On stability analysis of linear discrete-time switched systems using quadratic Lyapunov functions

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Abstract—The paper deals with the stability properties of linear discrete-time switched systems with polytopic sets of dynamics. The most classical and viable way of studying the uniform asymptotic stability of such a system is to check for the existence of a quadratic Lyapunov function. It is known from the literature that letting the Lyapunov function depend on the time-varying dynamic improves the chance that a quadratic Lyapunov function exists. We prove that the dependence on the dynamic can be actually assumed to be linear, with no prejudice on the effectiveness of the method. Moreover, we show that no gain in the sensibility is obtained if we allow the Lyapunov function to depend on the time as well. We conclude by showing that Lyapunov quadratic stability is a strictly stronger notion that uniform asymptotic stability.

I. INTRODUCTION

This paper is devoted to linear discrete-time parameter-varying polytopic systems of the type

\[ x(k + 1) = A_{\xi(k)}x(k). \]

These systems are characterized by the fact that the state-space matrices \( A_{\xi} \) depend on some time-varying parameters and evolve in a convex polytope. Control theorists have devoted considerable attention to the study of this class of dynamical systems. The main interest for this quite general class is its capability of representing system dynamics for which a static state-space matrix would be too rough and unnatural. For instance, when the parameters denote physical quantities real-time measurable, these systems are called linear parameter-varying systems for which parameter dependent controllers can be designed [1]. They also correspond to uncertain time-varying systems in the case of uncertain parameters, that is, parameters that cannot be considered precisely known, [16]. Finally, switching systems can also be described by polytopic systems and in this case the parameters play the role of switching functions [14].

Stability analysis is a fundamental question widely investigated in the literature. The increase of interest in testing the stability of a polytope of matrices in general is demonstrated by the large number of papers on the subject (see, e.g., [3], [13] and references therein). Historically, a particular attention has been given to a stability concept denominated quadratic stability, which we shall call static quadratic stability to avoid confusion with what follows. This notion was inspired by [2] where Lyapunov functions quadratic in the state and independent of the parameters where used for the first time. The main advantage in using such a particular Lyapunov function is the fact that necessary and sufficient conditions for quadratic stability can be formulated in terms of algebraic Riccati equations or linear matrix inequalities (LMI) [7]. The available solvers make the solutions proposed in this context numerically tractable. Looking for more general Lyapunov functionals has received special attention during the last decades in order to derive stability conditions that are weaker than static quadratic stability. A way to achieve this goal consists in using quadratic Lyapunov functions with an explicit dependence on the parameters leading to what are called parameter dependent Lyapunov functions (PDLF) [12].

Regarding linear discrete-time parameter-varying polytopic systems, which constitute the main concern of this note, LMI stability conditions using Lyapunov functions quadratic in the state and polytopic in the parameters have been developed in [8]. These conditions are proved to be necessary and sufficient for the existence of this kind of Lyapunov functions and can also be used for design problems (control, state reconstruction, etc). Recently, stability analysis has also been carried out in the framework of the so-called joint spectral radius: a measure of the maximal asymptotic growth rate [4]. Despite its natural interpretation, the joint spectral radius is difficult to compute. A procedure for approximating the joint spectral radius with arbitrary high accuracy is provided for the case of finite sets of matrices; however, of course, higher accuracy comes at larger computational cost.

A question of interest is the following: can one expect an improvement of the results in [8] by considering other quadratic PDLF (not necessarily polytopic with respect to the parameters and not necessarily time-independent)? In the case of linear time-varying (LTV) systems \((k \mapsto \xi(k) \text{ fixed})\) it is known that stability is equivalent to the existence of a time-varying quadratic Lyapunov function (see [17]). For LTV’s, therefore, time-varying quadratic Lyapunov functions are not equivalent to time-invariant ones as a tools to check stability. Answering whether this is still the case for linear discrete-time parameter-varying polytopic systems does not seem to be immediate. To this end, we focus in this paper on three criterions of stability. The first one is called Parameter Dependent quadratic stability (PD-quadratic stability). It refers to checking stability by mean of Lyapunov function quadratic in the state and parameter dependent without any
specified structure. The second one called Parameter and Time Dependent quadratic stability (PTD-quadratically stability). It refers to Lyapunov functions that are quadratic in the state and depend explicitly on both the time and the parameters. The last one, and a priori the less costlier to check, is the so called poly-quadratically stability used in [8] and which refers to Lyapunov functions quadratic in the state and polytopic in the parameters. Our main contribution is to prove that all these criterions are equivalent. The result can be rephrased as: quadratic stability needs only to be tested on quadratic polytopic Lyapunov functions. A partial result of the same type, guaranteeing the equivalence of PD- and poly-quadratically stability under a further technical restriction on the corresponding LMI’s (namely, \( G_i = G \) for every \( i \), using the notations of [8]) can be found in [11].

A related question which naturally arises is whether uniform asymptotic stability is equivalent to quadratic stability. The counterpart of this question for continuous-time switched systems was addressed by Dayawansa and Martin in [10], where it was shown that uniformly asymptotically stable switched systems exist which do not admit quadratic Lyapunov functions (see also [6]). The result has been further strengthened in [15] where it is proved that, even if for every uniformly asymptotically stable time-continuous switched system a polynomial Lyapunov function can be found, this becomes impossible if we impose a uniform bound on its degree. We extend here the result in [10] to the case of discrete-time switched systems, showing that a system can be uniformly asymptotically stable without being poly-quadratically stable (or PD-quadratically stable, nor PTD-quadratically stable).

Recall that the polytopic approach presented in [8] was applied to stability analysis and switched output feedback control design of discrete time switched linear systems in [9]. Owing to the results presented here, one can see that the stability analysis problems investigated in [8] and [9] are equivalent.

The paper is organized as follows. In Section II we introduce the main definitions and we discuss the equivalence of asymptotic stability under convexification of the set of state-space matrices. Section III forms the core of the paper and is devoted to the proof of the equivalence between the three stability criterions introduced above. Section IV shows that uniformly asymptotically stable polytopic systems are not necessarily poly-quadratically stable. We end the paper by a conclusion.

II. NOTATIONS AND MAIN DEFINITIONS

For every \( n \in \mathbb{N} \), let \( \{e_1, \ldots, e_n\} \) be the canonical basis of \( \mathbb{R}^n \) and denote by \( \mathcal{M}^{n \times n} \) the set of all real \( n \times n \) matrices. Given \( A, B \in \mathcal{M}^{n \times n} \), we write \( A \preceq B \) to denote that \( A - B \) is negative semidefinite. A function with values in \( \mathcal{M}^{n \times n} \) is convex if it is so with respect to such (partial) order. The Euclidian norm in \( \mathbb{R}^n \) and that induced in \( \mathcal{M}^{n \times n} \) are both denoted by \( \| \cdot \| \). A function \( w : \varepsilon \mapsto w(\varepsilon) \in \mathcal{M}^{n \times n} \) defined for all \( \varepsilon > 0 \) is said to be of order \( k \) (\( k \in \mathbb{N} \)) if \( \lim \sup_{\varepsilon \to 0} \| w(\varepsilon) \varepsilon^{-k} \) is finite. In this case we write \( w(\varepsilon) = O(\varepsilon) \).

Let \( n, m \) be two integer numbers, \( \Xi \) be a subset of \( \mathbb{R}^m \) and \( \mathcal{A} : \xi \mapsto \mathcal{A}_\xi \) be a map from \( \Xi \) to \( \mathcal{M}^{n \times n} \). Consider the system

\[
x(k + 1) = \mathcal{A}_\xi(k)x(k),
\]

where \( \xi(k) \in \Xi \) for every \( k \in \mathbb{N} \). We will refer to \( k \mapsto \xi(k) \) as a switching function.

**Definition 1:** We say that (1) is uniformly asymptotically stable (UAS) if for every \( x(0) \in \mathbb{R}^n \) the solution to (1) converges to zero uniformly with respect to \( \{\xi(k)\}_{k \in \mathbb{N}} \subset \Xi \) (i.e., for every \( \varepsilon > 0 \) there exists \( K \in \mathbb{N} \) such that for every \( \{\xi(k)\}_{k \in \mathbb{N}} \subset \Xi \) we have \( \|x(k)\| < \varepsilon \) for \( k \geq K \)) and if, moreover, for every \( R > 0 \) there exists \( r > 0 \) such that \( \|x(k)\| < R \) for every \( \{\xi(k)\}_{k \in \mathbb{N}} \subset \Xi \) and every \( k \in \mathbb{N} \), provided that \( \|x(0)\| < r \).

Due to the linear nature of the (discrete) dynamics of (1), it is well known that (1) is UAS stable if and only if it is uniformly exponentially stable. The most widespread tool for proving that a system of the form (1) is UAS is to exhibit a quadratic Lyapunov function, which in general can depend on \( \xi \).

**Definition 2:** We say that (1) is parameter-dependent quadratically stable (PD-quadratically stable) if there exist three positive constants \( \alpha_0, \alpha_1, \alpha_2 \) and a Lyapunov function

\[
V(x, \xi) = x^T \mathcal{P}(\xi)x
\]

such that

\[
\alpha_1 \|x\|^2 \leq V(x, \xi) \leq \alpha_2 \|x\|^2
\]

and whose difference along the solutions of (1) is negative definite decreasing, that is, for every \( x(0) \in \mathbb{R}^n \), every \( \{\xi(k)\}_{k \in \mathbb{N}} \subset \Xi \), and every \( k \in \mathbb{N} \), we have

\[
V(x(k + 1), \xi(k + 1)) - V(x(k), \xi(k)) \leq -\alpha_0 \|x(k)\|^2.
\]

Motivated by the characterization of asymptotic stability of LTV’s in terms of existence of time-varying quadratic Lyapunov functions (see [17]), we introduce the following (a priori weaker) notion.

**Definition 3:** We say that (1) is parameter- and time-dependent quadratically stable (PTD-quadratically stable) if there exist three positive constants \( \alpha_0, \alpha_1, \alpha_2 \) and a Lyapunov function

\[
V(k, x, \xi) = x^T \mathcal{P}(k, \xi)x
\]

such that

\[
\alpha_1 \|x\|^2 \leq V(k, x, \xi) \leq \alpha_2 \|x\|^2
\]

and for every \( x(0) \in \mathbb{R}^n \), every \( \{\xi(k)\}_{k \in \mathbb{N}} \subset \Xi \), and every \( k \in \mathbb{N} \), we have

\[
V(k + 1, x(k + 1), \xi(k + 1)) - V(k, x(k), \xi(k)) \leq -\alpha_0 \|x(k)\|^2.
\]
Most of the paper will focus on the case where $\mathcal{A}(\Xi)$ is a convex polytope, i.e., $\mathcal{A}(\Xi) = \text{conv}\{A_1, \ldots, A_M\}$ for some $M \in \mathbb{N}$ and $A_1, \ldots, A_M \in \mathcal{M}^{n \times n}$. Without loss of generality, this is equivalent to say that $m = M$, $\Xi$ is the simplex $\text{conv}\{e_1, \ldots, e_M\}$, and that $\mathcal{A}$ is the linear map satisfying $A_i = A(e_i)$. In this case we will say that the system (1) is convex.

**Definition 4:** Let (1) be convex. We say that (1) is poly-quadratically stable if a Lyapunov function $V$ as in (2) can be found with $P(\cdot)$ linear with respect to $\xi$.

**Remark 5:** Let (1) be convex. Then (1) is stable if and only if the system where $\Xi$ is replaced by $\{e_1, \ldots, e_M\}$ is stable. Indeed, let us recall that for a bounded set of matrices $\mathcal{A}(\Xi)$ the uniform asymptotic stability of (1) is equivalent to the property that the joint spectral radius

$$\rho(\mathcal{A}(\Xi)) = \limsup_{h \to \infty} \max_{\xi \in \Xi} \|A_{\xi_1} \cdots A_{\xi_h}\|^{1/h}$$

is strictly smaller than one. We want to see that $\rho(\text{conv}\{A_1, \ldots, A_M\}) = \rho(\{A_1 \ldots A_M\})$. For this purpose it is enough to observe that for any positive integer $N$ and any choice of convex combinations $B_j = \sum_{i=1}^N \lambda_i^{(j)} A_i$, $j = 1, \ldots, N$, the matrix product $B_N \cdots B_1$ is the convex combination

$$B_N \cdots B_1 = \sum_{i_1, \ldots, i_N=1}^M \lambda_i^{(1)} \cdots \lambda_{i_N}^{(N)} A_{i_N} \cdots A_{i_1}.$$

Therefore,

$$\|B_N \cdots B_1\| \leq \sum_{i_1, \ldots, i_N=1}^M \lambda_i^{(1)} \cdots \lambda_{i_N}^{(N)} \|A_{i_N} \cdots A_{i_1}\| \leq \max_{1 \leq i_1, \ldots, i_N \leq M} \|A_{i_N} \cdots A_{i_1}\|.$$

From the definition of the joint spectral radius it trivially follows that $\rho(\text{conv}\{A_1, \ldots, A_M\}) \leq \rho(\{A_1 \ldots A_M\})$, which proves the claim.

**III. EQUIVALENCE BETWEEN DIFFERENT NOTIONS OF QUADRATIC STABILITY**

**Theorem 6:** Let (1) be convex. Then (1) is PTD-quadratically stable if and only if it is PD-quadratically stable if and only if it is poly-quadratically stable.

**Proof.** It is clear by the definitions given in the previous section that if (1) is poly-quadratically stable then it is PD-quadratically stable and that if it is PD-quadratically stable then it is PTD-quadratically stable. We are therefore left to prove that a PTD-quadratically stable system is poly-quadratically stable.

Assume that (1) is PTD-quadratically stable and fix $V(k, \xi, x)$ and $P(k, \xi)$ as in Definition 3.

The first step consists in showing that $P$ can be assumed to be linear with respect to $\xi$. To this extent, define $P_{k,i} = P(k, e_i)$ for $k \in \mathbb{N}$ and $i = 1, \ldots, M$. Let, moreover, $\Pi : \mathbb{N} \times \Xi \to \mathcal{M}^{n \times n}$ be defined by

$$\Pi(k, \xi) = \sum_{i=1}^M \xi_i P_{k,i},$$

where the $\xi_i$'s denote the components of $\xi$, i.e., $\xi = \sum_{i=1}^M \xi_i e_i$.

Let us restrict our attention for a moment to switching functions with values in the set of vertices of $\Xi$. Requiring that for every $\{\xi(k)\}_{k \in \mathbb{N}} \subset \{e_1, \ldots, e_M\}$, every $k \in \mathbb{N}$, and every $x(0) \in \mathbb{R}^n$, we have

$$-\alpha_0 \|x(k)\|^2 \geq x(k+1)^T P(k+1, \xi(k+1)) x(k+1) - x(k)^T P(k, \xi(k)) x(k) = x(k)^T (A_{\xi}^T(k) P(k+1, \xi(k+1)) A_{\xi} - P(k, \xi(k))) x(k)$$

is equivalent to asking that, for every $k \in \mathbb{N}$ and $i, j \in \{1, \ldots, M\}$, the following matrix relation holds

$$A_{\xi}^T P_{k+1,i} A_i - P_{k,i} \leq -\alpha_0 \text{Id}. \quad (5)$$

Fix $j \in \{1, \ldots, M\}$ and let $F^j : \mathbb{N} \times \Xi \to \mathcal{M}^{n \times n}$ be defined by

$$F^j(k, \xi) = A_{\xi}^T P_{k,j} A_{\xi}.$$

Notice that $F^j$ is convex with respect to $\xi$, since it is the composition of a linear map with $f : A \mapsto A^T P_{k,j} A$, whose convexity is proved by the matrix relation

$$f(\lambda A + (1 - \lambda) B) - \lambda f(A) - (1 - \lambda) f(B) = -\lambda(1 - \lambda) f(A - B) \leq 0,$$

which holds for every $A, B \in \mathcal{M}^{n \times n}$ and every $\lambda \in [0, 1]$.

Therefore,

$$F^j(k+1, \xi) = F^j\left(k + 1, \sum_{i=1}^M \xi_i e_i\right) \leq \sum_{i=1}^M \xi_i F^j(k + 1, e_i) = \sum_{i=1}^M \xi_i A_{\xi}^T P_{k+1,j} A_i \leq -\alpha_0 \text{Id} + \Pi(k, \xi),$$

where the last inequality follows from (5) and the definition of $\Pi$.

We are ready to prove that $\Pi$ is a time-dependent Lyapunov function stabilizing (1). Indeed, for every $\xi^1, \xi^2 \in \Xi$ and every $k \in \mathbb{N}$, we have

$$A_{\xi^1}^T \Pi(k+1, \xi^2) A_{\xi^1} - \Pi(k, \xi^1) = \sum_{j=1}^M \xi_j^2 (A_{\xi^1}^T P_{k+1,j} A_{\xi^1} - \Pi(k, \xi^1)) \leq -\alpha_0 \text{Id}. \quad (5)$$

Define now $\Omega(k) = (P_{k,1}, \ldots, P_{k,M})$ for every $k \in \mathbb{N}$. Notice that if there exist $k_1, k_2 \in \mathbb{N}$ such that $k_1 < k_2$
and \( \Omega(k_1) = \Omega(k_2) \), then the theorem is proved. Indeed, by defining
\[
\Pi^*(\xi) = \sum_{k=k_1}^{k_2-1} \Pi(k, \xi),
\]
we have, for every \( \xi^1, \xi^2 \in \Xi \),
\[
A^T_\xi \Pi^*(\xi^2) A_{\xi^2} - \Pi^*(\xi^1) = 
A^T_\xi (\Pi(k_1, \xi^2) A_{\xi^1} - \Pi(k_2-1, \xi^1)) + 
+ \sum_{k=k_1}^{k_2-2} (A^T_\xi \Pi(k+1, \xi^2) A_{\xi^1} - \Pi(k, \xi^1)) \leq -(k_2 - k_1)\alpha_0 \text{Id}.
\]
Notice now that \( \{\Omega(k)\}_{k \in \mathbb{N}} \) is a bounded sequence in \( (A^{n \times n})^M \), due to (4). We can thus extract a converging subsequence \( \{\Omega(k_l)\}_{l \in \mathbb{N}} \). The idea is to take as time-independent Lyapunov function that corresponding to
\[
\Pi_l^*(\xi) = \sum_{k=k_1}^{k_{l+1}-1} \Pi(k, \xi),
\]
with \( l \) large enough. Indeed, let \( l \) be such that
\[
\frac{-\alpha_0}{2} \text{Id} \leq A^T_l (P_{k_{l+1}} - P_k) A_l \leq \frac{-\alpha_0}{2} \text{Id}
\]
for every \( i, j \in \{1, \ldots, M\} \). Then the same computations as above show that
\[
A^T_l \Pi_l^*(\xi^2) A_{\xi^2} - \Pi_l^*(\xi^1) \leq \frac{-\alpha_0}{2} \text{Id},
\]
and the proof of Theorem 6 is finished.

IV. ASYMPTOTIC VS QUADRATIC STABILITY

Lemma 7: There exist stable systems of type (1) which are not PD-quadratically stable.

Proof. The proof works by contradiction. The idea is to consider a continuous-time switched system with suitable properties and to time-discretize it. For every time-step the system that is obtained is stable, but the assumption that all of them are PD-quadratically stable leads to a contradiction when the time-step goes to zero (and the discrete systems converge, morally, to the continuous one).

Take \( n = 2 \) and \( \Xi = \{(1, 0), (0, 1)\} \) and assume by contradiction that if (1) is stable, then there exist two positive definite matrices \( P_1, P_2 \) such that, setting \( V(x, \xi) = x^T P_2 x \), we have that \( V \) is strictly decreasing along every \( \xi \)-trajectory \( (x(k), \xi(k)) \) for every \( x(0) \in \mathbb{R}^2 \) and every \( \{\xi(k)\}_{k \in \mathbb{N}} \subset \{1, 2\} \).

Consider a bidimensional continuous switched system of the type
\[
x = u(t)C_1 x + (1 - u(t))C_2 x, \quad u(t) \in \{0, 1\},
\]
and assume that \( C_1 \) and \( C_2 \) are Hurwitz matrices with non-real eigenvalues. It follows from the results by Dayawansa and Martin (see [10]) that we can assume in addition that the system is uniformly exponentially stable and that there exist no quadratic function which is positive definite and whose derivative along the trajectories of (6) is always non-positive. (Notice that we are not restating the exact result appearing in [10], which proves the possible nonexistence, for a uniformly exponentially stable system, of a quadratic Lyapunov function. Here we ask a slightly stronger property, since we want to rule out the possibility of a positive definite quadratic function which is non-increasing along trajectories of (6). The result, however, directly follows from the reasoning in [10].)

Let us define, for every \( \varepsilon > 0 \),
\[
A^*_1 = e^{\varepsilon C_1}, \quad A^*_2 = e^{\varepsilon C_2},
\]
and consider the corresponding family of discrete systems
\[
x(k + 1) = A^*_\xi(k) x(k), \quad \xi(k) \in \{1, 2\}. \quad (7)
\]
Every such system is obviously stable, because of the uniform exponential stability of (6). According to the contradiction hypothesis, let us assume the existence, for every \( \varepsilon > 0 \), of two positive definite matrices \( P_1^*, P_2^* \) such that
\[
(A^*_1)^T P_i^* A^*_i - P_i^* < 0 \quad i = 1, 2, \quad \text{(8)}
\]
\[
(A^*_2)^T P_i^* A^*_i - P_i^* < 0 \quad i \neq j. \quad \text{(9)}
\]
Up to a rescaling we can assume \( \max_{i=1,2} \|P_i^*\| = 1 \) for every \( \varepsilon > 0 \). By a Taylor expansion with respect to \( \varepsilon \) we have
\[
(\varepsilon C_i)^T P_i^* e^{\varepsilon C_i} =
(1 + \varepsilon C_i^T P_i^* (1 + \varepsilon C_i + O(\varepsilon^2)) =
P_i^* + \varepsilon (C_i^T P_i^* + P_i^* C_i) + O(\varepsilon^2)
\]
so that, according to inequality (8),
\[
(C_i^T P_i^* + P_i^* C_i) + O(\varepsilon) = \frac{(A^*_i)^T P_i^* A^*_i - P_i^*}{\varepsilon} < 0. \quad \text{(10)}
\]
By compactness of the set of matrices \( P_i^* \) we can find a suitable sequence \( \varepsilon_h \to 0 \) such that \( P_i^* \to P_i \) for \( i = 1, 2 \) for some positive semidefinite matrices \( P_1, P_2 \) and \( P_i \). Moreover, at least one among \( P_1 \) and \( P_2 \) has norm equal to one. Then, passing to the limit as \( \varepsilon_h \) goes to 0 in (10), we obtain
\[
C_i^T P_i + P_i C_i \leq 0, \quad i = 1, 2.
\]
Similarly, from (9) we get that
\[
P_i^* - P_i^* + O(\varepsilon) < 0 \quad \text{if} \ i \neq j \quad \text{i.e.} \ P_i^* - P_j^* = O(\varepsilon)
\]
and therefore
\[
P_1 = P_2 =: \Lambda.
\]
Observe that \( P \) is not only semidefinite, but it must be positive definite. Indeed, since \( \|P\| = 1 \), if it is semidefinite then there exists one dimensional subspace \( \Lambda \) such that \( x^T P x > 0 \) if and only if \( x \in \Lambda \). Then the only way not to increase \( x^T P x \) along a trajectory starting from \( \Lambda \) would
be to stay on \( \Lambda \), which is impossible because the \( C_1 \) and \( C_2 \) have non-real eigenvalues.

Thus \( P \) is positive definite and satisfies \( C_i^T P + P C_i \leq 0 \) for \( i = 1, 2 \). This is impossible due the initial assumption made on \( C_1, C_2 \). We reached a contradiction and the lemma is proved.

**Conclusion**

We investigated the equivalence between different notions of stability for discrete-time switched systems of type (1). We first proved (Section III) that looking for a quadratic Lyapunov function in its more general form \( V(k, \xi, x) = x^T P(k, \xi) x \) (that is, in a class which depends on an infinite number of parameters) is equivalent to looking for it in the much smaller class \( V(\xi, x) = x^T \left( \sum_{i=1}^{M} \xi_i P_i \right) x \) (which depends on a finite family of parameters). Recall that this last problem can be formulated in terms of LMI ([8]). Therefore, quadratic stability (however defined) turns out to be equivalent to a set of LMI's.

On the other hand, as suggested by [5], uniform asymptotic stability is a strictly stronger notion that quadratic stability (Section IV).

**References**


