Robustness analysis for the boundary control of the string equation

Martin GUGAT, Mario SIGALOTTI, and Marius TUCSNAK

I. INTRODUCTION AND MAIN RESULTS.

In this paper we consider the infinite dimensional system determined whose state is determined by the equations

\[
\frac{\partial^2 z}{\partial t^2}(x,t) - \frac{\partial^2 z}{\partial x^2}(x,t) = w(t), \quad x \in (0,1), t > 0 \tag{1.1}
\]

\[
z(x,0) = z_0(x), \quad \frac{\partial z}{\partial t}(x,0) = z_1(x), \quad x \in (0,1), \tag{1.2}
\]

\[
z(0,t) = 0, \quad t > 0, \tag{1.3}
\]

\[
\frac{\partial z}{\partial x}(1,t) = -a \frac{\partial z}{\partial t}(1,t) - bz(1,t), \quad t > 0. \tag{1.4}
\]

The above system is a model for the vibrations of a string which is fixed at the end \(x = 0\) and which is subjected to a boundary damping (depending on the velocity and on the position) at the end \(x = 1\). The term \(w\) in the right hand side of (1.1) stands for a constant (with respect to \(x\)) perturbing force.

In the case \(w \equiv 0\), the initial and boundary value problem (1.1)-(1.4) has been studied, in particular, in [1] and in [2]. In [2] the authors studied the Riesz basis generation and the optimal decay properties for \(a = 1\) and \(b = 0\), whereas in [1] it has been shown that, for any \(a > 0\) and \(b \geq 0\), the system determined by (1.1)-(1.4) is exponentially stable in the state space \(V = V \times L^2(0,1)\), where

\[
V = \{ \varphi \in H^1(0,1) | \varphi(0) = 0 \},
\]

and \(H^1(0,1)\) stands for the Sobolev space of functions in \(L^2(0,1)\) with generalized derivative in \(L^2(0,1)\). One of the main results in [1] asserts that, at given \(a\), the best decay rate of the solutions of (1.1)-(1.4) is achieved for \(b = 0\). This means that the presence of the “position term” \(-bz(1,t)\) in the feedback law (1.4) does not improve the decay rate of the system. The aim of this work is to show that the position term \(-bz(1,t)\) can improve the robustness properties of the considered system with respect to time dependent perturbations acting in the right hand side of (1.1).

In order to give a precise statement of our results we consider an output law given by

\[
y(t) = \frac{\partial z}{\partial x}(0,t). \tag{1.5}
\]

Definition 1.1: For \(\gamma > 0\) we say that the system (1.1)-(1.5) has the \(\gamma\)-robustness property if its solutions with \(z_0 = z_1 = 0\) satisfy the estimate

\[
\|y\|_{L^2(0,\infty)} \leq \gamma \|w\|_{L^2(0,\infty)}, \quad \text{for all } w \in L^2(0,\infty). \tag{1.6}
\]

For given \(a > 0\) and \(b \geq 0\) the infinum of the set of those \(\gamma\) satisfying (1.6), denoted by \(\gamma_0(a, b)\), is called the robustness coefficient associated to the pair \((a, b)\).

One of our main results shows that the introduction of the position term in the feedback law improves the robustness properties of our system. More precisely, the following result holds.

Theorem 1.2: For \(a \in \left(1, \frac{2}{\sqrt{2 + \sqrt{2}}} \right)\), the robustness coefficient defined above satisfies

\[
1 = \gamma_0(a, 0) \geq \inf_{b \geq 0} \gamma_0(a, b) > \frac{27}{4\pi^2}.
\]

The above result gives, by analytical methods, sufficient conditions for \(a\) and \(b\) in order to have \(\gamma_0(a, b) < 1\). We conjecture that this condition is satisfied for a larger class of pairs \((a, b)\) and we give numerical simulations supporting this assertion.

An important question tackled in this work consists in determining a pair \((a_0, b_0)\) minimizing \(\gamma_0\). We show below that such a pair exists. Moreover, we give, by numerical simulations, approximate values for \(a_0\), \(b_0\) and \(\gamma_0(a_0, b_0)\).

II. COMPUTATION OF THE TRANSFER FUNCTION

By applying the Laplace transform with respect to the time to the above system, we get that

\[
s^2 \hat{z}(x,s) - \frac{\partial^2 \hat{z}}{\partial x^2}(x,s) = \hat{w}(s), \quad x \in (0,1), s \in \mathbb{C}_0 \tag{2.1}
\]

\[
\hat{z}(0,s) = 0, \quad s \in \mathbb{C}_0, \tag{2.2}
\]

\[
\frac{\partial \hat{z}}{\partial x}(1,s) = (as - b)\hat{z}(1,s), \quad s \in \mathbb{C}_0. \tag{2.3}
\]

From (2.1) it follows that

\[
\hat{z}(x,s) = A(s) \sinh(sx) + B(s) \cosh(sx) + \frac{\hat{w}(s)}{s^2},
\]

for \(x \in (0,1)\) and \(s \in \mathbb{C}_0\). From the above relation and (2.2) it follows that \(B = -\frac{\hat{w}(s)}{s^2}\) so that

\[
\hat{z}(x,s) = A(s) \sinh(sx) - \frac{\hat{w}(s)}{s^2} \cosh(sx) + \frac{\hat{w}(s)}{s^2}, \tag{2.4}
\]

for \(x \in (0,1)\) and \(s \in \mathbb{C}_0\). As a consequence,

\[
\frac{\partial \hat{z}}{\partial x}(x,s) = sA(s) \cosh(sx) - \frac{\hat{w}(s)}{s} \sinh(sx), \tag{2.5}
\]

M. Gugat is with Lehrstuhl II für Angewandte Mathematik, Friedrich-Alexander Universität Nürnberg-Erlangen, Martensstr. 3, Erlangen, Germany, gugat@am.uni-erlangen.de

M. Sigalotti and M. Tucsnak are with Institut Élie Cartan UMR7502, Université Henri Poincaré Nancy 1, BP239, 54506 Vandœuvre-lès-Nancy Cedex, France, mario.sigalotti@inria.fr, tucsnak@iecn.u-nancy.fr
for $x \in (0,1)$ and $s \in \mathbb{C}_0$. From (2.3), (2.4) and (2.5) it follows that, for every $s \in \mathbb{C}_0$, we have
\[
sA(s) \cosh(s) - \frac{\tilde{w}(s)}{s} \sinh(s) = (as - b) \left( A(s) \sinh(s) - \frac{\tilde{w}(s)}{s^2} \cosh(s) + \frac{\tilde{w}(s)}{s^2} \right),
\]
which implies that
\[
A(s) (s \cosh s - (as - b) \sinh(s)) = \left[ \frac{as - b}{s^2} - (1 - \cosh s) + \frac{\sinh s}{s} \right] \tilde{w}(s).
\]
It follows that
\[
A(s) = \frac{\frac{as - b}{s} (1 - \cosh s) + \sinh(s)}{s \cosh(s) - (as - b) \sinh(s)} \tilde{w}(s) \quad (s \in \mathbb{C}_0).
\]
The above relation, combined to (1.5) and (2.5) yields that the output $y$ is related to the perturbation $w$ by the relation
\[
\hat{y}(s) = H(s) \hat{w}(s) \quad (s \in \mathbb{C}_0),
\]
with the transfer function $G$ given by
\[
G(s) = \frac{\frac{as - b}{s} (1 - \cosh s) + \sinh(s)}{s \cosh(s) - (as - b) \sinh(s)} \hat{w}(s) \quad (s \in \mathbb{C}_0). \quad (2.6)
\]

III. ROBUSTNESS ANALYSIS

We want to choose among those $a$ and $b$ for which the system (1.1)-(1.4) is exponentially stable for which the norm of $G$ (defined in (2.6)) in the Hardy space $H^\infty(\mathbb{C}_0)$ is the smallest possible. By the maximum principle, the maximum of $G$ on $\mathbb{C}_0$ will be attained on the imaginary axis. By using this fact and the Paley-Wiener theorem it follows that:

**Proposition 3.1:** For every $a$, $b > 0$ we have
\[
\gamma_0(a, b) = \sup_{\omega \in \mathbb{R}} |G(i\omega)|.
\]

By using the above facts, it follows that the relevant function we have to investigate is $f : \mathbb{R} \to \mathbb{R}$ defined by
\[
f(\omega) = |G(i\omega)|^2 \quad (\omega \in \mathbb{R}).
\]
After some simple calculations we obtain that
\[
f(\omega) = \frac{a^2 (1 - \cos \omega)^2 + \left[ \sin \omega + \frac{b}{\omega} (1 - \cos \omega) \right]^2}{a^2 \omega^2 \sin^2 \omega + (\omega \cos \omega + b \sin \omega)^2}.
\]
It can be easily checked that the above formula implies that
\[
f(\omega) = \sin^2 \left( \frac{\omega}{2} \right) \frac{a^2 \sin^2 \left( \frac{\omega}{2} \right) + \left( \cos \left( \frac{\omega}{2} \right) + \frac{b}{\omega} \sin \left( \frac{\omega}{2} \right) \right)^2}{a^2 \sin^2 \omega + (\omega \cos \omega + b \sin \omega)^2},
\]
where, for $x \in \mathbb{R}$, we have used the notation
\[
\sin \left( \frac{x}{2} \right) = \frac{\sin(x)}{x}.
\]
Notice that for $a = 1$ and $b = 0$ we have $f(\omega) = \sin^2 \left( \frac{\omega}{2} \right)$ so that $\gamma_0(1,0) = 1$.

In the sequel we discuss conditions for the inequality
\[
\sup_{\omega \in \mathbb{R}} f(\omega) < 1. \quad (3.1)
\]
and we prove Theorem 1.2.

Notice first that
\[
f(0) = \frac{(1 + \frac{b}{2})^2}{(1 + b)^2}.
\]
In particular, for $b = 0$ we have $f(0) = 1$. Thus the inequality $b > 0$ is a necessary condition for (3.1). Moreover, we see that $f(0) > 1/4$.

We have
\[
f(\pi) = \frac{4}{\pi^2} \left( a^2 + \frac{b^2}{\pi^2} \right).
\]
So another necessary condition for (3.1) is the inequality
\[
\frac{b^2}{\pi^2} < \frac{\pi^2}{4} - a^2. \quad (3.2)
\]
This implies in particular
\[
a < \frac{\pi}{2} \quad \text{and} \quad b < \frac{\pi^2}{2}.
\]
For all $b < \frac{\pi^2}{2}$, we have
\[
f(0) > \left( \frac{4 + \pi^2}{4 + 2\pi^2} \right)^2.
\]
We have
\[
f(2\pi/3) = \frac{9a^2 + \left( \sqrt{3} + \frac{9}{\sqrt{3}}b \right)^2}{\pi^2 a^2 + \left( \frac{\pi}{3} - \sqrt{3}b \right)^2}.
\]
Since this value is strictly decreasing as a function of $a$ and $a < \pi/2$, this implies
\[
f(2\pi/3) \geq \frac{\frac{9\pi^2}{4} + \left( \frac{\sqrt{3}}{2} + \frac{9}{4\sqrt{3}}b \right)^2}{\frac{\pi^2}{4} + \left( \frac{\pi}{3} - \sqrt{3}b \right)^2} \geq \frac{27}{4\pi^2} = 0.6839.
\]

It is easy to check that Theorem 1.2 follows from the following sequence of lemmas.

**Lemma 3.2:** Assume that $a \in [1, \pi/2)$, $b \in (0, \pi/4)$ and
\[
b \left( \frac{2}{\pi} + \pi + \frac{b}{\pi^2} \right) < \frac{\pi^2}{4} - a^2. \quad (3.3)
\]
Then $\sup_{\omega \geq \pi} f(\omega) < 1$.

**Proof.** Define
\[
P(\omega) = a^2 \sin^2 \left( \frac{\omega}{2} \right) + \cos \left( \frac{\omega}{2} \right) \left( b \sin \left( \frac{\omega}{2} \right) \right)^2 = \frac{a^2}{a^2 - (a^2 - 1) \cos^2 \left( \frac{\omega}{2} \right) + \frac{b^2}{4} \sin^2 \left( \frac{\omega}{2} \right)},
\]
\[
Q(\omega) = a^2 \sin^2 \omega + (\cos \omega + b \sin \omega)^2 = 1 + (a^2 - 1) \sin^2 \omega + 2b \cos \omega \sin \left( \frac{\omega}{2} \right) \sin \left( \frac{\omega}{2} \right) + b^2 \cos^2 \left( \frac{\omega}{2} \right) \sin^2 \left( \frac{\omega}{2} \right).
\]
Hence for all $\omega \geq \pi$ and $a \geq 1$ we have
\[
P(\omega) \leq a^2 + b \left\{ \frac{b^2}{4} \sin^2 \left( \frac{\omega}{2} \right) \right\},
\]
\[
Q(\omega) \geq 1 - 2b \sin \left( \frac{\omega}{2} \right).
\]
Let $S = \sin \left( \frac{\pi}{2} \right)$, and $S_1 = \sin \left( \frac{\pi}{4} \right) = \frac{2}{\sqrt{2}}$. Then
\[
f(\omega) = S^2 \frac{P(\omega)}{Q(\omega)} \leq S^2 \frac{a^2 + bS + (b^2/4)S^2}{1 - 2bS} \leq S_1^2 \frac{a^2 + bS_1 + (b^2/4)S_1^2}{1 - 2bS_1} = \frac{4\pi^2}{a^2 + \frac{2b}{\sqrt{2}} + \frac{4b^2}{\pi^2}} \]
\[
\text{The inequality}
\]
\[
\frac{4\pi^2}{a^2 + \frac{2b}{\sqrt{2}} + \frac{4b^2}{\pi^2}} < 1
\]
is equivalent to (3.3), and the assertion follows. □

Remark 3.3: Note that for $b < \pi/4$ we have
\[
f(0) > (1 + \pi/8)/(1 + \pi/4) = 0.78...
\]

Lemma 3.4: Let $S_1 = 4\sqrt{2}/(\sqrt{2} + 3\pi)$. Assume that $a \in [1,1/S_1)$, $b \in (0, \pi/4)$ and
\[
b \left\{ \sqrt{1 - \frac{\omega}{2} + \frac{\sqrt{2}}{2} S_1^2} bS_1 + \frac{b^2}{4} S_1^2 \right\} < 1 - a^2 S_1^2.
\]
Then
\[
\sup_{\omega \in [\pi/4, \pi]} f(\omega) < 1.
\]
Proof. For all $\omega \in [\pi/4, \pi]$ we have
\[
P(\omega) \leq a^2 + \frac{\sqrt{2} - \sqrt{2}}{2} b \left\{ \sin \left( \frac{\omega}{2} \right) \right\} + \frac{b^2}{4} \sin^2 \left( \frac{\omega}{2} \right),
\]
\[
Q(\omega) \geq 1 - \sqrt{1 - \frac{\omega}{2} b \sin \left( \frac{\omega}{2} \right)}.
\]
Let $S = \sin \left( \frac{\pi}{2} \right)$, and $S_1 = \sin \left( \frac{\pi}{4} \right) = \frac{4}{\sqrt{2\pi}} \sqrt{2} + \sqrt{2}$. Then
\[
f(\omega) = S^2 \frac{P(\omega)}{Q(\omega)} \leq S_1^2 \frac{a^2 + \frac{b^2}{4} S_1^2}{1 - \sqrt{1 - \frac{\omega}{2} b \sin \left( \frac{\omega}{2} \right)}}.
\]
The inequality
\[
S_1^2 \frac{a^2 + \frac{b^2}{4} S_1^2}{1 - \sqrt{1 - \frac{\omega}{2} b \sin \left( \frac{\omega}{2} \right)}} < 1
\]
is equivalent to (3.4), and the assertion follows. □

We have
\[
f\left( \frac{\pi}{2} \right) = \frac{4\pi^2}{a^2 + \frac{2b}{\sqrt{2}} + \frac{4b^2}{\pi^2}}.
\]
So another necessary condition for (3.1) is the inequality
\[
b \left\{ \frac{4}{\pi} - \left( 1 - \frac{4}{\pi^2} \right) b \right\} < \left( \frac{\pi^2}{4} - 1 \right) a^2 - 1.
\]

Lemma 3.5: Let $S_1 = \frac{4}{\sqrt{2\pi}}$. Assume that $a \in [1, \pi/2)$, $b \in (0, \pi/4)$ and
\[
2b \left\{ \frac{S_1^4}{\sqrt{2}} + b \left( \frac{1}{4} - \frac{8}{\pi} \right) \right\} < \left[ 1 - S_1^2 \left( 1 - \frac{1}{\sqrt{2}} \right) \right] - a^2 S_1^2 \left[ \frac{1}{1 + \frac{1}{\sqrt{2}}} - 1 \right].
\]
Then
\[
\sup_{\omega \in [\pi/2, 3\pi/4]} f(\omega) < 1.
\]
Proof. For all $\omega \in [\pi/2, 3\pi/4]$ we have
\[
P(\omega) \leq a^2 - (a^2 - 1) \left( \frac{2 - \sqrt{2}}{4} \right) + \frac{b^2}{\sqrt{2}} \sin \left( \frac{\omega}{2} \right) + \frac{4}{\pi^2} \sin^2 \left( \frac{\omega}{2} \right),
\]
\[
Q(\omega) \geq \frac{a^2 + \frac{1}{2} - b \sin \left( \frac{\omega}{2} \right)}{2 - \sqrt{2}} \sin \left( \frac{\omega}{2} \right).
\]
Let $S = \sin \left( \frac{\pi}{2} \right)$, and $S_1 = \sin \left( \frac{\pi}{4} \right) = \frac{1}{\sqrt{2\pi}}$. We have
\[
S^2 \geq 16(2 + \sqrt{2})/(9\pi).
\]
Then
\[
f(\omega) = S^2 \frac{P(\omega)}{Q(\omega)} \leq S^2 \frac{a^2 - (a^2 - 1) \left( \frac{2 - \sqrt{2}}{4} \right) + \frac{b^2}{\sqrt{2}} S + \frac{b^2}{4} S^2}{a^2 + \frac{1}{2} - b \sin \left( \frac{\omega}{2} \right)} \frac{1}{a^2 + \frac{1}{2} - b \sin \left( \frac{\omega}{2} \right)} \sin \left( \frac{\omega}{2} \right).
\]
The inequality
\[
S_1^2 \frac{a^2 - (a^2 - 1) \left( \frac{2 - \sqrt{2}}{4} \right) + \frac{b^2}{\sqrt{2}} S_1 + \frac{b^2}{4} S_1^2}{a^2 + \frac{1}{2} - b S_1 + \frac{b^2}{4} S_1^2} < 1
\]
is equivalent to (3.6), and the assertion follows. □

Lemma 3.6: Let $S_1 = \sin \left( \frac{\pi}{4} \right)$. Assume that $a \in [1, \pi/2)$, $b \in (0, \pi/4)$ and
\[
b S_1^3 \left\{ \cos \left( \frac{1}{2} \right) + \frac{1}{4} S_1 \right\} \leq - \left[ \frac{1}{2} S_1^2 + \sin^2 (1) - 1 \right] + \frac{2}{1} \sin^2 (1) \frac{1}{\sqrt{2}} a^2.
\]
Then
\[
\sup_{\omega \in [1, \pi/2]} f(\omega) < 1.
\]
Proof. For all $\omega \in [1, \pi/2]$ we have
\[
P(\omega) \leq \frac{(a^2 + 1)}{2} + b \cos \left( \frac{1}{2} \right) \sin \left( \frac{\omega}{2} \right) + \frac{4b^2}{\pi^2} \sin^2 \left( \frac{\omega}{2} \right),
\]
\[
Q(\omega) \geq 1 + (a^2 - 1) \sin^2 (1) + \frac{b^2}{4} \sin^2 \left( \frac{\omega}{2} \right).
\]
Let $S = |\text{sinc} \left( \frac{\omega}{2} \right)|$ and $S_1 = \left| \text{sinc} \left( \frac{\omega}{2} \right) \right| = 2 \sin \left( \frac{\omega}{2} \right)$. Then
\[
f(\omega) = S^2 \frac{P(\omega)}{Q(\omega)} \leq S^2 \frac{(a^2+1) + b \cos \left( \frac{\omega}{2} \right) S + \frac{b^2}{2} S^2}{1 + (a^2 - 1) \sin^2(1) + b^2 \frac{1}{2} S^2} \leq S_1^2 \frac{(a^2+1) + b \cos \left( \frac{\omega}{2} \right) S_1 + \frac{b^2}{2} S_1^2}{1 + (a^2 - 1) \sin^2(1)}
\]
The inequality
\[
S_1^2 \frac{(a^2+1) + b \cos \left( \frac{\omega}{2} \right) S_1 + \frac{b^2}{2} S_1^2}{1 + (a^2 - 1) \sin^2(1)} < 1
\]
is equivalent to (3.7), and the assertion follows. □

**Lemma 3.7:** Assume that $a \in [1, \pi/2)$, $b \in (0, \pi/4)$ and $\frac{1}{12} b^2 < a^2 - 1$ and
\[
\frac{4}{5} b \left( 1 + \frac{5}{3} b \right) < a^2 - 1.
\]
Then $\sup_{\omega \in [0, 1]} f(\omega) < 1$.

**Proof.** For all $\omega \in [0, 1]$ we have
\[
P(\omega) \leq 1 + b + \frac{b^2}{4} + \frac{(a^2 - 1)}{4} \omega^2,
\]
\[
Q(\omega) \geq 1 + 2b + b^2 + \left( a^2 - 1 - \frac{4}{3} b - \frac{1}{3} b^2 \right) \omega^2 + \left( \frac{1}{3} a^2 + \frac{1}{3} + \frac{1}{36} b^2 \right) \omega^4
\]
Then
\[
f(\omega) = S^2 \frac{P(\omega)}{Q(\omega)} \leq \frac{\bar{p}}{\bar{q}}
\]
where
\[
\bar{p} = 1 + b + \frac{b^2}{4} + \frac{(a^2 - 1)}{4} \omega^2
\]
\[
\bar{q} = 1 + 2b + b^2 + \left( a^2 - 1 - \frac{4}{3} b - \frac{1}{3} b^2 \right) \omega^2 + \left( \frac{1}{3} a^2 + \frac{1}{3} + \frac{1}{36} b^2 \right) \omega^4
\]
The inequality
\[
\frac{\bar{p}}{\bar{q}} < 1
\]
is equivalent to
\[
0 < b + \frac{3}{4} b^2 + \left( \frac{3}{4} (a^2 - 1) - \frac{4}{3} b - \frac{1}{3} b^2 \right) \omega^2 + \left( \frac{1}{3} a^2 + \frac{1}{3} + \frac{1}{36} b^2 \right) \omega^4
\]
which is valid for $\omega = 0$. For $\omega = 1$, the inequality is equivalent to (3.8), and the assertion follows. □

**IV. NUMERICAL SIMULATIONS AND CONCLUDING REMARKS**

In Theorem 1.2 we gave a sufficient condition on $a$ in order to have
\[
\inf_{b \geq 0} \gamma_0(a, b) < 1.
\]
In order to check if this condition is sharp we numerically evaluated the set of those $(a, b) \in (0, \infty) \times (0, \infty)$ for which
\[
\gamma_0(a, b) < 1.
\]
The result, shown in Figure 1, suggests that the range of $a$ for which (4.1) holds is larger than the range obtained in Theorem 1.2 by analytical methods. Another conclusion which can be drawn from our numerical simulations is that, at given $a > 0$, the range of those $b > 0$ satisfying the condition (4.2) may be bounded away from zero (see Figure 2).

![Figure 1](image1.png)

Fig. 1. In black, the region $\{(a, b) \mid \gamma_0(a, b) < 1\}$.

![Figure 2](image2.png)

Fig. 2. Detail of the region described in Figure 1.
The main conclusion of this work is that a position damping term in the feedback can improve the robustness properties of the boundary control of the wave equation. The optimal choice of the damping coefficient rests, in general, an open problem. An interesting open problem is the generalization of our approach to more general feedbacks or output laws.

REFERENCES
