Weak Dynamic Programming Principle for Viscosity Solutions *

Bruno Bouchard[†] and Nizar Touzi[‡]

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Abstract

We prove a weak version of the dynamic programming principle for standard stochastic control problems and mixed control-stopping problems, which avoids the technical difficulties related to the measurable selection argument. In the Markov case, our result is tailor-made for the derivation of the dynamic programming equation in the sense of viscosity solutions.

Key words: Optimal control, Dynamic programming, discontinuous viscosity solutions.

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1 Introduction

Consider the standard class of stochastic control problems in the Mayer form

$$V(t,x) \quad := \quad \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[f(X_T^\nu) | X_t^\nu = x\right],$$

where $\mathcal{U}$ is the controls set, X^ν is the controlled process, f is some given function, $0 < T \leq \infty$ is a given time horizon, $t \in [0,T)$ is the time origin, and $x \in \mathbb{R}^d$ is some given initial condition. This framework includes the general class of stochastic control problems under the so-called Bolza formulation, the corresponding singular versions, and optimal stopping problems.

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[†]CEREMADE, Université Paris Dauphine and CREST-ENSAE, bouchard@ceremade.dauphine.fr

[‡]Ecole Polytechnique Paris, Centre de Mathématiques Appliquées, touzi@cmap.polytechnique.fr

A key-tool for the analysis of such problems is the so-called dynamic programming principle (DPP), which relates the time$-t$ value function $V(t,.)$ to any later time$-\tau$ value $V(\tau,.)$ for any stopping time $\tau \in [t,T)$ a.s. A formal statement of the DPP is:

$$''V(t,x) \;=\; v(t,x) \;:=\; \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[V(\tau, X^\nu_\tau)|X^\nu_t = x\right].'' \tag{1.1}$$

In particular, this result is routinely used in the case of controlled Markov jump-diffusions in order to derive the corresponding dynamic programming equation in the sense of viscosity solutions, see Lions [9, 10], Fleming and Soner [7], Touzi [14], for the case of controlled diffusions, and Oksendal and Sulem [11] for the case of Markov jump-diffusions.

The statement (1.1) of the DPP is very intuitive and can be easily proved in the deterministic framework, or in discrete-time with finite probability space. However, its proof is in general not trivial, and requires on the first stage that V be measurable.

When the value function V is known to be continuous, the abstract measurability arguments are not needed and the proof of the dynamic programming principle is significantly simplified. See e.g. Fleming and Soner [7], or Kabanov [8] in the context of a special singular control problem in finance. Our objective is to reduce the proof to this simple context in a general situation where the value function has no a priori regularity.

The inequality "$V \leq v$" is the easy one but still requires that V be measurable. Our weak formulation avoids this issue. Namely, under fairly general conditions on the controls set and the controlled process, it follows from an easy application of the tower property of conditional expectations that

$$V(t,x) \;\leq\; \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[V^*(\tau, X^\nu_\tau)|X^\nu_t = x\right],$$

where V^* is the upper semicontinuous envelope of the function V.

The proof of the converse inequality "$V \geq v$" in a general probability space turns out to be difficult when the function V is not known a priori to satisfy some continuity condition. See e.g. Bertsekas and Shreve [2], Borkar [3], and El Karoui [6].

Our weak version of the DPP avoids the non-trivial measurable selection argument needed to prove the inequality $V \geq v$ in (1.1). Namely, in the context of a general control problem presented in Section 2, we show in Section 3 that:

$$V(t,x) \;\geq\; \sup\nolimits_{\nu \in \mathcal{U}} \mathbb{E}\left[\varphi(\tau, X^\nu_\tau)|X_t = x\right]$$

for every upper-semicontinuous minorant φ of V.

We also show that an easy consequence of this result is that

$$V(t,x) \;\geq\; \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[V_*(\tau^\nu_n, X^\nu_{\tau^\nu_n})|X_t = x\right],$$

where $\tau^\nu_n := \tau \wedge \inf\{s > t : |X^\nu_s - x| > n\}$, and V_* is the lower semicontinuous envelope of V.

This result is weaker than the classical DPP (1.1). However, in the controlled Markov jump-diffusions case, it turns out to be tailor-made for the derivation of the dynamic programming equation in the sense of viscosity solutions. Section 5 reports this derivation in the context of controlled jump diffusions.

Finally, Section 4 provides an extension of our argument in order to obtain a weak dynamic programming principle for mixed control-stopping problems.

2 The stochastic control problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T > 0$ a finite time horizon, and $\mathbb{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$ a given filtration of $\mathcal{F}$, satisfying the usual assumptions. For every $t \geq 0$, we denote by $\mathbb{F}^t = (\mathcal{F}^t_s)_{s \geq 0}$ the right-continuous filtration where, for every $s \geq 0$, $\mathcal{F}^t_s$ is the $\sigma-$algebra of events in $\mathcal{F}_s$ that are independent of $\mathcal{F}_t$. In particular, for $s \leq t$, $\mathcal{F}^t_s$ is the trivial degenerate $\sigma-$algebra.

We denote by $\mathcal{T}$ the collection of all $\mathbb{F}-$stopping times. For $\tau_1, \tau_2 \in \mathcal{T}$ with $\tau_1 \leq \tau_2$ a.s., the subset $\mathcal{T}_{[\tau_1,\tau_2]}$ is the collection of all $\tau \in \mathcal{T}$ such that $\tau \in [\tau_1, \tau_2]$ a.s. When $\tau_1 = 0$, we simply write $\mathcal{T}_{\tau_2}$. We use the notations $\mathcal{T}^t_{[\tau_1,\tau_2]}$ and $\mathcal{T}^t_{\tau_2}$ to denote the corresponding sets of stopping times that are independent of $\mathcal{F}_t$.

For $\tau \in \mathcal{T}$ and a subset A of a finite dimensional space, we denote by $\mathbb{L}^0_\tau(A)$ the collection of all $\mathcal{F}_\tau-$measurable random variables with values in A. $\mathbb{H}^0(A)$ is the collection of all $\mathbb{F}-$progressively measurable processes with values in A, and $\mathbb{H}^0_{\text{rcll}}(A)$ is the subset of all processes in $\mathbb{H}^0(A)$ which are right-continuous with finite left limits.

In the following, we denote by $B_r(z)$ (resp. $\partial B_r(z)$) the open ball (resp. its boundary) of radius $r > 0$ and center $z \in \mathbb{R}^\ell$, $\ell \in \mathbb{N}$.

Througout this note, we fix an integer $d \in \mathbb{N}$, and we introduce the sets:

$$\mathbf{S} := [0,T] \times \mathbb{R}^d \quad \text{and} \quad \mathcal{S}_0 := \left\{(\tau, \xi) : \tau \in \mathcal{T}_T \text{ and } \xi \in \mathbb{L}^0_\tau(\mathbb{R}^d)\right\}.$$

We also denote by USC($\mathbf{S}$) (resp. LSC($\mathbf{S}$)) the collection of all upper-semicontinuous (resp. lower-semicontinuous) functions from $\mathbf{S}$ to $\mathbb{R}$.

The set of *control processes* is a given subset $\mathcal{U}_0$ of $\mathbb{H}^0(\mathbb{R}^k)$, for some integer $k \geq 1$, so that the *controlled state process* defined as the mapping:

$$(\tau, \xi; \nu) \in \mathcal{S} \times \mathcal{U}_0 \longmapsto X^\nu_{\tau,\xi} \in \mathbb{H}^0_{\text{rcll}}(\mathbb{R}^d) \quad \text{for some } \mathcal{S} \text{ with } \quad \mathbf{S} \subset \mathcal{S} \subset \mathcal{S}_0$$

is well-defined and satisfies:

$$\left(\theta, X^\nu_{\tau,\xi}(\theta)\right) \in \mathcal{S} \quad \text{for all} \quad (\tau, \xi) \in \mathcal{S} \text{ and } \theta \in \mathcal{T}_{[\tau,T]}.$$

A suitable choice of the set $\mathcal{S}$ in the case of jump-diffusion processes driven by Brownian motion is given in Section 5 below.

Given a Borel function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ and $(t,x) \in \mathbf{S}$, we introduce the reward function $J : \mathbf{S} \times \mathcal{U} \longrightarrow \mathbb{R}$:

$$J(t,x;\nu) \;:=\; \mathbb{E}\left[f\left(X^{\nu}_{t,x}(T)\right)\right] \tag{2.1}$$

which is well-defined for controls ν in

$$\mathcal{U} \;:=\; \Big\{\nu \in \mathcal{U}_0 :\; \mathbb{E}\big[|f(X^{\nu}_{t,x}(T))|\big] < \infty,\; \forall\,(t,x) \in \mathbf{S}\Big\}. \tag{2.2}$$

We say that a control $\nu \in \mathcal{U}$ is *t-admissible* if it is independent of $\mathcal{F}_t$, and we denote by $\mathcal{U}_t$ the collection of such processes. The stochastic control problem is defined by:

$$V(t,x) \;:=\; \sup_{\nu \in \mathcal{U}_t} J(t,x;\nu) \quad \text{for} \quad (t,x) \in \mathbf{S}. \tag{2.3}$$

Remark 2.1 The restriction to control processes independent on $\mathcal{F}_t$ in the definition of $V(t,\cdot)$ is natural and consistent with the case where $t=0$, since $\mathcal{F}_0$ is assumed to be trivial, and is actually commonly used, compare with e.g. [15]. It will be technically important in the proof of the inequality (3.2) of Theorem 3.1. When the filtration is generated by a process with independent increments, it is moreover not restrictive as we will show in the context of SDEs driven by a Brownian motion and a compound Poisson process in Remark 5.2 below.

3 Dynamic programming for stochastic control problems

For the purpose of our weak dynamic programming principle, the following assumptions are crucial.

Assumption A *For all $(t,x) \in \mathbf{S}$ and $\nu \in \mathcal{U}_t$, the controlled state process satisfies:*

A1 (Independence) *The process $X^{\nu}_{t,x}$ is independent of $\mathcal{F}_t$.*

A2 (Causality) *For $\tilde{\nu} \in \mathcal{U}_t$, $\tau \in \mathcal{T}^t_{[t,T]}$ and $A \in \mathcal{F}^t_\tau$, if $\nu = \tilde{\nu}$ on $[t,\tau]$ and $\nu\mathbf{1}_A = \tilde{\nu}\mathbf{1}_A$ on $(\tau,T]$, then $X^{\nu}_{t,x}\mathbf{1}_A = X^{\tilde{\nu}}_{t,x}\mathbf{1}_A$.*

A3 (Stability under concatenation) *For every $\tilde{\nu} \in \mathcal{U}_t$, and $\theta \in \mathcal{T}^t_{[t,T]}$:*

$$\nu\mathbf{1}_{[0,\theta]} + \tilde{\nu}\mathbf{1}_{(\theta,T]} \;\in\; \mathcal{U}_t\,.$$

A4 (Consistency with deterministic initial data) *For all $\theta \in \mathcal{T}^t_{[t,T]}$, we have:*

a. *For* $\mathbb{P}$*-a.e* $\omega \in \Omega$*, there exists* $\tilde{\nu}_\omega \in \mathcal{U}_{\theta(\omega)}$ *such that*

$$\mathbb{E}\left[f\left(X^\nu_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) \leq J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde{\nu}_\omega).$$

b. *For* $t \leq s \leq T$*,* $\theta \in \mathcal{T}^t_{[t,s]}$*,* $\tilde{\nu} \in \mathcal{U}_s$*, and* $\bar{\nu} := \nu \mathbf{1}_{[0,\theta]} + \tilde{\nu}\mathbf{1}_{(\theta,T]}$*, we have:*

$$\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) = J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde{\nu}) \quad \textit{for} \quad \mathbb{P}-a.e.\ \omega \in \Omega.$$

Remark 3.1 Assumption A2 above means that the process $X^\nu_{t,x}$ is defined (caused) by the control ν pathwise.

Remark 3.2 In Section 5 below, we show that Assumption A4-a holds with equality in the jump-diffusion setting. Although we have no example of a control problem where the equality does not hold, we keep Assumption A4-a under this form because the proof only needs this requirement.

Remark 3.3 Assumption A3 above implies the following property of the controls set which will be needed later:
A5 (Stability under bifurcation) *For* $\nu_1, \nu_2 \in \mathcal{U}_t$*,* $\tau \in \mathcal{T}^t_{[t,T]}$ *and* $A \in \mathcal{F}^t_\tau$*, we have:*

$$\bar{\nu} := \nu_1 \mathbf{1}_{[0,\tau]} + (\nu_1 \mathbf{1}_A + \nu_2 \mathbf{1}_{A^c})\,\mathbf{1}_{(\tau,T]} \quad \in \quad \mathcal{U}_t.$$

To see this, observe that $\tau_A := T\mathbf{1}_A + \tau \mathbf{1}_{A^c}$ is a stopping time in $\mathcal{T}^t_{[t,T]}$, and $\bar{\nu} = \nu_1 \mathbf{1}_{[0,\tau_A]} + \nu_2 \mathbf{1}_{(\tau_A,T]}$ is the concatenation of ν_1 and ν_2 at the stopping time τ_A.

Iterating the above property, we see that for $0 \leq t \leq s \leq T$ and $\tau \in \mathcal{T}^t_{[t,T]}$, we have the following extension: for a finite sequence $(\nu_1, \ldots, \nu_n)$ of controls in $\mathcal{U}_t$ with $\nu_i = \nu_1$ on $[0, \tau]$, and for a partion $(A_i)_{1 \leq i \leq n}$ of Ω with $A_i \in \mathcal{F}^t_\tau$ for every $i \leq n$:

$$\bar{\nu} := \nu_1 \mathbf{1}_{[0,\tau]} + \mathbf{1}_{(\tau,T]} \sum_{i=1}^n \nu_i \mathbf{1}_{A_i} \quad \in \quad \mathcal{U}_t.$$

Our main result is the following weak version of the dynamic programming principle which uses the following notation:

$$V_*(t,x) := \liminf_{(t',x') \to (t,x)} V(t',x'), \quad V^*(t,x) := \limsup_{(t',x') \to (t,x)} V(t',x'), \quad (t,x) \in \mathbf{S}.$$

Theorem 3.1 *Let Assumptions A hold true. Then for every* $(t,x) \in \mathbf{S}$*, and for every family of stopping times* $\{\theta^\nu,\ \nu \in \mathcal{U}_t\} \subset \mathcal{T}^t_{[t,T]}$*, we have*

$$V(t,x) \quad \leq \quad \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[V^*(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right]. \tag{3.1}$$

Assume further that $J(.;\nu) \in \mathrm{LSC}(\mathbf{S})$ *for every* $\nu \in \mathcal{U}_0$*. Then, for any function* $\varphi : \mathbf{S} \longrightarrow \mathbb{R}$*:*

$$\varphi \in \mathrm{USC}(\mathbf{S}) \text{ and } V \geq \varphi \quad \Longrightarrow \quad V(t,x) \quad \geq \quad \sup_{\nu \in \mathcal{U}^\varphi_t} \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right], \tag{3.2}$$

where $\mathcal{U}^\varphi_t = \left\{\nu \in \mathcal{U}_t :\ \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))^+\right] < \infty \text{ or } \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))^-\right] < \infty\right\}$.

Before proceeding to the proof of this result, we report the following consequence.

Corollary 3.1 *Let the conditions of Theorem 3.1 hold, and assume $V_* > -\infty$. For $(t,x) \in \mathbf{S}$, let $\{\theta^\nu,\ \nu \in \mathcal{U}_t\} \subset \mathcal{T}^t_{[t,T]}$ be a family of stopping times such that $X^\nu_{t,x}\mathbf{1}_{[t,\theta^\nu]}$ is $\mathbb{L}^\infty-$bounded for all $\nu \in \mathcal{U}_t$. Then,*

$$\sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[V_*(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right] \le V(t,x) \le \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[V^*(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right]. \tag{3.3}$$

Proof The right-hand side inequality is already provided in Theorem 3.1. Fix $r > 0$. It follows from standard arguments, see e.g. Lemma 3.5 in [12], that we can find a sequence of continuous functions $(\varphi_n)_n$ such that $\varphi_n \le V_* \le V$ for all $n \ge 1$ and such that φ_n converges pointwise to V_* on $[0,T] \times B_r(0)$. Set $\phi_N := \min_{n\ge N} \varphi_n$ for $N \ge 1$ and observe that the sequence $(\phi_N)_N$ is non-decreasing and converges pointwise to V_* on $[0,T]\times B_r(0)$. Applying (3.2) of Theorem 3.1 and using the monotone convergence Theorem, we then obtain:

$$V(t,x) \ \ \ge \ \lim_{N\to\infty} \mathbb{E}\left[\phi_N(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right] = \mathbb{E}\left[V_*(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right].$$

□

Remark 3.4 Notice that the value function $V(t,x)$ is defined by means of $\mathcal{U}_t$ as the set of controls. Because of this, the lower semicontinuity of $J(.,\nu)$ required in the second part of Theorem 3.1 does not imply that V is lower semicontinuous. See however Remark 5.3 below.

Proof of Theorem 3.1 **1.** Let $\nu \in \mathcal{U}_t$ be arbitrary and set $\theta := \theta^\nu$. The first assertion is a direct consequence of Assumption A4-a. Indeed, it implies that, for $\mathbb{P}$-almost all $\omega \in \Omega$, there exists $\tilde{\nu}_\omega \in \mathcal{U}_{\theta(\omega)}$ such that

$$\mathbb{E}\left[f\left(X^\nu_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) \le J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde{\nu}_\omega).$$

Since, by definition, $J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde{\nu}_\omega) \le V^*(\theta(\omega), X^\nu_{t,x}(\theta)(\omega))$, it follows from the tower property of conditional expectations that

$$\mathbb{E}\left[f\left(X^\nu_{t,x}(T)\right)\right] \ = \ \mathbb{E}\left[\mathbb{E}\left[f\left(X^\nu_{t,x}(T)\right)|\mathcal{F}_\theta\right]\right] \ \le \ \mathbb{E}\left[V^*\left(\theta, X^\nu_{t,x}(\theta)\right)\right].$$

2. Let $\varepsilon > 0$ be given. Then there is a family $(\nu^{(s,y),\varepsilon})_{(s,y)\in\mathbf{S}} \subset \mathcal{U}_0$ such that:

$$\nu^{(s,y),\varepsilon} \in \mathcal{U}_s \quad \text{and} \quad J(s,y;\nu^{(s,y),\varepsilon}) \ge V(s,y) - \varepsilon, \quad \text{for every} \quad (s,y) \in \mathbf{S}. \tag{3.4}$$

By the lower-semicontinuity of $(t',x') \mapsto J(t',x';\nu^{(s,y),\varepsilon})$, for fixed $(s,y) \in \mathbf{S}$, together with the upper-semicontinuity of φ, we may find a family $(r_{(s,y)})_{(s,y)\in\mathbf{S}}$ of positive scalars so that, for any $(s,y) \in \mathbf{S}$,

$$\varphi(s,y) - \varphi(t',x') \ge -\varepsilon \ \text{ and } \ J(s,y;\nu^{(s,y),\varepsilon}) - J(t',x';\nu^{(s,y),\varepsilon}) \le \varepsilon \text{ for } (t',x') \in B(s,y;r_{(s,y)}), \tag{3.5}$$

where, for $r > 0$ and $(s, y) \in \mathbf{S}$,

$$B(s, y; r) := \{(t', x') \in \mathbf{S} \; : \; t' \in (s - r, s), \; |x' - y| < r\} \; .$$

Clearly, $\{B(s, y; r) \; : \; (s, y) \in \mathbf{S}, 0 < r \le r_{(s,y)}\}$ forms an open covering of $[0, T) \times \mathbb{R}^d$. It then follows from the Lindelöf covering Theorem, see e.g. [13] Theorem 6.3 Chap. VIII, that we can find a countable sequence $(t_i, x_i, r_i)_{i \ge 1}$ of elements of $\mathbf{S} \times \mathbb{R}$, with $0 < r_i \le r_{(t_i, x_i)}$ for all $i \ge 1$, such that $\mathbf{S} \subset \{T\} \times \mathbb{R}^d \cup (\cup_{i \ge 1} B(t_i, x_i; r_i))$. Set $A_0 := \{T\} \times \mathbb{R}^d$, $C_{-1} := \emptyset$, and define the sequence

$$A_{i+1} := B(t_{i+1}, x_{i+1}; r_{i+1}) \setminus C_i \quad \text{where} \quad C_i := C_{i-1} \cup A_i, \; i \ge 0.$$

With this construction, it follows from (3.4), (3.5), together with the fact that $V \ge \varphi$, that the countable family $(A_i)_{i \ge 0}$ satisfies

$$(\theta, X^\nu_{t,x}(\theta)) \in \cup_{i \ge 0} A_i \; \mathbb{P}\text{−a.s.}, \; A_i \cap A_j = \emptyset \text{ for } i \ne j \in \mathbb{N}, \text{ and } J(\cdot; \nu^{i,\varepsilon}) \ge \varphi - 3\varepsilon \text{ on } A_i \text{ for } i \ge 1, \tag{3.6}$$

where $\nu^{i,\varepsilon} := \nu^{(t_i, x_i), \varepsilon}$ for $i \ge 1$.

3. We now prove (3.2). We fix $\nu \in \mathcal{U}_t$ and $\theta \in \mathcal{T}^t_{[t,T]}$. Set $A^n := \cup_{0 \le i \le n} A_i$, $n \ge 1$. Given $\nu \in \mathcal{U}_t$, we define

$$\nu^{\varepsilon,n}_s := \mathbf{1}_{[t,\theta]}(s) \nu_s + \mathbf{1}_{(\theta,T]}(s) \Big(\nu_s \mathbf{1}_{(A^n)^c}(\theta, X^\nu_{t,x}(\theta)) + \sum_{i=1}^n \mathbf{1}_{A_i}(\theta, X^\nu_{t,x}(\theta)) \nu^{i,\varepsilon}_s \Big), \quad \text{for} \quad s \in [t, T].$$

Notice that $\{(\theta, X^\nu_{t,x}(\theta)) \in A_i\} \in \mathcal{F}^t_\theta$ as a consequence of the independence Assumption A1. Then, it follows from the stability under concatenation Assumption A3 and Remark 3.3 that $\nu^{\varepsilon,n} \in \mathcal{U}_t$. Then, using Assumptions A4-b, A2, and (3.6), we deduce that:

$$\begin{aligned}
\mathbb{E}\left[f\left(X^{\nu^{\varepsilon,n}}_{t,x}(T) \right) | \mathcal{F}_\theta \right] \mathbf{1}_{A^n}\left(\theta, X^\nu_{t,x}(\theta) \right) &= V\left(T, X^{\nu^{\varepsilon,n}}_{t,x}(T) \right) \mathbf{1}_{A_0}\left(\theta, X^\nu_{t,x}(\theta) \right) \\
&+ \sum_{i=1}^n J(\theta, X^\nu_{t,x}(\theta); \nu^{i,\varepsilon}) \mathbf{1}_{A_i}\left(\theta, X^\nu_{t,x}(\theta) \right) \\
&\ge \sum_{i=0}^n \left(\varphi(\theta, X^\nu_{t,x}(\theta)) - 3\varepsilon \right) \mathbf{1}_{A_i}\left(\theta, X^\nu_{t,x}(\theta) \right) \\
&= \left(\varphi(\theta, X^\nu_{t,x}(\theta)) - 3\varepsilon \right) \mathbf{1}_{A^n}\left(\theta, X^\nu_{t,x}(\theta) \right),
\end{aligned}$$

which, by definition of V and the tower property of conditional expectations, implies

$$\begin{aligned}
V(t, x) &\ge J(t, x; \nu^{\varepsilon,n}) \\
&= \mathbb{E}\left[\mathbb{E}\left[f\left(X^{\nu^{\varepsilon,n}}_{t,x}(T) \right) | \mathcal{F}_\theta \right] \right] \\
&\ge \mathbb{E}\left[\left(\varphi\left(\theta, X^\nu_{t,x}(\theta) \right) - 3\varepsilon \right) \mathbf{1}_{A^n}\left(\theta, X^\nu_{t,x}(\theta) \right) \right] + \mathbb{E}\left[f\left(X^\nu_{t,x}(T) \right) \mathbf{1}_{(A^n)^c}\left(\theta, X^\nu_{t,x}(\theta) \right) \right] \; .
\end{aligned}$$

Since $f\left(X_{t,x}^{\nu}(T)\right) \in \mathbb{L}^1$, it follows from the dominated convergence theorem that:

$$
\begin{aligned}
V(t,x) &\geq -3\varepsilon + \liminf_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X_{t,x}^{\nu}(\theta))\mathbf{1}_{A^n}\left(\theta, X_{t,x}^{\nu}(\theta)\right)\right] \\
&= -3\varepsilon + \lim_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X_{t,x}^{\nu}(\theta))^{+}\mathbf{1}_{A^n}\left(\theta, X_{t,x}^{\nu}(\theta)\right)\right] \\
&\qquad - \lim_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X_{t,x}^{\nu}(\theta))^{-}\mathbf{1}_{A^n}\left(\theta, X_{t,x}^{\nu}(\theta)\right)\right] \\
&= -3\varepsilon + \mathbb{E}\left[\varphi(\theta, X_{t,x}^{\nu}(\theta))\right],
\end{aligned}
$$

where the last equality follows from the left-hand side of (3.6) and from the monotone convergence theorem, due to the fact that either $\mathbb{E}\left[\varphi(\theta, X_{t,x}^{\nu}(\theta))^{+}\right] < \infty$ or $\mathbb{E}\left[\varphi(\theta, X_{t,x}^{\nu}(\theta))^{-}\right] < \infty$. The proof of (3.2) is completed by the arbitrariness of $\nu \in \mathcal{U}_t$ and $\varepsilon > 0$. □

Remark 3.5 *(Lower-semicontinuity condition I)* It is clear from the above proof that it suffices to prove the lower-semicontinuity of $(t,x) \mapsto J(t,x;\nu)$ for ν in a subset $\tilde{\mathcal{U}}_0$ of $\mathcal{U}_0$ such that $\sup_{\nu\in\tilde{\mathcal{U}}_t} J(t,x;\nu) = V(t,x)$. Here $\tilde{\mathcal{U}}_t$ is the subset of $\tilde{\mathcal{U}}_0$ whose elements are independent of $\mathcal{F}_t$. In most applications, this allows to reduce to the case where the controls are essentially bounded or satisfy a strong integrability condition.

Remark 3.6 *(Lower-semicontinuity condition II)* In the above proof, the lower-semicontinuity assumption is only used to construct the balls B_i on which $J(t_i, x_i; \nu^{i,\varepsilon}) - J(\cdot; \nu^{i,\varepsilon}) \leq \varepsilon$. Clearly, it can be alleviated, and it suffices that the lower-semicontinuity holds in time from the left, i.e.

$$
\liminf_{(t',x')\to(t_i,x_i),\ t'\leq t_i} J(t', x'; \nu^{i,\varepsilon}) \geq J(t_i, x_i; \nu^{i,\varepsilon}).
$$

Remark 3.7 *(The Bolza and Lagrange formulations)* Consider the stochastic control problem under the so-called Lagrange formulation:

$$
V(t,x) \;:=\; \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[\int_t^T Y_{t,x,1}^{\nu}(s) g\left(s, X_{t,x}^{\nu}(s), \nu_s\right) ds + Y_{t,x,1}^{\nu}(T) f\left(X_{t,x}^{\nu}(T)\right)\right],
$$

where

$$
dY_{t,x,y}^{\nu}(s) \;=\; -Y_{t,x,y}^{\nu}(s) k\left(s, X_{t,x}^{\nu}(s), \nu_s\right) ds \;\;,\;\; Y_{t,x,y}^{\nu}(t) = y > 0\;.
$$

Then, it is well known that this problem can be converted into the Mayer formulation (2.3) by augmenting the state process to (X, Y, Z), where

$$
dZ_{t,x,y,z}^{\nu}(s) \;=\; Y_{t,x,y}^{\nu}(s) g\left(s, X_{t,x}^{\nu}(s), \nu_s\right) ds \;\;,\;\; Z_{t,x,y,z}^{\nu}(t) = z \in \mathbb{R}\;,
$$

and considering the value function

$$
\bar{V}(t,x,y,z) := \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[Z_{t,x,y,z}^{\nu}(T) + Y_{t,x,y}^{\nu}(T) f\left(X_{t,x}^{\nu}(T)\right)\right] = yV(t,x) + z\;.
$$

In particular, $V(t,x) = \bar{V}(t,x,1,0)$. The first assertion of Theorem 3.1 implies

$$V(t,x) \;\; \le \;\; \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[Y^{\nu}_{t,x,1}(\theta^{\nu})V(\theta^{\nu},X^{\nu}_{t,x}(\theta^{\nu})) + \int_t^{\theta^{\nu}} Y^{\nu}_{t,x,1}(s)g\left(s,X^{\nu}_{t,x}(s),\nu_s\right)ds\right]. \quad (3.7)$$

Given a upper-semicontinuous minorant φ of V, the function $\bar{\varphi}$ defined by $\bar{\varphi}(t,x,y,z) := y\varphi(t,x)+z$ is an upper-semicontinuous minorant of $\bar{V}$. From the second assertion of Theorem 3.1, we see that for a family $\{\theta^{\nu},\ \nu\in\mathcal{U}_t\} \subset \mathcal{T}^t_{[t,T]}$,

$$\begin{aligned} V(t,x) \;\; &\ge \;\; \sup_{\nu\in\mathcal{U}^{\bar{\varphi}}_t} \mathbb{E}\left[\bar{\varphi}\left(\theta^{\nu},X^{\nu}_{t,x}(\theta^{\nu}),Y^{\nu}_{t,x,1}(\theta^{\nu}),Z^{\nu}_{t,x,1,0}(\theta^{\nu})\right)\right] \\ &= \;\; \sup_{\nu\in\mathcal{U}^{\bar{\varphi}}_t} \mathbb{E}\left[Y^{\nu}_{t,x,1}(\theta^{\nu})\varphi(\theta^{\nu},X^{\nu}_{t,x}(\theta^{\nu})) + \int_t^{\theta^{\nu}} Y^{\nu}_{t,x,1}(s)g\left(s,X^{\nu}_{t,x}(s),\nu_s\right)ds\right]. \quad (3.8)\end{aligned}$$

Remark 3.8 *(Infinite Horizon)* Infinite horizon problems can be handled similarly. Following the notations of the previous Remark 3.7, we introduce the infinite horizon stochastic control problem:

$$V^{\infty}(t,x) \;\; := \;\; \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[\int_t^{\infty} Y^{\nu}_{t,x,1}(s)g\left(s,X^{\nu}_{t,x}(s),\nu_s\right)ds\right].$$

Then, it is immediately seen that V^{∞} satisfies the weak dynamic programming principle (3.7)-(3.8).

4 Dynamic programming for mixed control-stopping problems

In this section, we provide a direct extension of the dynamics programming principle of Theorem 3.1 to the larger class of mixed control and stopping problems.

In the context of the previous section, we consider a Borel function $f:\mathbb{R}^d \longrightarrow \mathbb{R}$, and we assume $|f| \le \bar{f}$ for some continuous function $\bar{f}$. For $(t,x)\in\mathbf{S}$ the reward $\bar{J}:\mathbf{S}\times\bar{\mathcal{U}}\times\mathcal{T}_{[t,T]} \longrightarrow \mathbb{R}$:

$$\bar{J}(t,x;\nu,\tau) \;\; := \;\; \mathbb{E}\left[f\left(X^{\nu}_{t,x}(\tau)\right)\right], \quad (4.1)$$

which is well-defined for every control ν in

$$\bar{\mathcal{U}} \;\; := \;\; \left\{\nu\in\mathcal{U}_0:\ \mathbb{E}\Big[\sup_{t\le s\le T}\bar{f}(X^{\nu}_{t,x}(s))\Big] < \infty\ \forall\ (t,x)\in\mathbf{S}\right\}.$$

The mixed control-stopping problem is defined by:

$$\bar{V}(t,x) \;\; := \;\; \sup_{(\nu,\tau)\in\bar{\mathcal{U}}_t\times\mathcal{T}^t_{[t,T]}} \bar{J}(t,x;\nu,\tau)\ , \quad (4.2)$$

where $\bar{\mathcal{U}}_t$ is the subset of elements of $\bar{\mathcal{U}}$ that are independent of $\mathcal{F}_t$.

The key ingredient for the proof of (4.6) is the following property of the set of stopping times $\mathcal{T}_T$:

$$\text{For all } \theta, \tau_1 \in \mathcal{T}_T^t \text{ and } \tau_2 \in \mathcal{T}_{[\theta,T]}^t, \quad \text{we have} \quad \tau_1 \mathbf{1}_{\{\tau_1 < \theta\}} + \tau_2 \mathbf{1}_{\{\tau_1 \geq \theta\}} \in \mathcal{T}_T^t. \tag{4.3}$$

In order to extend the result of Theorem 3.1, we shall assume that the following version of A4 holds:

Assumption A4' *For all $(t,x) \in \mathbf{S}$, $(\nu,\tau) \in \bar{\mathcal{U}}_t \times \mathcal{T}_{[t,T]}^t$ and $\theta \in \mathcal{T}_{[t,T]}^t$, we have:*

a. *For $\mathbb{P}$-a.e $\omega \in \Omega$, there exists $(\tilde{\nu}_\omega, \tilde{\tau}_\omega) \in \bar{\mathcal{U}}_{\theta(\omega)} \times \mathcal{T}_{[\theta(\omega),T]}^{\theta(\omega)}$ such that*

$$\mathbf{1}_{\{\tau \geq \theta\}}(\omega) \mathbb{E}\left[f\left(X_{t,x}^\nu(\tau)\right) | \mathcal{F}_\theta \right](\omega) \leq \mathbf{1}_{\{\tau \geq \theta\}}(\omega) J\left(\theta(\omega), X_{t,x}^\nu(\theta)(\omega); \tilde{\nu}_\omega, \tilde{\tau}_\omega\right)$$

b. *For $t \leq s \leq T$, $\theta \in \mathcal{T}_{[t,s]}^t$, $(\tilde{\nu}, \tilde{\tau}) \in \bar{\mathcal{U}}_s \times \mathcal{T}_{[s,T]}^s$, $\bar{\tau} := \tau \mathbf{1}_{\{\tau < \theta\}} + \tilde{\tau} \mathbf{1}_{\{\tau \geq \theta\}}$, and $\bar{\nu} := \nu \mathbf{1}_{[0,\theta]} + \tilde{\nu} \mathbf{1}_{(\theta,T]}$, we have for $\mathbb{P}-$a.e. $\omega \in \Omega$:*

$$\mathbf{1}_{\{\tau \geq \theta\}}(\omega) \mathbb{E}\left[f\left(X_{t,x}^{\bar{\nu}}(\bar{\tau})\right) | \mathcal{F}_\theta \right](\omega) \quad = \quad \mathbf{1}_{\{\tau \geq \theta\}}(\omega) J(\theta(\omega), X_{t,x}^\nu(\theta)(\omega); \tilde{\nu}, \tilde{\tau}).$$

Theorem 4.1 *Let Assumptions A1, A2, A3 and A4' hold true. Then for every $(t,x) \in \mathbf{S}$, and for all family of stopping times $\{\theta^\nu,\ \nu \in \bar{\mathcal{U}}_t\} \subset \mathcal{T}_{[t,T]}^t$:*

$$\bar{V}(t,x) \quad \leq \sup_{(\nu,\tau) \in \bar{\mathcal{U}}_t \times \mathcal{T}_{[t,T]}^t} \mathbb{E}\left[\mathbf{1}_{\{\tau < \theta^\nu\}} f(X_{t,x}^\nu(\tau)) + \mathbf{1}_{\{\tau \geq \theta^\nu\}} \bar{V}^*(\theta^\nu, X_{t,x}^\nu(\theta^\nu)) \right]. \tag{4.4}$$

Assume further that the map $(t,x) \longmapsto \bar{J}(t,x;\nu,\tau)$ satisfies the following lower-semicontinuity property

$$\liminf_{t' \uparrow t, x' \to x} \bar{J}(t',x';\nu,\tau) \geq \bar{J}(t,x;\nu,\tau) \quad \text{for every} \quad (t,x) \in \mathbf{S} \text{ and } (\nu,\tau) \in \bar{\mathcal{U}} \times \mathcal{T}. \tag{4.5}$$

Then, for any function $\varphi \in \mathrm{USC}(\mathbf{S})$ with $\bar{V} \geq \varphi$:

$$\bar{V}(t,x) \quad \geq \sup_{(\nu,\tau) \in \bar{\mathcal{U}}_t^\varphi \times \mathcal{T}_{[t,T]}^t} \mathbb{E}\left[\mathbf{1}_{\{\tau < \theta^\nu\}} f(X_{t,x}^\nu(\tau)) + \mathbf{1}_{\{\tau \geq \theta^\nu\}} \varphi(\theta^\nu, X_{t,x}^\nu(\theta^\nu)) \right], \tag{4.6}$$

where $\bar{\mathcal{U}}_t^\varphi = \left\{ \nu \in \bar{\mathcal{U}}_t : \ \mathbb{E}\left[\varphi(\theta^\nu, X_{t,x}^\nu(\theta^\nu))^+ \right] < \infty \text{ or } \mathbb{E}\left[\varphi(\theta^\nu, X_{t,x}^\nu(\theta^\nu))^- \right] < \infty \right\}$.

For simplicity, we only provide the proof of Theorem 4.1 for optimal stopping problems, i.e. in the case where $\bar{\mathcal{U}}$ is reduced to a singleton. The dynamic programming principle for mixed control-stopping problems is easily proved by combining the arguments below with those of the proof of Theorem 3.1.

Proof (*for optimal stopping problems*) We omit the control ν from all notations, thus simply writing $X_{t,x}(\cdot)$ and $\bar{J}(t,x;\tau)$. Inequality (4.4) follows immediately from the tower property together with Assumptions A4'-a, recall that $\bar{J} \le \bar{V}^*$.

We next prove (4.6). Arguying as in Step 2 of the proof of Theorem 3.1, we first observe that, for every $\varepsilon > 0$, we can find a countable family $\bar{A}_i \subset (t_i - r_i, t_i] \times A_i \subset \mathbf{S}$, together with a sequence of stopping times $\tau^{i,\varepsilon}$ in $\mathcal{T}^{t_i}_{[t_i,T]}$, $i \ge 1$, satisfying $\bar{A}_0 = \{T\} \times \mathbb{R}^d$ and

$$\cup_{i\ge 0}\bar{A}_i = \mathbf{S},\ \ \bar{A}_i \cap \bar{A}_j = \emptyset \text{ for } i \ne j \in \mathbb{N}, \ \text{ and } \ \bar{J}(\cdot;\tau^{i,\varepsilon}) \ge \varphi - 3\varepsilon \text{ on } \bar{A}_i \text{ for } i \ge 1\ . \tag{4.7}$$

Set $\bar{A}^n := \cup_{i\le n}\bar{A}_i$, $n \ge 1$. Given two stopping times $\theta, \tau \in \mathcal{T}^t_{[t,T]}$, it follows from (4.3) (and Assumption A1 in the general mixed control case) that

$$\tau^{n,\varepsilon} := \tau \mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\ge\theta\}} \left(T\mathbf{1}_{(\bar{A}^n)^c}\left(\theta, X_{t,x}(\theta)\right) + \sum_{i=1}^{n} \tau^{i,\varepsilon} \mathbf{1}_{\bar{A}_i}\left(\theta, X_{t,x}(\theta)\right) \right)$$

is a stopping time in $\mathcal{T}^t_{[t,T]}$. We then deduce from the tower property together with Assumptions A4'-b and (4.7) that

$$\begin{aligned} \bar{V}(t,x) &\ge \bar{J}(t,x;\tau^{n,\varepsilon}) \\ &\ge \mathbb{E}\left[f\left(X^\nu_{t,x}(\tau)\right)\mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\ge\theta\}}\left(\varphi(\theta, X_{t,x}(\theta)) - 3\varepsilon\right)\mathbf{1}_{\bar{A}^n}(\theta, X_{t,x}(\theta))\right] \\ &\quad + \mathbb{E}\left[\mathbf{1}_{\{\tau\ge\theta\}} f(X_{t,x}(T))\mathbf{1}_{(\bar{A}^n)^c}(\theta, X_{t,x}(\theta))\right]\ . \end{aligned}$$

By sending $n \to \infty$ and arguing as in the end of the proof of Theorem 3.1, we deduce that

$$\bar{V}(t,x) \ \ge\ \mathbb{E}\left[f\left(X_{t,x}(\tau)\right)\mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\ge\theta\}}\varphi(\theta, X_{t,x}(\theta))\right] - 3\varepsilon,$$

and the result follows from the arbitrariness of $\varepsilon > 0$ and $\tau \in \mathcal{T}_{[t,T]}$. $\square$

5 Application to controlled Markov jump-diffusions

In this section, we show how the weak DPP of Theorem 3.1 allows to derive the correponding dynamic programming equation in the sense of viscosity solutions. We refer to Crandal, Ishii and Lions [5] and Fleming and Soner [7] for a presentation of the general theory of viscosity solutions.

For simplicity, we specialize the discussion to the context of controlled Markov jump-diffusions driven by a Brownian motion and a compound Poisson process. The same technology can be adapted to optimal stopping and impulse control or mixed problems, see e.g. [4].

5.1 Problem formulation and verification of Assumption A

We shall work on the product space $\Omega := \Omega_W \times \Omega_N$ where Ω_W is the set of continuous functions from $[0,T]$ into $\mathbb{R}^d$, and Ω_N is the set of integer-valued measures on $[0,T] \times E$ with $E := \mathbb{R}^m$ for some $m \geq 1$. For $\omega = (\omega^1, \omega^2) \in \Omega$, we set $W(\omega) = \omega^1$ and $N(\omega) = \omega^2$ and define $\mathbb{F}^W = (\mathcal{F}_t^W)_{t \leq T}$ (resp. $\mathbb{F}^N = (\mathcal{F}_t^N)_{t \leq T}$) as the smallest right-continuous filtration on Ω_W (resp. Ω_N) such that W (resp. N) is optional. We let $\mathbb{P}_W$ be the Wiener measure on $(\Omega_W, \mathcal{F}_T^W)$ and $\mathbb{P}_N$ be the measure on $(\Omega_N, \mathcal{F}_T^N)$ under which N is a compound Poisson measure with intensity $\tilde{N}(de, dt) = \lambda(de)dt$, for some finite measure λ on E, endowed with its Borel tribe $\mathcal{E}$. We then define the probability measure $\mathbb{P} := \mathbb{P}_W \otimes \mathbb{P}_N$ on $(\Omega, \mathcal{F}_T^W \otimes \mathcal{F}_T^N)$. With this construction, W and N are independent under $\mathbb{P}$. Without loss of generality, we can assume that the natural right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ induced by (W, N) is complete. In the following, we shall slightly abuse notations and sometimes write $N_t(\cdot)$ for $N(\cdot, (0,t])$ for simplicity.

We let U be a closed subset of $\mathbb{R}^k$, $k \geq 1$, and $\mu : \mathbf{S} \times U \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbf{S} \times U \longrightarrow \mathbb{M}^d$ be two Lipschitz continuous functions, and $\beta : \mathbf{S} \times U \times E \longrightarrow \mathbb{R}^d$ a measurable function, Lipschitz-continuous with linear growth in (t, x, u) uniformly in $e \in E$. Here $\mathbb{M}^d$ denotes the set of d-dimensional square matrices.

By $\mathcal{U}_0$, we denote the collection of all square integrable predictable processes with values U. For every $\nu \in \mathcal{U}_0$, the stochastic differential equation:

$$dX(r) = \mu\left(r, X(r), \nu_r\right) dr + \sigma\left(r, X(r), \nu_r\right) dW_r + \int_E \beta(r, X(r-), \nu_r, e) N(de, dr), \quad t \leq r \leq T, \tag{5.1}$$

has a unique strong solution $X_{\tau,\xi}^\nu$ satisfying $X_{\tau,\xi}^\nu(\tau) = \xi$, for any initial condition $(\tau, \xi) \in \mathcal{S} := \{(\tau, \xi) \in \mathcal{S}_0 : \mathbb{E}\left[|\xi|^2\right] < \infty\}$, satisfying

$$\mathbb{E}\left[\sup_{\tau \leq r \leq T} |X_{\tau,\xi}^\nu(r)|^2\right] \quad < \quad C(1 + \mathbb{E}\left[|\xi|^2\right]), \tag{5.2}$$

for some constant C which may depend on ν.

Remark 5.1 Clearly, less restrictive conditions could be imposed on β and N. We deliberately restrict here to this simple case, in order to avoid standard technicalities related to the definition of viscosity solutions for integro-differential operators, see e.g. [1] and the references therein.

From the independence of the increments of the Brownian motion and the compound Poisson measure, it follows that, for $0 \leq t \leq s$, the $\sigma-$algebra $\mathcal{F}_s^t$ is generated by $\{W_r - W_t,\ N_r - N_t : t \leq r \leq s\}$. The following remark shows that in the present case, it is not necessary to restrict the control processes ν to $\mathcal{U}_t$ in the definition of the value function $V(t, x)$.

Remark 5.2 Let $\tilde{V}$ be defined by

$$\tilde{V}(t,x) := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[f(X^{\nu}_{t,x}(T))\right] .$$

The difference between $\tilde{V}(t,\cdot)$ and $V(t,\cdot)$ comes from the fact that all controls in $\mathcal{U}$ are considered in the former, while we restrict to controls independent of $\mathcal{F}_t$ in the latter. We claim that

$$\tilde{V} = V ,$$

so that both problems are indeed equivalent. Clearly, $\tilde{V} \geq V$. To see that the converse holds true, fix $(t,x) \in [0,T)\times\mathbb{R}^d$ and $\nu \in \mathcal{U}$. Then, ν can be written as a measurable function of the canonical process $\nu((\omega_s)_{0\leq s\leq t},(\omega_s-\omega_t)_{t\leq s\leq T})$, where, for fixed $(\omega_s)_{0\leq s\leq t}$, the map $\nu_{(\omega_s)_{0\leq s\leq t}} : (\omega_s-\omega_t)_{t\leq s\leq T} \mapsto \nu((\omega_s)_{0\leq s\leq t},(\omega_s-\omega_t)_{t\leq s\leq T})$ can be viewed as a control independent on $\mathcal{F}_t$. Using the independence of the increments of the Brownian motion and the compound Poisson process, and Fubini's Lemma, it thus follows that

$$J(t,x;\nu) \;=\; \int \mathbb{E}\left[f(X^{\nu_{(\omega_s)_{0\leq s\leq t}}}_{t,x}(T))\right] d\mathbb{P}((\omega_s)_{0\leq s\leq t}) \leq \int V(t,x)d\mathbb{P}((\omega_s)_{0\leq s\leq t})$$

where the latter equals $V(t,x)$. By arbitrariness of $\nu\in\mathcal{U}$, this implies that $\tilde{V}(t,x)\leq V(t,x)$.

Remark 5.3 By the previous remark, it follows that the value function V inherits the lower semicontinuity of the performance criterion required in the second part of Theorem 3.1, compare with Remark 3.4. This simplification is specific to the simple stochastic control problem considered in this section, and may not hold in other control problems, see e.g. [4]. Consequently, we shall deliberately ignore the lower semicontinuity of V in the subsequent analysis in order to show how to derive the dynamic programming equation in a general setting.

Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a lower semicontinuous function with linear growth, and define the performance criterion J by (2.1). Then, it follows that $\mathcal{U} = \mathcal{U}_0$ and, from (5.2) and the almost sure continuity of $(t,x)\mapsto X^{\nu}_{t,x}(T)$, that $J(.,\nu)$ is lower semicontinuous, as required in the second part of Theorem 3.1.

The value function V is defined by (2.3). Various types of conditions can be formulated in order to guarantee that V is locally bounded. For instance, if f is bounded from above, this condition is trivially satisfied. Alternatively, one may restrict the set U to be bounded, so that the linear growth of f implies corresponding bounds for V. We do not want to impose such a constraint because we would like to highlight the fact that our methodology applies to general singular control problems. We then leave this issue as a condition which is to be checked by specific arguments to the case in hand.

Proposition 5.1 *In the above controlled diffusion context, assume further that V is locally bounded. Then, the value function V satisfies the weak dynamic programming principle* (3.1)-(3.2).

Proof Conditions A1, A2 and A3 from Assumption A are obviously satisfied in the present context. It remains to check that A4 holds true. For $\omega \in \Omega$ and $r \geq 0$, we denote $\omega_\cdot^r := \omega_{\cdot \wedge r}$ and $\mathbf{T}_r(\omega)(\cdot) := \omega_{\cdot \vee r} - \omega_r$ so that $\omega_\cdot = \omega_\cdot^r + \mathbf{T}_r(\omega)(\cdot)$. Fix $(t,x) \in \mathbf{S}$, $\nu \in \mathcal{U}_t$, $\theta \in \mathcal{T}_{[t,T]}^t$, and observe that, by the flow property,

$$\begin{aligned}
\mathbb{E}\left[f\left(X_{t,x}^\nu(T)\right)|\mathcal{F}_\theta\right](\omega) &= \int f\left(X_{\theta(\omega),X_{t,x}^\nu(\theta)(\omega)}^{\nu(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\omega))}(T)(\mathbf{T}_{\theta(\omega)}(\omega))\right)d\mathbb{P}(\mathbf{T}_{\theta(\omega)}(\omega)) \\
&= \int f\left(X_{\theta(\omega),X_{t,x}^\nu(\theta)(\omega)}^{\nu(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\tilde\omega))}(T)(\mathbf{T}_{\theta(\omega)}(\tilde\omega))\right)d\mathbb{P}(\tilde\omega) \\
&= J(\theta(\omega), X_{t,x}^\nu(\theta)(\omega); \tilde\nu_\omega)
\end{aligned}$$

where, $\tilde\nu_\omega(\tilde\omega) := \nu(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\tilde\omega))$ is an element of $\mathcal{U}_{\theta(\omega)}$. This already proves A4-a. As for A4-b, note that if $\bar\nu := \nu \mathbf{1}_{[0,\theta]} + \tilde\nu \mathbf{1}_{(\theta,T]}$ with $\tilde\nu \in \mathcal{U}_s$ and $\theta \in \mathcal{T}_{[t,s]}^t$, then the same computations imply

$$\mathbb{E}\left[f\left(X_{t,x}^{\bar\nu}(T)\right)|\mathcal{F}_\theta\right](\omega) = \int f\left(X_{\theta(\omega),X_{t,x}^\nu(\theta)(\omega)}^{\tilde\nu(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\tilde\omega))}(T)(\mathbf{T}_{\theta(\omega)}(\tilde\omega))\right)d\mathbb{P}(\tilde\omega),$$

where we used the flow property together with the fact that $X_{t,x}^\nu = X_{t,x}^{\bar\nu}$ on $[t,\theta]$ and that the dynamics of $X_{t,x}^{\bar\nu}$ depends only on $\tilde\nu$ after θ. Now observe that $\tilde\nu$ is independent of $\mathcal{F}_s$ and therefore on $\omega^{\theta(\omega)}$ since $\theta \leq s$ $\mathbb{P}-$a.s. It follows that

$$\begin{aligned}
\mathbb{E}\left[f\left(X_{t,x}^{\bar\nu}(T)\right)|\mathcal{F}_\theta\right](\omega) &= \int f\left(X_{\theta(\omega),X_{t,x}^\nu(\theta)(\omega)}^{\tilde\nu(\mathbf{T}_s(\tilde\omega))}(T)(\mathbf{T}_{\theta(\omega)}(\tilde\omega))\right)d\mathbb{P}(\tilde\omega) \\
&= J(\theta(\omega), X_{t,x}^\nu(\theta)(\omega); \tilde\nu)\ .
\end{aligned}$$

$\square$

Remark 5.4 It can be similarly proved that A4' holds true, in the context of mixed control-stopping problems.

5.2 PDE derivation

We can now show how our weak formulation of the dynamic programming principle allows to characterize the value function as a discontinuous viscosity solution of a suitable Hamilton-Jacobi-Bellman equation.

Let C^0 denote the set of continuous maps on $[0,T] \times \mathbb{R}^d$ endowed with the topology of uniform convergence on compact sets. To $(t,x,p,A,\varphi) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M}^d \times C^0$, we associate the Hamiltonian of the control problem:

$$H(t,x,p,A,\varphi) := \inf_{u \in U} H^u(t,x,p,A,\varphi),$$

where, for $u \in U$,

$$\begin{aligned} H^u(t,x,p,A,\varphi) \;&:=\; -\langle \mu(t,x,u),p\rangle - \frac{1}{2}\mathrm{Tr}\left[(\sigma\sigma')(t,x,u)A\right] \\ &\quad - \int_E \left(\varphi(t,x+\beta(t,x,u,e)) - \varphi(t,x)\right)\lambda(de), \end{aligned}$$

and σ' is the transpose of the matrix σ.

Notice that the operator H is upper-semicontinuous, as an infimum over a family of continuous maps (note that β is locally bounded uniformly with respect to its last argument and that λ is finite, by assumption). However, since the set U may be unbounded, it may fail to be continuous. We therefore introduce the corresponding lower-semicontinuous envelope:

$$H_*(z) \;:=\; \liminf_{z' \to z} H(z') \quad \text{for} \quad z = (t,x,p,A,\varphi) \in \mathbf{S} \times \mathbb{R}^d \times \mathbb{M}^d \times C^0.$$

Corollary 5.1 *Assume that V is locally bounded. Then:*
(i) V^* *is a viscosity subsolution of*

$$-\partial_t V^* + H_*(.,DV^*,D^2V^*,V^*) \;\le\; 0 \quad \textit{on} \quad [0,T) \times \mathbb{R}^d.$$

(ii) V_* *is a viscosity supersolution of*

$$-\partial_t V_* + H(.,DV_*,D^2V_*,V_*) \;\ge\; 0 \quad \textit{on} \quad [0,T) \times \mathbb{R}^d.$$

Proof 1. We start with the supersolution property. Assume to the contrary that there is $(t_0,x_0) \in [0,T) \times \mathbb{R}^d$ together with a smooth function $\varphi : [0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying

$$0 = (V_* - \varphi)(t_0,x_0) < (V_* - \varphi)(t,x) \quad \text{for all} \quad (t,x) \in [0,T) \times \mathbb{R}^d,\; (t,x) \neq (t_0,x_0),$$

such that

$$\left(-\partial_t\varphi + H(.,D\varphi,D^2\varphi,\varphi)\right)(t_0,x_0) \;<\; 0. \tag{5.3}$$

For $\varepsilon > 0$, let ϕ be defined by

$$\phi(t,x) \;:=\; \varphi(t,x) - \varepsilon(|t-t_0|^2 + |x-x_0|^4),$$

and note that ϕ converges uniformly on compact sets to φ as $\varepsilon \to 0$. Since H is upper-semicontinuous and $(\phi,\partial_t\phi,D\phi,D^2\phi)(t_0,x_0) = (\varphi,\partial_t\varphi,D\varphi,D^2\varphi)(t_0,x_0)$, we can choose $\varepsilon > 0$ small enough so that there exist $u \in U$ and $r > 0$, with $t_0 + r < T$, satisfying

$$\left(-\partial_t\phi + H^u(.,D\phi,D^2\phi,\phi)\right)(t,x) \;<\; 0 \quad \text{for all} \quad (t,x) \in B_r(t_0,x_0). \tag{5.4}$$

Let $(t_n,x_n)_n$ be a sequence in $B_r(t_0,x_0)$ such that $(t_n,x_n,V(t_n,x_n)) \to (t_0,x_0,V_*(t_0,x_0))$, and let $X^n_\cdot := X^u_{t_n,x_n}(\cdot)$ denote the solution of (5.1) with constant control $\nu = u$ and initial condition $X^n_{t_n} = x_n$, and consider the stopping time

$$\theta_n := \inf\left\{s \ge t_n \;:\; (s,X^n_s) \notin B_r(t_0,x_0)\right\}.$$

Note that $\theta_n < T$ since $t_0 + r < T$. Applying Itô's formula to $\phi(\cdot, X^n)$, and using (5.4) and (5.2), we see that

$$\phi(t_n, x_n) \;=\; \mathbb{E}\left[\phi(\theta_n, X^n_{\theta_n}) - \int_{t_n}^{\theta_n} \left[\partial_t \phi - H^u(., D\phi, D^2\phi, \phi)\right](s, X^n_s) ds\right] \;\le\; \mathbb{E}\left[\phi(\theta_n, X^n_{\theta_n})\right].$$

Now observe that $\varphi \ge \phi + \eta$ on $([0,T] \times \mathbb{R}^d) \setminus B_r(t_0, x_0)$ for some $\eta > 0$. Hence, the above inequality implies that $\phi(t_n, x_n) \le \mathbb{E}\left[\varphi(\theta_n, X^n_{\theta_n})\right] - \eta$. Since $(\phi - V)(t_n, x_n) \to 0$, we can then find n large enough so that

$$V(t_n, x_n) \;\le\; \mathbb{E}\left[\varphi(\theta_n, X^n_{\theta_n})\right] - \eta/2 \quad \text{for sufficiently large} \quad n \ge 1.$$

On the other hand, it follows from (3.2) that:

$$V(t_n, x_n) \;\ge\; \sup_{\nu \in \mathcal{U}_{t_n}} \mathbb{E}\left[\varphi(\theta_n, X^\nu_{t_n, x_n}(\theta_n))\right] \;\;\ge\;\; \mathbb{E}\left[\varphi(\theta_n, X^n_{\theta_n})\right],$$

which is the required contradiction.

2. We now prove the subsolution property. Assume to the contrary that there is $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$ together with a smooth function $\varphi : [0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying

$$0 = (V^* - \varphi)(t_0, x_0) > (V^* - \varphi)(t, x) \quad \text{for all} \quad (t, x) \in [0, T) \times \mathbb{R}^d, \; (t, x) \ne (t_0, x_0),$$

such that

$$\left(-\partial_t \varphi + H_*(., D\varphi, D^2\varphi, \varphi)\right)(t_0, x_0) \;\;>\;\; 0. \tag{5.5}$$

For $\varepsilon > 0$, let ϕ be defined by

$$\phi(t, x) \;\;:=\;\; \varphi(t, x) + \varepsilon(|t - t_0|^2 + |x - x_0|^4),$$

and note that ϕ converges uniformly on compact sets to φ as $\varepsilon \to 0$. By the lower-semicontinuity of H_*, we can then find $\varepsilon, r > 0$ such that $t_0 + r < T$ and

$$\left(-\partial_t \phi + H^u(., D\phi, D^2\phi, \phi)\right)(t, x) \;>\; 0 \quad \text{for every} \quad u \in U \text{ and } (t, x) \in B_r(t_0, x_0). \tag{5.6}$$

Since (t_0, x_0) is a strict maximizer of the difference $V^* - \phi$, it follows that

$$\sup_{([0,T] \times \mathbb{R}^d) \setminus B_r(t_0, x_0)} (V^* - \phi) \le -2\eta \;\; \text{for some} \;\; \eta > 0\,. \tag{5.7}$$

Let $(t_n, x_n)_n$ be a sequence in $B_r(t_0, x_0)$ such that $(t_n, x_n, V(t_n, x_n)) \to (t_0, x_0, V^*(t_0, x_0))$. For an arbitrary control $\nu^n \in \mathcal{U}_{t_n}$, let $X^n := X^{\nu^n}_{t_n, x_n}$ denote the solution of (5.1) with initial condition $X^n_{t_n} = x_n$, and set

$$\theta_n \;\;:=\;\; \inf\left\{s \ge t_n \;:\; (s, X^n_s) \notin B_r(t_0, x_0)\right\}.$$

Notice that $\theta_n < T$ as a consequence of the fact that $t_0 + r < T$. We may assume without loss of generality that

$$|(V-\phi)(t_n, x_n)| \;\leq\; \eta \quad \text{for all} \quad n \geq 1. \tag{5.8}$$

Applying Itô's formula to $\phi(\cdot, X^n)$ and using (5.6) leads to

$$\phi(t_n, x_n) \;\; = \;\; \mathbb{E}\left[\phi(\theta_n, X^n_{\theta_n}) - \int_{t_n}^{\theta_n} \left[\partial_t \phi - H^{\nu^n}(., D\phi, D^2\phi, \phi)\right](s, X^n_s) ds\right] \;\geq\; \mathbb{E}\left[\phi(\theta_n, X^n_{\theta_n})\right].$$

In view of (5.7), the above inequality implies that $\phi(t_n, x_n) \geq \mathbb{E}\left[V^*(\theta_n, X^n_{\theta_n})\right] + 2\eta$, which implies by (5.8) that:

$$V(t_n, x_n) \;\geq\; \mathbb{E}\left[V^*(\theta_n, X^n_{\theta_n})\right] + \eta \quad \text{for} \quad n \geq 1.$$

Since $\nu^n \in \mathcal{U}_{t_n}$ is arbitrary, this contradicts (3.1) for $n \geq 1$ fixed. □

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