Optimal Production Policy under the Carbon Emission Market *

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Abstract

The aim of this paper is to address the effect of the carbon emission market on the production policy of the emitting production firms. We investigate this effect in the cases where there is no large carbon producer, where there is a large producer who can not affect the risk premia, and where there is a large producer who can change the risk premia by its production. We ignore any possible investment of the production firm in pollution reducing technologies. We formulate optimal production policy by a stochastic optimization problem. Then, we show that the market reduces the optimal production policy of the small producer and the large producer who does not affect the risk premia of the market. However, dependent on the way the large producer activities change the market risk premia, the large producer can optimally produce more than what she used to do before the existence of the emission market.

Key words: EU ETS, Carbon allowance paper, Optimal production policy, HJB equations

AMS 2000 subject classifications: 93E20, 91B60, 91B70.

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1 Introduction

The long term costs of global warming is believed to be significantly more than the cost of controlling it by reducing the pollution due to greenhouse gases (see [8]). One direct way to reduce the emission is to impose the taxation on the installations whose production increases the pollution. One can propose the standard taxation system which imposes a limitation level on the production of each installation over a time period and any amount of production above this level will be penalized. This taxation method has some significant disadvantages. First, there is no change in the production of the installations whose current optimal production policy does not reach the level. Second, there is no benefit for those who are below their level to keep their position. This effect also creates incentive to merge with other installation who needs to produce above their level.

The Kyoto protocol in 1997 concerns with the reduction of the greenhouse gases including CO_2 and is accepted by several countries e.g. European Union members. In 2000, the European Commission launch European Climate Change Program (ECCP) to implement Kyoto protocol in Europe. As an alternative to standard taxation, ECCP proposed European Union Emission Trading Scheme (EU ETS) which provides a way to control the emission of CO_2 within carbon polluters through trading the papers which allows them extra emission. More precisely, ETS imposes a cap over the total carbon emission. Within ETS, certain industrial installations with intensive carbon pollution are given free allowances. If any installation wants to produce more than her initial allowance, she should buy allowance through EU ETS. However, the allowances will be needed if the total carbon emission per member state violates imposed cap. On the other hand, if such installations, are far away from their production limit, they could sell their allowance through the market.

First phase of the program was run from January 2005 to the end of 2007. All the included installations who violate their limits, were supposed to provide enough allowances, if the cap on total emission is reached. The cap for the second phase (2008–2012) has been revised after the collapse in the first phase in April 2006 due to the release of the information about the unreachability to total carbon emission cap. Moreover in the second phase, ECCP proposed to relevant installations to put off execution of the first phase emission allowance to the second phase by paying 40 euros per tone. The same mechanism is determined for second phase and third phase by the cost of 100 euros per tone. This mechanism, which is referred to as *banking*, proposes an option for the allowance holder to execute the allowance to offset the excess production or to keep it for the next phase. For more details see [2], [3], [4], [5] [6] and [8].

Nowadays, there are other regional markets implementing similar schemes as EU ETS, e.g. the US REgenial CLean Air Incentive Market (RECLAIM) or Regional Greenhouse Gas Incentive (RGGI). Throughout this paper, by emission market we mean the emission trading scheme EU ETS.

In this paper, we analyze the effect of emission market in reducing the carbon emission through the change on production policy of the relevant firms. The firm's objective is to maximize her utility on her wealth which is made of both the profit of her production and the value of her carbon allowance portfolio over her production and her portfolio strategy. We solve the utility maximization problem on portfolio strategy by the duality argument and then on the production by the use of Hamilton–Jacobi–Bellman (HJB) equations.

We observe that the market always reduces the optimal production policy of the small producers and large producers who can not affect the risk premia. However, under certain cases, the large producer can have a larger optimal production in the market. The comparison is based on the fact that negative of the derivative of the value function with respect to the total emission imposed by the firm is equal to the price of the carbon allowance, under some assumptions.

2 Small producer with one-period carbon emission market

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a one-dimensional Brownian motion W. We denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the completed canonical filtration of the Brownian motion W, and by $\mathbb{E}_t := \mathbb{E}[\cdot|\mathcal{F}_t]$ the conditional expectation operator given $\mathcal{F}_t$.

We consider a production firm with preferences described by the utility function $U : \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$ assumed to be strictly increasing, strictly concave and C^1 over $\{U < \infty\}$. We denote by $\pi_t(\omega, q)$ the (random) time t

rate of profit of the firm for a production rate q. Here $\pi : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is an $\mathbb{F}-$progressively measurable map. As usual we shall omit ω from the notations. For fixed (t, ω), we assume that the function $\pi_t(\cdot) := \pi(t, \cdot)$ is strictly concave, C^1 in q and satisfies

$$\pi'_t(0+) > 0 \quad \text{and} \quad \pi'_t(\infty) < 0.$$

Let us denote by $e_t(q_t)$ the rate of carbon emissions generated by a production rate q. Here, $e_{\cdot}(.) : \Omega \times [0, T] \times \mathbb{R}_+$ is an $\mathbb{F}-$progressively measurable map and C^1 in $q \in \mathbb{R}_+$. Then the total quantity of carbon emissions induced by a production policy $\{q_t, t \in [0, T]\}$ is given by

$$E_T^q := \int_0^T e_t(q_t) dt. \tag{2.1}$$

The aim of the carbon emission market is to incur this cost to the producer so as to obtain an overall reduction of the carbon emissions.

From now on, we analyze the effect of the presence of the carbon emission market within the *cap-and-trade* scheme.

In order to model the carbon emission market, we introduce an (unobservable) state variable Y defined by the dynamics:

$$dY_t \quad = \quad \mu_t dt + \gamma_t dW_t, \tag{2.2}$$

where μ and γ are two bounded $\mathbb{F}-$adapted processes and $\gamma > 0$.

We assume that there is one single period $[0, T]$ during which the carbon emission market is in place. The case of multiple successive periods will be analyzed later. At each time $t \geq 0$, the random variable Y_t indicates the market view of the cumulated carbon emissions. At time T, $Y_T \geq \kappa$ (resp. $Y_T < \kappa$) means that the cumulated total emission have (resp. not) exceeded the cap κ, fixed by the trading scheme. Let α be the penalty per unit of carbon emission. Then, the value of the carbon emission contract at time T is:

$$S_T \quad := \quad \alpha \mathbb{1}_{\{Y_T \geq \kappa\}}.$$

The carbon emission allowance can be viewed as a derivative security defined by the above payoff. The carbon emission market allows for trading this contract in continuous-time throughout the time period $[0, T]$. Assuming that the market is frictionless, it follows from the classical no-arbitrage valuation theory that the price of the carbon emission contract at each time t is given by

$$S_t \ := \ \mathbb{E}_t^{\mathbb{Q}}\left[S_T\right] \ = \ \alpha \mathbb{Q}_t\left[Y_T \geq \kappa\right], \tag{2.3}$$

where $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$, the so-called equivalent martingale measure, $\mathbb{E}_t^{\mathbb{Q}}$ and $\mathbb{Q}_t$ denote the conditional expectation and probability given $\mathcal{F}_t$. Given market prices of the carbon allowances, the risk-neutral measure may be inferred from the market prices. Since the market is frictionless, the value of the initial holdings in (free) allowances, $E^{\max}$, can be expressed equivalently in terms of their value in cash $S_0 E^{\max}$.

In the present context, and in contrast with an otherwise standard taxation based regulation (Remark 2.1), production firms have a clear incentive to reduce emissions as they have the possibility to sell their allowances on the emission market. Hence, the financial market induces a mutualization of carbon emissions. In particular, there is no incentive to merge for the single objective of avoiding the carbon taxes. We will see however that large producers can have a negative impact.

We now formulate the objective function of the firm in the presence of the emission market. The primary activity of the firm is the production modeled by the rate q_t at time t. This generates a gain $\pi_t(q_t)$. The resulting carbon emissions are given by $e_t(q_t)$. Given that the price of the externality is available on the market, the profit on the time interval $[0,T]$ is given by:

$$\int_0^T \pi_t(q_t)dt - S_T \int_0^T e_t(q_t)dt. \tag{2.4}$$

In addition to the production activity, the company trades continuously on the carbon emissions market. Let $\{\theta_t, t \geq 0\}$ be an $\mathbb{F}-$adapted process which is $S-$integrable. For every $t \geq 0$, θ_t indicates the number of contracts of carbon emissions held by the company at time t. Under the self-financing condition, the wealth accumulated by trading on the emission market is:

$$x + \int_0^T \theta_t dS_t, \tag{2.5}$$

where x is the initial capital of the company, including the market value of its free emission allowances contracts. By (2.4) and (2.5), together with an integration by parts, the total wealth of the firm at time T is

$$X_T^\theta + B_T^q \tag{2.6}$$

where

$$X_T^\theta := x + \int_0^T \theta_t dS_t, \qquad B_T^q := \int_0^T (\pi_t(q_t) - S_t e_t(q_t))\, dt - \int_0^T E_t^q dS_t,$$

and

$$E_t^q := \int_0^t e_u(q_u) du, \quad \text{for all} \quad t \in [0,T].$$

We assume that the firm is allowed to trade without any constraint. Then, the objective of the manager is:

$$V^{(1)} \quad := \quad \sup \left\{ \mathbb{E}\left[U\left(X_T^\theta + B_T^q \right)\right] : \ \theta \in \mathcal{A}, \ q \in \mathcal{Q} \right\}, \tag{2.7}$$

where $\mathcal{A}$ is the collection of all $\mathbb{F}-$progressively measurable processes such that the process X is bounded from below by a martingle, and $\mathcal{Q}$ is the collection of all non-negative $\mathbb{F}-$progressivey measurable processes.

Notice that the stochastic integrals with respect to S can be collected together in the expression of $X_T^\theta + B_T^q$. since $\mathcal{A}$ is a linear subspace, it follows that the maximization with respect to q and θ are completely decoupled, this problem is easily solved by optimizing successively with respect to q and θ. The partial maximization with respect to q provides an optimal production level $q^{(1)}$ defined by the first order condition:

$$\frac{\partial \pi_t}{\partial q}(q_t^{(1)}) \quad = \quad S_t \frac{\partial e_t}{\partial q}(q_t^{(1)}). \tag{2.8}$$

Because of the assumptions on $\pi_t(.)$ and $e_t(.)$, we deduce immediately that $q_t^{(1)}$ is less than the *business-as-usual* optimal production q^{bas} of the firm in the absence of any restriction on the emission, which is determined by trhe first order condition $(\partial \pi_t / \partial q)(q_t^{\text{bas}}) = 0$. In other words, the emission market leads to a reduction of the production, and therefore a reduction of the carbon emissions.

We next turn to the optimal trading strategy by solving:

$$\sup_\theta \mathbb{E}\left[U\left(X_T^{x,\theta - E^{q^{(1)}}} + B^{q^{(1)}} \right)\right] \quad \text{where} \quad B^q := \int_0^T (\pi_t(q_t) - S_t e_t(q_t))\, dt.$$

In the present context of a complete market, the solution is given by:

$$x + \int_0^T \left(\theta_t^{(1)} - E_t^{q^{(1)}} \right) dS_t + B^{q^{(1)}} \quad = \quad (U')^{-1}\left(y^{(1)} \frac{d\mathbb{Q}}{d\mathbb{P}} \right)$$

where the Lagrange multiplier $y^{(1)}$ is defined by:

$$\mathbb{E}^{\mathbb{Q}}\left[(U')^{-1}\left(y^{(1)}\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \quad = \quad x + \mathbb{E}^{\mathbb{Q}}\left[B^{q^{(1)}}\right].$$

Let us sum up the present context of a small firm:

- the trading activity of the company has no impact on its optimal production policy,
- the firm's optimal production $q^{(1)}$ is smaller than that of the business-as-usual situation, so that the emission market is indeed a good tool for the reduction of carbon emissions,
- the emission market assigns a price to the externality that the firm manager can use in order to optimize his production scheme.

Remark 2.1. Let us examine the case where there is no possibility to trade the carbon emission allowances. This is the standard taxation system where α is the amount of tax to be paid at the end of period per unit of carbon emission. Assuming again that the firm's horizon coincides with this end of period, its objective is:

$$V_0 \quad := \quad \sup_{q_\cdot \in \mathcal{Q}} \mathbb{E}\left[U\left(\int_0^T \pi_t(q_t)dt - \alpha\left(E_T^q - E^{\max}\right)^+\right)\right]$$

where $E^{\max}$ is the free allowances of the market. Direct calculation leads to the following characterization of the optimal production level:

$$\frac{\partial \pi_t}{\partial q}\left(q_t^{(0)}\right) \quad = \quad \alpha\frac{\partial e_t}{\partial q}\left(q_t^{(0)}\right)\mathbb{E}_t^{\mathbb{Q}^{(0)}}\left[\mathbb{1}_{\mathbb{R}_+}\left(E_T^{q_t^{(0)}} - E^{\max}\right)\right] \tag{2.9}$$

where

$$\frac{d\mathbb{Q}^{(0)}}{d\mathbb{P}} \quad = \quad \frac{U'\left(\int_0^T \pi_t(q_t^{(0)})dt - \alpha\left(E_T^{q_t^{(0)}} - E^{\max}\right)^+\right)}{\mathbb{E}\left[U'\left(\int_0^T \pi_t(q_t^{(0)})dt - \alpha\left(E_T^{q_t^{(0)}} - E^{\max}\right)^+\right)\right]}. \tag{2.10}$$

The natural interpretation of (2.9) and(2.10) is that the production firm assigns an individual price to its emissions:

$$S_t \quad := \quad \alpha\mathbb{E}_t^{\mathbb{Q}^{(0)}}\left[\mathbb{1}_{\mathbb{R}_+}\left(E_T^{q_t^{(0)}} - E^{\max}\right)\right], \tag{2.11}$$

i.e. the expected value of the amount of tax to be paid under the measure $\mathbb{Q}^{(0)}$ defined by her marginal utility as a density. The probability measure $\mathbb{Q}^{(0)}$ is the so-called risk-neutral measure in financial mathematics, or the stochastic discount factor of the firm. Given this evaluation, the firm optimizes her adjusted profit function, $\pi_t(q) - e_t(q)S_t$;

$$\frac{\partial \pi_t}{\partial q}(q^{(0)}) \quad = \quad \frac{\partial e_t}{\partial q}(q^{(0)})S_t.$$

We continue by commenting on the optimal production policy defined by (2.9)-(2.10):

- assuming that the firms know the nature of their utility functions, the system of equations (2.9)-(2.10) is still a nontrivial nonlinear fixed point problem.

- This problem would be considerably simplified if the manager were to know the market price for carbon emissions (2.11). But of course, in the present context, this is an individual subjective price which is not quoted on any financial market.

- The present situation, based on a classical taxation policy, offers no incentive to reduce emissions beyond $E^{\max}$. Indeed, if the optimal production in the absence of taxes produces carbon emissions below the level $E^{\max}$, then it is indeed the same as the business-as-usual situation. So, the taxation does not contribute to reduce the carbon emissions. As a consequence, the only way to benefit from having carbon emissions below the level $E^{\max}$ is to merge with another firm whose emissions are above its given free emissions allowances. Hence, such a policy puts a clear incentive to mergers. $\square$

The emission market provides an evaluation of the externality of carbon emissions by firms. Given this information there is no more need to know precisely the utility function of the firm in order to solve the nonlinear system (2.9)-(2.10). The quoted price of the externality is then very valuable for the managers as it allows them to better optimize their production scheme.

3 Large producer with one-period carbon emission market

In this section, we consider the case of a large carbon emitting production firm. We shall see that this leads to different considerations as the trading activity will have an impact on the production policy of the company.

We model this situation by assuming that the state variable Y is affected by the production policy of the firm:

$$dY_t^q = (\mu_t + \beta e_t(q_t))\, dt + \gamma_t dW_t \tag{3.1}$$

where $\beta > 0$ is a given impact coefficient. The price process S of the carbon emission allowances is, as in the previous section, given by the no-arbitrage valuation principle:

$$S_t^q = \alpha \mathbb{Q}_t^q \left[Y_T^q \geq \kappa \right], \tag{3.2}$$

and is also affected by the production policy q. The equivalent martingale measure $\mathbb{Q}^q$ is characterized by its Radon-Nykodim density which can be represented as a Doléans-Dade exponential martingale generated by some risk premium process λ. In general, the risk premium process λ may depend on the path of the control process q. For technical reasons, we shall restrict our anaysis to those risk-neutral probability measures with risk premium process depending on the current value of the control process:

$$\left.\frac{d\mathbb{Q}^q}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left(-\int_0^T \lambda_t(q_t) dW_t - \frac{1}{2}\int_0^T \lambda_t(q_t)^2 dt \right) \tag{3.3}$$

where $\lambda : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is an $\mathbb{F}-$progressively measurable map. The dynamics of the price process S are given by

$$\frac{dS_t^q}{S_t^q} = \sigma_t^q \left(dW_t + \lambda_t(q_t) dt \right), \; t < T, \tag{3.4}$$

where the volatility function σ_t^q is progressively measurable and depends on the control process $\{q_s, 0 \leq s \leq T\}$. As in the previous section, the wealth process of the company is given by:

$$X_T^{x,\theta} := x + \int_0^T \theta_t dS_t^q \quad \text{and} \quad B_T^q := \int_0^T \pi_t(q_t) dt - S_T^q \int_0^T e_t(q_t) dt$$

3.1 Large Carbon emission with no impact on risk premia

In this subsection, we restrict our attention to the case of large emitting firm with no impact on the risk premia, i.e.

$$\lambda_t(q) \quad \text{is independent of } q \text{ for any } t \geq 0. \tag{3.5}$$

The objective of the large emitting firm is:

$$V_0^{(2)} \quad := \sup_{q.\in\mathcal{Q},\ \theta\in\mathcal{A}} \mathbb{E}\left[U\left(X_T^{x,\theta} + B_T^q\right)\right].$$

Proposition 3.1. *Assume* (3.5)*, and that the market is complete with unique risk-neutral measure* $\mathbb{Q}$*. Then, the optimal production policy is independent of the utility function of the producer* U*, and obtained by solving:*

$$\sup_{q.\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left[B_T^q\right]. \tag{3.6}$$

Moreover, if $q^{(2)}$ *is an optimal production scheme, then the optimal investment strategy* $\theta^{(2)}$ *is characterized by*

$$X_T^{x,\theta^{(2)}} + B_T^{q^{(2)}} = (U')^{-1}\left(y^{(2)}\frac{d\mathbb{Q}}{d\mathbb{P}}\right), \quad x + \mathbb{E}^{\mathbb{Q}}\left[B_T^{q^{(2)}}\right] = \mathbb{E}^{\mathbb{Q}}\left[(U')^{-1}\left(y^{(2)}\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]. \tag{3.7}$$

Proof. We first fix some production strategy q. Since the market is complete, the partial maximization with respect to θ can be performed by the classical duality method:

$$X_T^{x,\theta^q} + B_T^q = (U')^{-1}\left(y^q\frac{d\mathbb{Q}}{d\mathbb{P}}\right), \tag{3.8}$$

where the Lagrange multiplier y^q is defined by

$$\mathbb{E}^{\mathbb{Q}}\left[(U')^{-1}\left(y^q\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \quad = \quad x + \mathbb{E}^{\mathbb{Q}}\left[B_T^q\right]. \tag{3.9}$$

This reduces the problem to:

$$\sup_{q.\geq 0} \mathbb{E}\left[U \circ (U')^{-1}\left(y^q\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]. \tag{3.10}$$

Notice that $U \circ (U')^{-1}$ is decreasing and the density $\frac{d\mathbb{Q}}{d\mathbb{P}} > 0$. Then (3.10) reduces to

$$\inf \{y^q : \ q_\cdot \geq 0\} .$$

Since $(U')^{-1}$ is also decreasing, (3.9) converts the problem into

$$\sup \left\{ \mathbb{E}^{\mathbb{Q}} \left[B_T^q \right] : \ q_\cdot \in \mathcal{Q} \right\} .$$

Finally, given the optimal strategy $q^{(2)}$, the optimal investment policy is characterized by (3.8). $\square$

In order to push further the characterization of the optimal production policy $q^{(2)}$, we specialize the discussion to the Markov case by assuming that $\pi_t(q) = \pi(t,q)$, $e_t(q) = e(t,q)$, and $\lambda_t(q) = \lambda(t)$ for some deterministic functions $\pi, e : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ in $C^{0,1}(\mathbb{R}_+ \times \mathbb{R}_+)$, $\lambda : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ in $C^0(\mathbb{R}_+)$, and

$$dY_t^q = (\mu(t, Y_t^q) + \beta e(t, q_t))\, dt + \gamma(t, Y_t^q) dW_t,$$

for some continuous deterministic functions $\mu, \gamma : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$.

The state variable E is now defined by the dynamics

$$dE_t^q \quad = \quad e(t, q_t) dt \tag{3.11}$$

which records the cumulated carbon emissions of the company. The dynamic version of the producer planning problem (3.6) is given by:

$$V^{(2)}(t,e,y) \quad := \quad \sup_{q_\cdot \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}_{t,e,y} \left[\int_t^T \pi(t, q_t) dt - \alpha E_T^q \mathbb{1}_{\{Y_T^q > 0\}} \right] . \tag{3.12}$$

Then, $V^{(2)}$ is a viscsity solution of the dynamic programming equation:

$$\begin{aligned} 0 \quad = \quad & \frac{\partial V^{(2)}}{\partial t} + (\mu - \lambda\gamma) V_y^{(2)} + \frac{1}{2}\gamma^2 V_{yy}^{(2)} \\ & + \max_{q \geq 0} \left\{ \pi(t,q) + e(t,q) V_e^{(2)} + \beta e(t,q) V_y^{(2)} \right\} , \end{aligned} \tag{3.13}$$

together with the terminal condition

$$V^{(2)}(T,e,y) \quad = \quad -\alpha e \mathbb{1}_{\{y>0\}} . \tag{3.14}$$

For the moment assume that the value function $V^{(2)}$ is smooth. Then, the optimal strategy is given by

$$\frac{\partial \pi}{\partial q}\left(t, q^{(2)}\right) \quad = \quad -\frac{\partial e}{\partial q}\left(t, q^{(2)}\right)\left(V_e^{(2)} + \beta V_y^{(2)}\right)(t, e, y).$$

By the definition of the value function $V^{(2)}$ in (3.12), we expect that

$$-V_e^{(2)}(t, E_t, Y_t) = S_t. \tag{3.15}$$

Then

$$\frac{\partial \pi}{\partial q}\left(t, q_t^{(2)}\right) \quad = \quad \frac{\partial e}{\partial q}\left(t, q_t^{(2)}\right)\left(S_t - V_y^{(2)}(t, E_t^{q^{(2)}}, Y_t^{q^{(2)}})\right) \tag{3.16}$$

Also, it is clear that $V^{(2)}$ is non-increasing in y. Then, comparing the previous expression with (2.8), it follows from the assumption on π and e that:

$$q^{(2)} \quad < \quad q^{(1)}.$$

In other words, the impact of the production firm on the prices of carbon emission allowances increases the cost of the externality for the firm. This immediately affects the profit function of the firm and leads to a decrease of the level of optimal production. Hence, the presence of the emission market is playing a positive role in terms of reducing the carbon emissions.

The following result shows that under certain assumptions, the above formal calculation is valid in our model.

Theorem 3.1. *Suppose that μ_t is continuous and deterministic, γ is constant, $\lambda(q) = \lambda_0$, and $e(q) = e_1 q + e_0$ where λ_0, e_1 and e_0 are non–negative constants. Assume that π is $C^{0,1}([0,T] \times \mathbb{R}_+)$, strictly concave in q and*

$$\frac{\partial \pi}{\partial q}(t, 0+) > 0 \quad \textit{and} \quad \frac{\partial \pi}{\partial q}(t, \infty) < 0.$$

Then $V_e^{(2)}$ exists and (3.15) holds true. In addition, if problem (3.13)–(3.14) has a bounded solution in $C^{1,1,2}([0,T) \times \mathbb{R}_+ \times \mathbb{R})$, then there exists an optimal production strategy satisfying (3.16).

Proof. Since V is concave in e, it has left and right partial derivatives with respect to e everywhere, and the partial gradient V_e exists almost everywhere. Under our conditions, the additional differentiabiity property together with the remaining characterizations of the theorem follow from Proposition B.1 and Lemma B.1.

For the last assertion of the Theorem, notice that by Lemma (A.1), V is the unique bounded viscosity solution of (3.13)–(3.14). Therefore, by the assumption of the Theorem, $V \in C^{1,1,2}([0,T) \times \mathbb{R}_+ \times \mathbb{R})$ and one can use the dynamic programming principle to deduce $q^{(2)}$ obtained from (3.16) is an optimal strategy. □

3.2 Large Carbon emission Impacting the Risk-Neutral Measure

We now consider the general case where the risk premium process is impacted by the emissions of the production firm:

$$\frac{d\mathbb{Q}^q}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \exp\left(-\int_0^T \lambda(q_t)dW_t - \frac{1}{2}\int_0^T \lambda(q_t)^2 dt\right).$$

The partial maximization with respect to θ, as in the proof of Proposition 3.1, is still valid in this context, and reduces the production firm's problem to

$$\sup_{q.\in\mathcal{Q}} \mathbb{E}\left[U \circ (U')^{-1}\left(y^q \frac{d\mathbb{Q}^q}{d\mathbb{P}}\right)\right] \tag{3.17}$$

where y^q is defined by

$$\mathbb{E}^{\mathbb{Q}^q}\left[(U')^{-1}\left(y^q \frac{d\mathbb{Q}^q}{d\mathbb{P}}\right)\right] = x + \mathbb{E}^{\mathbb{Q}^q}\left[B_T^q\right]. \tag{3.18}$$

We also assume that the preferences of the production firm are defined by an exponential utility function

$$U(x) := -e^{-\eta x}, \quad x \in \mathbb{R}.$$

Then $U \circ (U')^{-1}(y) = -y/\eta$, and (3.17) reduces to

$$\inf_{q.\geq 0} \mathbb{E}\left[y^q \frac{d\mathbb{Q}^q}{d\mathbb{P}}\right] = \inf_{q.\geq 0} y^q. \tag{3.19}$$

Finally, the budget constraint (3.18) is in the present case:

$$\begin{aligned} x + \mathbb{E}^{\mathbb{Q}^q}\left[B_T^q\right] &= \frac{-1}{\eta}\mathbb{E}^{\mathbb{Q}^q}\left[\ln\left(\frac{y^q}{\eta}\frac{d\mathbb{Q}^q}{d\mathbb{P}}\right)\right] \\ &= \frac{-1}{\eta}\left\{\ln\left(\frac{y^q}{\eta}\right) + \mathbb{E}^{\mathbb{Q}^q}\left[\ln\left(\frac{d\mathbb{Q}^q}{d\mathbb{P}}\right)\right]\right\}, \end{aligned}$$

so that the optimization problem (3.19) is equivalent to:

$$\begin{aligned} &\sup_{q.\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}^q}\left[B_T^q + \frac{1}{\eta}\ln\left(\frac{d\mathbb{Q}^q}{d\mathbb{P}}\right)\right] \\ = &\sup_{q.\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}^q}\left[\int_0^T \left(\pi + \frac{\lambda^2}{2\eta}\right)(t,q_t)dt - S_T^q\int_0^T e_t(q_t)dt\right]. \end{aligned} \tag{3.20}$$

Notice the difference between the above optimization problem, which determines the optimal production policy of the production firm, and the problem (3.6). In the present situation where the risk premium process is impacted by the carbon emissions of the firm, the firm's optimization criterion is penalized by the entropy of the risk-neutral measure with respect to the statistical measure.

The firm's optimal production problem (3.20) is a standard stochastic control problem. We continue our discussion by considering the Markov case, and introducing the dynamic version of (3.20):

$$V^{(3)}(t,e,y) := \sup_{q.\in\mathcal{Q}} \mathbb{E}_{(t,e,y)}^{\mathbb{Q}^q}\left[\int_t^T \left(\pi + \frac{\lambda^2}{2\eta}\right)(t,q_t)dt - E_T^q\alpha\mathbb{1}_{\{Y_T^q\geq 0\}}\right], \tag{3.21}$$

where the controlled state dynamics is given by:

$$\begin{aligned} dY_t^q &= \left(\mu(t,Y_t^q) + \beta e(t,q_t) - \gamma(t,Y_t^q)\lambda(t,q_t)\right)dt + \gamma(t,Y_t^q)dW_t^q, \\ dE_t^q &= e(t,q_t)dt, \end{aligned}$$

W^q is a Brownian motion under $\mathbb{Q}^q$, and μ and γ are $C^{1,2}$ functions in (t,y), and μ, e and λ are $C^{1,2}$ functions in (t,q).

By classical arguments, we then see that $V^{(3)}$ solves the dynamic programming equation:

$$\begin{aligned} 0 = &\frac{\partial V^{(3)}}{\partial t} + \mu V_y^{(3)} + \frac{1}{2}\gamma^2 V_{yy}^{(3)} \\ &+ \max_{q\in\mathbb{R}_+}\left\{\pi(t,q) + \frac{1}{2\eta}\lambda(t,q)^2 + e(t,q)(V_e^{(3)} + \beta V_y^{(3)}) - \gamma\lambda(t,q)V_y^{(3)}\right\} \end{aligned} \tag{3.22}$$

together with the terminal condition

$$V^{(3)}(T,e,y) \quad = \quad -\alpha e \mathbb{1}_{\{y>0\}}. \tag{3.23}$$

In terms of the value function $V^{(3)}$, the optimal production policy is obtained as the maximizer in the above equation. Under the technical Assumption (A.2) below, an interior maximum occurs, and if $V^{(3)}$ is regular enough, then the first order condition is:

$$\frac{\partial \pi}{\partial q}(q^{(3)}) + \frac{1}{\eta}(\lambda \frac{\partial \lambda}{\partial q})(q^{(3)}) + \frac{\partial e}{\partial q}(q^{(3)})(V_e^{(3)} + \beta V_y^{(3)}) - \gamma \frac{\partial \lambda}{\partial q}(q^{(3)}) V_y^{(3)} = 0, \tag{3.24}$$

where the dependence with respect to (t,e,y) has been omitted for simplicity. As before, we expect that the value function (3.21) is regular enough and that the price of the carbon emissions allowance contract, as observed on the emission market, is given by:

$$S_t \quad = \quad -V_e^{(3)}(t,E_t,Y_t). \tag{3.25}$$

Then, it follows that the optimal production policy of the firm is defined by:

$$\begin{aligned} \frac{\partial \pi}{\partial q}(t,q^{(3)}) \quad &= \quad \frac{\partial e}{\partial q}(t,q^{(3)}) \left(S_t - \beta V_y^{(3)}(t,Y_t,E_t) \right) \\ &\quad + \frac{\partial \lambda}{\partial q}(t,q^{(3)}) \left(\gamma V_y^{(3)}(t,Y_t,E_t) - \frac{1}{\eta}\lambda(t,q^{(3)}) \right). \end{aligned} \tag{3.26}$$

The latter expression is the main formula for our financial interpretation and our subsequent numerical experiments. In contrast with the previous case where the risk-premium process was not impacted by the carbon emissions of the large firm, we can not conclude from the above formula that $q^{(3)}$ is smaller than $q^{(1)}$; recall that the optimal production policy in the absence of a financial market defined by

$$\frac{\partial \pi}{\partial q}(t,q^{(1)}) = \frac{\partial e}{\partial q}(t,q^{(1)}) S_t.$$

This is due to the fact that the difference term

$$-\frac{\partial e}{\partial q}(t,q^{(3)}) \beta V_y^{(3)}(t,Y_t,E_t) + \frac{\partial \lambda}{\partial q}(t,q^{(3)}) \left(\gamma V_y^{(3)}(t,Y_t,E_t) - \frac{1}{\eta}\lambda(t,q^{(3)}) \right)$$

has no known sign, and there is no economic argument supporting that it should have some specific sign. The economic intuition hidden in this term is that the large producer may take advantage of his impact on the emission market by manipulating the prices so as to achieve a profit from its trading activity which compensates a higher production activity inducing larger carbon emissions. In the present situation, we see that the emission market has a negative effect on the carbon emissions: the large firm may optimally choose to increase its carbon emissions thus increasing its profit by means of its ability to manipulate the financial market.

The next result shows that for some choice of the coefficients, (3.25) holds true and we have the relation (3.26).

Theorem 3.2. *Suppose that μ_t is continuous and deterministic, γ is constant, $e(q) = e_1 q + e_0$ and $\lambda(q) = \lambda_1 q + \lambda_0$, and $\tilde{\pi}_t(q) := \pi_t(q) + \frac{\lambda(q)^2}{2\eta}$ is deterministic and strictly concave in q with*

$$\tilde{\pi}_t'(0) > 0 \quad \textit{and } \tilde{\pi}_t'(-\infty) < 0.$$

Then $V_e^{(3)}$ exists and (3.25) holds true. In addition, if problem (3.22)–(3.23) has a solution in $C^{1,1,2}([0,T) \times \mathbb{R}_+ \times \mathbb{R})$, then there exists an optimal production strategy satisfying (3.26).

Proof. The proof follows the same line of argument as the proof of Theorem 3.1. □

4 Multiperiod model with banking

The analysis of the previous sections are restricted to the case where the carbon allowances market is organized over one single period. In this section, we discuss how to extend our results to a multiperiod model with *banking*. We then assume that there are n maturities for the carbon allowances market

$$T_1 < \ldots < T_n$$

instead of a single one. According to the banking rule, the carbon emission allowance can serve for the next periods if not used for the current one. Then, the allowance can be viewed as a derivative security with payoff:

$$S_{T_n} := \alpha \left(\mathbb{1}_{\{Y^q_{T_1} \geq \kappa\}} + \mathbb{1}_{\{Y^q_{T_1} < \kappa\}} \mathbb{1}_{\{Y^q_{T_2} \geq \kappa\}} + \ldots + \mathbb{1}_{\{Y^q_{T_i} < \kappa,\ i<n\}} \mathbb{1}_{\{Y^q_{T_n} \geq \kappa\}} \right).$$

Following the same argument as in the previous section, the no-arbitrage market price at each time $t \le T_n$ is given by

$$S_t := \mathbb{E}_t^{\mathbb{Q}}\left[S_{T_n}\right] \quad \text{for all} \quad t \le T_n, \tag{4.1}$$

where $\mathbb{Q}$ is the risk-neutral measure. Now it is clear that all the analysis of the previous sections apply by just replacing the price formula (2.3) by the above market price (4.1).

5 Numerical results

5.1 A linear-quadratic example

The main goal of the numerical results is to understand the behavior of the optimal strategy

$$\begin{aligned} \frac{\partial \pi}{\partial q}(t, q^{(3)}) \quad &= \quad \frac{\partial e}{\partial q}(t, q^{(3)}) \left(S_t - \beta V_y^{(3)}(t, Y_t, E_t)\right) \\ &\quad + \frac{\partial \lambda}{\partial q}(t, q^{(3)}) \left(\gamma V_y^{(3)}(t, Y_t, E_t) - \frac{1}{\eta}\lambda(t, q^{(3)})\right) \end{aligned} \tag{5.1}$$

and more precisely find an example where $q^{(3)} > q^{(1)}$.
We consider the Dynamic Programming Equation

$$V_t + \mu V_y + \frac{1}{2}\gamma^2 V_{yy} + \max_{q \ge 0} \theta(q, V_e, V_y) = 0 \tag{5.2}$$

where θ is defined by

$$\theta(q, V_e, V_y) = \pi(t, q) + \frac{1}{2\eta}\lambda(t, q)^2 + e(t, q)(V_e^{(3)} + \beta V_y^{(3)}) - \gamma \lambda(t, q) V_y^{(3)},$$

and with the terminal boundary condition

$$V(T, e, y) = -\alpha e \mathbb{1}_{\{y \ge 0\}}.$$

Here, we consider a simple case where

$$\pi(q) = q(1 - q), \;\; e(q) = \lambda(q) = q, \;\; \beta = 1, \text{ and } \alpha = 1.$$

Note that this example satisfies the assumption of Theorem 3.2. So, $V_e = -S_t$ and therefore one can compare $q^{(1)}$, $q^{(2)}$ and $q^{(3)}$. It follows that

$$\theta(q, V_e, V_y) = -\left(1 - \frac{1}{2\eta}\right) q^2 + \left(1 + V_e + (1 - \gamma) V_y\right) q.$$

We next assume that $\eta > \frac{1}{2}$ so that the function θ is strictly concave in the q variable. Then, it follows from the first order condition that the optimal production policy is given by:

$$q^{(3)} = \frac{1}{2\rho}(1 + V_e + (1-\gamma)V_y)$$

with $\rho = \left(1 - \frac{1}{2\eta}\right)$, and

$$\max_{q\geq 0} \theta(q, V_e, V_y) = \frac{1}{4\rho}\left(1 + V_e + (1-\gamma)V_y\right)^2.$$

Then, the Dynamic Programming Equation (5.2) reduces to:

$$V_t + \mu V_y + \frac{1}{2}\gamma^2 V_{yy} + \frac{1}{4\rho}\left(1 + V_e + (1-\gamma)V_y\right)^2 = 0. \tag{5.3}$$

Note that, in order to to compare with $q^{(1)}$, optimal strategy (5.1) could be written as:

$$\pi'\left(q^{(3)}\right) = e'\left(q^{(3)}\right)S_t - \tau(e, y),$$

where the correction term $\tau(e, y)$ is defined by

$$\tau(e, y) \quad = \quad \frac{2\eta(1-\gamma)}{2\eta - 1}V_y + \frac{1}{2\eta - 1}(1 + V_e).$$

The main objective of our numerical implementation is to exhibit examples of parameters which induce $\tau(e, y) < 0$, or equivalently in terms of the optimal strategy $q^{(3)} > q^{(1)}$.

5.2 Numerical scheme

The first step is to set a computational bounded domain $[0, L_e] \times [-L_y, L_y]$ for the (e, y) space domain and discretize the computational domain by the grid $\{(e_i, y_j)\}_{i,j}$. Since we deal with non-linear advection and diffusion phenomena, it is natural to consider Neumann boundary conditions.

Let Δt be the time step and $t^{(k)} = k\Delta t$, for $k = 0, \cdots, n := \lfloor \frac{T}{\Delta t} \rfloor$. We set the discrete terminal data $V_{ij}^{t^{(n)}} = -e_i \mathbb{1}_{\{y_j \geq 0\}}$.

The main difficulty in solving the equation (5.3) is the semi-linear terms. In order to overcome this difficulty, we used a time-splitting discretization which divides our scheme into two steps:

- Step 1: we use an implicit finite-differences scheme to solve the diffusion part of the model. This means that on a time step $[t^{(n)}, t^{(n+1)}]$, we solve

$$V_t + \frac{1}{2}\gamma^2 V_{yy} = 0. \tag{5.4}$$

- Step 2: we solve the coupling between the advection part with the non-linear effects

$$V_t + \mu V_y + \frac{1}{4\rho}\left(1 + V_e + (1-\gamma)V_y\right)^2 = 0. \tag{5.5}$$

  In this important part, we used a relaxation scheme introduced by C. Besse [1]. The scheme is constructed as follow: We rewrite (5.5) as the system of two equations:

$$V_t + \mu V_y + \frac{1}{4\rho}\left(1 + V_e + (1-\gamma)V_y\right)\varphi = 0, \tag{5.6}$$

$$\varphi = 1 + V_e + (1-\gamma)V_y \tag{5.7}$$

  which are solved using a leap-frog scheme in time.

Compared to the Crank-Nicholson scheme, which is also based on a time-centering method, this scheme allows us to avoid a costly numerical treatment of the nonlinearity and to preserve the flexibility of spatial discretization choice.

5.3 Results

For parameters $\mu = 0.1$, $\gamma = 0.65$, $\eta = 5$ and the final time is $T = 10$ we produced the following results.

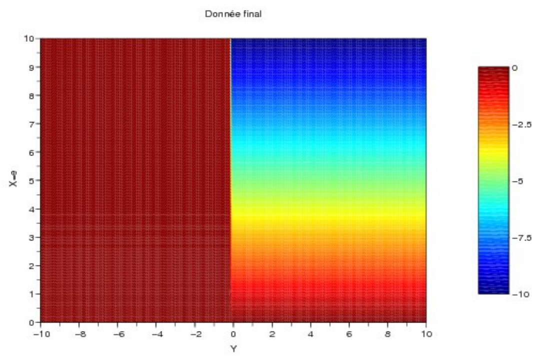

Figure 1: Terminal boudary condition $V^{(3)}(T = 10, e, y)$

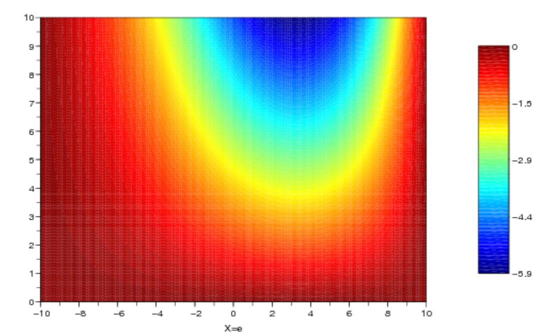

Figure 2: The solution of the dynamic programming equation $V^3(e, y)$ at time $t = 0.2$

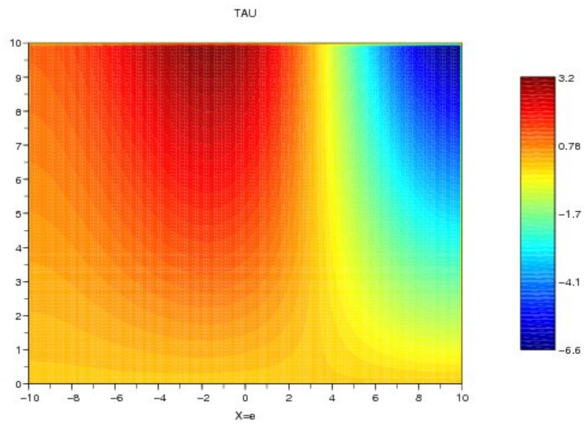

Figure 3: The difference term $\tau(e,y)$ at time $t=0.2$

The blue region shows the couples (e,y) for which we have $q^{(3)} > q^{(2)}$ and therefore within this region the large producer optimally increases her production.

A Uniqueness and verification

Let

$$V(t,e,y) \quad = \quad \sup_{q.\in\mathcal{Q}} \mathbb{E}_{t,e,y}\left[\int_t^T \tilde{\pi}(s,q_s)\mathrm{d}s - \alpha E_T^{q,e}\mathbb{1}_{\{Y_T^{q,y}\geq\kappa\}}\right], \qquad \text{(A.1)}$$

where

$$\begin{aligned} dY_t^q &= \Big(\mu(t,Y_t^q) + \beta e(t,q_t) + \gamma(t,Y_t)\lambda(t,q_t)\Big)dt + \gamma(t,Y_t^q)dW_t, \\ dE_t^q &= e_t(q)dt \end{aligned}$$

with $\pi, e : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ in $C^{0,1}(\mathbb{R}_+ \times \mathbb{R}_+)$, $\lambda : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ are in $C^0(\mathbb{R}_+)$, $\mu, \gamma : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous in t and Lipschitz in y, and $\gamma \geq 0$.

Notice that $V = V^{(2)}$ or $V^{(3)}$ when $\tilde{\pi} := \pi$ or $\pi + \frac{\lambda^2}{2\eta}$, respectively. Also for simplicity, the dependency of martingale measure with respect to q in the definition of $V^{(2)}$ or $V^{(3)}$ is absorbed in the dynamic of Y_t^q. Therefore in the current Appendix the reference expectation $\mathbb{E}$ is with respect to the measure $\mathbb{P}$ under which the dynamic of Y_t^q is as in the above.

Throughout the Appendix, we suppose

$$\begin{aligned} &(i) \quad \tilde{\pi},\ e,\ \text{and } \lambda \text{ are in } C^{0,1}([0,T]\times\mathbb{R}_+), \\ &(ii) \quad e \text{ is convex and, } \lambda \text{ and } e \text{ are increasing in } q, \\ &(iii) \quad \tilde{\pi} \text{ is strictly concave in } q\ ,\ \frac{\partial\tilde{\pi}}{\partial q}(t,0+)>0 \ \text{ and } \ \frac{\partial\tilde{\pi}}{\partial q}(t,\infty)<0. \end{aligned} \tag{A.2}$$

The following Lemma is needed for the proof of Theorems 3.2 and 3.1.

Lemma A.1. *There exists some $\overline{q}$ such that:*

$$V(t,e,y) = \sup_{q.\in\overline{\mathcal{Q}}} \mathbb{E}_{t,e,y}\left[\int_t^T \tilde{\pi}(t,q_t)dt - E_T^q\alpha \mathbb{1}_{\{Y_T^q\geq 0\}}\right], \tag{A.3}$$

where $\overline{\mathcal{Q}}$ is the collection of all $q.\in\mathcal{Q}$ with $0\leq q\leq\overline{q}$.

Proof. By (A.2)(i), we can introduce $\overline{q}$ such that $\tilde{\pi}(\overline{q})<0$ and $\tilde{\pi}$ is decreasing in $q\in[\overline{q},\infty)$. Therefore, if $\tilde{q}:=q\wedge\overline{q}$, then $E^{\tilde{q},e}\leq E^{q,e}$ and $\tilde{\pi}(\tilde{q})\geq\tilde{\pi}(q)$. On the other hand, by Theorem 1.1 in [7], $Y_T^{\tilde{q},y}\leq Y_T^{q,y}$ a.s.. Therefore,

$$J(\tilde{q}) \geq J(q) \text{ a.s.,}$$

where $J(q) := \int_t^T \tilde{\pi}(t,q_t)dt - E_T^q\alpha\mathbb{1}_{\{Y_T^q\geq 0\}}$. □

The next result states that V can be characterized by the PDE Therefore, V solves the dynamic programming equation:

$$\begin{aligned} 0 \;=\; & \frac{\partial V}{\partial t} + \mu V_y + \frac{1}{2}\gamma^2 V_{yy} \\ & + \max_{0\leq q\leq\overline{q}} \{\tilde{\pi}(t,q) + e(t,q)(V_e+\beta V_y) - \gamma\lambda(t,q)V_y\} \end{aligned} \tag{A.4}$$

together with the terminal condition

$$V(T,e,y) \;=\; -\alpha e\mathbb{1}_{\{y>0\}}. \tag{A.5}$$

Theorem A.1. *Let* (A.2) *hold true. Then V is the unique bounded viscosity solution of* (A.4)-(A.5) *on* $[0,T]\times\mathbb{R}_+\times\mathbb{R}$.

Proof. Notice that one can write (A.4) as

$$0 = \frac{\partial V}{\partial t} + H(t, y, V_y, V_e, V_{yy})$$

where

$$\begin{aligned} H(t, y, v_1, v_2, v_{11}) &:= \mu(t, y)v_1 + \frac{1}{2}\gamma^2(t, y)v_{11} \\ &\quad + \max_{\overline{q} \geq q \geq 0} \{\tilde{\pi}(t, q) + e(t, q)(v_2 + \beta v_1) - \gamma(t, y)\lambda(t, q)v_1\}. \end{aligned}$$

By continuity of H, one can apply Theorem 7.4 in [10] to obtain that V satisfies (A.4) in viscosity sense on $[0, T) \times \mathbb{R}_+ \times \mathbb{R}$.
On the other hand, for any $q \in \overline{\mathcal{Q}}$, $\mathbb{1}_{\{Y_T^{t,(q,y)} \geq \kappa\}}$ and $E_T^{t,(q,e)}$ converges to $\mathbb{1}_{\{y \geq \kappa\}}$ and e a.s. as $t \to T$, respectively. Therefore, by Lebesgue dominated convergence Theorem

$$\lim_{t \to T} V(t, e, y) = -\alpha e \mathbb{1}_{\{y \geq \kappa\}} = V(T, e, y).$$

Consequently, we can deduce that V is the bounded viscosity solution of the boundary value problem (A.4)–(A.5).
The uniqueness follows from the comparison principle for viscosity solutions in [10]. □

B Existence of optimal production policy

We first show that the existence of an optimal production policyallows to relate the value function V to the market price of carbon allowance S_t.

Lemma B.1. *Let the assumption* (A.2) *hold true. If there exists an optimal control q^* for any (t, e, y) then $\frac{\partial V}{\partial e}(t, e, y) = -\alpha \mathbb{E}[\mathbb{1}_{\{Y_T^{t,y,q^*} \geq \kappa\}}]$.*

Remark B.1. *Lemma B.1 is crucial for the comparison between $q^{(3)}$ and $q^{(2)}$ or $q^{(1)}$. Notice that $S_t = \alpha \mathbb{E}_t[\mathbb{1}_{\{Y_T^{q^*} \geq \kappa\}}]$ is market price which is observable and $\left(\pi + \frac{\lambda^2}{2\eta}\right)$ is concave in q. Therefore, one can replace V_e by $-S_t$ in* (3.24) *and examine the sign of V_y to establish comparison.*

Proof. Notice that by the concavity of V in e, $\frac{\partial V}{\partial e}$ exists almost everywhere. Suppose that $e > e'$. Then, by direct calculations one can write

$$V(t, e, y) - V(t, e', y) + (e - e')\alpha \mathbb{E}\left[\mathbb{1}_{\{Y_T^{t,y,q^*} \geq \kappa\}}\right] \leq 0,$$

where q^* is an optimal strategy for $V(t,e,y)$. This implys that

$$\frac{V(t,e,y)-V(t,e',y)}{e-e'}+\mathbb{E}\left[\mathbb{1}_{\{Y_T^{t,y,q^*}\geq\kappa\}}\right] \quad\leq\quad 0.$$

By passing to the limite as $e'\to e$,

$$V_e(t,e,y) \quad\leq\quad -\mathbb{E}\left[\mathbb{1}_{\{Y_T^{t,y,q^*}\geq\kappa\}}\right].$$

For the other side inequality use $e'>e$. □

We next provide a sufficient condition for the existence of an optimal production policy.

Proposition B.1. *Let μ be deterministic, γ be constant and*

$$e(t,q):=e_1q+e_0 \quad \textit{and} \quad \lambda(t,q):=\lambda_1q+\lambda_0, \quad q\geq 0, \tag{B.1}$$

where $e_0,\lambda_0,e_1,\lambda_1$ are nonnegative constants. Then the control problem (A.1) *has an optimal control q^* in $\overline{\mathcal{Q}}$.*
In particular, in this setting we have $V_e(t,E_t^{q^*},Y_t^{q^*})=-S_t$.

Proof. If $e_1=\lambda_1=0$, the result is trivial. Therefore we suppose that at least one of them is non–zero. Notice that when μ and γ are deterministic, one can write

$$Y_t^q := Y_t^0+\int_0^t(\beta e(q_s)+\gamma\lambda(q_s))\,dt \quad \text{with} \quad Y_t^0:=y+\int_0^t(\mu_s\mathrm{d}s+\gamma W_s).$$

By Girsanov theorem, we notice that, for every $q\in\overline{\mathcal{Q}}$, the random variable Y_T^q has a Gaussian distribution under the equivalent probability measure $\frac{d\mathbb{Q}}{d\mathbb{P}}:=\mathcal{E}\big(-(\beta e(q_t)+\gamma\lambda q_t+\mu_t)\gamma^{-1}dW_t\big)$. Here $\mathcal{E}$ is the Doleans-Dade exponential. Then, the distribution of Y_T^q is absolutely continuous with respect to the Lebesgue measure on $[0,T]$ for all $q\in\overline{\mathcal{Q}}$.

In other words, the distribution of Y_T^q has no atoms, and the cumulative distribution function of the random variable Y_T^q is continuous.
Let $(q^n)_{n\geq1}$ be a maximizing sequence of V_0, i.e.

$$q^n\in\mathcal{Q} \quad \text{for all } n\geq 1 \text{ and} \quad J(q^n)\longrightarrow V_0.$$

Step 1. Since the processes q^n are uniformly bounded, we deduce from weak convergence and Mazur's lemma that, after possibly passing to a subsequence, there exists a convex combination $\hat{q}^n$ of $(q^j, j \geq n)$ such that:

$$\hat{q}^n := \sum_{j \geq n} \lambda_j^n q^j \quad \longrightarrow \quad q^* \text{ in } \mathbb{L}^1(\Omega \times [0,T]) \text{ and } m \otimes \mathbb{P} - \text{a.s.} \tag{B.2}$$

where m is the Lebesgue measure on $[0,T]$. Here $\lambda_j^n \geq 0$ and $\sum_{j \geq n} \lambda_j^n = 1$. Clearly $q^* \in \mathcal{Q}$. Since Y^q is linear in q, this implies that

$$\hat{Y}_T^n := \sum_{j \geq n} \lambda_j^n Y_T^{q^j} \quad \longrightarrow \quad Y_T^{q^*}, \text{ a.s.} \tag{B.3}$$

Step 2. By direct estimation and use of Hölder inequality, $Y_T^{q^n}$ is tight under $\mathbb{P}$ and therefore under any equivalent probability measure $\hat{\mathbb{P}}$ with density in $L^2(\mathbb{P})$. Hence after passing to a subsequence, it should converge in distribution to a $\mathcal{F}_T$ random variable Y_T^* which must be equal to $Y_T^{q^*}$;

$$Y_T^{q^n} \longrightarrow Y_T^{q^*} \quad \text{in distribution under} \quad \hat{\mathbb{P}}.$$

Since the convergence in distribution is equivalent to convergence of the corresponding cumulative density functions at all points of continuity, because the probability distribution of Y_T^q is absolutely continuous with respect to Lebesgue measure, it follows that for any positive random variable Z with $\mathbb{E}[Z] = 1$ and $\mathbb{E}[Z^2] < \infty$,

$$\begin{aligned} \mathbb{E}\left[Z \mathbb{1}_{\{Y_T^{q^n} \geq \kappa\}}\right] &= \hat{\mathbb{P}}\left[Y_T^{q^n} \geq \kappa\right] \\ &\longrightarrow \hat{\mathbb{P}}\left[Y_T^{q^*} \geq \kappa\right] = \mathbb{E}\left[Z \mathbb{1}_{\{Y_T^{q^*} \geq \kappa\}}\right]. \end{aligned} \tag{B.4}$$

Step 3. Notice that because e and λ are affine, One can write:

$$\int_0^T e(q_s) \mathrm{d}s = \delta \left(Y_T^{q^j} - Y_T^0 - c \right),$$

where $\delta := (\beta e_1 + \gamma \lambda_1)^{-1}$ and $c := \beta e_0 + \gamma \lambda_0$. By the concavity condition (A.2), we see that:

$$\begin{aligned} \sum_{j \geq n} \lambda_j^n J\left(q^j\right) &\leq \mathbb{E}\left[\int_0^T \tilde{\pi}(t, \hat{q}_t^n) dt - \alpha \sum_{j \geq n} \lambda_j^n \mathbb{1}_{\{Y_T^{q^j} \geq \kappa\}} \int_0^T e(q_s^j) ds\right], \\ &= \mathbb{E}\left[\int_0^T \tilde{\pi}(t, \hat{q}_t^n) dt - \alpha \sum_{j \geq n} \lambda_j^n \delta \left(Y_T^{q^j} - Y_T^0 - c\right) \mathbb{1}_{\{Y_T^{q^j} \geq \kappa\}}\right] \end{aligned}$$

Observe that $Y_T^{q^j} - Y_T^0 - c = \left(Y_T^{q^j} - \kappa\right)^+ + Z^+ - Z^-$ on $\{Y_T^{q^j} \geq \kappa\}$ where $Z^\pm := (Y_T^0 + c - \kappa)^\pm + 1$.

$$\begin{aligned}\sum_{j\geq n} \lambda_j^n J\left(q^j\right) \quad &\leq \quad \mathbb{E}\Big[\int_0^T \tilde{\pi}(t, \hat{q}_t^n)dt - \alpha \sum_{j\geq n} \lambda_j^n \delta \left(Y_T^{q^j} - \kappa\right)^+\Big] \\ &\quad + \alpha\delta \sum_{j\geq n} \lambda_j^n \mathbb{E}\left[Z^+ \mathbb{1}_{\{Y_T^{q^j} \geq \kappa\}}\right] - \alpha\delta \sum_{j\geq n} \lambda_j^n \mathbb{E}\left[Z^- \mathbb{1}_{\{Y_T^{q^j} \geq \kappa\}}\right].\end{aligned}$$

By the convexity of the function $y \longmapsto y^+$

$$\begin{aligned}\sum_{j\geq n} \lambda_j^n J\left(q^j\right) \quad &\leq \quad \mathbb{E}\Big[\int_0^T \tilde{\pi}(t, \hat{q}_t^n)dt - \alpha\delta \left(Y_T^{\hat{q}^n} - \kappa\right)^+\Big] \\ &\quad + \alpha\delta \sum_{j\geq n} \lambda_j^n \mathbb{E}\left[Z^+ \mathbb{1}_{\{Y_T^{q^j} \geq \kappa\}}\right] \\ &\quad - \alpha\delta \sum_{j\geq n} \lambda_j^n \mathbb{E}\left[Z^- \mathbb{1}_{\{Y_T^{q^j} \geq \kappa\}}\right],\end{aligned}$$

Finally, by applying Step 2 successively to $Z := Z^+$ and Z^-, one can write

$$V(t, e, y) \quad = \quad \lim_{n\to\infty} \sum_{j\geq n} \lambda_j^n J\left(q^j\right) \leq \mathbb{E}\left[\int_0^T \tilde{\pi}(t, q^*)dt - \alpha Y_T^{q^*} \mathbb{1}_{\{Y_T^{q^*} \geq \kappa\}}\right]$$

by dominated convergence. Since $q^* \in \overline{\mathcal{Q}}$, we deduce that $J(q^*) = V_0$. □

Remark B.2. *Proposition* (B.1) *is also valid if we replace Condition* (B.1) *by* $\lambda(q) = a + be(q)$ *and* $\tilde{\pi}(t, e^{-1}(q))$ *is convex on* q. *The modification is straightforward.*

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