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EXISTENCE OF STEADY TWO-PHASE FLOWS WITH DISCONTINUOUS BOILING EFFECTS

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Abstract. We aim at characterizing the existence and uniqueness of steady solutions to hyperbolic balance laws with source terms depending discontinuously on the unknown. We exhibit conditions for such differential equations to be well-posed and apply it to a model describing boiling flows.

1. Introduction.

The aim of this paper is to present a framework for the study of steady states of 1D balance laws with sources defined as a discontinuous function of the unknown. Such steady states satisfy systems of the form

$$\frac{d}{dx}F(U)(x) = S(U(x)), \tag{1a}$$

where the source jumps when a certain function h reaches a threshold, *i.e.*

$$S(U) = \begin{cases} S^-(U) & \text{if} \quad h(U) < 0, \\ S^+(U) & \text{if} \quad h(U) \geq 0. \end{cases} \tag{1b}$$

The discontinuity of S with respect to the unknown leads to both theoretical and numerical difficulties. Especially, Picard-Lindelöf theory is unavailable and extensions are required.

The application we have in mind is the study of boiling flows. We aim at studying the homogenized two-phase flow model based on a drift-flux model ([11, 10, 9]) used for the developpement of the FLICA4 code ([15, 3, 14])

$$\partial_t U + \partial_x F(U) = S(U), \tag{2a}$$

$$U = \left(\alpha\rho_v,\ \rho,\ \rho u,\ \rho\left(e + \frac{u^2}{2}\right)\right)^T, \tag{2b}$$

$$F(U) = \left(\alpha\rho_v u,\ \rho u,\ \rho u^2 + p,\ \rho\left((e + \frac{u^2}{2} + \frac{p}{\rho})u\right)\right)^T, \tag{2c}$$

$$S(U) = \begin{cases} (0,\ 0,\ 0,\ \phi)^T & \text{if} \quad h(U) < h^b, \\ (K\phi,\ 0,\ 0,\ \phi)^T & \text{if} \quad h(U) \geq h^b, \end{cases} \tag{2d}$$

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with a constant $K > 0$. Here, $\alpha\rho_v$ is the density of vapor alone, and ρ, ρu, ρe are the density, momentum and energy of the homogenized flow, *i.e.* of liquid and vapor together. The source term models the heating of the fluid, through the term $\phi > 0$ in the energy equation, and the creation of vapor (in the first equation) when the enthalpy h is above a boiling threshold h^b.

In the next two sections, we first present a framework that guarantees the existence and uniqueness of solution of (1), first on a very simple scalar case, then on a more general vectorial framework. This is applied to the problem (2) in Section 4. Section 5 is devoted to conclusion and outlooks.

2. **Preliminaries.**

Consider the Cauchy problem

$$\frac{dU}{dx} = S(U, x), \qquad U(0) = U_0. \tag{3}$$

Here, $S : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is a function of $U \in \mathbb{R}^N$ and $x \in \mathbb{R}$ that may be discontinuous. As S is not continuous, we need a definition of solutions to (3) in a weak sense.

Definition 2.1. Let I be an open interval of $\mathbb{R}$ containing 0. A function $U : I \subset \mathbb{R} \to \mathbb{R}$ is a Carathéodory solution to (3) if it is absolutely continuous and satisfies

$$\forall x \in I, \qquad U(x) = U_0 + \int_0^x S(U(y), y)dy.$$

In order to illustrate the difficulties emerging with discontinuous right-hand-side (RHS) in (3), let us first consider the following simple scalar case (inspired by [12, 8])

$$\frac{d}{dx}u = \begin{cases} s^- & \text{if} \quad u < 0, \\ s^+ & \text{if} \quad u \geq 0, \end{cases} \qquad u(0) = u_0. \tag{4}$$

The behavior of u away from 0 is well understood. Difficulties arise when u reaches 0. We can list three types of behavior (represented on Fig. 1):

1. If $s^- \geq 0$ and $s^+ \leq 0$, then for all $u_0 \in \mathbb{R}$

$$u(x) = \begin{cases} u_0 + s^- x & \text{if} \quad u_0 \leq 0 \quad \text{and} \quad x \leq \dfrac{-u_0}{s^-}, \\ u_0 + s^+ x & \text{if} \quad u_0 \geq 0 \quad \text{and} \quad x \leq \dfrac{-u_0}{s^+}. \end{cases} \tag{5a}$$

However this solution can not be extended for x larger than $u_0/s^\pm$.

2. If $s^- \leq 0$ and $s^+ \geq 0$, then for all $u_0 \in \mathbb{R}$

$$u(x) = \begin{cases} u_0 + s^- x & \text{if} \quad u_0 \leq 0, \\ u_0 + s^+ x & \text{if} \quad u_0 \geq 0. \end{cases} \tag{5b}$$

Remark that, if $u_0 = 0$, the functions $x \mapsto s^- x$ and $x \mapsto s^+ x$ are two Carathéodory solutions of (4).

3. If s^- and s^+ have strictly the same sign, say positive, then for all $x \geq 0$,

$$u(x) = \begin{cases} u_0 + s^+ x & \text{if} \quad u_0 \geq 0, \\ u_0 + s^- x & \text{if} \quad u_0 \leq 0 \quad \text{and} \quad x \leq \dfrac{-u_0}{s^-}, \\ u_0 + s^- \dfrac{-u_0}{s^-} + s^+ \left(x - \dfrac{-u_0}{s^-} \right) & \text{if} \quad u_0 \leq 0 \quad \text{and} \quad x \geq \dfrac{-u_0}{s^-}. \end{cases} \tag{5c}$$

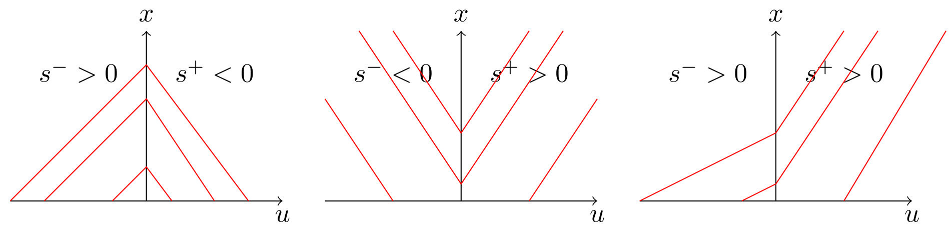

FIGURE 1. Solutions of (5) depending on the signs of s^- and s_+: from left to right, solutions of (5a), (5b) and (5c)

The solutions defined in these three cases are depicted in the phase space (x, u) on Fig. 1. Remark that on this simple example, neither existence nor uniqueness of a solution is guaranteed. Thus, further considerations are necessary to obtain the well-posedness of (3) in a general case or of (1) for our applications.

In the next section, we focus on a vectorial ODE. We prove its well-posedness under a condition corresponding to a vectorial version of the third case (5c).

3. **A framework for ODE with Heaviside RHS.**

Consider now the problem

$$\frac{d}{dx}U(x) = \begin{cases} S^-(U(x), x) & \text{if} \quad h(U(x)) < 0, \\ S^+(U(x), x) & \text{if} \quad h(U(x)) \geq 0, \end{cases} \qquad U(0) = U_0, \tag{6}$$

where the unknown $U(x) \in \mathbb{R}^N$ is vectorial and the enthalpy $h(U)$ is scalar.

We seek a natural framework for (6) to be well-posed. The result below could be obtained as a corollary of *e.g.* [13, 4, 5] or through Filippov's theory ([7, 1, 6]). Here we present a simple condition on the surface $h(U) = 0$ under which any solution changes sign at most once. The solution is then obtained by gluing together two solutions obtained with the Picard-Lindelöf theorem.

Lemma 3.1. *Suppose that*

- $h \in C^1(\mathbb{R}^N, \mathbb{R})$,
- *Both S^- and S^+ satisfy the hypothesis of the Picard-Lindelöf theorem: continuity with respect to x and locally Lipschitz continuity with respect to U,*
- $\forall x \in \mathbb{R}$, *and* $\forall V \in \mathbb{R}^N$, *such that* $h(V) = 0$,

$$(\nabla_U h(V).S^-(V, x)) > 0 \quad \textit{and} \quad (\nabla_U h(V).S^+(V, x)) > 0. \tag{7}$$

Then, for any Carathéodory solutions $\bar{U}$ to (6), there exists at most one point $x_0 \in \mathbb{R}$ such that $h(\bar{U})(x_0) = 0$, and $h(\bar{U})$ is strictly negative on $x < x_0$ and strictly positive on $x > x_0$.

Remark 1. The vector $\nabla_U h(V)$ is normal to the hypersurface $\{U \in \mathbb{R}^N, \text{ s.t. } h(U) = 0\}$. Thus, the condition (7) imposes that the vector fields S^- and S^+ are both pushing the solution toward the same side of $h(U) = 0$. The solution is then constructed by following S^- until it reaches $h(U) = 0$, and then following S^+ (see Fig. 2).

Proof. First, we remark that, as h is $C^1(\mathbb{R}^N, \mathbb{R})$ and the Carathéodory solution $\bar{U}$ is absolutely continuous, then $h(\bar{U})$ is continuous and has a derivative almost everywhere which is

$$\frac{d}{dx}h(\bar{U})(x) = \nabla_U h(\bar{U})(x).S(\bar{U}(x), x). \tag{8}$$

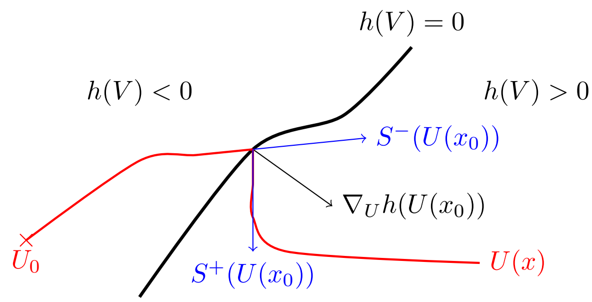

FIGURE 2. Representation, for a problem of the form (6) in $\mathbb{R}^2$, of the solution $U(x) \in \mathbb{R}^2$, the hypersurface $\{V \in \mathbb{R}^2 \text{ s.t. } h(V) = 0\}$ and the vectors $S^-(U(x_0))$, $S^+(U(x_0))$ and $\nabla_U h(U(x_0))$

Then, assume there exists a point x_0 such that $h(\bar{U}(x_0)) = 0$. Then, for all $y \geq 0$ (we may reason similarly for $y < 0$),

$$\begin{aligned} h(\bar{U})(x_0 + y) &= \int_{x_0}^{x_0+y} \nabla_U h(\bar{U})(x).S(\bar{U}(x), x)dx \\ &\geq \int_{x_0}^{x_0+y} \min(\nabla_U h(\bar{U})(x).S^-(\bar{U}(x), x), \nabla_U h(\bar{U})(x).S^+(\bar{U}(x), x))dx, \end{aligned}$$

The function in the last integral is continuous and strictly positive at $x = x_0$ by (7). Thus there exists $\epsilon > 0$ such that

$$\forall x \in]x_0, x_0 + \epsilon[, \quad h(\bar{U})(x) > 0, \quad \text{and} \quad \forall x \in]x_0 - \epsilon, x_0[, \quad h(\bar{U})(x) < 0. \tag{9}$$

Suppose by contradiction that there exists $x_1 > x_0$ such that $h(\bar{U})(x_1) = 0$. The continuity of $h(\bar{U})$ and (9) yield the existence of x_2 in (x_0, x_1), such that $h(\bar{U}(x_2)) = 0$. Repeating this operation, we construct a sequence $(x_i)_{i\in\mathbb{N}}$ of distinct points where $h(\bar{U})$ is null, and that converges towards a limit denoted by x_∞ by dichotomy. Considering that

$$\begin{aligned} |h(\bar{U})(x_\infty)| &= \left| h(\bar{U})(x_i) + \int_{x_i}^{x_\infty} \nabla_U h(\bar{U})(x).S(\bar{U}(x), x)dx \right| \\ &\leq |x_\infty - x_i| \|\nabla_U h(\bar{U})(x)\|_{\infty,[x_0,x_1]} \|S(\bar{U}(x), x)\|_{\infty,[x_0,x_1]}. \end{aligned}$$

we obtain $x_i \to_{i\to+\infty} x_\infty$ and $h(\bar{U}(x_\infty)) = 0$, which contradicts the existence of the interval (9). Once we know that $h(\bar{U})$ has at most one zero, (9) gives the sign of $h(\bar{U})$ on both sides. □

Proposition 1. *Under the hypothesis of Lemma 3.1, for all initial conditions $U_0 \in \mathbb{R}^N$, there exists a unique maximal solution U to (6) that is absolutely continuous. Furthermore, this solution U depends continuously on U_0.*

Proof. We prove the case $h(U_0) < 0$, the other one being completely similar. According to Lemma 3.1, there is at most one point x_0 where $h(U)$ switches sign, and as $h(U(0)) < 0$ it is larger than 0. Thus, any Carathéodory solution U takes the form

$$U(x) = U_0 + \begin{cases} \displaystyle\int_0^x S^-(U(y), y)dy \quad \text{if} \quad x < x_0, \\ \displaystyle\int_0^{x_0} S^-(U(y), y)dy + \int_{x_0}^x S^+(U(y), y)dy \quad \text{otherwise.} \end{cases} \tag{10}$$

The existence and uniqueness follows from the Picard-Lindelöf theory. Indeed on $x < x_0$ the solution coincides with the solution of the Cauchy problem

$$V'(x) = S^-(V(x), x), \qquad V(0) = U_0$$

which exists and is unique as S^- is continuous and locally Lipschitz continuous with respect to its first variable. Then on $x \geq x_0$, it coincides with the solution of the Cauchy problem

$$V'(x) = S^+(V(x), x), \qquad V(x_0) = U(x_0).$$

To conclude the proof it remains to show that x_0 is a continuous function of the initial data U_0. Fix U_0 and x_0 such that

$$\varphi(U_0, x_0) = h(U(x_0)) = h\left(U_0 + \int_0^{x_0} S^-(U(y), y)dy\right) = 0$$

As $\frac{\partial\varphi}{\partial x_0}(U_0, x_0) = \nabla_U h\left(U(x_0)\right) \cdot S^-(U(x_0), x_0)$ is not null by (7), the implicit function theorem yields the result. $\square$

4. **Application to homogenized two-phase fluid models.**

First, we rewrite Proposition 1, then we apply it to a reformulation of (2).

4.1. **With a non-linear flux.**

When the flux function F in (1) is non-linear, we may simply adapt Proposition 1 into the following result.

Corollary 1. *Suppose that*

- $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$,
- $h \in C^1(\mathbb{R}^N, \mathbb{R})$,
- S^- *and* S^+ *are continuous w.r.t.* x *and locally Lipschitz continuous w.r.t.* U,
- $\forall x \in [0, L]$, *and* $\forall V \in \mathbb{R}^N$, *s.t.* $h(V) = 0$,

$$\nabla_U h(V).(DF(V))^{-1}.S^-(V, x) > 0, \text{ and } \nabla_U h(V).(DF(V))^{-1}.S^+(V, x) > 0. \tag{11}$$

Then, for all initial conditions $U_0 \in \mathbb{R}^N$ *satisfying* $\det(DF(U_0)) \neq 0$, *there exists a unique maximal solution* U *to* (1) *absolutely continuous and satisfying* $\det(DF(U)) \neq 0$. *Furthermore, this solution depends continuously on* U_0.

Remark 2. Requiring that $DF(U)$ is invertible corresponds to imposing that the flows remains subsonique and admissible, which is commonly admitted for practical applications. This condition may restrict the size of the spatial domain.

Proof. Any Carathéodory solution U to (1) is differentiable almost everywhere. Thus, as $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, then $F(U)$ is absolutely continuous and differentiable almost everywhere, and its derivative equals almost everywhere

$$\frac{d}{dx}F(U)(x) = DF(U)(x).\frac{d}{dx}U(x).$$

Thus any solution to (1) of such regularity and satisfying $\det(DF(U)) \neq 0$, also solves the Cauchy problem

$$\frac{d}{dx}U(x) = \begin{cases} (DF(U)(x))^{-1}.S^{-}(U(x),x) & \text{if} \quad h(U)(x) < 0, \\ (DF(U)(x))^{-1}.S^{+}(U(x),x) & \text{if} \quad h(U)(x) \geq 0. \end{cases} \tag{12}$$

Using Proposition 1 and the hypothesis, (12) has a unique solution U and it depends continuously on U_0. $\square$

4.2. **On the boiling flow model.** Now, we aim to apply this result to (1). In order to apply Corollary 1, we rewrite the problem with a new set of unknowns $\tilde{U}$ such that

- we can perform the computations required in (11) ;
- it has a physical interpretation.

We chose for variables

$$\tilde{U} = (c_v,\ q,\ p,\ h),$$

where c_v is the volume fraction of vapor, q is the momentum. The enthalpy h is chosen among the variables to simplify the definition of $\nabla_U h$ and q to simplify the definition of $D\tilde{F}$. These variables $\tilde{U}$ are commonly defined based on U as

$$\tilde{U} = \phi^{-1}(U) = \left(\frac{\alpha\rho_v}{\rho},\ \rho u,\ p,\ e + \frac{p}{\rho}\right), \qquad U = \phi(\tilde{U}) = \left(\frac{c_v}{\tau},\ \frac{1}{\tau},\ q,\ \frac{h}{\tau} - p + \frac{\tau q^2}{2}\right),$$

where $\tau = \frac{1}{\rho}$ is the specific volume. We close the new system, not by expressing p as a function of U (it is a variable in the new system), but by fixing

$$\tau = c_v\tau_v + (1 - c_v)\tau_l,$$

as a convex combination of the vapor and liquid specific volumes τ_v and τ_l, where $\tau_v(p,h)$ and $\tau_l(p,h)$ are given $C^1(\mathbb{R}^2, \mathbb{R})$ functions of p and h, and independent of q and c_v. These functions are commonly tabulated.

Rewriting the steady state of (2) in terms of $\tilde{U}$ reads

$$\frac{d}{dx}\tilde{F}(\tilde{U}) = \tilde{S}(\tilde{U}) \tag{13}$$

$$\tilde{F}(\tilde{U}) = F \circ \phi(\tilde{U}) = \left(c_v q,\ q,\ \tau q^2 + p,\ \frac{\tau^2 q^3}{2} + qh\right),$$

$$\tilde{S}(\tilde{U}) = S \circ \phi(\tilde{U}) = \begin{cases} (0, & 0, & 0, & \phi) & \text{if } h < h^b, \\ (K\phi, & 0, & 0, & \phi) & \text{if } h \geq h^b. \end{cases}$$

We obtain in the end the following requirement.

Proposition 2. *Suppose that*

$$\forall p \in \mathbb{R}^+, \qquad q^2 \frac{\partial \tau}{\partial p}(p, h^b) + 1 \ > \ Kq^2[\tau(\tau_v - \tau_l)](p, h^b) \ > \ 0. \tag{14a}$$

Then, for all boundary conditions $\tilde{U}(0) = \tilde{U}_0 = (c_{v,0}, q_0, p_0, h_0)$ satisfying

$$q_0 \neq 0, \qquad \text{and} \qquad q_0^2 \left(\frac{\partial \tau}{\partial p} + \tau \frac{\partial \tau}{\partial h}\right)(p_0, h_0) + 1 \ \neq \ 0, \tag{14b}$$

there exists a unique maximal solution U absolutely continuous to (13). Furthermore, this solution depends continuously on $\tilde{U}_0$.

Remark 3. Condition (14b) corresponds to imposing that the flow remains subsonique. This formula is obtained by imposing the invertibility of $D\tilde{F}(\tilde{U})$ which is necessary and sufficient to ensure the uniqueness of a steady solution $\tilde{U}$. Of course, one also need ϕ to be a bijection to ensure the existence of a unique solution U to the original equation.

The formula (14b) refers not directly to the speed of sound, because in a non-steady framework, $\tilde{U}$ is not transported, but U is. The speed of sound would be obtained from the eigenvalues of $DF(U) = D\tilde{F}(\phi^{-1}(U)).D\phi^{-1}(U)$. In the incompressible case $\partial_p \tau = 0$, one finds after computations that those eigenvalues are $\tau q \pm \sqrt{\tau / \frac{\partial \tau}{\partial h}}$ and twice τq, where one identifies the velocity $u = \tau q$ and the speed of sound yields $c = \sqrt{\tau / \frac{\partial \tau}{\partial h}}$.

Proof. First, one verifies that $\frac{dq}{dx} = 0$, thus $q \neq 0$ is constant and (13) reduces to

$$\frac{d}{dx}\bar{F}(\bar{U}) = \bar{S}(\bar{U}) \qquad \bar{S}(\bar{U}) = \begin{cases} (0, \quad 0, \quad \phi) & \text{if} \quad h < h^b, \\ (K\phi, \quad 0, \quad \phi) & \text{if} \quad h \geq h^b, \end{cases}$$

$$\bar{U} = (c_v,\ p,\ h), \qquad \bar{F}(\bar{U}) = \left(c_v q,\ \tau q^2 + p,\ q\left(\frac{\tau^2 q^2}{2} + h\right)\right).$$

One computes

$$D\bar{F}(\bar{U}) = \begin{pmatrix} q & 0 & 0 \\ q^2(\tau_v - \tau_l) & q^2 \frac{\partial \tau}{\partial p} + 1 & q^2 \frac{\partial \tau}{\partial h} \\ q^3 \tau(\tau_v - \tau_l) & q^3 \tau \frac{\partial \tau}{\partial p} & q\left(q^2 \tau \frac{\partial \tau}{\partial h} + 1\right) \end{pmatrix}, \tag{15}$$

the determinant of which yields

$$Det := \det\left(D\bar{F}(\bar{U})\right) = q^2\left[q^2\left(\frac{\partial \tau}{\partial p} + \tau \frac{\partial \tau}{\partial h}\right) + 1\right],$$

which is non-zero at the boundary by hypothesis. Inverting (15) yields

$$(D\bar{F}(\bar{U}))^{-1} = \frac{1}{Det}\begin{pmatrix} q\left(1 + q^2\left(\frac{\partial \tau}{\partial p} + \tau \frac{\partial \tau}{\partial h}\right)\right) & 0 & 0 \\ -q^3(\tau_v - \tau_l) & q^2\left(q^2 \tau \frac{\partial \tau}{\partial h} + 1\right) & -q^3 \frac{\partial \tau}{\partial h} \\ -q^3 \tau(\tau_v - \tau_l) & -q^4 \tau \frac{\partial \tau}{\partial p} & q\left(q^2 \frac{\partial \tau}{\partial p} + 1\right) \end{pmatrix}.$$

Multiplying it by the source term and by $\nabla_{\bar{U}} h(\bar{U}) = (0,\ 0,\ 1)$ leads to

$$\nabla_{\bar{U}} h(V).(D\bar{F}(V))^{-1}.\bar{S^-}(V,x) = \frac{q\phi}{Det}\left(1 + q^2 \frac{\partial \tau}{\partial p}\right),$$

$$\nabla_{\bar{U}} h(V).(D\bar{F}(V))^{-1}.\bar{S^+}(V,x) = \frac{q\phi}{Det}\left(1 + q^2\left(\frac{\partial \tau}{\partial p} - K\tau(\tau_v - \tau_l)\right)\right).$$

If (14b) holds, these two values are positive and we may apply Corollary 1. □

5. **Conclusion and outlook.** We have described, in a theoretical framework, a set of conditions providing the existence and uniqueness of a steady solution, in the sense of Carathodory, to hyperbolic systems of balance laws. We have applied it for the study of a boiling flow model. The resulting conditions on the physical parameters for such steady flows to exists are twofold. First, the flow needs to remain subsonic in the whole spatial domain, this constrains the domain length and the boundary conditions. Second, if the source is discontinuous along an hypersurface in the phase space, then the source and the flux on both sides need to be defined

in such a way that the flow may only cross the discontinuity hypersurface in one direction.

In the present work, we have only considered boundary conditions on one sides, which suffice to study time independent flow. Though, it is more common in this field to use two boundaries with further requirements (see typically [2] for unsteady flows).

At the numerical level, capturing equilibrium states such as steady states for balance laws has been widely studied. Though, the discontinuity of source terms of the form (1) brings new difficulties, the study of which is left for future work.

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REFERENCES

[1] J.-P. Aubin and A. Cellina. *Differential Inclusions, Set-Valued Maps And Viability Theory*. Grundl. der Math. Wiss. Springer, 1984.

[2] C. Bardos, A.-Y. Le Roux, and J.-C. Nédélec. First order quasilinear equations with boundary conditions. *Comm. Partial Diff. Equ.*, 4:1017–1034, 1979.

[3] A. Bergeron and I. Toumi. Assessment of the flica-iv code on rod bundle experiments. *Proceedings of ICONE-6, San Diego, California, USA*, 1998.

[4] A. Bressan. Unique solutions for a class of discontinuous differential equations. *Proceedings of the American Mathematical Society*, 104(3):772–778, 1988.

[5] A. Bressan and G. Colombo. Existence and continuous dependence for discontinuous o.d.e.'s. *Boll. Un. Mat. Ital. 4-B*, pages 295–311, 1909.

[6] H. Brézis. *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland publishing company, 1973.

[7] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Springer, 1988.

[8] O. Hajek. Discontinuous differential equations. *Int. J. Differential Equations*, 32:149–170, 1979.

[9] T. Hibiki and M. Ishii. One-dimensional drift-flux model and constitutive equations for relative motion between phases in various two-phase flow regimes. *Int. J. Heat Mass Transfer*, 46:4935–4948, 2003.

[10] M. Ishii. One dimensional drift-flux model and constitutive equations for relative motion between phases in various two-phase flow. Technical report, ANL, 1977.

[11] M. Ishii and T. Hibiki. *Thermo-fluid dynamics of two-phase flows*. Springer, 2011.

[12] C. Lobry and T. Sari. Équations différentielles à second membre discontinu. *Contrôle non linéaire et Applications*, 64:255–289, 2005.

[13] A. Pucci. Sistemi di equazioni differenziali con secondo membro discontinuo rispetto all'incognita. *Rend. Ist. Mat. Univ. Trieste*, III:75–80, 1971.

[14] E. Royer, S. Aniel, A. Bergeron, P. Fillion, D. Gallo, F. Gaudier, O. Grégoire, M. Martin, E. Richebois, P. Salvadore, S. Zimmer, T. Chataing, P. Clément, and F. François. Flica4: status of numerical and physical models and overview of applications. *Proceedings of NURETH-11, Avignon, France*, 2005.

[15] I. Toumi, A. Bergeron, D. Gallo, E. Royer, and D. Caruge. Flica-4: A three-dimensional two-phase flow computer code with advanced numerical methods for nuclear applications. *Nuclear Engineering and Design*, 200, 2000.

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