Achieving robustness through coarse space enrichment

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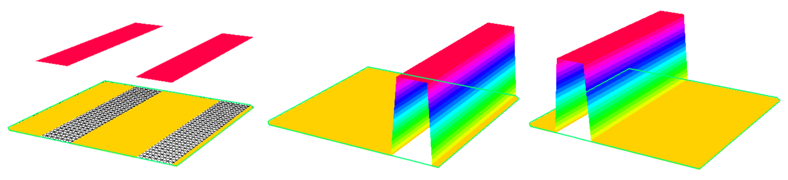
I) Some cases where the 'best' coarse space is known

IA) Darcy in Heterogeneous media: $-\nabla \cdot (\alpha \nabla u) = 1$

One coarse function per subdomain per high contrast layer.



C Vuik, A Segal, and J.A. Meijerink, JCP (1999).

One coarse function per subdomain.

I. G. Graham, P. O. Lechner, and R. Scheichl, Numerische Mathematik (2007).

Hou, Wu, and Cai, Mathematics of Computation (1999).

IB) Elasticity in the incompressible limit: $\nu \to 1/2$ with discontinuous pressure

Non-overlapping DD: one coarse vector per subdomain (preserves the volume).

B. Vereecke, H. Bavestrello, D. Dureisseix, CMAME (2003).

S. Gippert, A. Klawonn, M. Lanser, P. Radtke, and O. Rheinbach, DD21 Proceedings (2013).

Overlapping DD One coarse vector per face.

C. R. Dohrmann and O. B. Widlund, IJNME (2010).

Others

P. Le Tallec, J. Mandel and M. Vidrascu, SIAM J. Numer. Anal. (1998). (plates & shells)

C. Farhat, P.S. Chen and J. Mandel, INJME (1995). (time dependent)

II) Automatic construction of coarse spaces based on Generalized Eigenvalue Problems in the overlaps (GenEO)

Abstract Schwarz framework for the problem: Find $\mathbf{x}_* \in V$ such that $A\mathbf{x}_* = \mathbf{b}$.

Assume that $A = \Sigma_\tau(A_\tau)$ is spd, A_τ are spsd.
Choose
1. local subspaces $V_j \subset V, \quad j = 1, \dots, N$
2. interpolators $\tilde{R}_j^\top : V_j \to V$
3. local solvers $\tilde{A}_j : V_j \to V_j$
4. the coarse space $V_H \subset V$, $R_H^\top : V_H \to V$

Requirement: $V = \Sigma_{j=1}^N \tilde{R}_j^\top V_j + R_H^\top V_H$.

Two level preconditioner

$$M_{2level}^{-1} = P_0 + (I - P_0)\Sigma_{j=1}^N \tilde{R}_j^\top \tilde{A}_j^{-1} \tilde{R}_j (I - P_0)^\top,$$

where P_0 is the A-orthogonal projection onto V_H.

Two Crucial assumptions Stable splitting: $\forall u \in$ range$(I - P_0)$ there exists $(z_1, \dots, z_N) \in V_1 \times \dots V_N$ such that

(1) $\quad u = \Sigma_{j=1}^N \tilde{R}_j^\top z_j$ and $\Sigma_{j=1}^N \langle \tilde{A}_j z_j, z_j \rangle \leq C_0^2 \langle Au, u \rangle$.

Stability of the local solver w.r.t. the exact solver: for all $u_j \in$ range$(\tilde{A}_j^{-1}\tilde{R}_j(I - P_0))$,

(2) $\quad \langle A\tilde{R}_j^\top u_j, \tilde{R}_j^\top u_j \rangle \leq \omega \langle \tilde{A}_j u_j, u_j \rangle$.

Then, $\begin{cases} eig(M_{2level}^{-1}A) \geq C_0^2 \\ eig(M_{2level}^{-1}A) \leq \mathcal{N}\omega \end{cases}$ so $cond(M_{2level}^{-1}A) \leq \mathcal{N}\,\omega\, C_0^{-2}$, where $\mathcal{N}$ measures the maximal number of neighbours of a subdomain.

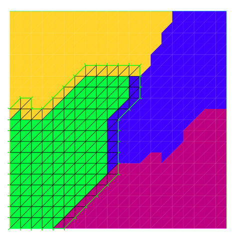
Additive Schwarz

N. S., V. Dolean, P. Hauret, F. Nataf, C. Pechstein, and R. Scheichl, Numerische Mathematik (2013).

N. S., V. Dolean, P. Hauret, F. Nataf, C. Pechstein, and R. Scheichl, C.R. Mathématiques (2011).

N. S., PhD thesis, (coming very soon!).

Definition in the Abstract framework:



1. V_j: N Overlapping subdomains
2. $R_j^\top$: Assembly operators
3. $\tilde{A}_j = R_j A R_j^\top$ (Exact local solvers)

(2) holds with $\omega = 1$ on the whole of V:

$$\langle \tilde{A}_j u_j, u_j \rangle = \langle A R_j^\top u_j, R_j^\top u_j \rangle.$$

(1) can be rewritten as N local conditions for any given C_0:
- Choose partition of unity operators: $\Xi_j : V_h(\Omega_j) \to V_{h,0}(\Omega_j)$.
- Define the components in the stable splitting: $z_j = \Xi_j(u_{|\Omega_j})$.
- Now $(1) \Leftarrow \left(\Sigma_{j=1}^N \langle A R_j^\top \Xi_j(u_{|\Omega_j}), R_j^\top \Xi_j(u_{|\Omega_j})\rangle\right) \leq C_0^2 \langle Au, u \rangle$.
- Or, locally (since $\mathcal{N}\langle Au, u \rangle \leq \Sigma_{j=1}^N \langle A_{|\Omega_j} u_{|\Omega_j}, u_{|\Omega_j} \rangle$)

$$(1) \Leftarrow \left| \langle \tilde{A}_j \Xi_j(u_{|\Omega_j}), \Xi_j(u_{|\Omega_j}) \rangle \leq \frac{C_0^2}{\mathcal{N}} \langle A_{|\Omega_j} u_{|\Omega_j}, u_{|\Omega_j} \rangle, \quad \forall j \right|.$$

Definition: Schwarz GenEO coarse space

Find generalized eigenpairs $(\Lambda_j^k, p_j^k) \in (\mathbb{R}^+ \cup \{+\infty\}) \times V_h(\Omega_j)$ of

$$A_{|\Omega_j} p_j^k = \Lambda_j^k \Xi_j \tilde{A}_j \Xi_j p_j^k.$$

Choose a threshold τ and define

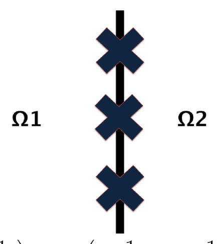
$$V_H = \text{span}\{R_j^\top \Xi_j(p_j^k); \Lambda_j^k < \tau, j = 1, \dots, N\}.$$

Then

$$cond(M_{2level}^{-1}A) \leq \mathcal{N}\left(1 + \frac{\mathcal{N}}{\tau}\right) \quad \begin{cases} \mathcal{N}: \text{'number of neighbours'} \\ \tau: \text{threshold chosen by user} \end{cases}$$

Notation for Schwarz & FETI

Space of FE functions in Ω_j:
$V_{h,0}(\Omega_j) = \{u_{|\Omega_j}; u \in V_h \text{ and } \text{supp}(u) \subset \bar{\Omega}_j\}$.
Restrictions to Ω_j: $V_h(|\Omega_j) = \{u_{|\Omega_j}; u \in V_h\}$.
Problem restricted to Ω_j: $A_{|\Omega_j} = \Sigma_{\tau \subset \bar{\Omega}_j} A_\tau$.
Schur complement: $S_j = A_{|\Omega_j}^{\Gamma\Gamma} + A_{|\Omega_j}^{\Gamma I} A_{|\Omega_j}^{II\,-1} A_{|\Omega_j}^{I\Gamma}$.



- $R_1^\top \begin{pmatrix} x_1^1 \\ x_1^2 \\ x_1^3 \end{pmatrix} + R_2^\top \begin{pmatrix} x_2^1 \\ x_2^2 \\ x_2^3 \end{pmatrix} = \begin{pmatrix} x_1^1 + x_2^1 \\ x_1^2 + x_2^2 \\ x_1^3 + x_2^3 \end{pmatrix}$ **assembly**
- $B_1 \begin{pmatrix} x_1^1 \\ x_1^2 \\ x_1^3 \end{pmatrix} + B_2 \begin{pmatrix} x_2^1 \\ x_2^2 \\ x_2^3 \end{pmatrix} = \begin{pmatrix} x_1^1 - x_2^1 \\ x_1^2 - x_2^2 \\ x_1^3 - x_2^3 \end{pmatrix}$ **jump**

Bibliography: Related methods

Schwarz:

J. Galvis and Y. Efendiev, Multiscale Modeling & Simulation (2010).

Y. Efendiev, J. Galvis, R. Lazarov, and J. Willems, ESAIM (2012).

V. Dolean, F. Nataf, R. Scheichl, and N. Spillane, CMAM (2012).

BDD and FETI:

J. Mandel and B. Sousedík, CMAME (2007).

J. Šístek, , B. Sousedík, and J. Mandel (2013).

Spectral AMG:

M. Brezina, C. Heberton, J. Mandel, and P. Vaněk (2001).

T. Chartier, R. D. Falgout, V. E. Henson, J. Jones, T. Manteuffel, S. McCormick, J. Ruge, and P. S. Vassilevski, SIAM J. Sci. Comput (2003).

J. Xu, L. Zikatanov, Computing and Vizualization in Schience (2003).

FETI

N. S., D.J. Rixen, Automatic Spectral coarse spaces for robust FETI and BDD methods, IJNME (2013).

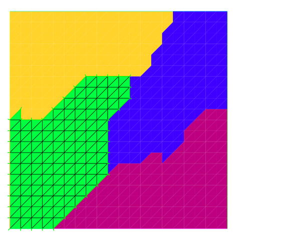
Find λ in the set of admissible constraints:

$$PM_{FETI}^{-1}P^\top \underbrace{(\Sigma_{i=1}^N B_j S_j^\dagger B_j^\top)}_{:=F} = PM_{FETI}^{-1}P^\top d$$

where (with diagonal scaling matrices $D = \text{diag}(D_1, \dots, D_N)$)
$M_{FETI}^{-1} = \Sigma_{i=1}^N (BD^{-1}B^\top)^{-1} B_j D_j^{-1} S_j D_j^{-1} B_j (BD^{-1}B^\top)^{-1}$.

Spectrum of $M_{FETI}^{-1}F$ = spectrum FM_{FETI}^{-1}.

We write $\mathbf{F}\mathbf{M}_{\mathbf{FETI}}^{-1}$ in the Abstract Framework: $A = M_{FETI}^{-1}$



1. V_j = {dofs on the boundary of Ω_j}
2. $\tilde{R}_j^\top = B_j$ jump operator
3. $\tilde{A}_j = S_j$

Indeed, $\Sigma_{j=1}^N \tilde{R}_j^\top \tilde{A}_j^\dagger \tilde{R}_j := \Sigma_{j=1}^N B_j S_j^\dagger B_j^\top := F$.
(1) holds with $C_0^2 = 1$ on the whole of V: Given $u \in V$, let $z_j := D_j^{-1} B_j (BD^{-1}B^\top)^{-1} u$ then

$$\Sigma_{j=1}^N B_j z_j = \Sigma_{j=1}^N B_j D_j^{-1} B_j (BD^{-1}B^\top)^{-1} u = u,$$

and
$\Sigma_{j=1}^N \langle \tilde{A}_j z_j, z_j \rangle := \Sigma_{j=1}^N \langle S_j z_j, z_j \rangle = \langle M_{FETI}^{-1} u, u \rangle := \langle Au, u \rangle$.
(2) rewrites: $\langle M_{FETI}^{-1} B_j u_j, B_j u_j \rangle \leq \omega \langle S_j u_j, u_j \rangle$.

Definition: FETI GenEO coarse space

Find generalized eigenpairs $(\Lambda_j^k, p_j^k) \in \mathbb{R}^+ \times V_j$ of

$$S_j p_j^k = \Lambda_j^k B_j^\top M_{FETI}^{-1} B_j p_j^k.$$

Choose a threshold τ and define

$$V_H = \text{span}\{M_{FETI}^{-1} B_j p_j^k; 0 < \Lambda_j^k < \tau, j = 1, \dots, N\}.$$

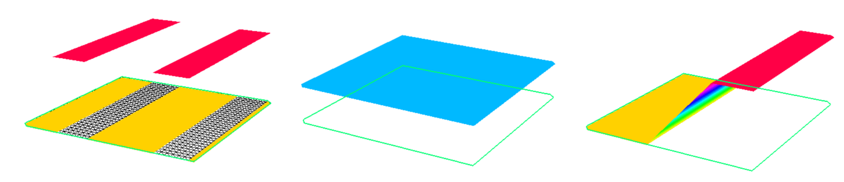
Then

$$cond(M_{2level}^{-1} P^\top F_{|\text{Im}(P)}) \leq \frac{\mathcal{N}}{\tau} \quad \begin{cases} \mathcal{N}: \text{'number of neighbours'} \\ \tau: \text{threshold chosen by user} \end{cases}$$

III) Does the Automatic coarse space find the 'best' coarse space?

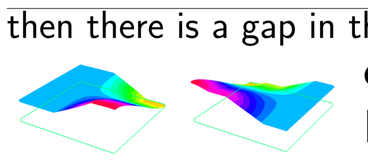
IIIA) Darcy in Heterogeneous media: $-\nabla \cdot (\alpha \nabla u) = 1$ with Schwarz-GenEO

Layers: Number of basis functions = Number of high coefficient layers.



$\alpha \qquad \Lambda^1 = 0 \qquad \Lambda^2 = 6 \cdot 10^{-7}$

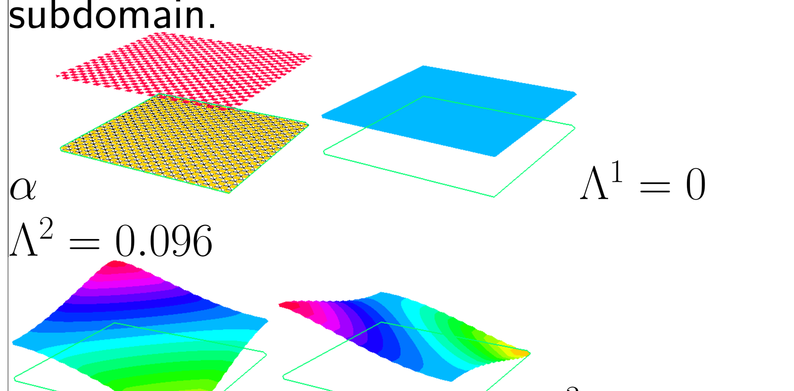
then there is a gap in the spectrum:



$\Lambda^3 = 0.11 \quad \Lambda^4 = 0.14$

eigenvectors 3 and 4 are NOT related to α and NOT in the coarse space.

Islands: One coarse function per subdomain.



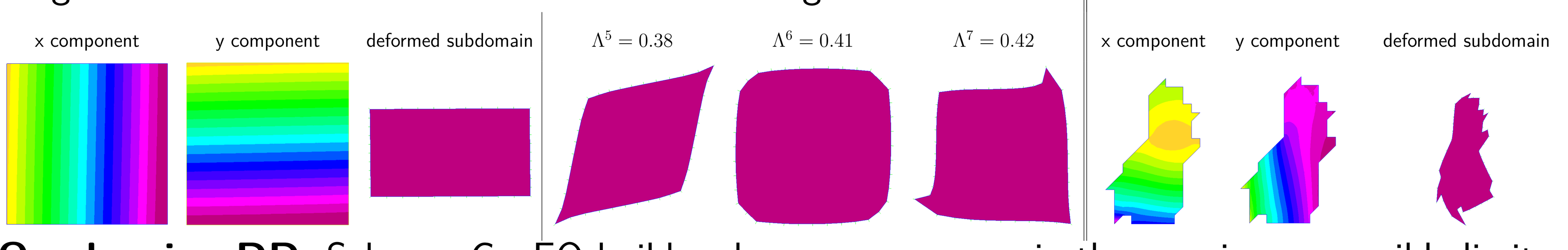
$\alpha \qquad \Lambda^1 = 0$

$\Lambda^2 = 0.096 \qquad \Lambda^3 = 0.096$

IIIB) Elasticity in the incompressible limit: $\nu = 0.4999$ ($\mathbb{P}_2 - \mathbb{P}_0$ pressure eliminated)

Non-overlapping DD:

Regular Subdomain: $\Lambda^4 = 0.003$ — Next 3 eigenvectors — Metis Subdomain: $\Lambda^4 = 0.04$

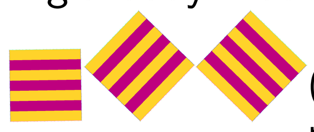


Overlapping DD: Schwarz-GenEO builds a huge coarse space in the near incompressible limit. There is no easy fix without going back to the PDEs and building a new "mass" matrix.

IIIC) 2D Elasticity (Pink: $E = 10^6$; Yellow: $E = 1$) with FETI-GenEO

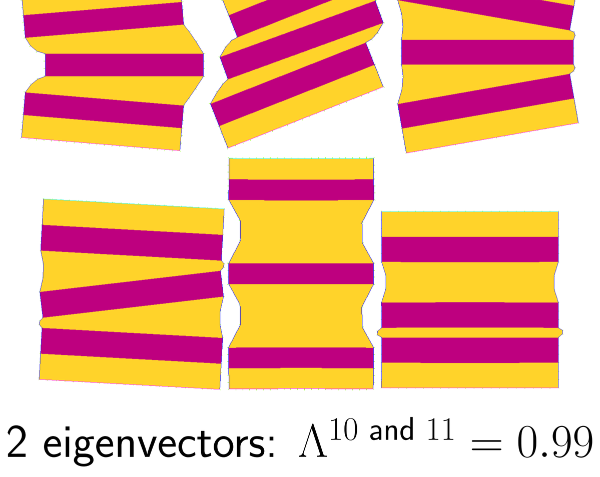
Layers: Number of basis functions = (Number of hard layers - 1) × Number of rigid body modes

- Rigid body modes $\Lambda^{1 \text{ to } 3} = 0$.

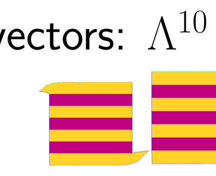


(the rigid body modes are already taken care of by FETI.)

- Coarse space: $\Lambda^{4 \text{ to } 9} = \{8 \cdot 10^{-4}; 9 \cdot 10^{-4}; 3 \cdot 10^{-3}; \cdot 10^{-2}; 4 \cdot 10^{-2}; 0.11\}$.

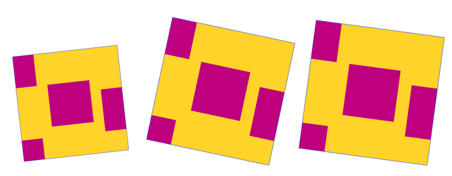


- Next 2 eigenvectors: $\Lambda^{10 \text{ and } 11} = 0.99$

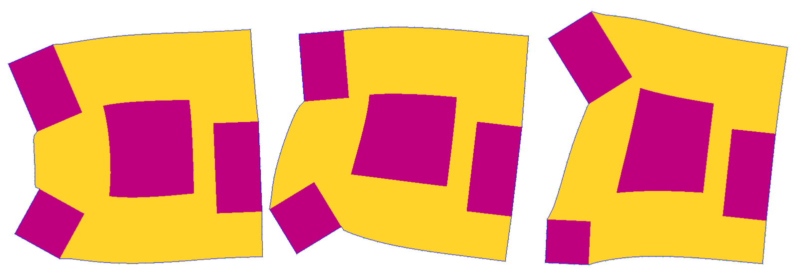


Islands: Only corner islands are picked up

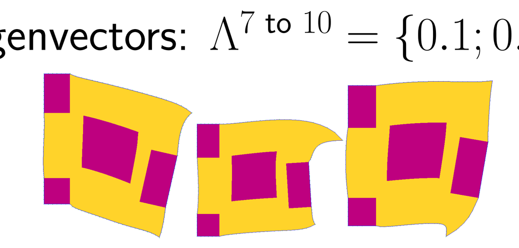
- Rigid body modes $\Lambda^{1 \text{ to } 3} = 0$.



- Coarse space: $\Lambda^{4 \text{ to } 6} = \{1 \cdot 10^{-6}; 1 \cdot 10^{-5}; 2 \cdot 10^{-5}\}$.



- Next 3 eigenvectors: $\Lambda^{7 \text{ to } 10} = \{0.1; 0.3; 0.4\}$



Remark: this is in agreement with a heuristic generalization to elasticity of

C. Pechstein, R. Scheichl, Numerische Mathematik (2011).