An arbitrage-free interest rate model consistent with economic constraints for long-term asset liability management

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Abstract

This paper develops a Heath-Jarrow-Morton model of the yield curve which fits the particular requirements of long-term asset and liability management (ALM). In particular, the proposed HJM model can reproduce expected long-term statistical properties of any two interest rates yields with different maturities, while still satisfying the no-arbitrage constraints. We provide constraints on the expected statistical properties so that the model can be calibrated.

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1 Introduction

Long-term asset and liability management (ALM) models are designed for financial institutions (insurance companies, pension funds...), governmental institutions or private corporates facing financial commitments on maturities for which markets give little information (50 years to a century). In the previous literature, the development of interest rate models for ALM was focused on reproducing long-term statistical properties with discrete time series methods. Examples of this modeling methodology are given by Wilkie's seminal work on UK pension fund assessment [15, 16] and the TY model by Yakoubov et al [17].

Motivated by the extensive use of quantitative models in risk management analysis and strategic asset allocation, the recent literature focused on developing models which are consistent with both short term arbitrage-free constraints and long-term realistic dynamics, see Cairns [2]. Such ALM models are widely used [18], as for instance the Towers Perrin ALM model [8, 9] and the Russel-Yasuda Kasai model [3, 4].

However, an ALM model which is market consistent and statistically plausible (as in fixed-income desks for derivatives pricing) is not sufficient. As argued by Mulvey et al [8], the parameters should result from a compromise between statistical calibration methods and expertise considerations. Indeed, since stakeholders of financial institutions dealing with long-term commitments are aware of the impact of economic hypothesis on risk perception, they prefer to discuss them with economists rather than letting a numerical procedure determining some strategic parameters. Hence, decision-makers want to have the ability to introduce their

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views in the long-term behavior of the model. These views can be seen either as long-term economic expectations for the most probable forecast given by a chief-economist, or the decision-maker's way to introduce its risk aversion in the model ("I want to see what is going to happen for the value of our portfolio if the short-term interest rate is around 3% in the long-term").

In this paper, our objective is to build an ALM model which is

- consistent with the statistical properties of financial data and the no-arbitrage market hypothesis,
- and allows for the exogenous introduction of long-term economic expectations.

A quantitative ALM model involves generally the dynamics of the inflation, the term structure of interest rates, equities, and the three main currencies (USD, EUR, YEN). This induces a high dimensional stochastic dynamic system with a huge number of parameters. Among these parameters, only a small subset would be concerned by the experts exogenous views, namely the average value of the short-term interest rate and its volatility, equity market prices of risk, purchasing power parity long-term value, and the correlation between long-term and short term interest rates. The main difficulty is due to the fact that once the views are reflected in the corresponding parameters, it is not clear whether the model is still consistent with the market no-arbitrage hypothesis.

In particular, an important role is played by the term structure of interest rates which lies at the beginning of the cascade modeling, see [4, 9, 17], and has a paramount impact on the discount rate for liabilities. In Mulvey et al. [9], an interest rate model is designed based on a mean-reverting process for two maturities. This model allows a straightforward and easy computation for any other interest rate maturity by a simple interpolation, and the introduction of the experts views by means of bounds on various parameters. Unfortunately, there is no guarantee that the resulting dynamics will still be arbitrage free.

The model developed in the present paper complements the preceding works by ensuring market consistency and accounting for the exogenous experts views. We start from the Heath-Jarrow-Morton (HJM) framework which guarantees that no-arbitrage conditions are satisfied [7], and given a set of exogenously given predictions for the mean and the variance of a pair of interest rates yields with different maturities, we provide conditions under which a consistent model can be designed.

The paper is structured as follow. Section 2 contains the the description of the modeling requirements, the set of exogenous constraints, and the justification for our modeling choices. In Section 3, we report our solution approach to the nonlinear calibration problem, and provides the condition on the set of exogenous expert views so as to be consistent with an arbitrage-free HJM model. Section 4 provides some non-trivial numerical consequences, and illustrates the different dynamics obtained whether or not the no-arbitrage conditions are satisfied. Section 5 contains some concluding remarks and some perspectives.

2 Problem formulation

In an ALM model, the term structure of interest rates plays an essential role since it links asset returns and discount factor of interest rates. While cash return is related to the short-term interest rate, the evolution of the discount factor should be consistent with the evolution of some long-maturity interest rate yield. This illustrates the existence of a strong consistency between all financial variables and raise the difficulty of reproducing this consistency in a stochastic model with a limited number of parameters. The difficulties of designing such a stochastic interest rate model can be summarized in two main points: ensuring the absence of arbitrage opportunities (both static and dynamic) and allowing for a calibration procedure of all the parameters. In this section, we first describe the general requirements that an interest rate model is expected to satisfy for the purpose of long-term ALM applications. Subsection 2.2 reports the specific exogenous constraints corresponding the expert views. Finally, Subsection 2.3 justifies our HJM framework and formulates the calibration problem in our setting.

2.1 Modeling requirements

Following the reference book on ALM modeling by Ziemba & Mulvey [18], an interest rates model designed for long-term ALM applications should allow for a reasonable risk management analysis and for the introduction of decision-makers economic predictions. According to [18], such an interest rates model needs to satisfy the following general properties:

(i) *no arbitrage condition.* An arbitrage free model is necessary for derivatives pricing or short-term dynamic hedging. It could be argued that this property is less relevant for long-term ALM decision problems, since hedging derivatives is not the core of ALM problems. As seen in the introduction, some ALM models do not make it a strong requirement (see Mulvey et al [9]). Nevertheless, an arbitrage-free model is more natural and makes decision-makers more comfortable about its use.

(ii) *Term structure realism.* The model must be able to generate all the possible forms of the term structure of interest rates since, over a long period of time, one should expect all possible economic situations to occur. This excludes for instant, the use of a pure Vasicek interest rate model based on only the instantaneous short-term rate [14] (as in Munk et al [10] for instance), since these models cannot reproduce shifts from contango to backwardation term structure. Generally, a two-factor model will be at least needed to provide a compromise between tractability and description of possible dynamics of the interest rate term structure.

(iii) *Integration capacity.* Since the interest rate term structure is at the beginning of the cascade modeling of many other asset prices (bond price, equity returns), its modeling should be compliant with the efficient simulation of the remaining financial variables.

(iv) *Simplicity.* Long-term ALM risk management studies imply the computation of a very large number of scenarios on a period of sometimes a century. Hence, it is an important feature of a model to be able to perform this intensive computation in an efficient way. Closed-form solutions for bond prices and interest rates, at any time, are essential to ensure this efficiency requirement.

(v) *Calibration on historical data.* Calibration on historical data must be feasible to measure the capacity of the model to reproduce past situations.

(vi) *Exogenous economic constraints.* The model must also be able to satisfy specific characteristics given by experts and which are different from what has been observed in the past. This degree of freedom can also be used to study different characteristics of the future in order to assess "what if" scenarios.

These above requirements requirements (i) to (iv) are intuitive, and refer to the compromise between realism and tractability. Requirements (v) and (vi) are concerned by the calibration capabilities of the model, an essential property for the purpose of generating realistic future scenarios. In a short-term study, following the intuition that "what will happen in the near future is close to what happened in the near past", a predictive model is needed to satisfy Requirement (v), i.e. to be calibrated on historical data. Instead, in long-term studies, the global model should be able to be reproduce long-term target values provided by economists and financial experts who can convert qualitative information on the evolution of financial institutions and regulation on the long-run into quantitative variables.

The main financial variables for which target values are generally given are expected stock return and volatilities, inflation trend and volatility and expected returns, volatility and correlation for a small set of bonds.

2.2 Exogenous long-term ALM constraints

In this paper, we fix a pair of interest rates yields $\big(B_t(t+m), B_t(t+m')\big), t \geq 0$, with time-to-maturity $m, m' > 0$, and we require that the interest rates model has the capacity to take as exogenous the following values:

- the long-term expectations $\lim_{t\to\infty} \mathbb{E}\{B_t(t+m)\}$ and $\lim_{t\to\infty} \mathbb{E}\{B_t(t+m)\}$,
- the long-term standard deviations $\lim_{t\to\infty} \mathbb{V}\{B_t(t+m)\}$ and $\lim_{t\to\infty} \mathbb{V}\{B_t(t+m')\}$
- the long-term covariance $\lim_{t\to\infty} \mathbb{C}ov\{B_t(t+m), B_t(t+m')\}$.

Hence, there are 5 exogenous constraints based on the experts forecasts. We will see in section 2.3 that a sixth constraint will have to be introduced to ensure consistency with the other variables modeled in the global scenario generation model.

2.3 A Gaussian HJM model with long-term ALM constraints

We formulate the Gaussian multifactor HJM model, and we translate the long-term ALM constraints highlighted in the previous subsection. Notice that one

Of course, the Gaussian feature of our model induces negative interest rates with positive probability, and one would rather prefer a multifactor CIR [5] model as in Duffie & Kan [6]. The drawback of such a setting is that the calibration constraints are more complex, and the simulation needs more advanced and numerically consuming methods.

We also observe that the recent attractive Cairns' model [2] allows to reproduce the main expected properties of the interest rates term structure. But, it is also not straightforward to see how parameters can be modified to fit the above long-term ALM constraints. Moreover, the computation of bond prices is not analytical and requires a numerical approximation of a one-dimensional integral. Therefore, in general, the simulation under the historical probability becomes quickly difficult.

The Gaussian multifactor HJM model is instead more tractable and is well-known to produce a rich family of curves for the interest rates term structure [1, 12]. Furthermore, we shall show that it can be designed so as to incorporate our long-term ALM constraints, while still being tractable for the simulation aspects.

We denote by $B_t(T)$ the time$-t$ price of the zero-coupon bond with maturity T. We consider a Gaussian HJM model where the volatility of zero-coupon bonds is a deterministic function of time $\Gamma_t(T)$. The absence of arbitrage opportunity between zero-coupon bonds of different maturities implies the existence of a vector of market prices of risk λ_t which allows writing the relation between the different zero-coupon bond yields as:

$$\frac{dB_t(T)}{B_t(T)} = r_t dt + \Gamma_t(T) \cdot (dW_t + \lambda_t dt), \tag{1}$$

where $\cdot$ is the scalar product and r_t represents the instantaneous rate. Integrating the above stochastic differential equation and using the fact that $B_t(t) = 1$, we deduce the expression of $B_t(T)$ in terms of the volatility, avoiding the unobserved variable r_t:

$$\begin{aligned} B_t(T) = \frac{B_0(T)}{B_0(t)} \exp \Bigg\{ & \int_0^t [\Gamma_s(T) - \Gamma_s(t)] \cdot (dW_s + \lambda_s ds) \\ & - \frac{1}{2} \int_0^t |\Gamma_s(T)|^2 - |\Gamma_s(t)|^2 \, ds \Bigg\}. \end{aligned}$$

The associated discount rate $R_t(T)$ is deduced using:

$$B_t(T) = e^{-(T-t)R_t(T)}, \tag{2}$$

which leads to:

$$\begin{aligned} R_t(T) = \frac{TR_0(T) - tR_0(t)}{T-t} & - \frac{1}{T-t} \int_0^t [\Gamma_s(T) - \Gamma_s(t)] \cdot (dW_s + \lambda_s ds) \\ & + \frac{1}{2(T-t)} \int_0^t |\Gamma_s(T)|^2 - |\Gamma_s(t)|^2 \, ds \end{aligned} \tag{3}$$

From now on, we will consider the class of Gaussian two-factor models with Vasicek type volatilities. In this case, the 2 stochastic factors impacting the evolution of the yield curve are modeled by the $\mathbb{R}^2$-valued Brownian motion $W_t = (W_t^{(1)}, W_t^{(2)})$. The volatility functions have the following form:

$$\Gamma_t^{(i)}(T) = \sigma_i \frac{1 - e^{-a_i(T-t)}}{a_i}, \quad i = 1, 2 \tag{4}$$

and we will consider that the market prices of risk are constant in time $\lambda_t^{(i)} = \lambda_i$. In this context, it is possible to compute the long-term expectation and covariance of two different interest rates yields with constant time to maturity m et m', i.e. $R_t(t+m)$ and $R_t(t+m')$. Considering Vasicek type volatility functions and constant market prices of risk, focusing on a constant time to maturity $m = T - t$ and $m' = T' - t$, and taking the limit $t \to \infty$, we find the following relations for the limit values:

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}\left\{R_t(t+m)\right\} =& R_0^\infty - \frac{1}{m}\sum_{i=1}^{2} \frac{\sigma_i \lambda_i}{a_i^2}(1 - e^{-a_i m}) \\ &+ \frac{1}{2m}\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3}\left[1 - e^{-a_i m} + \frac{1}{2}(1 - e^{-a_i m})^2\right], \end{aligned} \tag{5}$$

$$\lim_{t\to\infty} \mathbb{C}ov\left\{R_t(t+m), R_t(t+m')\right\} = \sum_{i=1}^{2} \frac{\sigma_i^2}{2a_i^3} \frac{(1 - e^{-a_i m})(1 - e^{-a_i m'})}{m\, m'} \tag{6}$$

where R_0^∞ is the value for the interest rate of "infinite" maturity observed on the initial yield curve. In practice, it corresponds to a maturity of 30 years. In the following we will denote $\mu := \lim_{t\to\infty} \mathbb{E}\left\{R_t(t+m)\right\}$ and $\mu' := \lim_{t\to\infty} \mathbb{E}\left\{R_t(t+m')\right\}$ the long term expectation of interest rate of maturities m and m', $\Sigma^2 := \lim_{t\to\infty} \mathbb{C}ov\left\{R_t(t+m), R_t(t+m)\right\}$ and $\Sigma'^2 := \lim_{t\to\infty} \mathbb{C}ov\left\{R_t(t+m'), R_t(t+m')\right\}$ their long term variances and $\rho\Sigma\Sigma' := \lim_{t\to\infty} \mathbb{C}ov\left\{R_t(t+m), R_t(t+m')\right\}$ their covariance.

It is well known that one can consider that the two Brownian factors are independent without loss of generality. The only restriction of this model is the choice of the volatility function. In the following we then assume that the two Brownian factors are independent. But, as stated by requirements (iii) (see section 2.1) the yield curve simulation is integrated in a global generation process of all the financial variables. The generation of coherent scenarios of all the variables is performed through the correlation between the yield curves points and the other assets prices. Therefore, to avoid potential inconsistency between historical estimation of the correlation matrix and the parameter calibration of the interest model, the correlation between the two bond yields has also to be considered as an input of the calibration process. Hence, we introduce a sixth relation to assess this global coherence requirement. This sixth relation is the correlation between the bond yields of maturities m and m'. The expression of this correlation as a function of the HJM model parameters is given by the following expressions:

$$\begin{aligned} \mathbb{C}ov\left(\frac{dB_t}{B_t}(t+m), \frac{dB_t}{B_t}(t+m')\right) &= \mathbb{V}ar\left(r_t dt\right) + \Gamma_t(t+m)\Gamma_t^T(t+m')dt \\ &+ \mathbb{C}ov\left(r_t dt, \Gamma_t(t+m)dW_t\right) \\ &+ \mathbb{C}ov\left(r_t dt, \Gamma_t(t+m')dW_t\right) \end{aligned}$$

This expression involves terms of different orders and unknown values like the variance of the instantaneous rate or the covariance between HJM factors and the instantaneous rate. However, these terms are of order dt^α with $\alpha > 1$. Using standard Itô's calculus and noting ρ_{dB} the correlation between two bond yields of maturities m and m' and using the previous expression with Vasicek type volatilities, we obtain:

$$\rho_{dB} = \frac{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1 - e^{-a_i m})(1 - e^{-a_i m'})}{\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1 - e^{-a_i m})^2}\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1 - e^{-a_i m'})^2}} \tag{7}$$

We are now in a position to state the non-linear system corresponding to the long-term ALM calibration problem. We assume that the following expert views are given:

- the long-term expectations μ and μ' of the two interest rate yields,
- the long-term standard deviations Σ and Σ' of the two interest rate yields.
- the long-term correlation ρ between the two interest rates yields,
- the instantaneous correlation ρ_{dB} between the two interest rates yields.

Our calibration problem is to find, for given values of μ, μ', Σ, Σ', ρ and ρ_{dB}, conditions under which there exists a set of values σ_i, λ_i and a_i, $i = 1, 2$ satisfying the following non-linear system:

$$\sum_{i=1}^{2} \left(\frac{\sigma_i^2}{a_i^3} - 2\frac{\sigma_i \lambda_i}{a_i^2} \right) (1 - e^{-a_i m}) = 2m(\mu - R_0^\infty) - m^2\Sigma^2 \tag{8a}$$

$$\sum_{i=1}^{2} \left(\frac{\sigma_i^2}{a_i^3} - 2\frac{\sigma_i \lambda_i}{a_i^2} \right) (1 - e^{-a_i m'}) = 2m'(\mu' - R_0^\infty) - m'^2\Sigma'^2 \tag{8b}$$

$$\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3} (1 - e^{-a_i m})^2 = 2m^2\Sigma^2 \tag{8c}$$

$$\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3} (1 - e^{-a_i m})(1 - e^{-a_i m'}) = 2mm'\rho\Sigma\Sigma' \tag{8d}$$

$$\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3} (1 - e^{-a_i m'})^2 = 2m'^2\Sigma'^2 \tag{8e}$$

$$\frac{\displaystyle\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2} (1 - e^{-a_i m})(1 - e^{-a_i m'})}{\sqrt{\displaystyle\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2} (1 - e^{-a_i m})^2} \sqrt{\displaystyle\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2} (1 - e^{-a_i m'})^2}} = \rho_{dB} \tag{8f}$$

Since the left-hand side of equation (8d) must be positive, the correlation coefficients are subject to the restrictions:

$$0 < \rho < 1, \tag{9}$$

$$0 < \rho_{dB} < 1. \tag{10}$$

This necessary condition for a solution to exist is related to the particular class of models we have chosen, and is not satisfied in the context of a general HJM model. However, in practice, the correlation between the two interest rates yields under consideration, is observed to be highly positive. Hence, this constraint will always be satisfied.

3 Calibration method

In this section we describe our solution approach (Section 3.1), and we determine the precise restriction that the parameters have to satisfy in order to ensure the existence of a solution to the long-term ALM calibration problem (Section A.1).

3.1 Solution approach

Our approach is based on two remarks: (i) the market prices of risk λ_1 and λ_2 appear only in equations (8a) and (8b), and (ii) conditionally to the other parameters, the system relating the market prices of risk to the predictions of expectations μ and μ' is linear. Therefore the global system (8) can be solved in two steps:

- Solving equations (8c-8f):

$$\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3}(1-e^{-a_i m})^2 = 2m^2\Sigma^2 \tag{11a}$$

$$\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3}(1-e^{-a_i m})(1-e^{-a_i m'}) = 2mm'\rho\Sigma\Sigma' \tag{11b}$$

$$\sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3}(1-e^{-a_i m'})^2 = 2m'^2\Sigma'^2 \tag{11c}$$

$$\frac{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m})(1-e^{-a_i m'})}{\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m})^2}\sqrt{\sum_{i=1,2} \frac{\sigma_i^2}{a_i^2}(1-e^{-a_i m'})^2}} = \rho_{dB} \tag{11d}$$

  Given Σ, Σ', ρ and ρ_{dB}, the resolution of this first set of equations will provide a solution for $(a_1, a_2, \sigma_1, \sigma_2)$.

- Once a solution to the previous problem is found, we solve the subsystem composed of equations (8a-8b):

$$\sum_{i=1}^{2} \left(\frac{\sigma_i^2}{a_i^3} - 2\frac{\sigma_i\lambda_i}{a_i^2}\right)(1-e^{-a_i m}) = 2m(\mu - R_0^\infty) - m^2\Sigma^2, \tag{12a}$$

$$\sum_{i=1}^{2} \left(\frac{\sigma_i^2}{a_i^3} - 2\frac{\sigma_i\lambda_i}{a_i^2}\right)(1-e^{-a_i m'}) = 2m'(\mu' - R_0^\infty) - m'^2\Sigma'^2, \tag{12b}$$

  given μ, μ' and R_0^∞ and where the remaining unknowns are only λ_1 and λ_2. This last step is very easy since the system is linear w.r.t. λ_1 and λ_2.

3.1.1 Calibration of the long-term covariances

We set the following notations:

$$s_i = \frac{\sigma_i^2}{a_i^3}, \qquad i = 1, 2$$

$$A = 2m^2\Sigma^2, \qquad B = 2mm'\rho\Sigma\Sigma', \qquad C = 2m'^2\Sigma'^2.$$

The equations (11a), (11b) and (11c) can be rewritten in the equivalent following hierarchical structure:

$$s_2 = \frac{A - s_1(1-e^{-a_1 m})^2}{(1-e^{-a_2 m})^2} \tag{13a}$$

$$\frac{1-e^{-a_2 m'}}{1-e^{-a_2 m}} = \frac{B - s_1(1-e^{-a_1 m})(1-e^{-a_1 m'})}{A - s_1(1-e^{-a_1 m})^2} \tag{13b}$$

$$s_1 = \frac{AC - B^2}{A(1-e^{-a_1 m'})^2 + C(1-e^{-a_1 m})^2 - 2B(1-e^{-a_1 m})(1-e^{-a_1 m'})} \tag{13c}$$

Equation (13c) gives the expression of the set of solutions represented by a relation between a_1 and s_1. Any pair (a_1, s_1) satisfying (13c) is an admissible solution of the subsystem. The choice of one particular value may be based on the resolution of equation (11d). The resolution of this equation is not possible analytically. However, as we will show in appendix A, the set of admissible solutions for a_1 is a bounded interval. Therefore it is possible to determine an approximation of a solution by an optimization algorithm:

$$a_1 = \underset{a_1 \in \mathcal{A}_1}{\arg\min} \|\rho_{dB} - h(a_1)\| \tag{14}$$

where $\mathcal{A}_1$ is the set of admissible values for a_1 and $h(a_1)$ is the expression of the left hand side of equation (11d) where a_2, σ_1 and σ_2 are replaced by their value as a function of a_1, given expressions (13a)-(13c). The fact that $\mathcal{A}_1$ will be a bounded interval (see section A.1) and that $h(a_1)$ is sufficiently regular implies that an exhaustive research of the minimum is numerically tractable.

Once the solution a_1 has been computed, we can deduce the associated value of s_1, a_2 and s_2 using respectively equations (13c), (13b) and (13a). Note that the determination of a_2 needs the inversion of a particular function:

$$\phi(a) = \frac{(1 - e^{-am'})}{(1 - e^{-am})} \tag{15}$$

This function is not analytically invertible but has the particular advantage of being constantly decreasing from $\frac{m'}{m} > 1$. Indeed, computing the differential of ϕ and, for example, noting that $\frac{m'e^{-am'}}{1-e^{-am'}} < \frac{me^{-am}}{1-e^{-am}}$ for any $a > 0$ and $m' > m$, we can show that ϕ is strictly decreasing. Calculating the limit at 0 and $+\infty$ the set of possible values for ϕ is $\left(1\ ,\ \frac{m'}{m}\right)$. Therefore, the inversion of equation (13b) can be numerically done by means of a simple dichotomy, for example. The existence of a solution is conditioned by the fact that the right hand side of equation (13b) must be in $\left(1\ ,\ \frac{m'}{m}\right)$.

3.1.2 Calibration of the long-term expectations

Once we have determined the parameters (a_1, a_2, s_1, s_2), it remains to estimate the two market prices of risk λ_1 and λ_2 using equations (12a) and (12b). This subsystem of two equations is linear with respect to λ_1 and λ_2. Hence the estimation of theses parameters is trivial once we have ensured that the system is invertible. To see that it is always the case, we rewrite the system as follows:

$$\begin{bmatrix} \psi(a_1, \sigma_1, m) & \psi(a_2, \sigma_2, m) \\ \psi(a_1, \sigma_1, m') & \psi(a_2, \sigma_2, m') \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \varphi(a_1, a_2, \sigma_1, \sigma_2, R_0^\infty, m) \\ \varphi(a_1, a_2, \sigma_1, \sigma_2, R_0^\infty, m') \end{pmatrix} \tag{16}$$

with

$$\begin{aligned} \psi(a, \sigma, m) &= \frac{\sigma}{a^2}(1 - e^{-am}) \\ \varphi(a_1, a_2, \sigma_1, \sigma_2, R_0^\infty, m) &= -\frac{1}{2}\left[2m(\mu - R_0^\infty) - m^2\Sigma^2 - \sum_{i=1}^{2} \frac{\sigma_i^2}{a_i^3}(1 - e^{-a_i m})\right] \end{aligned} \tag{17}$$

Since the function $\phi(\cdot)$ is strictly decreasing, it is trivial to show that this system is always invertible whenever $a_2 > a_1$ and $m' > m$.

3.2 Calibration solution synthesis

The parameter estimation procedure constructed above can be summarized in the following way.

$$a_1 = \underset{a_1 \in \mathcal{A}_1}{\arg\min} \|\rho_{dB} - h(a_1)\| \qquad \text{cf. eq.(14)}$$

$$\sigma_1^2 = \frac{a_1^2(AC - B^2)}{A(1-e^{-a_1 m'})^2 + C(1-e^{-a_1 m})^2 - 2B(1-e^{-a_1 m})(1-e^{-a_1 m'})} \qquad \text{cf. eq.(13c)}$$

$$a_2 = \phi^{-1}\left(\frac{B - s_1(1-e^{-a_1 m})(1-e^{-a_1 m'})}{A - s_1(1-e^{-a_1 m})^2}\right) \qquad \text{cf. eq.(13b)}$$

$$\sigma_2^2 = \frac{a_2^2\left(A - s_1(1-e^{-a_1 m})^2\right)}{(1-e^{-a_2 m})^2} \qquad \text{cf. eq. (13a)}$$

$$\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = \begin{bmatrix}\psi(a_1,\sigma_1,m) & \psi(a_2,\sigma_2,m) \\ \psi(a_1,\sigma_1,m') & \psi(a_2,\sigma_2,m')\end{bmatrix}^{-1} \begin{pmatrix}\varphi(a_1,a_2,\sigma_1,\sigma_2,R_0^\infty) \\ \varphi'(a_1,a_2,\sigma_1,\sigma_2,R_0^\infty)\end{pmatrix} \qquad \text{cf. eq.(16)}$$

Obviously the existence of a global solution of this system is guaranteed only when prediction values respect some constraints. Actually these constraints are only on Σ, Σ', ρ for several reasons:

- There are no constraints on μ and μ' since any value can be reached by solving the linear subsystem (16) in λ_1 and λ_2. One can notice that even if any value of μ and μ' can be mathematically reached, these values must be chosen with respect to some economic considerations. For example in a global financial variable generation process these values must be related to economic expectation of inflation.

- There is a constraint on ρ_{dB} which can not be analytically determined but only numerically (see appendix A). However this constraint is not considered in this paper since we relax the strong equality of (8f), replacing it by a minimization strategy (14). Therefore the global solution may not give exactly the prediction value ρ_{dB}. We proposed this relaxation because it is well known that two-factor models produce significant correlation between two bond yields (cf. [12], p. 31), sometimes higher than what is observed in the market. Therefore, the exact equality of equation (11d) is very restrictive and may even lead to infeasibilities. This issue will be further discussed in Section 4.

A complete study of the feasibility set is done in appendix A and leads to the following constraints:

$$\boxed{\begin{array}{c} \frac{m}{m'}\Sigma \ < \ \Sigma' \ < \ \Sigma \\[2ex] \rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \end{array}}$$

These constraints have non-trivial consequences for exogenous prediction values. However we can deduce some necessary properties of the prediction values. First, the volatility of interest rates must be decreasing with the maturity. This property has long been observed on the market and hence, this constraint is always consistent with the observations. Second, the closer m and m' are, the closer the volatilities of the corresponding interest rates are. Moreover, their correlation gets closer to one. However, even if m and m' are significantly different, the model imposes a high value for the correlation. Actually this is observed in the market. For example, the empirical correlation between rates of maturity 1 month and 10 year, computed on historical data from January 2000 to May 2008, is about 80%.

4 Numerical tests

In this section the numerical calibration process described previously is illustrated on a case where the two interest rate maturities for which long-term economic expectation are provided, are the 1 month and 10 year maturity Euro interest rates. Their respective long-term expectation μ and μ', volatility Σ and Σ' and their correlation ρ are given in Table 1.

	1 month	10 years
expectation (μ and μ')	3%	4%
volatility (Σ and Σ')	1%	0.6%
correlation between rates (ρ)	80 %	
correlation between yields (ρ_{dB})	7.5%	

Table 1: Long term expected economic value for the 1 month and 10 year Euro interest rate.

These values are arbitrary chosen but are realistic since they are those observed in the Euro market from January 2000 to May 2008. Since long term expectations of interest rates depend on anticipated inflation, strategic allocations committees generally prefer to work with *real* interest rates and inflation anticipations. Nevertheless it is straightforward to deduce the nominal interest rates that should be expected once real interest rate and inflation are being fixed. Since the purpose of these numerical tests are illustrative of our calibration methodology, we work with nominal interest rates.

4.1 Interpolation of the initial rate curve

We consider the initial yield curve in Euro as the one observed at 2008/10/01 issued from Datastream [1]. In order to obtain the initial rate curve at any point, we choose a Nelson-Siegel type interpolation method [11]:

$$R_0(T) = \beta_0 + \beta_1 \left(\frac{1 - e^{\frac{T}{\tau}}}{\frac{T}{\tau}} \right) + \beta_2 \left(\frac{1 - e^{\frac{T}{\tau}}}{\frac{T}{\tau}} - e^{\frac{T}{\tau}} \right)$$

with the parameter τ set *a priori* to 1.5, following the practice of Banque de France [13] and β_0, β_1 and β_2 are obtained by linear regression from historical data. This interpolation is made necessary because the simulation of the interest rate curve requires the initial rates for all maturities, (see the term $R_0(t)$ in equation (3)). The interpolation leads to the limit value $R_0^\infty = \beta_0$, see eq. (8).

4.2 Calibration

The solution of the non-linear system (8) obtained by the algorithm summarized in Section 3.2 is given in Table 2.

[1] www.datastream.com

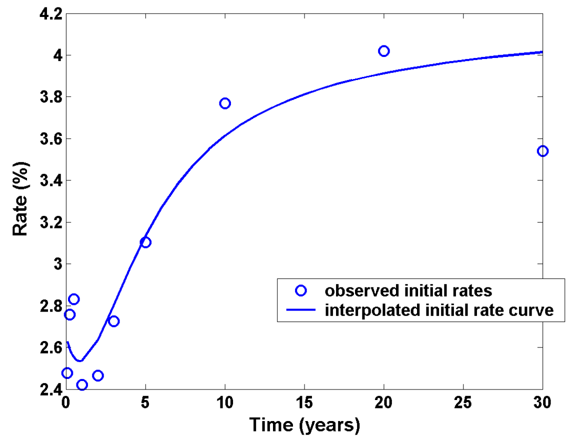

Figure 1: Initial rate curve: the observed initial rates are 1, 3 and 6 months, 1, 2, 3, 5, 10, 20 and 30 years. The interpolation method is of Nelson-Siegel type with parameter $\tau = 1.5$.

a_1	0.0595	σ_1	0.0027	λ_1	0.0880
a_2	23.4268	σ_2	0.00947	λ_2	5.1726

Table 2: Model parameter values for associated economic expectations.

Notice that the set of prediction values satisfies the existence constraints given by (33) and (34), which means that the resulting parameters of Table 2 allow us to produce interest rates with the prediction values defined in Table 1. Indeed, the economic expectations given in Table 1 imply that the correlation between rates, ρ, must be up to 60%. The set $\mathcal{A}_1$ of possible values of a_1 is therefore $\big(0\ ,\ 0.086\big)$, and Figure 2 shows all the admissible values for the bonds yield correlation ρ_{dB}. Because $\rho_{dB} = 7.5\%$ cannot be reached, the resulting calibrated parameters do not exactly match all expected predictions: the obtained yield correlation is 25.6%. This is essentially due to our modeling restriction to only two factors as already specified in section 3.2 and also observed in [12].

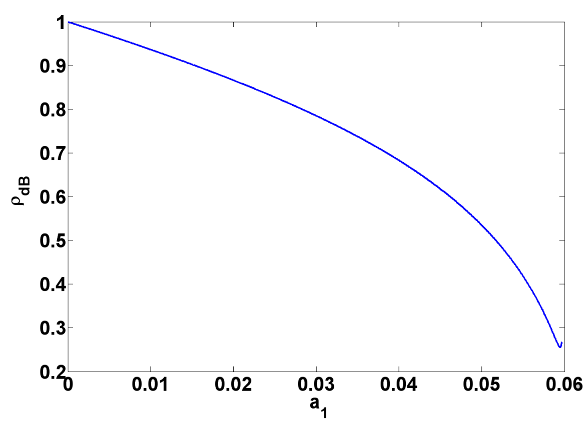

Figure 2: Admissible values of ρ_{dB} as a function of the value of a_1.

4.3 Simulation of interest rates

We have simulated 5,000 trajectories of 1-month and 10-year interest rates during one century, with a discretization step of one week. From these trajectories we determined the mean values and the quantiles 5% and 95% to define a confidence interval at 90% during all the simulated period.

(a) 1-month interest rate (b) 10-year interest rate

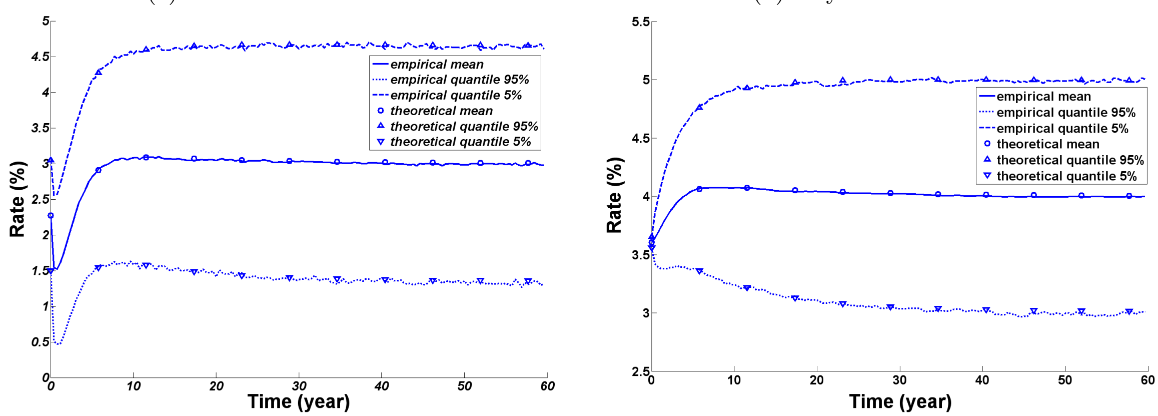

Figure 3: Simulation result of (a) 1-month and (b) 10-year interest rates. The dotted and solid lines represent empirical statistics of the rates (means and confidence interval at 90%) while the different symbols represent theoretical statistics.

(a) initial rate curve at 2008/10/01 (b) initial rate curve at 2010/06/03

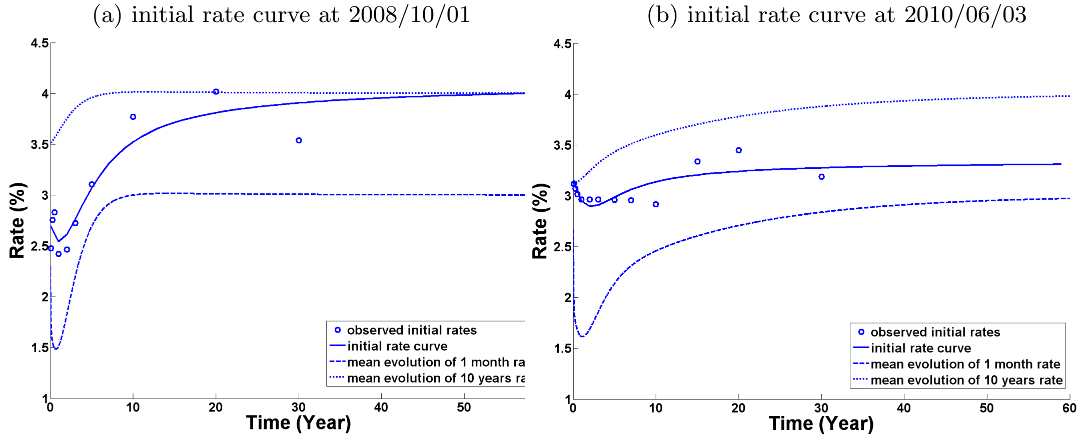

Figure 4: Comparison between initial rate curve and mean evolution of 1 month and 10 year interest rates.

Figure 3 shows the results of these simulations with the parameters defined in Table 2. In each graph the dotted and solid lines represent the empirical statistics of 1-month and 10-year interest rates by the mean value and the confidence interval at 90%. The lines drawn by symbols "o", "▵" and "▿" represent the theoretical statistics of these rates. The correspondence between theoretical and empirical results shows that the model produces the right expected predictions in terms of mean and volatility. It can be seen also that it takes about 30 years for the interest rate to reach its expected value. The way interest rates converge to their long-term expected value depends crucially on the initial rate curve. Indeed the evolution of both rates follows the pattern given by the initial rate curve. During the first 30 years, the interest rates evolve according to the information provided by the initial rate curve. And after 30 years the rates tend to the only available information which is the set of expected value to be reproduced.

This result shows that the initial rate curve plays an essential role in the dynamic evolution of the interest rates. Therefore the importance of the initial date of study and the interpolation method cannot be underestimated. Figure 4 highlights theses points. Mean statistics are presented for the 10-year interest rates when the initial rate correspond to the observed rate curve at 2008/10/01 and at 2010/06/03. We can see that the convergence time to the expected value, as well as the way it reaches this value, significantly depends on the form of the initial rate curve.

Finally we can see a positive probability to have negative interest rates, in particular for the 1-month rate. Indeed, the proposed model does not ensure the positivity of the interest rates. Because the case of negative rates appears mostly in a short time, particularly in the beginning of the period, the impact of these negative values is negligible in long-term ALM studies.

4.4 Parameter sensitivity

In this section we focus on the sensitivity of the estimated parameters due to changes in prediction values. The method is: starting from prediction values of table 1 we vary only one parameter. In the following we show two illustrations of this sensitivity. The parameters which have been varied are μ and Σ'. In both cases we show the evolution of estimated parameters a_1, a_2, σ_1, σ_2, λ_1 and λ_2. However, due to the fact that our proposed method is an inversion and then the interpretation of parameter evolution is difficult, we also show the evolution of the factors of equations (8a) and (8c):

$$F_i(m) = \frac{\sigma_i^2}{a_i^3}(1 - e^{-a_i m})^2 \tag{18}$$

Figure 5 shows parameter sensitivity to changes in μ. Because Σ, Σ', ρ and ρ_{dB} did not change, only the market prices of risk are impacted. Their evolution are linear with respect to μ which confirms the linear property of the subsystem (16). λ_1 remain quasi constant whereas λ_2 is decreasing with μ. This is explained by the negativity of the function φ defined in (17). We can also see that the market price of risk can be negative. Even if the proposed algorithm always gives mathematical solution of the system, one must then be careful about this undesirable effect.

Figure 6 shows parameter sensitivity to changes in Σ'. The factors $F_1(m')$ and $F_2(m')$ are obviously increasing with Σ' whereas factors $F_1(m)$ and $F_2(m)$ remain quasi constant to keep constant Σ. We can see that σ_1 and σ_2 are decreasing, which shows that the increasing of factors $F_1(m')$ and $F_2(m')$ is due to a decreasing of a_1 and a_2. Because these parameters can be interpreted as mean reverting coefficients, this shows that the increasing of long term volatility will imply a model with weak mean reverting property. This may be an undesirable effect because if the user wants to choose a high long term volatility, the resulting model will respect this property but with scenarios of interest rates that can stay far from its mean value for a long time, since the resulting mean reverting coefficients are weak.

5 Conclusion

In this paper we proposed a model of the yield curve within the HJM framework that can fit the specific requirements of long-term asset and liability management. In particular, the proposed HJM model can reproduce expected long-term statistical properties of any pair of interest rates yields-to-maturity, while still satisfying the no-arbitrage constraints. Moreover, the choice of a two factor model with Vasicek type volatility functions leads to an easy computation of interest rate derivatives. This simplicity allows building a quasi-analytical procedure to calibrate the model, i. e. to find the model parameters as a function of the expected statistical properties. Precise constraints on these expected values were given so as to ensure the existence of a solution. The numerical example highlights the essential role of the form of the initial yield curve, in the dynamics of the two interest rates to be controlled. It also highlights that the proposed

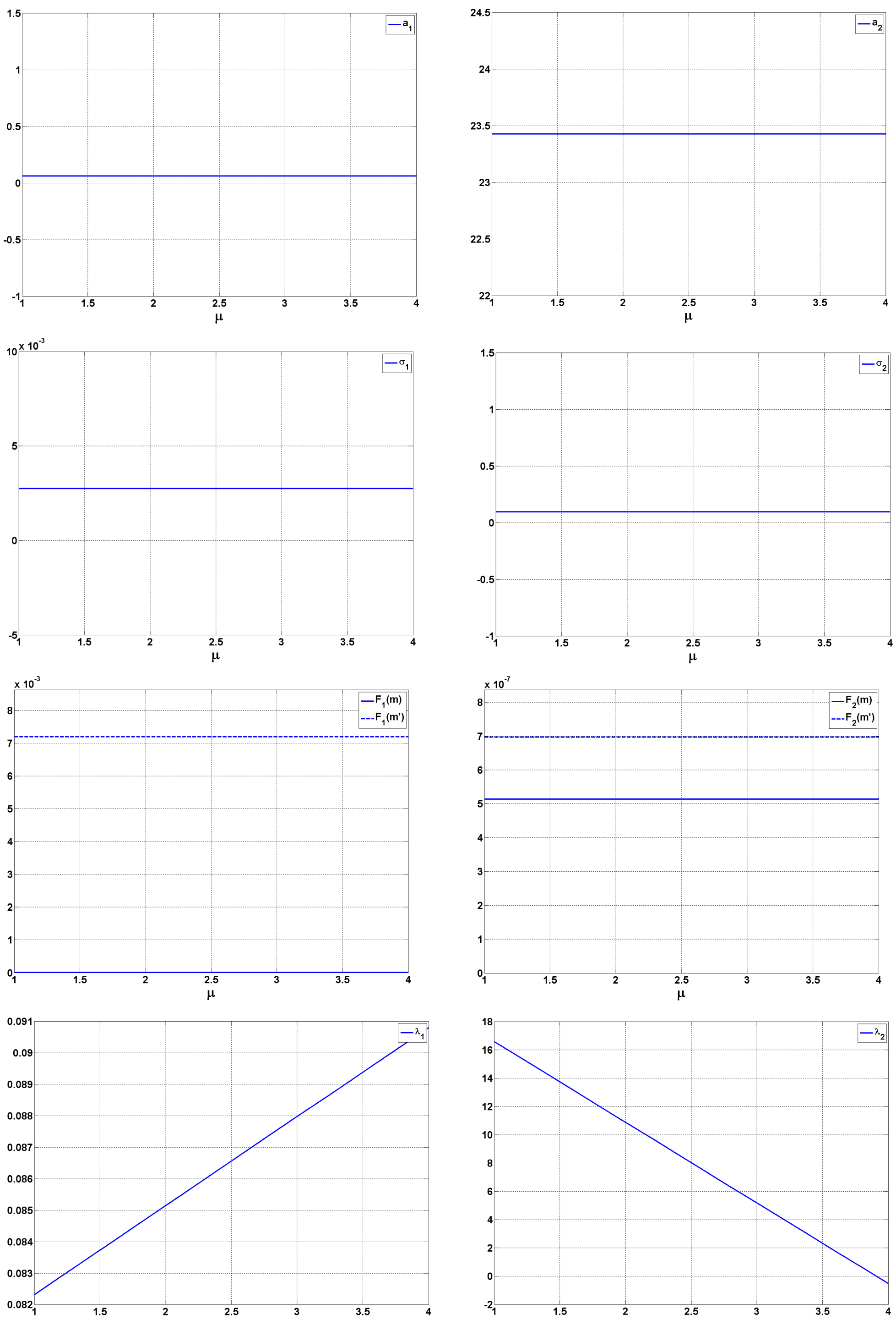

Figure 5: Evolution of estimated parameters with changes in μ

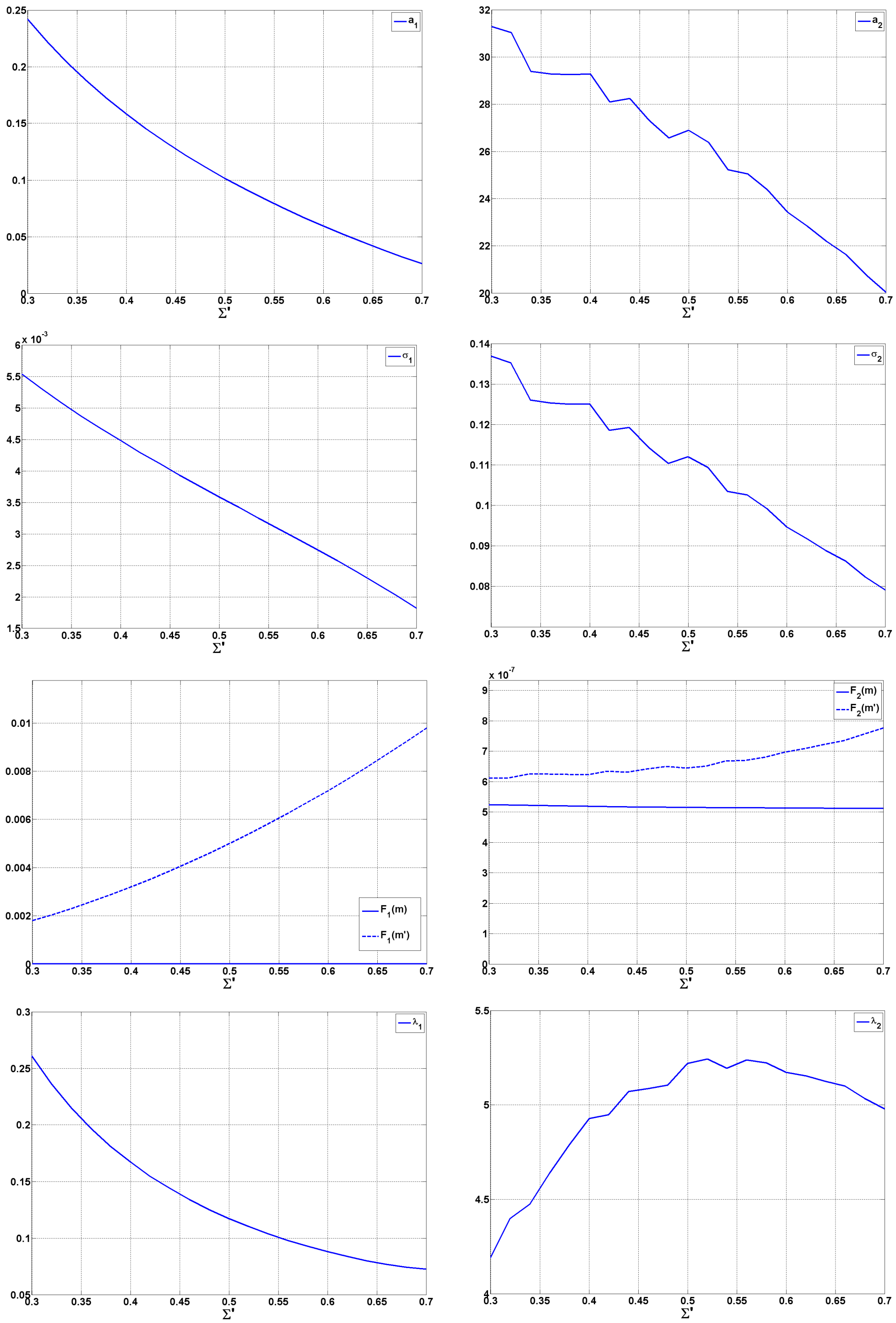

Figure 6: Evolution of estimated parameters with changes in Σ'

algorithm is helpful for automatically determining model parameters but the user must take into account economic consideration to confirm or reject the results of inversion.
The proposed interest rate model assumes constant market prices of risk. The introduction of non constant market prices of risk will lead to a higher flexibility in the parameter estimation process. In particular, this might extend the calibration process to other prediction values. Indeed this might allow considering a deterministic $\mu(t)$ trajectory of the expectation of interest rates. This might be done by the same algorithm, solving the subsystem on $\lambda_1(t)$ and $\lambda_2(t)$ at each time. However this subsystem looses its linear feature and is time-dependent. This extension is left to future work.

References

[1] D. Brigo and F. Mercurio, *Interest Rate Models Theory and Practice*, Springer-Verlag, 2001.

[2] A. J. G. Cairns, *A family of term-structure models for long-term risk management and derivative pricing*, Mathematical Finance **14** (2004), no. 3, 415–444.

[3] D. R. Carino and Z. T. Ziemba, *Formulation of the Russel-Yasuda Kasai financial planning model*, Operations Research **46** (1998), no. 4, 433–449.

[4] D. V. Carino, T. Kent, D. H. Meyers, C. Stacy, M. Sylvanus, A. Turner, K. Watanabe, and W. T. Ziemba, *The Russel-Yasuda Kasai model: an asset liability model for Japanese insurance company using multistage stochastic programming*, World Wide Asset and Liability Modeling (W. Ziemba and J. Mulvey, eds.), Cambridge University Press, 1998, pp. 609–633.

[5] J. Cox, J. Ingersoll, and S. Ross, *A theory of the term-structure of interest rates*, Econometrica **53** (1995), 385–408.

[6] D. Duffie and R. Kan, *A yield-factor model of interest rates*, Mathematical Finance **6** (1996), no. 4, 379–406.

[7] D. Heath, R. Jarrow, and A. Morton, *Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation*, Econometrica **60** (1992), no. 1, 77–105.

[8] J. Mulvey, G. Gould, and C. Morgan, *An asset and liability management system for Towers Perrin-Tillinghast*, Interfaces **30** (2000), no. 1, 96–114.

[9] J. M. Mulvey and A. E. Thorlacius, *The Towers Perrin global capital market scenario generation system*, World Wide Asset and Liability Modeling (W. Ziemba and J. Mulvey, eds.), Cambridge University Press, 1998, pp. 286–313.

[10] C. Munk, C. Sorensen, and T. N. Vinther, *Dynamic asset allocation under mean-reverting returns, stochastic interest rates, and inflation uncertainty: Are popular recommendations consistent with rational behavior?*, International Review of Economics and Finance **13** (2004), no. 2, 141–166.

[11] C. R. Nelson and A. F. Siegel, *Parsimonious modeling of yield curves*, Journal of Business **60** (1987), no. 4, 473–489.

[12] R. Rebonato, *Term-structure models: a review*, Tech. report, QUARC - Royal Bank of Scotland, 2003.

[13] R. Ricard and P. Sicsic, *Estimation d'une structure par termes des taux d'intérêt sur données françaises*, Tech. report, Banque de France, 1995.

[14] O. A. Vasicek, *An equilibrium characterization of the term structure*, Journal of Financial Economics **5** (1977), 177–188.

[15] A. D. Wilkie, *A stochastic investment model for actuarial use*, Transactions of the Faculty of Actuaries **39** (1986), 341–403.

[16] ______, *More on a stochastic asset model for actuarial use (with discussion)*, Brit. Actuarial J. **1** (1995), 777–964.

[17] Y. Yakoubov, M. Teeger, and D. Duval, *A stochastic investment model for asset and liability management*, 9th International AFIR Colloquium, August 1999, pp. 237–266.

[18] W. T. Ziemba and J. M. Mulvey, *Worldwide Asset and Liability Modeling*, Cambridge University Press, 2004.

A Appendix: Feasibility set

In this appendix we determine the constraints ensuring the existence of a global solution of system (8). This study is done by the following steps:

- We show that the existence of a solution of subsystem (8c)–(8e) results in the existence of a set $\mathcal{A}_1$ of admissible values for a_1.
- The existence of a non-empty set $\mathcal{A}_1$ result in constraints on the predicted values Σ, Σ' and ρ.
- The set of admissible values for a_1 will lead to a set of reachable values for ρ_{dB}. In this paper we do not consider this constraint because we relaxed equation (8f) by minimizing a criterion. Therefore the global solution may not exactly reach the predicted value ρ_{dB}.

A.1 Admissible set for a_1

Recalling equations (13a), (13b) and (13c) and the notations:

$$\begin{aligned} s_i &= \frac{\sigma_i^2}{a_i^3}, \qquad i = 1, 2 \\ A = 2m^2\Sigma^2, \qquad B &= 2mm'\rho\Sigma\Sigma', \qquad C = 2m'^2\Sigma'^2, \end{aligned}$$

we directly obtain the following four constraints:

- In equation (13a), since s_2 must be positive, we must have:

$$A - s_1(1 - e^{-a_1 m})^2 > 0 \tag{19}$$

- In equation (13b), since $\phi(a_2)$ must be in $\left(1\ ,\ \frac{m'}{m}\right)$, we must have the two restrictions:

$$\frac{B - s_1(1 - e^{-a_1 m})(1 - e^{-a_1 m'})}{A - s_1(1 - e^{-a_1 m})^2} > 1 \tag{20}$$

$$\frac{B - s_1(1 - e^{-a_1 m})(1 - e^{-a_1 m'})}{A - s_1(1 - e^{-a_1 m})^2} < \frac{m'}{m} \tag{21}$$

- In equation (13c), since s_1 must be positive, we must have:

$$\frac{AC - B^2}{(\sqrt{A}(1 - e^{-a_1 m'}) - \sqrt{C}(1 - e^{-a_1 m}))^2 + 2(\sqrt{AC} - B)(1 - e^{-a_1 m})(1 - e^{-a_1 m'})} > 0 \tag{22}$$

Actually we have only two real constraints: if the inequations (20) and (21) hold true, then there exists a solution of the subsystem. Indeed replacing s_1 by its value defined on (13c) we show that $A - s_1(1 - e^{-a_1 m})^2$ is always strictly positive. Also the inequation (22) is always verified, since $AC - B^2 > 0$ because of the definition of A, B and C.

Separating the parameters and the predicted values in the constraints (20) and (21) and replacing s_1 using equation (13c), we obtain these two expressions:

$$f(a_1) < B - A, \tag{23}$$

$$g(a_1) < \frac{m'}{m}A - B, \tag{24}$$

with

$$
\begin{aligned}
f(a_1) &= \frac{(AC-B^2)\left((1-e^{-a_1 m})(1-e^{-a_1 m'})-(1-e^{-a_1 m})^2\right)}{A(1-e^{-a_1 m'})^2+C(1-e^{-a_1 m})^2-2B(1-e^{-a_1 m})(1-e^{-a_1 m'})} \\
g(a_1) &= \frac{(AC-B^2)\left(\frac{m'}{m}(1-e^{-a_1 m})^2-(1-e^{-a_1 m})(1-e^{-a_1 m'})\right)}{A(1-e^{-a_1 m'})^2+C(1-e^{-a_1 m})^2-2B(1-e^{-a_1 m})(1-e^{-a_1 m'})}
\end{aligned}
$$

In the following we study independently the two functions f and g to define the respective sets $\mathcal{A}_f$ and $\mathcal{A}_g$ of admissible solutions for a_1 and finally we define the intersection $\mathcal{A}_1 = \mathcal{A}_f \cap \mathcal{A}_g$ which defines the set of values for a_1 that ensure the existence of the global solution. Each function study follows the same steps: we begin by defining the values of a_1 that reach the bound ($B-A$ for f and $\frac{m'}{m}A-B$ for g), and then, we compute the limit values of the function to find the set of a_1 that respects the constraint.

A.1.1 Study of f

Here we study the first constraint with the resolution of:

$$
f(a_1) - (B-A) = 0. \tag{25}
$$

This equation leads to a polynomial of degree 2 on the variable $\phi(a_1)$ defined in equation (15). Indeed equation (25) is equivalent to:

$$
A(A-B)\phi(a_1)^2 + \left[B(B-A)+A(C-B)\right]\phi(a_1) + B(B-C) = 0.
$$

The solutions of this polynomial lead to the final result:

$$
f(a_1) = (B-A) \Leftrightarrow \phi(a_1) = \frac{B}{A} \quad \text{or} \quad \phi(a_1) = \frac{C-B}{B-A}
$$

Since the range of ϕ is $\left(1\ ,\ \frac{m'}{m}\right)$, these solutions exist only if $\frac{B}{A}$ and $\frac{C-B}{B-A}$ belong to this interval. Recalling the definition of A, B and C we have:

$$
\begin{aligned}
\frac{B}{A} &= \frac{\rho m'\Sigma'}{m\Sigma} \\
\frac{C-B}{B-A} &= \frac{m'\Sigma'(m'\Sigma'-\rho m\Sigma)}{m\Sigma(\rho m'\Sigma'-m\Sigma)}
\end{aligned}
$$

Actually as the constraints $B-A>0$ and $\frac{m'}{m}A-B>0$ lead to the relation:

$$
\frac{m\Sigma}{m'\Sigma'} < \rho < \frac{\Sigma}{\Sigma'},
$$

it is obvious that under this constraint, the ratio $\frac{B}{A}$ is always in $\left(1\ ,\ \frac{m'}{m}\right)$. It is also trivial to see that:

$$
\frac{C-B}{B-A} > \frac{B}{A},
$$

by direct computing, for example the difference between these two parts and remembering that $AC-B^2$ and $B-A$ are positive. Therefore we have $\frac{C-B}{B-A} > 1$. We must then define the constraint ensuring that $\frac{C-B}{B-A} < \frac{m'}{m}$, which leads to the final result:

$$
\frac{C-B}{B-A} < \frac{m'}{m} \Leftrightarrow \rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}
$$

In addition, we can compute the limiting value of f:

$$\lim_{a_1 \to +\infty} f(a_1) = 0$$

By the fact that f is continuous and ϕ is decreasing, we deduce the set of a_1 that respects the constraint (23) depending on the prediction values:

$$\mathcal{A}_f = \begin{cases} \left(\phi^{-1}\left(\frac{B}{A}\right) \ , \ \infty\right) & \text{if } \ \rho \leq \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \\ \left(0 \ , \ \phi^{-1}\left(\frac{C-B}{B-A}\right)\right) \bigcup \left(\phi^{-1}\left(\frac{B}{A}\right) \ , \ \infty\right) & \text{if } \ \rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \end{cases}$$

A.1.2 Study of g

We apply the same reasoning as above to the function g. We begin by solve the equation:

$$g(a_1) - \left(\frac{m'}{m}A - B\right) = 0. \tag{26}$$

This equation leads also to a polynomial of degree 2 on the variable $\phi(a_1)$:

$$\left(AB - \frac{m'}{m}A^2\right)\phi(a_1)^2 + \left(2\frac{m'}{m}AB - AC - B^2\right)\phi(a_1) + \left(BC - \frac{m'}{m}B^2\right) = 0.$$

The solutions of this polynomial lead to the final result:

$$g(a_1) = (B - A) \Leftrightarrow \phi(a_1) = \frac{B}{A} \quad \text{or} \quad \phi(a_1) = \frac{\frac{m'}{m}B - C}{\frac{m'}{m}A - B}$$

The study of f showed that $\frac{B}{A} \in \left(1 \ ; \ \frac{m'}{m}\right)$ under the existing constraints on the predictions. It remains to study the value of the second solution. First it is trivial to show that:

$$\frac{\frac{m'}{m}B - C}{\frac{m'}{m}A - B} < \frac{B}{A},$$

calculating the difference between the two parts and remembering that $AC - B^2$ and $\frac{m'}{m}A - B$ are positive. Therefore, the existence of this solution is only conditioned by the lower bound of the interval, which leads to the final result:

$$\frac{\frac{m'}{m}B - C}{\frac{m'}{m}A - B} > 1 \Leftrightarrow \rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}$$

In addition, we can compute the limiting value of g:

$$\lim_{a_1 \to 0} g(a_1) = 0$$

By the fact that g is continuous and ϕ is decreasing, we deduce the set $\mathcal{A}_g$ of a_1 that respects the constraint (24) depending on the value of the predictions:

$$\mathcal{A}_g = \begin{cases} \left(0 \ , \ \phi^{-1}\left(\frac{B}{A}\right)\right) & \text{if } \ \rho \leq \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \\ \left(0 \ , \ \phi^{-1}\left(\frac{B}{A}\right)\right) \bigcup \left(\phi^{-1}\left(\frac{\frac{m'}{m}B - C}{\frac{m'}{m}B - A}\right) \ , \ \infty\right) & \text{if } \ \rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \end{cases}$$

A.1.3 Final admissible set for a_1

The remarkable result of the two previous studies is that a common condition appears on the prediction values for determining the set of admissible values of a_1. From these previous results we deduce that the only way for having a non-empty set of admissible values is when:

$$\rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \tag{27}$$

and, in this case, $a_1 \in \mathcal{A}_f \cap \mathcal{A}_g$. Since $\frac{\frac{m'}{m}B-C}{\frac{m'}{m}B-A} < \frac{B}{A} < \frac{C-B}{B-A}$ and ϕ is decreasing, we obtain:

$$a_1 \in \left(0\ ,\ \phi^{-1}(\frac{C-B}{B-A})\right) \bigcup \left(\phi^{-1}(\frac{\frac{m'}{m}B-C}{\frac{m'}{m}B-A})\ ,\ \infty\right) \tag{28}$$

In addition, we can show the specific relation:

$$\phi(a_2) = \frac{B\phi(a_1)-C}{A\phi(a_1)-B},$$

Therefore if a_1 belongs to the first part of $\mathcal{A}_f \cap \mathcal{A}_g$, i.e. $a_1 < \phi^{-1}\left(\frac{C-B}{B-A}\right)$ then a_2 necessarily belongs to the second part, i.e. $a_2 > \phi^{-1}\left(\frac{\frac{m'}{m}B-C}{\frac{m'}{m}B-A}\right)$. By the symmetric definition of a_1 and a_2 we can restrict the set of admissible values for a_1 and assume:

$$a_1 \in \left(0\ ,\ \phi^{-1}(\frac{C-B}{B-A})\right).$$

In particular, the inversion of the system will then lead to a short-term factor and a long-term factor since a_1 and a_2 will belong to distinct intervals.

A.2 Consequences for possible economic prediction values

Since f and g are always strictly positive, we must necessarily have $B - A > 0$ and $\frac{m'}{m}A - B > 0$ (cf. eq. (23) and (24)). Adding the constraints (27) of existence of a solution, we can summarize the constraints on the exogenous prediction values as follows:

$$\rho > \frac{m\Sigma}{m'\Sigma'}, \tag{29}$$

$$\rho < \frac{\Sigma}{\Sigma'}, \tag{30}$$

$$\rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}, \tag{31}$$

In particular, this imply that $\frac{m\Sigma}{m'\Sigma'} < 1$ and then $\Sigma' > \frac{m}{m'}\Sigma$. Under this assumption the resulting restriction of ρ is:

$$\rho \in \left(\frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'}\ ,\ \frac{\Sigma}{\Sigma'}\right) \tag{32}$$

Actually it is trivial to show that this interval is non-empty only when $\Sigma' < \Sigma$. Finally, the constraints of existence of a solution, in terms of restriction of acceptable values of the exogenous predictions are:

$$\frac{m}{m'}\Sigma < \Sigma' < \Sigma \tag{33}$$

$$\rho > \frac{m'\Sigma'^2 + m\Sigma^2}{(m+m')\Sigma\Sigma'} \tag{34}$$