M) = \min \{ t > 0 | t \text{ is a conjugate time for a geodesic } \gamma : [0, +\infty) \to M, \gamma(0) = p \},
\]

or \( 2t \) is the length of a geodesic loop passing through \( p \). \hfill (4.5)

Fix a compact set \( M_0 \) of \( M \) such that \( \{ p \in M | K(p) < 0 \} \subset M_0 \). We can assume that \( \text{Clos} M \setminus M_0 \) is a finite union of tubes of \( M \), that is, according to the definition of Busemann [3], subsets of \( M \) which are homeomorphic to half-cylinders and whose boundaries are simple closed geodesic polygons. Moreover, due to Cohn-Vossen theorem, we can suppose that the total curvature of each tube is smaller than \( \pi \).

Fix one of such tubes and denote it by \( T \). We have to prove that \( \inf_{p \in T} i_p(M) > 0 \). Let \( \gamma \) be a simple geodesic loop contained in the interior of \( T \). Then \( \gamma \) identifies two connected regions of \( T \), a bounded and an unbounded one. The bounded region \( B \) must contain the boundary of \( T \), otherwise Gauss-Bonnet theorem would constrain the total curvature of \( B \) to be greater than or equal to \( \pi \), contradicting the assumptions made on \( T \). Property (44.16) in [3] (originally stated in the general framework of \( G \)-surfaces) implies that there exists a curve in \( T \), freely homotopic to \( \partial T \), of minimal length. In particular, the length of \( \gamma \) is bounded from below by a positive constant.

On the other hand, since \( K \leq K_\infty \), the first conjugate time for any half geodesic contained in \( M \) is bounded from below by \( \pi/\sqrt{K_\infty} \). The lemma follows from (4.5).

Let us show how the integral smallness of \( K \) on the covering domains \( Q_{L,d}(q) \), ensured — at infinity — by (4.3), allows to estimate the evolution of geodesics in coordinates.

Let \( p(\cdot) \) be a geodesic in \( Q_{L,d}(q) \), parameterized by arc-length; then \( (p(\cdot), \dot{p}(\cdot)) \) is a curve
in \(U Q_{L,d}(q)\), whose coordinates satisfy

\[
\begin{align*}
\dot{x} &= \frac{\cos \theta}{B}, \\
\dot{y} &= \sin \theta, \quad (4.6) \\
\dot{\theta} &= F \cos \theta, \quad (4.8)
\end{align*}
\]

which is a particular case of (2.8)--(2.10).

Denote by \(\sigma\), the curve in \(U Q_{L,d}(q)\), which is solution of the system (4.6)--(4.8) with initial condition \(\sigma(z) = \varepsilon \in U Q_{L,d}(q)\), and let \([-T_{1,\varepsilon}, T_{2,\varepsilon}]\) be its maximal interval of definition.

Let

\[
N^0 = \{ q \in N | d(\pi(q), K^{-1}((-\infty, 0))) > L + d \},
\]

and remark that the complement of \(N^0\) in \(N\) is compact. The set \(N^0\) is defined in such a way that, for every \(q_0 \in N^0\), \(K\) is nonnegative on \(Q_{L,d}(q_0)\).

Fix \(\pi < \theta < \frac{\pi}{2}\) (for instance, \(\theta = \frac{\pi}{4}\)).

**Lemma 4.3.** There exists \(\delta_0 > 0\), depending only on \(L\) and \(K_{\infty}\), such that, for every \(q_0 \in N^0\), if \(Q = Q_{L,d}(q_0)\) and \(\delta > 0\) verify \(\int_Q K\ dA \leq \delta \leq \delta_0\), then, for every \(z_0 = (0, y_0, \theta_0)\) with \(y_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]\) and \(\theta_0 \in [-\pi, \pi]\), the corresponding \(\sigma_{\theta_0}(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))\) satisfies

(i) for every \(t\) in \([-T_{1,\varepsilon}, T_{2,\varepsilon}\],

\[
\begin{align*}
|x(t) - \cos(\theta_0)t| &\leq (2L + d)\delta, \\
|y(t) - y_0 - \sin(\theta_0)t| &\leq 2L\delta, \\
|\theta(t) - \theta_0| &\leq \delta.
\end{align*}
\]

Moreover, if \(|\theta_0| \leq \frac{d}{\pi L}\), then

(ii) \(x(-T_{1,\varepsilon}) = -L\) and \(x(T_{2,\varepsilon}) = L\).

**Proof of Lemma 4.3.** As it was done for (3.5) in Section 3, one obtains from (4.2) that, for every \((x, y) \in Q\),

\[
\frac{1}{2} \leq B(x, y) \leq 1. \quad (4.12)
\]

Fix \(z_0 = (0, y_0, \theta_0)\) and \(\sigma_{\theta_0}(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))\) as in the statement of the lemma. Let \([-T_1, T_2] = [-T_{1,\varepsilon}, T_{2,\varepsilon}\] and denote by \((-t_1, t_2)\) the maximal open neighborhood of zero (in \([-T_1, T_2]\)) such that, for every \(t \in (-t_1, t_2),

\[
|\theta(t)| < \frac{\pi}{3}. \quad (4.13)
\]

Using (4.12) and (4.13) in (4.6), one deduces that, for every \(t \in (-t_1, t_2),

\[
\frac{1}{2} \leq \dot{x}(t) \leq 2. \quad (4.14)
\]

Thus, \(t_1, t_2 \leq 2L\). Furthermore, we can define a map \(\tau : (x(-t_1), x(t_2)) \to (-t_1, t_2)\) by means of the relation \(x(\tau(\xi)) = \xi\). Notice that \(\tau\) is continuous, as well as the function \(\eta : (x(-t_1), x(t_2)) \to [-d, d]\) given by \(\eta(\xi) = y(\tau(\xi))\).
Let $G$ be the open region of $Q$ defined by
\[ G = \bigcup_{\xi \in (x(-t_1), x(t_2))} I(\eta(\xi)), \]
where $I(l)$ denotes the open interval with 0 and $l$ as boundary points.

Using (4.6) and (4.8), we have, for every $t \in (-t_1, t_2)$,
\[
|\theta(t) - \theta_0| = |\theta(x(t)) - \theta_0| = \left| \int_{I(x(t))} \frac{\dot{\theta}(\tau(\xi))}{\dot{x} (\tau(\xi))} d\xi \right|
\leq \int_{I(x(t))} d\xi \left( \int_{I(\eta(\xi))} -B_{y\eta}(\xi, v) d\nu \right)
= \int_{I(x(t))} \int_{I(\eta(\xi))} K(\xi, v) B(\xi, v) d\nu d\xi,
\]
where the last equality follows from (2.2). Notice that the above computations are justified by the assumption that $q_0 \in N^0$. Since the surface element of $Q$ is given by $B(\xi, v) d\nu d\xi$, we have
\[
|\theta(t) - \theta_0| \leq \int_G K \, dA \leq \delta . \tag{4.15}
\]

If $\delta_0$ is small enough, i.e., $0 < \delta_0 < \frac{\pi}{2} - \bar{\sigma}$, then $|\theta(\cdot)|$ is separated from $\frac{\pi}{2}$ on $(-t_1, t_2)$ and, therefore, $t_1 = T_1$ and $t_2 = T_2$.

Integrating (4.7) leads to
\[
|y(t) - y_0 - \sin(\theta_0)t| \leq \int_{I(t)} |\theta(s) - \theta_0| \, ds \leq |t| \delta .
\]
Then, for every $t \in [-T_1, T_2]$,
\[
|y(t)| \leq \frac{d}{2} + |t| (|\theta_0| + \delta) \leq \frac{d}{2} + 2L (|\theta_0| + \delta),
\]
which implies that the endpoints of $\sigma_{x_0}$ must be characterized by the relations $x(-T_1) = -L$ and $x(T_2) = L$, provided that $|\theta_0|, \delta \leq \frac{d}{2L}$. Point (ii) is thus proved. It remains to establish (4.9). Integrating (4.6) we get, for every $t \in [-T_1, T_2]$,
\[
|x(t) - \cos(\theta_0)t| = |x(t) - \cos(\theta_0)x(t)| = \left| \int_{I(x(t))} \left( 1 - \cos(\theta_0) \frac{B(\xi, \eta(\xi))}{\cos(\theta(\tau(\xi)))} \right) d\xi \right|
\leq \int_{I(x(t))} (1 - B(\xi, \eta(\xi))) d\xi + 2 \int_{I(x(t))} | \cos(\theta_0) - \cos(\theta(\tau(\xi))) | \, d\xi.
\]
The second integral is bounded from $2L \max_{\xi \in [x(-T_1), x(T_2)]} |\theta(\tau(\xi)) - \theta_0|$, itself being bounded by $2L \delta$. As for the first integral,
\[
\int_{I(x(t))} (1 - B(\xi, \eta(\xi))) d\xi \leq \int_{I(x(t))} d\xi \left| \int_{I(\eta(\xi))} B_{y\eta}(\xi, v) \, d\nu \right| \leq \delta d ,
\]
where the last inequality can be recovered by performing the same computations as in the estimate of $\theta - \theta_0$ done previously. Gathering all the partial estimates, we obtain, for every $t \in [-T_1, T_2]$,
\[
|x(t) - \cos(\theta_0)t| \leq (2L + d) \delta . 
\]
For every $\delta > 0$, for every $q$ outside a compact subset of $N$ (depending, in general, on $\delta$),

$$K \geq 0 \text{ on } Q_{L,d}(q) \text{ and } \int_{Q_{L,d}(q)} K \, dA \leq \delta. \quad (4.16)$$

We can choose $\delta \leq \delta_0$ (defined as in Lemma 4.3), and fix $q_0 \in N$ such that (4.16) holds for every $q$ verifying $d(\pi(q), \pi(q_0)) \leq 2L$. We next build a Riemannian two-dimensional manifold, which will be a finite covering of a region of $M$ containing $\pi(q_0)$. The covering domain will be obtained by gluing together several rectangles of the type $Q_{L,d}(q)$. The purpose is to track a teardrop of size $r/\varepsilon$, $r > 1$ to be fixed, for the lifted Dubins problem and to eventually obtain that $q_0^* \in A_{q_0}$. In this perspective, since $L$ will measure the size of the covering domain, we fix

$$L = \frac{4r}{\varepsilon}. \quad (4.17)$$

The (UCC) property will follow, as in the asymptotic flat case, from the fact that the tracking operation can be performed for any $q_0$ outside a big enough compact subset of $N$ (depending, in general, on $\varepsilon$).

From now on, we assume, sometimes implicitly, that $\delta$ is as small as needed with respect to $\varepsilon$, $1/r$ and $1/K_\infty$. Notice that $r$ is considered independent of $\varepsilon$ and $K_\infty$. We will denote by $C(\cdot)$ any constant which is a function of the quantities appearing in its argument. For instance, $C(K_\infty, \varepsilon, r)$ denotes a constant which depends on $K_\infty$, $\varepsilon$, and $r$. Fix

$$\overline{\gamma} = \frac{d}{8}. \quad (4.18)$$

For every $q$ verifying (4.16), let $\sigma(q, \cdot) = (x(q, \cdot), y(q, \cdot), \theta(q, \cdot))$ be the solution in $U Q_{L,d}(q)$ of the system (4.6)–(4.8), with initial condition

$$(x(q, 0), y(q, 0), \theta(q, 0)) = (0, -\overline{\gamma}, 0).$$

Denote by $[-T_1(q), T_2(q)]$ the maximal interval of definition of $\sigma(q, \cdot)$. If $\delta$ is small enough, then we can assume that $y(q, \cdot)$ takes only negative values. Let $W(q)$ be the region of $Q_{L,d}(q)$ defined by

$$W(q) = \{(x(q, s), t) | s \in [-T_1(q), T_2(q)], t \in [y(q, s), 0]\}, \quad (4.19)$$

whose boundary is given by $[-L, L] \times \{0\}, [-L] \times [y(q, -T_1(q)), 0], [L] \times [y(q, T_2(q)), 0]$, and by $\Gamma(q)$, the support of the curve $s \mapsto (x(q, s), y(q, s))$. (See Figure 2.)

Set

$$l = \left\lceil \frac{L}{\overline{\gamma}} \right\rceil, \quad (4.20)$$

where $\lceil \cdot \rceil$ denotes the integer part. For every $k = 1, \ldots, l$, define

$$q_k = e^{2\pi k} (e^{(k-1)\pi/\gamma_0}) \quad (4.21)$$

and, correspondingly, $Q_k = Q_{L,d}(q_k)$, $\sigma_k(\cdot) = \sigma(q_k, \cdot)$, $[-T_{1,k}, T_{2,k}] = [-T_1(q_k), T_2(q_k)]$, $W_k = W(q_k)$ and $\Gamma_k = \Gamma(q_k)$. Let, moreover, $\Gamma_0$ be the segment $[-L, L] \times \{0\}$ contained in $W_1$.

For $k \in \{1, \ldots, l-1\}$, for every $s \in (-L, L) \cap (-T_{1,k}, T_{2,k})$, we want to identify the points $(x_k(s), y_k(s)) \in \Gamma_k \subset W_k$ and $(s, 0) \in W_{k+1}$. By construction, there exist a neighborhood $U_i^k$
of \((x_k(s), y_k(s))\) in \(Q_k\), a neighborhood \(U^k_2\) of \((s, 0)\) in \(Q_{k+1}\), and an isometry \(i^k : U^k_1 \rightarrow U^k_2\) (with respect to the metric induced by \(M\)) such that \(i^k(U^k_1 \cap \partial W_k) = U^k_2 \cap \partial W_{k+1}\). Consider the Riemannian two-dimensional manifold with boundary \(D(q_0)\), obtained from the abstract union \(\bigcup_{1 \leq k \leq l} W_k\) by identification, for every \(k \in \{1, \ldots, l\}\) and every \(s \in [-L, L] \cap [-T_{1,k}, T_{2,k}]\), of \((x_k(s), y_k(s)) \in W_k\) with \((s, 0) \in W_{k+1}\). Since the gluing of two adjacent strips is rendered isometric (by the isometries \(i^k\)), then the Riemannian structures induced by \(M\) on each of the strips \(W_k\) actually define a global Riemannian structure on \(D(q_0)\).

Let us assume, from now on, that

\[
\int_{D(q_0)} K dA \leq \delta .
\]

(4.22)

4.2. The fundamental tessellation

Our purpose is to define a tessellation on \(D(q_0)\), that is, a subdivision by geodesic polygons, in a checked pattern. Any such subdivision can be seen as a discrete system of coordinates globally defined on \(D(q_0)\), assigning to any point the polygon which contains it.

A tessellation is determined by a grid of vertical and horizontal lines. The candidate vertical lines are the curves \(\Gamma_k\) defined above. Next lemma provides the estimates which allow to complete the desired construction.

**Lemma 4.4.** There exist \(\delta'_0 \in (0, \delta_0)\), depending on \(K_\infty, \epsilon, \text{ and } r\), and two constants \(C(K_\infty)\) and \(C(\epsilon, r) > 0\) such that the following holds: Let \(\overline{q} \in N\) verifies (4.16) with \(\delta \leq \delta'_0\). Take \(t_0 \in \mathbb{R}, \theta_0 \in S^1\) such that \(|t_0| \leq L - \frac{\pi}{4}\) and \(|\theta_0 + \frac{\pi}{2}| \leq \delta\). Define \(z_0 = (t_0, 0, -\theta_0)\).

Let \((x(\cdot), y(\cdot), \theta(\cdot))\) be the coordinates of \(\sigma_{z_0}(\cdot)\) and denote by \([0, T]\) its maximal interval of definition in \(U W(\overline{q})\). Then \((x(T), y(T)) \in \Gamma(\overline{q})\) and \(|T - \overline{t}| \leq C(\epsilon, r)\delta\). Moreover, for every \(s \in [0, T]\), \(|x(s)| + |y(s)| + |s| + |\theta(s) + \frac{\pi}{2}| \leq C(K_\infty)\delta\).

**Proof of Lemma 4.4.** As it was done for estimates (3.5) and (3.6) of Section 3, we have, for every \((x, y) \in W(\overline{q})\),

\[
\frac{1}{2} \leq B(x, y) \leq 1, \quad |F(x, y)| \leq 2 \sqrt{K_\infty}.
\]

Let \(I_0 = [0, T_0]\) be the largest interval (in \([0, T]\)) so that \(|\theta(s) + \frac{\pi}{2}| < \frac{\pi}{4}\) for every \(s \in I_0\). Since \(\theta(0) = -\theta_0\), then \(T_0 > 0\). On \(I_0\), the function \(v(s) = \theta(s) + \pi/2\) verifies the differential inequality

\[
|\dot{v}| \leq 2 \sqrt{K_\infty} |v| ,
\]

(4.23)
which implies that
\[
\left| \theta(s) + \frac{\pi}{2} \right| \leq \delta e^{2\sqrt{K_\infty} s}. \tag{4.24}
\]

Since \( \dot{y} = -\cos \nu \) and \( |\dot{x}| \leq 2|\sin \nu| \), we have, for every \( s \in I_0 \),
\[
|y(s) + s| \leq \int_0^s \nu(\tau)^2 \, d\tau \leq \delta^2 s e^{4\sqrt{K_\infty} s}, \tag{4.25}
\]
and
\[
|x(s)| \leq 2\delta s e^{2\sqrt{K_\infty} s}. \tag{4.26}
\]

Recall that, due to Lemma 4.3, the coordinates \((x, y)\) of a point in \( \Gamma(\bar{q}) \) satisfy \( |y - \bar{y}| \leq 2L\delta \).
It follows from (4.24)–(4.26) that, if \( \delta \) is small enough with respect to \( d \), then \( T_0 = T \) and \( \sigma(T) \in \Gamma(\bar{q}). \) Moreover, \( |T - \bar{y}| \leq C(\varepsilon, r)\delta \). The estimates on \( x(\cdot), y(\cdot), \) and \( \dot{\theta}(\cdot) \) follow.

For every \( j \in \{-l+1, \ldots, l-1\} \), let \( \Delta_j \) be the support of the geodesic in \( D(q_0) \) which starts at \((j\bar{y}, 0) \in \Gamma_0 \subset W_1\), making an oriented angle \( -\pi/2 \) with \( \Gamma_0 \). Assume that (4.22) holds, with \( \delta \leq \delta'_0 \). For every \( j \in \{-l+1, \ldots, l-1\} \), a repeated application of Lemma 4.4 to \( \Delta_j \) and the successive \( \Gamma_k \) shows that, for every \( k \in \{0, \ldots, l\} \), \( \Delta_j \) intersects \( \Gamma_k \) transversally, provided that \( \delta \) is small enough with respect to \( \bar{y} \). Indeed, at every step of the iteration, the angle determined by \( \Delta_j \) and \( \Gamma_k \) at their point of intersection differs from \( \pi/2 \) by \( \int_{D_{j,k}} K \, dA \leq \delta \), where \( D_{j,k} \) is the geodesic quadrilateral of \( D(q_0) \) bounded by \( \Delta_0, \Delta_j, \Gamma_0, \) and \( \Gamma_k \). For every \( j \in \{-l+1, \ldots, l-1\} \) and \( k \in \{0, \ldots, l\} \), denote by \( z_{j,k} \) the point of intersection of \( \Delta_j \) and \( \Gamma_k \). Due to Lemma 4.4, the length of the portion of \( \Delta_j \) connecting \( z_{j,k} \) with \( z_{j,k+1} \) differs from \( \bar{y} \) by at most \( C(\varepsilon, r)\delta \).

For the same reason, the length of the portion of \( \Gamma_k \) which joins \( z_{j,k} \) and \( z_{j+1,k} \) differs from \( \bar{y} \) by at most \( C(\varepsilon, r)\delta \).

Denote by \( P_{j,k} \) the geodesic quadrilateral with vertices \( z_{j,k}, z_{j,k+1}, z_{j+1,k+1}, \) and \( z_{j+1,k} \), for \((j, k) \in \{-l + 1, \ldots, l - 2\} \times \{0, \ldots, l - 1\} \). The edges of \( P_{j,k} \) are portions of the horizontal and vertical lines \( \Delta_j, \Delta_{j+1}, \Gamma_k, \) and \( \Gamma_{k+1} \). The family of all such \( P_{j,k} \) is called a tessellation on \( D(q_0) \).

Correspondingly, a tessellation on the Euclidean plane, which covers the rectangle \([0, l\bar{y}] \times [-l+1, l-1] \times \{0, \ldots, l-1\} \), is given by the family of squares
\[
C_{j,k} = [j\bar{y}, (j+1)\bar{y}] \times [k\bar{y}, (k+1)\bar{y}], \quad (j, k) \in \{-l + 1, \ldots, l - 2\} \times \{0, \ldots, l - 1\}.
\]

Define \( \mathcal{T} \) as the union of all \( \partial C_{j,k} \subset \mathbb{R}^2 \), that is,
\[
\mathcal{T} = \left( \bigcup_{j=-l+1}^{l-1} [0, l\bar{y}] \times \{j\bar{y}\} \right) \cup \left( \bigcup_{k=0}^{l-1} \{k\bar{y}\} \times [-l+1, l-1] \times \{0, \ldots, l-1\} \right),
\]
and, similarly, \( \mathcal{T}' \) as the union of all \( \partial P_{j,k} \subset D(q_0) \). It follows from the above considerations that there exists a homeomorphism
\[
\Psi : \mathcal{T} \longrightarrow \mathcal{T}'
\]
such that \( \Psi(z_{j,k}) = z'_{j,k} \), whose restriction to any edge of any square \( C_{j,k} \) is \((1 + C(K_\infty, \varepsilon, r)\delta)\)-Lipschitz continuous.

Consider a teardrop of size \( \xi \), starting from the point \((0, 0) \in \mathbb{R}^2 \) in the direction \((1, 0) \), contained in the Euclidean rectangle \([0, (l+1)\bar{y}] \times [-l+1, l-1] \times \{0, \ldots, l-1\} \). Such teardrop trajectory
meets $T$ in a sequence of points $p_m = (x_m, y_m), 0 \leq m \leq V$, numbered according to the order of intersection. (See Figure 3.) Notice that

$$p_0 = p_V = (0, 0). \quad (4.27)$$

The idea is now to define the correspondents of the points $p_m$ on $T'$, to equip them with a direction, and to construct a control strategy steering $q_0$ to $q_V$, passing through all these intermediate states.

4.3. Reduction to the elementary problem

In the following discussion, it simplifies the presentation to assume that $r/\varepsilon$, the size of the teardrop, is large with respect to $\overline{y}$, the tessellation step. If this is the case, then each portion of teardrop which is contained in a square of the tessellation, considered as a planar curve, presents a small variation in its angular component. To this extent, we ask $\varepsilon$ to be small with respect to $\overline{y}$, $r$ being by hypothesis greater than one. Such requirement is not restrictive, because $\overline{y}$, according to its definition (4.18), depends only on $K_{\infty}$, and our final goal is to prove the complete controllability of $(D_\varepsilon)$ for every positive $\varepsilon$.

In detail, fix $\omega > 0$ and assume that, for every $r > 1$, the teardrop of size $r/\varepsilon$ intersects the grid of step $\overline{y}$ in a sequence of points $p_m, 0 \leq m \leq V$ ($V$ depending on $r$), such that

(a) the total variation of the angle component of the portion of teardrop connecting $p_m$ with $p_{m+1}$ is smaller than $\omega$, for every $0 \leq m < V$.

Assume that $\omega > 0$ is small enough, so that, from every sequence $p_m$ as above, we can extract a subsequence, still denoted by $p_m, 0 \leq m \leq V$, verifying

(b) (4.27) holds;
(c) the Euclidean distance between $p_m$ and $p_{m+1}$ is larger than $\overline{y}/2$, for every $0 \leq m < V$;
(d) the portion of teardrop connecting $p_m$ with $p_{m+1}$ intersects $T$ at most at one more point, for every $0 \leq m < V$.

Notice that (d) implies that
(a') the total variation of the angle component of the portion of teardrop connecting \( p_m \) with \( p_{m+1} \) is smaller than \( 2\omega \), for every \( 0 \leq m < V \).

An sequence in \( \mathcal{T} \) is associated with \( \{p_m\}_{m=0}^V \) by

\[ p'_m = \Psi(p_m). \]

Consider the broken geodesic in \( \mathbb{R}^2 \) obtained by connecting, for every \( 0 \leq m < V \), \( p_m \) to \( p_{m+1} \) with a segment \( S_m \). Such curve is characterized by the lengths \( |S_m| \) of the segments \( S_m \) and by the family \( \{\alpha_m\}_{m=0}^{V-1} \), where \( \alpha_m \) is the oriented angle between \( p_{m+1} - p_m \) and \( p_{m+2} - p_{m+1} \). According to (c) and (d), we have

\[ |S_m| \in \left[ y/2, \sqrt{5}y \right], \quad \text{(4.28)} \]

while an upper bound on \( \alpha_m \) follows from the remark that \( |\alpha_m| \) is maximal when \( S_m \) and \( S_{m+1} \) are two concatenated cords of length \( \sqrt{5}y \) of a circle of radius \( r/\varepsilon \). (See Figure 4.) By easy trigonometric considerations,

\[ |\alpha_m| \leq \arcsin \left( \frac{\sqrt{5}}{r} \frac{\varepsilon y}{y} \right) \leq C(K_\infty) \frac{\varepsilon}{r}. \quad \text{(4.29)} \]

A first approximation of the desired teardrop in \( \mathcal{D}(q_0) \) is obtained by constructing a broken geodesic connecting all points \( p'_m \).

**Lemma 4.5.** There exists a family \( \{S'_m\}_{m=0}^{V-1} \) of geodesic segments in \( \mathcal{D}(q_0) \) such that \( S'_m \) connects \( p'_m \) and \( p'_{m+1} \), \( ||S'_m|| - |S_m| \leq C(K_\infty, \varepsilon, r) \delta \), and the angle \( \alpha'_m \) between \( S'_m \) and \( S'_{m+1} \) satisfies \( |\alpha'_m| \leq C(K_\infty) \frac{\varepsilon}{r} \).

**Proof of Lemma 4.5.** A principal coordinate strip in \( \mathcal{D}(q_0) \) is a portion of the domain \( \mathcal{D}(q_0) \) of the following type: Either it is bounded by two curves \( \Delta_j \) and \( \Delta_{j+4} \) (and we denote it by
\[ \Sigma_j \] or by two curves \( \Gamma_k \) and \( \Gamma_{k+1} \) (we write \( \Sigma^k \)). In \( \Sigma_j \) (respectively, \( \Sigma^k \)) we can consider the geodesic coordinates with base line \( \Delta_{j+2} \) (respectively \( \Gamma_{k+2} \)).

Let \( \theta_m \in S^1 \) be defined by

\[ e^{i\theta_m} = \frac{p_{m+1} - p_m}{|p_{m+1} - p_m|}. \]

Assume that \(-\bar{\theta} < \theta_m \leq \bar{\theta} \) or \( \pi - \bar{\theta} < \theta_m \leq \pi + \bar{\theta} \) and take \( j \) such that \( p'_m \) and \( p'_{m+1} \) belong to \( \Sigma_j \). (When \( \frac{\pi}{2} - \bar{\theta} < \theta_m \leq \frac{3\pi}{2} + \bar{\theta} \) the same procedure can be carried out on a strip of the type \( \Sigma^k \).) Let \((x'_m, y'_m), (x'_{m+1}, y'_{m+1})\) be the coordinates of \( p'_m, p'_{m+1} \) in \( \Sigma_j \). It follows from Lemma 4.3, Lemma 4.4, and the Lipschitz continuity of \( \Psi \) that

\[ \left| (x'_{m+1} - x'_m) - (x_{m+1} - x_m) \right| + \left| (y'_{m+1} - y'_m) - (y_{m+1} - y_m) \right| \leq C(K_\infty, \varepsilon, r)\delta. \]

Therefore,

\[ \frac{y'_{m+1} - y'_m}{x'_{m+1} - x'_m} - \tan \theta_m \leq C(K_\infty, \varepsilon, r)\delta, \]

and so

\[ \left| \arctan \left( \frac{y'_{m+1} - y'_m}{x'_{m+1} - x'_m} \right) - \theta_m \right| \leq C(K_\infty, \varepsilon, r)\delta. \]

Lemma 4.3 can be used to estimate the coordinate behavior of geodesics starting from \( p'_m \) or, to be more precise, of solutions of the system (4.6)--(4.8) with initial conditions of the type \((x'_m, y'_m, \theta'_m)\), with \(|\theta'_m| \) far from \( \pi/2 \). By standard continuity considerations, there exists a geodesic segment \( S'_m \) joining \( p'_m \) and \( p'_{m+1} \), whose initial direction is given by \((x'_m, y'_m, \theta'_m)\), with

\[ |\theta'_m - \theta_m| \leq C(K_\infty, \varepsilon, r)\delta \]

(4.30)

and

\[ ||S'_m| - |S_m|| \leq C(K_\infty, \varepsilon, r)\delta. \]

(4.31)

Moreover, we can assume that \( \theta'_m \) is the unique initial condition such that (4.31) holds. Indeed, geodesic segments satisfying (4.31) have no conjugate points, since \( \pi/\sqrt{K_\infty} \) is a uniform lower bound on the value of conjugate times. Therefore, \( S_m \) is defined independently of the choice of the principal coordinate strip.

In order to estimate the angle \( \alpha'_m \) between \( S'_m \) and \( S'_{m+1} \), we notice that it is not restrictive to assume that \(-\bar{\theta} < \theta_m, \theta_{m+1} \leq \bar{\theta} \) and that \( p'_m, p'_{m+1}, \) and \( p'_{m+2} \) belong to the same principal coordinate strip \( \Sigma_j \). Hence, there exists an angle \( \theta'_{m+1} \), which characterizes \( S'_{m+1} \), such that

\[ |\theta'_{m+1} - \theta_{m+1}| \leq C(K_\infty, \varepsilon, r)\delta. \]

In the coordinate system of \( \Sigma_j \), \( \alpha'_m \) is evaluated by \( \theta'_{m+1} - \bar{\alpha}'_m \), where \( \bar{\alpha}'_m \) is the angle coordinate of the tangent vector to \( S'_m \) at \( p'_{m+1} \). Notice that, according to Lemma 4.3, \(|\theta'_m - \bar{\theta}'_m| \leq \delta \).

The relation between the measure of an angle in coordinates and its Riemannian value has been discussed in Remark 2.1, where it was noticed that the conversion mapping is Lipschitz continuous, with Lipschitz constant depending continuously on the value of \( B \). Since \( B \) is, by hypothesis, uniformly separated from zero, it follows that there exists a universal constant \( C > 0 \) such that

\[
|\alpha'_m| \leq C(|\theta'_{m+1} - \bar{\theta}'_m| \leq C(\alpha_m) + C(K_\infty, \varepsilon, r)\delta \leq C(K_\infty)\varepsilon. \]

where the last equality follows from (4.29).

Let \( \beta_m \) be the oriented angle determined by the teardrop passing through \( p_m \) and the segment \( S_m \) (with the agreement that the teardrop is oriented in its running sense and the segment from \( p_m \) to \( p_{m+1} \)). Denote by \( v'_m \in U_{p_m} D(q_0) \) the unit vector which makes an angle \( \beta_m \) with \( S_m \). (When \( m = V \), let \( v'_V \) be defined by \( (p'_V, v'_V) = q_0 \).) In order to prove the existence of a trajectory in \( U D(q_0) \) which connects \( q_0 \) and \( q_0' \), admissible for the lifted Dubins problem, we will show that, for every \( m \), \( (p'_m, v'_m) \) is attainable from \( (p'_m', v'_m') \). The advantage is that all such controllability subproblems, which we will call elementary problems, are essentially equivalent and that each of them “lives” in a single coordinate chart.

4.4. Solution of the elementary problem

The \( m \)-th elementary problem is conveniently formulated in the geodesic coordinates of the rectangle whose base curve is \( S_m' \). In such a coordinate system we have \( p_m' = (0, 0) \) and \( p_{m+1}' = (|S_m'|, 0) \). We noticed in Remark 2.1 that, at the points of the base curve \( S_m' \), the coordinate angle measures the true Riemannian angle between the corresponding unitary vector and \( S_m' \). Therefore, what has to be solved is the motion planning problem defined by (2.8)–(2.10) with initial condition \( (0, 0, \beta_m) \) and final condition \( (|S_m'|, 0, \alpha'_m + \beta_{m+1}) \).

Fix a constant \( C_0 = C(K_\infty) \) such that,

\[
|\alpha'_m|, |\beta_m| \leq C_0 \frac{\varepsilon}{r} \tag{4.32}
\]

for every \( m \).

In the Euclidean plane, the portion of teardrop connecting \( p_m \) to \( p_{m+1} \) is a trajectory of

\[
\begin{align*}
\dot{x} &= \cos \theta, \\
\dot{y} &= \sin \theta, \\
\dot{\theta} &= u, \\
(x, y, \theta)(0) &= (0, 0, \beta_m),
\end{align*}
\tag{4.33}
\]

Recall that \( |S_m| \in \left[\frac{\pi}{2}, \sqrt{2}\pi\right] \). If \( \varepsilon \) is small enough with respect to \( d \), then every solution of (4.33) intersects the surface

\[
\{(x, y, \theta) \in \mathbb{R}^2 \times S^1 \mid x = |S_m|\}
\]

in a time close to \( |S_m| \). Fix \( T > 0 \) such that every admissible trajectory of (4.33) intersects \( \{x = |S_m|\} \) only once, transversally, within time \( T \). Let \( E(\cdot) \) be the map which associates with an admissible control \( u:[0, T] \to [-\varepsilon, \varepsilon] \) the coordinates \((y, \theta)\) of the trajectory corresponding to \( u(\cdot) \), evaluated at the point of intersection with \( \{x = |S_m|\} \). Notice that \( E \) is a continuous map from the space of admissible controls, endowed with the \( L^1 \) topology, into \( \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \).

The family of bang-bang control functions which are given by the concatenation of two arcs, the first one corresponding to control \( +\varepsilon \) and the second one to control \( -\varepsilon \), form a continuous curve in \( L^1((0, T)) \), joining the two constant control functions \( u \equiv \varepsilon \) and \( u \equiv -\varepsilon \). Taking into account also the two-bang concatenations where the controls are applied in the reversed order, it is clear that the family of all bang-bang control functions with at most two arcs forms a closed curve in the space of admissible controls. Choose a parameterization of such curve of the type \( \{u_s:[0, T] \to [-\varepsilon, \varepsilon]\}_{s \in \mathbb{S}} \). Then \( \gamma : s \mapsto E(u_s) \) is a continuous closed curve in \( \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \), which can be computed explicitly. In particular, straightforward computations show that there
exists $\rho = \rho(K_\infty) > 0$ such that $\gamma$ encircles the ball of center $E(0) = (\tan(\beta_m)|S_m|, \beta_m)$ and radius $\rho \varepsilon$.

Fix $r = r(K_\infty) > 1$ such that

$$\frac{C_0}{r} \leq \frac{\rho}{\delta}.$$  

Such a choice of $r$ implies that $\gamma$ encircles the segment

$$\Sigma = \left\{ (0, \theta) \mid |\theta| \leq 2C_0 \varepsilon \right\}.$$  

Moreover, the distance from the support of $\gamma$ to $\Sigma$ is bounded from below by $\rho \varepsilon / 2$, for $\delta$ small.

Consider now the nonflat elementary problem

$$\begin{cases}
\dot{x} = \cos \theta, \\
\dot{y} = \sin \theta, \\
\dot{\theta} = u + F \cos \theta, \\
(x, y, \theta)(0) = (0, 0, \beta_m).
\end{cases}$$ (4.34)

Fix any admissible control $u : [0, T] \to [-\varepsilon, \varepsilon]$. Denote by $(x(\cdot), y(\cdot), \theta(\cdot))$ (respectively, $(x'(\cdot), y'(\cdot), \theta'(\cdot))$) the solution of (4.33) [respectively, of (4.34)] corresponding to $u(\cdot)$. The same computations as in Lemma 4.3 imply that

$$\left| x(t) - x'(t) \right| + \left| y(t) - y'(t) \right| + \left| \theta(t) - \theta'(t) \right| \leq C(\varepsilon, K_\infty)\delta.$$  

In particular, we can assume that $(x'(\cdot), y'(\cdot), \theta'(\cdot))$, defined on $[0, T]$, intersects $\{x = |S'_m|\}$ once, transversally. Define $E'(u)$ as the pair of coordinates $(y'(\cdot), \theta'(\cdot)) \in \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ evaluated at the point of intersection. The map $E'$ verifies

$$\left| E(u) - E'(u) \right| \leq C(\varepsilon, K_\infty)\delta.$$  

Thus, the curve $\gamma' : s \mapsto E'(u_s)$, which is closed and continuous, encircles $\Sigma$, at least for $\delta$ small with respect to $\varepsilon$ and $1/K_\infty$. By standard degree theory considerations, the image via $E'$ of the space of admissible controls contains $\Sigma$. Hence, the elementary problem is solvable, i.e., $(p'_m + 1, v'_m + 1)$ is attainable from $(p'_m, v'_m)$ for the Dubins problem lifted on $D(q_0)$. It follows that $q_0^- \in A_{q_0}$ and Theorem 4.1 is finally proved.

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**References**


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