Dynamic Programming Approach to Principal-Agent Problems

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Abstract

We consider a general formulation of the Principal-Agent problem with a lump-sum payment on a finite horizon. Our approach is the following: we first find the contract that is optimal among those for which the agent's value process allows a dynamic programming representation and for which the agent's optimal effort is straightforward to find. We then show that, under technical conditions, the optimization over the restricted family of contracts represents no loss of generality. Moreover, the principal's problem can then be analyzed by the standard tools of control theory. Our proofs rely on the Backward Stochastic Differential Equations approach to non-Markovian stochastic control, and more specifically, on the recent extensions to the second order case.

Key words. Stochastic control of non-Markov systems, Hamilton-Jacobi-Bellman equations, second order Backward SDEs, Principal-Agent problem, Contract Theory.

1 Introduction

Optimal contracting between two parties – Principal ("she") and Agent ("he"), when Agent's effort cannot be contracted upon, is a classical moral hazard problem in microeconomics. It has applications in many areas of economics and finance, for example in corporate governance and portfolio management (see Bolton and Dewatripont (2005) for a book treatment, mostly in discrete-time models). In this paper we develop a general approach to solving such problems in continuous-time Brownian motion models, in the case in which Agent is paid only at the terminal time.

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The first, seminal paper on Principal-Agent problems in continuous-time is Holmström and Milgrom (1987), henceforth HM (1987). They consider Principal and Agent with CARA utility functions, in a model in which Agent's effort influences the drift of the output process, but not the volatility, and show that the optimal contract is linear. Their work was extended by Schättler and Sung (1993, 1997), Sung (1995, 1997), Müller (1998, 2000), and Hellwig and Schmidt (2002). The papers by Williams (2009) and Cvitanić, Wan and Zhang (2009) use the stochastic maximum principle and Forward-Backward Stochastic Differential Equations (FBSDEs) to characterize the optimal compensation for more general utility functions.

Our method provides a direct way to solving such problems, while at the same time allowing also Agent to control the volatility of the output process, and not just the drift. [1] In many important applications, such as, for example, delegated portfolio management, Agent, indeed, controls the volatility of the output process. This application is studied for the first time in a pre-cursor to this paper, Cvitanić, Possamaï and Touzi (2015), for the special case of CARA utility functions, showing that the optimal contract depends not only on the output value (in a linear way, because of CARA preferences), but also on the risk the output has been exposed to, via its quadratic variation. The present paper includes all the above cases as special cases, considering a multi-dimensional model with arbitrary utility functions and Agent's efforts affecting both the drift and the volatility of the output, that is, both the return and the risk. [2] Our novel method is also used in Aïd, Possamaï and Touzi (2015) for a problem of optimal electricity pricing, and has a potential to be applied to many other applications involving Principal-Agent problems.

In recent years a different continuous-time model has emerged and has been very successful in explaining contracting relationship in various settings - the infinite horizon problem in which Principal may fire/retire Agent and the payments are paid continuously, rather than as a lump-sum payment at the terminal time, as introduced in another seminal paper, Sannikov (2008). We leave for a future paper the analysis of the Sannikov's model using our approach.

The main approach taken in the literature is to characterize Agent's value process (also called continuation/promised utility) and his optimal actions given an arbitrary contract payoff, and then to analyze the maximization problem of the principal over all possible payoffs. [3] This approach may be hard to apply, because it may be hard to solve Agent's stochastic control problem given an arbitrary payoff, possibly non-Markovian, and it may also be hard for Principal to maximize over all such contracts. Furthermore, Agent's optimal control may depend on the given contract in a highly nonlinear manner, rendering Principal's optimization problem even harder. For these reasons, in its most general form the problem was approached in the literature also by means of the calculus of variations, thus adapting the tools of the stochastic version of the Pontryagin maximum principle; see Cvitanić and Zhang (2012). Our approach is different, much more direct, and it works in great generality. We restrict the family of admissible contracts to the contracts for which Agent's value process allows a dynamic

[1] This still leads to moral hazard in models with multiple risk sources, that is, driven by a multi-dimensional Brownian motion.

[2] See also recent papers by Mastrolia and Possamaï (2015), and Sung (2015)), which, though related to our formulation, work in frameworks different from ours

[3] For a recent different approach, see Evans, Miller and Yang (2015). For each possible Agent's control process, they characterize contracts that are incentive compatible for it. However, their setup is less general than ours, and it does not allow for volatility control, for example.

programming representation. For such contracts, it is easy for Principal to identify what the optimal policy for Agent is - it is the one that maximizes the corresponding Hamiltonian. Moreover, the admissible family is such that Principal can apply standard methods of stochastic control. Finally, we show that under relatively mild technical conditions, the supremum of Principal's expected utility over the restricted family is equal to the supremum over all feasible contracts. We accomplish that by representing Agent's value proccess by means of the so-called second order BSDEs as introduced by Soner, Touzi and Zhang (2011), see also Cheridito, Soner, Touzi and Victoir (2007), and using recent results of Possamaï, Tan and Zhou (2015), to bypass the regularity conditions in Soner, Touzi and Zhang (2011).

One way to provide the intuition for our approach is the following. In a Markovian framework, Agent's value is, under technical conditions, determined via its first and second derivatives with respect to the state variables. In a general non-Markovian framework, the role of these derivatives is taken over by the (first-order) sensitivity of Agent's value process to the output, and its (second-order) sensitivity to its quadratic variation process. Thus, it is possible to transform Principal's problem into the problem of choosing optimally those sensitivities. If Agent controls only the drift, only the first order sensitivity is relevant, and if he also controls the volatility, the second one becomes relevant, too. In the former case, this insight was used in a crucial way in Sannikov (2008). The insight implies that the appropriate state variable for Principal's problem (in Markovian models) is Agent's value. This has been known in discrete-time models already since Spear and Srivastava (1987). We arrive to it from a different perspective, the one of considering contracts which are a priori defined via the first and second order sensitivities.

The rest of the paper is structured as follows: We describe the model and the Principal-Agent problem in Section 2. We introduce the restricted family of admissible contracts in Section 3. In Section 4 we show, under technical conditions that the restriction is without loss of generality. Section 5 presents some examples. We conclude in Section 6.

2 Principal-Agent problem

We first introduce our mathematical model.

2.1 The canonical space of continuous paths

Let $T > 0$ be a given terminal time, and $\Omega := C^0([0,T], \mathbb{R}^d)$ the set of all continuous maps from $[0,T]$ to $\mathbb{R}^d$, for a given integer $d > 0$. The canonical process on Ω, representing the output Agent is in charge of, is denoted by X, *i.e.*

$$X_t(x) = x(t) = x_t \quad \text{for all} \quad x \in \Omega,\ t \in [0,T],$$

and the corresponding canonical filtration by $\mathbb{F} := \{\mathcal{F}_t, t \in [0,T]\}$, where

$$\mathcal{F}_t \quad := \quad \sigma(X_s,\ s \le t),\ t \in [0,T].$$

We denote by $\mathbb{P}_0$ the Wiener measure on $(\Omega, \mathcal{F}_T)$, and for any $\mathbb{F}-$stopping time τ, by $\mathbb{P}_\tau$ the regular conditional probability distribution of $\mathbb{P}_0$ w.r.t. $\mathcal{F}_\tau$ (see Stroock and Varadhan (1979)), which is independent of $x \in \Omega$ by independence and stationarity of the Brownian increments.

We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ is a semi-martingale measure if X is a semi-martingale under $\mathbb{P}$. Then, on the canonical space Ω, there is a $\mathbb{F}-$progressively measurable process (see e.g. Karandikar (1995)), denoted by $\langle X \rangle = (\langle X \rangle_t)_{0 \le t \le T}$, which coincides with the quadratic variation of X, $\mathbb{P}-$a.s. for all semi-martingale measure $\mathbb{P}$. We next introduce the $d \times d$ non-negative symmetric matrix $\widehat{\sigma}_t$ such that

$$\widehat{\sigma}_t^2 := \limsup_{\varepsilon \searrow 0} \frac{\langle X \rangle_t - \langle X \rangle_{t-\varepsilon}}{\varepsilon},\ t \in [0,T].$$

A map $\Psi : [0,T] \times \Omega \longrightarrow E$, taking values in any Polish space E will be called $\mathbb{F}-$progressive if $\Psi(t,x) = \Psi(t, x_{\cdot \wedge t})$, for all $t \in [0,T]$ and $x \in \Omega$.

2.2 Controlled state equation

A control process (Agent's effort/action) $\nu = (\alpha, \beta)$ is an $\mathbb{F}-$adapted process with values in $A \times B$ for some subsets A and B of finite dimensional spaces. The controlled process takes values in $\mathbb{R}^d$, and is defined by means of the controlled coefficients:

$$\lambda : \mathbb{R}_+ \times \Omega \times A \longrightarrow \mathbb{R}^n,\ \text{bounded, with } \lambda(\cdot, \alpha)\ \mathbb{F}-\text{progressive for any } \alpha \in A,$$
$$\sigma : \mathbb{R}_+ \times \Omega \times B \longrightarrow \mathcal{M}_{d,n}(\mathbb{R}),\ \ \text{bounded, with } \sigma(\cdot, \beta)\ \mathbb{F}-\text{progressive for any } \beta \in B,$$

for a given integer n, and where $\mathcal{M}_{d,n}(\mathbb{R})$ denotes the set of $d \times n$ matrices with real entries. For all control process ν, and all $(t,x) \in [0,T] \times \Omega$, the controlled state equation is defined by the stochastic differential equation driven by an $n-$dimensional Brownian motion W,

$$X_s^{t,x,\nu} = x(t) + \int_t^s \sigma_r(X^{t,x,\nu}, \beta_r)\big[\lambda_r(X^{t,x,\nu}, \alpha_r)dr + dW_r\big],\ s \in [t,T], \tag{2.1}$$

and such that $X_s^{t,x,\nu} = x(s),\ s \in [0,t]$.

A weak solution of (2.1) is a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\mathbb{P}[X_{\cdot \wedge t} = x_{\cdot \wedge t}] = 1$, and

$$X_\cdot - \int_t^\cdot \sigma_r(X, \beta_r)\lambda_r(X, \alpha_r)dr, \quad \text{and} \quad X_\cdot X_\cdot^\top - \int_t^\cdot (\sigma_r \sigma_r^\top)(X, \beta_r)dr,$$

are $(\mathbb{P}, \mathbb{F})-$martingales on $[t,T]$.

For such a weak solution $\mathbb{P}$, there is an $n-$dimensional $\mathbb{P}-$Brownian motion $W^{\mathbb{P}}$, as well as $\mathbb{F}-$adapted, and $A \times B-$valued processes $(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ such that[4]

$$X_s = x_t + \int_t^s \sigma_r(X, \beta_r^{\mathbb{P}}) \left[\lambda_r(X, \alpha_r^{\mathbb{P}})dr + dW_r^{\mathbb{P}}\right],\ s \in [t,T],\ \mathbb{P}-\text{a.s.} \tag{2.2}$$

In particular, we have

$$\widehat{\sigma}_t^2 \ \ = \ \ (\sigma_t \sigma_t^\top)(X, \beta_t^{\mathbb{P}}), \quad dt \otimes d\mathbb{P} - \text{a.s.}$$

[4]Brownian motion $W^{\mathbb{P}}$ is defined on a possibly enlarged space, if $\widehat{\sigma}$ is not invertible $\mathbb{P}-$a.s. We refer to Possamaï, D., Tan, X., Zhou, C. (2015) for the precise statements.

The next definition involves an additional map

$$c : \mathbb{R}_+ \times \Omega \times A \times B \longrightarrow \mathbb{R}_+, \text{ measurable, with } c(\cdot, u) \ \mathbb{F}-\text{progressive for all } u \in A \times B,$$

which represents Agent's cost of effort.

Throughout the paper we fix a real number $p > 1$.

Definition 2.1. *A control process ν is said to be admissible if SDE* (2.1) *has a weak solution, and for any such weak solution $\mathbb{P}$ we have*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \sup_{a\in A} |c_s(X, a, \beta_s^{\mathbb{P}})|^p ds\right] \quad < \quad \infty. \tag{2.3}$$

We denote by $\mathcal{U}(t,x)$ the collection of all admissible controls, $\mathcal{P}(t,x)$ the collection of all corresponding weak solutions of (2.1)*, and $\mathcal{P}_t := \cup_{x\in\Omega}\mathcal{P}(t,x)$.*

Notice that we do not restrict the controls to those for which weak uniqueness holds. Moreover, by Girsanov theorem, two weak solutions of (2.1) associated with (α, β) and (α', β) are equivalent. However, different diffusion coefficients induce mutually singular weak solutions of the corresponding stochastic differential equations.

For later use, we introduce an alternative representation of sets $\mathcal{P}(t,x)$. We first denote for all $(t,x) \in [0,T] \times \Omega$:

$$\Sigma_t(x,b) := \sigma_t\sigma_t^\top(x,b), \quad b \in B, \quad \text{and} \quad B_t(x,\Sigma) := \big\{b \in B : \sigma_t\sigma_t^\top(x,b) = \Sigma\big\}, \quad \Sigma \in \mathcal{S}_d^+.$$

For an $\mathbb{F}-$progressively measurable process β with values in B, consider then the SDE driven by a $d-$dimensional Brownian motion W

$$X_s^{t,x,\beta} = x_t + \int_t^s \Sigma_r^{1/2}(X, \beta_r)dW_r, \ s \in [t,T], \tag{2.4}$$

with $X_s^{t,x,\beta} = x_s$ for all $s \in [0,t]$. A weak solution of (2.4) is a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\mathbb{P}[X_{\cdot\wedge t} = x_{\cdot\wedge t}] = 1$, and

$$X_\cdot \ \text{ and } \ X_\cdot X_\cdot^\top - \int_t^\cdot \Sigma_r(X, \beta_r)dr,$$

are $(\mathbb{P}, \mathbb{F})-$martingales on $[t,T]$. Then, there is an $\mathbb{F}-$adapted process $\bar{\beta}^{\mathbb{P}}$ and some $d-$dimensional $\mathbb{P}-$Brownian motion $W^{\mathbb{P}}$ such that

$$X_s = x_t + \int_t^s \Sigma_r^{1/2}(X, \bar{\beta}_r^{\mathbb{P}})dW_r^{\mathbb{P}}, \ s \in [t,T], \ \mathbb{P}-\text{a.s.} \tag{2.5}$$

Definition 2.2. *A volatility control process β is said to be admissible if the SDE* (2.4) *has a weak solution, and for all such solution $\mathbb{P}$, we have*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \sup_{a\in A} |c_s(X, a, \beta_s^{\mathbb{P}})|^p ds\right] \quad < \quad \infty.$$

We denote by $\mathcal{B}(t,x)$ the collection of all volatility control processes, $\overline{\mathcal{P}}(t,x)$ the collection of all corresponding weak solutions of (2.4)*, and $\overline{\mathcal{P}}_t := \cup_{x\in\Omega}\overline{\mathcal{P}}(t,x)$.*

We emphasize that sets $\overline{\mathcal{P}}(t,x)$ are equivalent to sets $\mathcal{P}(t,x)$, in the sense that $\mathcal{P}(t,x)$ consists of probability measures which are equivalent to corresponding probability measures in $\overline{\mathcal{P}}(t,x)$, and vice versa. Indeed, for $(\alpha,\beta)\in\mathcal{U}(t,x)$ we claim that $\beta\in\mathcal{B}(t,x)$. To see this, denote by $\mathbb{P}^{\alpha,\beta}$ any of the associated weak solutions to (2.1). Then, there always is a $d\times n$ rotation matrix R such that, for any $(s,x,b)\in[0,T]\times\Omega\times B$,

$$\sigma_s(x,b) \quad = \quad \Sigma_s^{1/2}(x,b)R_s(x,b). \tag{2.6}$$

Since $d\leq n$, and in addition Σ may be degenerate, notice that there may be many (and even infinitely many) choices of R, and in this case we may choose any measurable one. We next define $\mathbb{P}^\beta$ by

$$\frac{d\mathbb{P}^\beta}{d\mathbb{P}^{\alpha,\beta}} := \mathcal{E}\left(-\int_t^T \lambda_s(X,\alpha_s^{\mathbb{P}^{\alpha,\beta}})\cdot dW_s^{\mathbb{P}^{\alpha,\beta}}\right).$$

By Girsanov theorem, X is then a $(\mathbb{P}^\beta,\mathbb{F})-$martingale, which ensures that $\beta\in\mathcal{B}(t,x)$. In particular, the polar sets of $\mathcal{P}_0$ and $\overline{\mathcal{P}}_0$ are the same. Conversely, let us fix $\beta\in\mathcal{B}(t,x)$ and denote by $\mathcal{A}$ the set of $A-$valued and $\mathbb{F}-$progressively measurable processes. Then, we claim that for any $\alpha\in\mathcal{A}$, we have $(\alpha,\beta)\in\mathcal{U}(t,x)$. Indeed, let us denote by $\overline{\mathbb{P}}^\beta$ any weak solution to (2.4) and define

$$\frac{d\overline{\mathbb{P}}^{\alpha,\beta}}{d\overline{\mathbb{P}}^\beta} := \mathcal{E}\left(\int_t^T R_s(X,\bar\beta_s^{\overline{\mathbb{P}}^\beta})\lambda_s(X,\alpha_s)\cdot dW_s^{\overline{\mathbb{P}}^\beta}\right).$$

Then, by Girsanov Theorem, we have

$$\begin{aligned}
X_s &= \int_t^s \Sigma_r^{1/2}(X,\bar\beta_r^{\overline{\mathbb{P}}^\beta})R_r(X,\bar\beta_r^{\overline{\mathbb{P}}^\beta})\lambda_r(X,\alpha_r)dr + \int_t^s \Sigma_r^{1/2}(X,\bar\beta_r^{\overline{\mathbb{P}}^\beta})d\overline{W}_r^{\overline{\mathbb{P}}^\beta} \\
&= \int_t^s \sigma_r(X,\bar\beta_r^{\overline{\mathbb{P}}^\beta})\lambda_r(X,\alpha_r)dr + \int_t^s \Sigma_r^{1/2}(X,\bar\beta_r^{\overline{\mathbb{P}}^\beta})d\overline{W}_r^{\overline{\mathbb{P}}^\beta},
\end{aligned}$$

where $\overline{W}^{\overline{\mathbb{P}}^\beta}$ is a $d-$dimensional $(\overline{\mathbb{P}}^{\alpha,\beta},\mathbb{F})-$Brownian motion. Hence, $(\alpha,\beta)\in\mathcal{U}(t,x)$. Moreover, setting

$$W_\cdot^{\overline{\mathbb{P}}^\beta} = R_\cdot(X,\bar\beta_\cdot^{\overline{\mathbb{P}}^\beta})W_\cdot^{\overline{\mathbb{P}}^{\alpha,\beta}} + \int_t^\cdot R_s(X,\bar\beta_s^{\overline{\mathbb{P}}^\beta})\lambda_s(X,\alpha_s)ds,$$

defines a Brownian motion under $\mathbb{P}^{\alpha,\beta}$. Since $\overline{\mathbb{P}}^\beta$ and $\overline{\mathbb{P}}^{\alpha,\beta}$ are equivalent, we have

$$\Sigma(X,\beta^{\overline{\mathbb{P}}^{\alpha,\beta}}) = \Sigma(X,\bar\beta^{\overline{\mathbb{P}}^\beta}),\ dt\otimes\overline{\mathbb{P}}^\beta - a.e.\ (\text{or } dt\otimes\overline{\mathbb{P}}^{\alpha,\beta} - a.e.),$$

that is $\beta_s^{\overline{\mathbb{P}}^{\alpha,\beta}}$ and $\bar\beta_s^{\overline{\mathbb{P}}^\beta}$ both belong to $B_s(X,\widehat\sigma_s^2(X))$, $dt\otimes\overline{\mathbb{P}}^\beta-$a.e. We can summarize everything by the following equality

$$\mathcal{P}(t,x) = \bigcup_{\alpha\in\mathcal{A}}\Big\{\mathcal{E}\Big(\int_t^T R_s(X,\bar\beta_s^{\mathbb{P}})\lambda_s(X,\alpha_s)\cdot dW_s^{\mathbb{P}}\Big)\cdot\mathbb{P} : \mathbb{P}\in\overline{\mathcal{P}}(t,x),\ R \text{ satisfying (2.6)}\Big\}. \tag{2.7}$$

2.3 Agent's problem

Let us fix $(t,x,\mathbb{P})\in[0,T]\times\Omega\times\mathcal{P}(t,x)$, together with the associated control $\nu^{\mathbb{P}} := (\alpha^{\mathbb{P}},\beta^{\mathbb{P}})$. The canonical process X is called the *output* process, and the control $\nu^{\mathbb{P}}$ is called Agent's *effort* or *action*. Agent is in charge of controlling the (distribution of the) output process by choosing the effort process $\nu^{\mathbb{P}}$ in the state equation (2.1), while subject to cost of effort at rate $c(X, \alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$. Furthermore, Agent has a fixed reservation utility $R \in \mathbb{R}$, i.e., he will not accept to work for Principal unless the contract is such that his expected utility is above R.

Agent is hired at time t and receives the compensation ξ from Principal at time T. Principal does not observe Agent's effort, only the output process. Consequently, the compensation ξ, which takes values in $\mathbb{R}$, can only be contingent on X, that is ξ is $\mathcal{F}_T-$measurable.

Random variable ξ is called a *contract*, and we write $\xi \in \mathcal{C}_t$ if the following integrability condition is satisfied:

$$\sup_{\mathbb{P} \in \mathcal{P}_t} \mathbb{E}^{\mathbb{P}}[|\xi|^p] < +\infty. \tag{2.8}$$

We now introduce Agent's objective function:

$$J^A(t, x, \mathbb{P}, \xi) := \mathbb{E}^{\mathbb{P}}\Big[\mathcal{K}^{\nu^{\mathbb{P}}}_{t,T}(X)\xi - \int_t^T \mathcal{K}^{\nu^{\mathbb{P}}}_{t,s}(X) c_s\big(X, \nu^{\mathbb{P}}_s\big) ds\Big],\ \mathbb{P} \in \mathcal{P}(t, x),\ \xi \in \mathcal{C}_t, \tag{2.9}$$

where

$$\mathcal{K}^{\nu}_{t,s}(X) \quad := \quad \exp\Big(- \int_t^s k_r(X, \nu_r) dr\Big), \quad s \in [t, T],$$

is a discount factor defined by means of a bounded measurable function

$$k : \mathbb{R}_+ \times \Omega \times A \times B \longrightarrow \mathbb{R}, \quad \text{with} \quad k(\cdot, u)\ \mathbb{F} - \text{progressive for all } u \in A \times B.$$

Notice that J^A is well-defined for all $(t, x) \in [0, T] \times \Omega$, $\xi \in \mathcal{C}_t$ and $\mathbb{P} \in \mathcal{P}(t, x)$. This is a consequence of the boundedness of k, the non-negativity of c, as well as the conditions (2.8) and (2.3).

Remark 2.3. *If Agent is risk-averse with utility function U_A, then we replace ξ with $\xi' = U_A(\xi)$ in J^A, and we replace ξ by $U_A^{-1}(\xi')$ in Principal's problem below. All the results remain valid.*

Remark 2.4. *Our approach can also accommodate an objective function for the agent of the form*

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\mathrm{sgn}(U_A)\int_t^T \mathcal{K}^{\nu^{\mathbb{P}}}_{t,s}(X) c_s\big(X, \nu^{\mathbb{P}}_s\big) ds\right) \mathcal{K}^{\nu^{\mathbb{P}}}_{t,T}(X) U_A(\xi)\right],$$

for a utility function U_A having constant sign. In particular, our framework includes exponential utilities, after appropriately modifying the assumptions.

Agent's goal is to choose optimally the effort, given the compensation contract ξ promised by Principal:

$$V^A(t, x, \xi) := \sup_{\mathbb{P} \in \mathcal{P}(t,x)} J^A(t, x, \mathbb{P}, \xi). \tag{2.10}$$

An admissible control $\mathbb{P}^\star \in \mathcal{P}(t, x)$ will be called optimal if

$$V^A(t, x, \xi) = J^A(t, x, \mathbb{P}^\star, \xi).$$

We denote by $\mathcal{P}^\star(t,x,\xi)$ the collection of all such optimal controls $\mathbb{P}^\star$.

In the literature, the value function V^A is called sometimes *continuation utility* or *promised utility*, and it turns out to play a crucial role as the state variable of Principal's optimization problem; see Sannikov (2008) for its use in continuous-time models and further references.

2.4 Principal's problem

We now state Principal's optimization problem.

At maturity T, Principal receives the final value of the output X_T and pays the compensation ξ promised to Agent. We will restrict the contracts that can be offered by Principal to those that admit an optimal solution to Agent's problem, i.e., we allow only the contracts ξ for which $\mathcal{P}^\star(t,x,\xi)\neq\emptyset$. Recall also that Agent's participation is conditioned on having his value above reservation utility R. Thus, Principal is restricted to choose a contract from the set

$$\Xi(t,x) := \big\{\xi\in\mathcal{C}_t,\ \mathcal{P}^\star(t,x,\xi)\neq\emptyset,\ V^A(t,x,\xi)\geq R\big\}. \tag{2.11}$$

As a final ingredient, we need to fix Agent's optimal strategy in the case in which set $\mathcal{P}^\star(t,x,\xi)$ contains more than one solutions. Following the standard convention, we assume that Agent, when indifferent between such solutions, implements the one that is the best for Principal.

In view of this, Principal's problem is given by

$$V^P(t,x) \quad := \quad \sup_{\xi\in\Xi(t,x)} J^P(t,x,\xi), \tag{2.12}$$

where

$$J^P(t,x,\xi) := \sup_{\mathbb{P}^\star\in\mathcal{P}^\star(t,x,\xi)} \mathbb{E}^{\mathbb{P}^\star}\Big[\mathcal{K}^P_{t,T}(X)U\big(\ell(X_T)-\xi\big)\Big],$$

where function $U:\mathbb{R}\longrightarrow\mathbb{R}$ is a given non-decreasing and concave utility function, $\ell:\mathbb{R}^d\longrightarrow\mathbb{R}$ is a liquidation function, and

$$\mathcal{K}^P_{t,s}(X) \quad := \quad \exp\Big(-\int_t^s k^P_r(X)dr\Big),\quad s\in[t,T],$$

is a discount factor, defined by means of a bounded measurable function

$$k^P:\mathbb{R}_+\times\Omega\longrightarrow\mathbb{R},$$

such that k^P is $\mathbb{F}-$progressive.

Remark 2.5. *Agent's and Principal's problems are non-standard stochastic control problems. First, ξ is allowed to be of non-Markovian nature. Second, Principal's optimization is over ξ, and is a priori not a control problem that may be approached by dynamic programming. The objective of this paper is to develop an approach that naturally reduces both problems to those that can be solved by dynamic programming.*

3 Family of restricted contracts

In this section we identify a restricted family of contract payoffs for which the standard stochastic control methods can be applied.

3.1 Agent's dynamic programming equation

In view of the definition of Agent's problem in (2.10), it is natural to introduce the Hamiltonian functional, for all $(t,x) \in [0,T) \times \Omega$ and $(y,z,\gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R})$:

$$H_t(x,y,z,\gamma) := \sup_{u \in A \times B} h_t(x,y,z,\gamma,u), \tag{3.1}$$

$$h_t(x,y,z,\gamma,u) := -c_t(x,u) - k_t(x,u)y + \sigma_t(x,\beta)\lambda_t(x,\alpha)\cdot z + \frac{1}{2}(\sigma_t\sigma_t^\top)(x,\beta) : \gamma, \tag{3.2}$$

for $u := (\alpha,\beta)$.

Remark 3.1. (i) *Mapping H plays an important role in the theory of stochastic control of Markov diffusions, see e.g. Fleming and Soner (1993). Indeed, suppose that*

- *the coefficients $\lambda_t, \sigma_t, c_t, k_t$ depend on x only through the current value $x(t)$,*
- *the contract ξ depends on x only through the final value $x(T)$, i.e. $\xi(x) = g(x(T))$ for some function $g : \mathbb{R}^d \longrightarrow \mathbb{R}$.*

Then, under fairly general conditions, the value function of Agent's problem is given by

$$V^A(t,x(t),\xi) = v(t,x(t)),$$

where the function $v : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ can be characterized as the unique viscosity solution (with appropriate growth at infinity) of the dynamic programming partial differential equation (called Hamilton-Jacobi-Bellman (HJB) equation)

$$-\partial_t v(t,\mathrm{x}) - H(t,\mathrm{x},v(t,\mathrm{x}),Dv(t,\mathrm{x}),D^2v(t,\mathrm{x})) = 0,\ (t,\mathrm{x}) \in [0,T) \times \mathbb{R}^d,\ v(T,\mathrm{x}) = g(\mathrm{x}),\ \mathrm{x} \in \mathbb{R}^d.$$

A recently developed theory of path-dependent partial differential equations extends the approach to the non-Markovian case; see Ekren, Touzi and Zhang (2014).

We note the obvious, but crucial fact that Agent's value process V_t at the terminal time is equal to the contract payoff,

$$\xi = V_T.$$

This provides the motivation for our approach: to find as general a representation of V_T as possible using dynamic programming, and consider payoffs $\xi = V_T$.

Such a representation is the main result of this section that we work on next, and it follows the line of the standard verification arguments in stochastic control theory. Fix some $(t,x) \in [0,T] \times \Omega$. Let

$$Z : [t,T] \times \Omega \longrightarrow \mathbb{R}^d \quad \text{and} \quad \Gamma : [t,T] \times \Omega \longrightarrow \mathcal{S}_d(\mathbb{R})$$

be $\mathbb{F}-$predictable processes with

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left[Z_s Z_s^\top : \widehat{\sigma}_s^2 + |\Gamma_s : \widehat{\sigma}_s^2|\right] ds\right)^{\frac{p}{2}}\right] < +\infty,\ \text{for all } \mathbb{P} \in \mathcal{P}(t,x),$$

We denote by $\mathcal{V}(t,x)$ the collection of all such process pairs (Z,Γ).

Given an initial condition $Y_t \in \mathbb{R}$, define a $\mathbb{F}-$progressively measurable process $Y^{Z,\Gamma}$, $\mathbb{P}-$a.s., for all $\mathbb{P} \in \mathcal{P}(t,x)$ by

$$Y_s^{Z,\Gamma} := Y_t - \int_t^s H_r\left(X, Y_r^{Z,\Gamma}, Z_r, \Gamma_r\right) dr + \int_t^s Z_r \cdot dX_r + \frac{1}{2}\int_t^s \Gamma_r : d\langle X\rangle_r,\ s \in [t,T]. \tag{3.3}$$

Notice that $Y^{Z,\Gamma}$ is well-defined as a consequence of the Lipschitz property of H in y, due to k being bounded. We want to see under which conditions Agent's value process is equal to $Y^{Z,\Gamma}$ for some $(Z,\Gamma) \in \mathcal{V}(t,x)$.

The next result follows the line of the classical verification argument in stochastic control theory, and requires the following condition.

Assumption 3.2. *For any $t \in [0,T]$, functional H has at least one measurable maximizer $u^\star = (\alpha^\star, \beta^\star) : [t,T) \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R}) \longrightarrow A \times B$, i.e. $H(.) = h\big(., u^\star(.)\big)$. Moreover, for all $(t,x) \in [0,T] \times \Omega$ and for any $(Z,\Gamma) \in \mathcal{V}(t,x)$, the control process*

$$\nu_s^{\star,Z,\Gamma}(\cdot) := u_s^\star\big(\cdot, Y_s^{Z,\Gamma}(\cdot), Z_s(\cdot), \Gamma_s(\cdot)\big),\ s \in [t,T],$$

is admissible, that is $\nu^{\star,Z,\Gamma} \in \mathcal{U}(t,x)$.

We are now in a position to define a subset of contracts for which Agent's value function coincides with the above process $Y^{Z,\Gamma}$, and for which it is straightforward to identify actions that are incentive compatible, that is, optimal for Agent.

Proposition 3.3. *For $(t,x) \in [0,T] \times \Omega$, $Y_t \in \mathbb{R}$ and $(Z,\Gamma) \in \mathcal{V}(t,x)$, we have:*

(i) $Y_t \geq V^A\big(t,x,Y_T^{Z,\Gamma}\big)$.

(ii) *Assuming further that Assumption 3.2 holds true, we have $Y_t = V_A\big(t,x,Y_T^{Z,\Gamma}\big)$. Moreover, given a contract payoff $\xi = Y_T^{Z,\Gamma}$, any weak solution $\mathbb{P}^{\star,Y,Z}$ of the SDE* (2.1) *with control $\nu^{\star,Z,\Gamma}$ is optimal for Agent's problem, i.e. $\mathbb{P}^{\nu^{\star,Z,\Gamma}} \in \mathcal{P}^\star\big(t,x,Y_T^{Z,\Gamma}\big)$.*

Proof. (i) Fix an arbitrary $\mathbb{P} \in \mathcal{P}(t,x)$, and denote the corresponding control process $\nu^{\mathbb{P}} := (\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$. Then, it follows from a direct application of Itô's formula that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{t,T}^{\nu^{\mathbb{P}}} Y_T^{Z,\Gamma}\right] \quad = \quad & Y_t + \mathbb{E}^{\mathbb{P}}\left[\int_t^T \mathcal{K}_{t,s}^{\nu^{\mathbb{P}}}(X)\Big(-k_s(X,\nu_s^{\mathbb{P}})Y_s^{Z,\Gamma} - H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s)\right. \\
& \left. + Z_s \cdot \sigma_s(X,\beta_s^{\mathbb{P}})\lambda(X,\alpha_s^{\mathbb{P}}) + \frac{1}{2}\widehat{\sigma}_s^2 : \Gamma_s\Big) ds\right],
\end{aligned}$$

where we have used the fact that $(Z,\Gamma) \in \mathcal{V}(t,x)$, together with the fact that the stochastic integral $\int_t^\cdot \mathcal{K}_{t,s}^{\nu^{\mathbb{P}}}(X) Z_s \cdot \widehat{\sigma}_s^2 dW_s^{\mathbb{P}}$ defines a martingale, by the boundedness of k and σ.

By the definition of Hamiltonian H in (3.1), we may re-write the last equation as

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{t,T}^{\nu^{\mathbb{P}}} Y_T^{Z,\Gamma}\right] &= Y_t + \mathbb{E}^{\mathbb{P}}\Big[\int_t^T \mathcal{K}_{t,s}^{\nu^{\mathbb{P}}}(X)\big(c_s(X,\nu^{\mathbb{P}}) - H_s\left(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s\right) \\
&\qquad + h_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s, \nu_s^{\mathbb{P}})\big) ds\Big] \\
&\leq Y_t + \mathbb{E}^{\mathbb{P}}\left[\int_t^T \mathcal{K}_{t,s}^{\nu^{\mathbb{P}}}(X) c_s(X, \alpha_s^{\mathbb{P}}, \beta_s^{\mathbb{P}}) ds\right],
\end{aligned}$$

and the result follows by arbitrariness of $\mathbb{P} \in \mathcal{P}(t,x)$.

(ii) Denote $\nu^\star := \nu^{\star,Z,\Gamma}$ for simplicity. Under Assumption 3.2, the exact same calculations as in (i) imply, for any weak solution $\mathbb{P}^{\nu^\star}$,

$$\mathbb{E}^{\mathbb{P}^{\nu^\star}}\left[\mathcal{K}_{t,T}^{\nu^{\mathbb{P}^{\nu^\star}},t,x} Y_T^{Z,\Gamma}\right] = Y_t + \mathbb{E}^{\mathbb{P}^{\nu^\star}}\left[\int_t^T \mathcal{K}_{t,s}^{\nu^{\mathbb{P}^{\nu^\star}}}(X) c_s(X, \alpha_s^{\mathbb{P}^{\nu^\star}}, \beta_s^{\mathbb{P}}) ds\right].$$

Together with (i), this shows that $Y_t = V_A\big(t,x,Y_T^{Z,\Gamma}\big)$, and $\mathbb{P}^{\nu^\star} \in \mathcal{P}^\star\big(t,x,Y_T^{Z,\Gamma}\big)$. □

3.2 Restricted Principal's problem

Recall the process $u_t^\star(x,y,z,\gamma) = (\alpha^\star,\beta^\star)_t(x,y,z,\gamma)$ introduced in Assumption 3.2. In this section, we denote

$$\lambda_t^\star(x,y,z,\gamma) := \lambda_t\big(x, \alpha_t^\star(x,y,z,\gamma)\big), \quad \sigma_t^\star(x,y,z,\gamma) := \sigma_t\big(x, \beta_t^\star(x,y,z,\gamma)\big). \tag{3.4}$$

Notice that Assumption 3.2 says that for all $(t,x) \in [0,T] \times \Omega$ and for all $(Z,\Gamma) \in \mathcal{V}(t,x)$, the stochastic differential equation, driven by a $n-$dimensional Brownian motion W

$$\begin{aligned} X_s^{t,x,u^\star} &= x(t) + \int_t^s \sigma_r^\star(X^{t,x,u^\star}, Y_r^{Z,\Gamma}, Z_r, \Gamma_r)\big[\lambda_r^\star(X^{t,x,u^\star}, Y_r^{Z,\Gamma}, Z_r, \Gamma_r) dr + dW_r\big],\ s \in [t,T], \\ X_s^{t,x,u^\star} &= x(s),\ s \in [0,t], \end{aligned} \tag{3.5}$$

has at least one weak solution $\mathbb{P}^{\star,Z,\Gamma}$. The following result on Principal's value process V^P when the contract payoff is $\xi = Y_T^{Z,\Gamma}$ is a direct consequence of Proposition 3.3.

Proposition 3.4. *For all $(t,x) \in [0,T] \times \Omega$, we have $V^P(t,x) \geq \sup_{Y_t \geq R} \underline{V}(t,x,Y_t)$, where, for $Y_t \in \mathbb{R}$:*

$$\underline{V}(t,x,Y_t) := \sup_{(Z,\Gamma)\in\mathcal{V}(t,x)} \ \sup_{\hat{\mathbb{P}}^{Z,\Gamma} \in \mathcal{P}^\star(t,x,Y_T^{Z,\Gamma})} \mathbb{E}^{\mathbb{P}^{\star,Z,\Gamma}}\big[\mathcal{K}_{t,T}^P U\big(\ell(X_T) - Y_T^{Z,\Gamma}\big)\big]. \tag{3.6}$$

In the ensuing sections, we identify conditions under which the lower bound $\underline{V}(t,x,y)$, representing Principal's value when the contracts are restricted to the $\mathcal{F}_T-$measurable random variables $Y_T^{Z,\Gamma}$ with given initial condition Y_t, is, in fact equal to the unrestricted Principal's value $V^P(t,x)$. In the remainder of this section, we recall how $\underline{V}(t,x,y)$ can be computed, in principle.

An advantage of our approach is that $\underline{V}$ is the value function of a standard stochastic control problem with control processes $(Z,\Gamma) \in \mathcal{V}(t,x)$, and controlled state process $(X, Y^{Z,\Gamma})$, the controlled dynamics of X given (in weak formulation) by (3.5), and those of $Y^{Z,\Gamma}$ given by (3.3):

$$dY_s^{Z,\Gamma} = \Big(Z_s \cdot \sigma_s^\star \lambda_s^\star + \frac{1}{2}\Gamma_s : \sigma_s^\star(\sigma_s^\star)^\top - H_s\Big)(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s) ds + Z_s \cdot \sigma_s^\star(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s) dW_s^{\mathbb{P}}. \tag{3.7}$$

In view of the controlled dynamics (3.5)-(3.7), the relevant optimization term for the dynamic programming equation corresponding to the control problem $\underline{V}$ is defined for $(t, \mathrm{x}, y) \in [0,T] \times$

$\mathbb{R}^d \times \mathbb{R}$ by:

$$
\begin{aligned}
&G(t,\mathrm{x},y,p,M)\\
&:= \sup_{(z,\gamma)\in\mathbb{R}\times\mathcal{S}_d(\mathbb{R})}\Big\{(\sigma_t^\star\lambda_t^\star)(\mathrm{x},y,z,\gamma)\cdot p_x + \Big(z\cdot(\sigma_t^\star\lambda_t^\star)+\frac{1}{2}\gamma:\sigma_t^\star(\sigma_t^\star)^\top - H_t\Big)(\mathrm{x},y,z,\gamma)p_y\\
&\qquad\qquad + \frac{1}{2}(\sigma_t^\star(\sigma_t^\star)^\top)(\mathrm{x},y,z,\gamma):\big(M_{\mathrm{xx}}+zz^\top M_{\mathrm{yy}}\big)+(\sigma_t^\star(\sigma_t^\star)^\top)(\mathrm{x},y,z,\gamma)z\cdot M_{\mathrm{xy}}\Big\},
\end{aligned}
$$

where $M =: \begin{pmatrix} M_{\mathrm{xx}} & M_{\mathrm{xy}} \\ M_{\mathrm{xy}}^\top & M_{\mathrm{yy}} \end{pmatrix} \in \mathcal{S}_{d+1}(\mathbb{R})$, $M_{xx}\in\mathcal{S}_d(\mathbb{R})$, $M_{yy}\in\mathbb{R}$, $M_{xy}\in\mathcal{M}_{d,1}(\mathbb{R})$ and $p =: \begin{pmatrix} p_x \\ p_y \end{pmatrix} \in \mathbb{R}^d\times\mathbb{R}$.

The next well-known theorem recalls how to compute $\underline{V}$ in the Markovian case, *i.e.* when the model coefficients are not path-dependent. A similar statement can be formulated in the path dependent case, by using the notion of viscosity solutions of path-dependent PDE's introduced in Ekren, Keller, Touzi & Zhang (2014), and further developed in Ekren, Touzi and Zhang (2014a) and (2014b), Ren, Touzi and Zhang (2014a) and (2014b). However, one then faces the problem of the controls (z,γ) possibly being unbounded, which typically leads to G being non-Lipschitz in variables (Dv, D^2v), unless additional conditions on the coefficients are imposed.

Theorem 3.5. *Let $\varphi_t(x,.)=\varphi_t(x_t,.)$ for $\varphi = k, k^P, \lambda^\star, \sigma^\star, H$, and let Assumption 3.2 hold. Assume further that the map $G:[0,T)\times\mathbb{R}^d\times\mathbb{R}^{d+1}\times\mathbb{S}_{d+1}(\mathbb{R})\longrightarrow\mathbb{R}$ is upper semicontinuous. Then, $\underline{V}(t,x,y)$ is a viscosity solution of the dynamic programming equation:*

$$
\begin{cases}
(\partial_t v - k^P v)(t,\mathrm{x},y) + G\big(t,\mathrm{x},v(t,\mathrm{x},y),Dv(t,\mathrm{x},y),D^2v(t,\mathrm{x},y)\big)=0,\ (t,\mathrm{x},y)\in[0,T)\times\mathbb{R}^d\times\mathbb{R},\\
v(T,\mathrm{x},y)=U(\ell(\mathrm{x})-y),\ (\mathrm{x},y)\in\mathbb{R}^d\times\mathbb{R}.
\end{cases}
$$

In general, we see that Principal's problem involves both x and y as state variables. We consider below conditions under which the number of state variables can be reduced.

4 Comparison with the unrestricted case

In this section we find conditions under which equality holds in Proposition 3.4, i.e. the value function of the restricted Principal's problem of Section 3.2 coincides with Principal's value function with unrestricted contracts. We start with the case in which the volatility coefficient is not controlled.

4.1 Fixed volatility of the output

We consider here the case in which Agent is only allowed to control the drift of the output process:

$$
B = \{\beta^o\} \quad \text{for some fixed} \quad \beta^o\in\overline{\mathcal{U}}(t,x). \tag{4.1}
$$

Let $\mathbb{P}^{\beta^o}$ be any weak solution of the corresponding SDE (2.4). The main tool for our results below is the use of Backward Stochastic Differential Equations, BSDE's. This requires introducing filtration $\mathbb{F}_+^{\mathbb{P}^{\beta^o}}$, defined as the $\mathbb{P}^{\beta^o}-$completion of the right-limit of $\mathbb{F}$,[5] under which the predictable martingale representation property holds true.

In the present setting, all probability measures $\mathbb{P} \in \mathcal{P}(t,x)$ are equivalent to $\mathbb{P}^{\beta^o}$. Consequently, equation (3.3) only needs to be considered under $\mathbb{P}^{\beta^0}$, and reduces to

$$Y_s^Z := Y_s^{Z,0} = Y_t - \int_t^s F_r^0 \left(X, Y_r^{Z,\Gamma}, Z_r\right) dr + \int_t^s Z_r \cdot dX_r, \;\; s \in [t,T], \;\; \mathbb{P}^{\beta^o} - a.s., \tag{4.2}$$

where the dependence on the process Γ gets simplified, and

$$F_t^0(x,y,z) \;\; := \;\; \sup_{\alpha \in A} \big\{ - c_t(x,\alpha,b) - k_t(x,\alpha,b)y + \sigma_t\big(x, \beta_t^o(x)\big)\lambda_t(x,\alpha)\cdot z \big\}. \tag{4.3}$$

Theorem 4.1. *Let Assumption 3.2 hold. Under assumption* (4.1)*, assuming in addition that* $(\mathbb{P}^{\beta^o}, \mathbb{F}_+^{\mathbb{P}^{\beta^o}})$ *satisfies the predictable martingale representation property and the Blumenthal zero-one law, we have*

$$V^P(t,x) \;=\; \sup_{y \geq R} \underline{V}(t,x,y), \quad \textit{for all} \quad (t,x) \in [0,T] \times \Omega.$$

Proof. For all $\xi \in \Xi(t,x)$, we observe that condition (2.8) guarantees that $\xi \in \mathbb{L}^p(\mathbb{P}^{\beta^0})$. To prove that the stated equality holds, it is sufficient to show that all such ξ can be represented in terms of a controlled diffusion $Y^{Z,0}$. We know that F is uniformly Lipschitz-continuous in (y,z) because k, σ and λ are bounded, hence, by definition of admissible contracts, we have

$$\mathbb{E}^{\mathbb{P}^{\beta^o}} \left[\int_0^T |F_t^0(X,0,0)|^p \right] \;\; < \;\; \infty,$$

Then, the standard theory (see for instance Possamaï, D., Tan, X., Zhou, C. (2015), henceforth PTZ (2015)) guarantees that the BSDE

$$Y_t \;\; = \;\; \xi + \int_t^T F_r^0(X, Y_r, Z_r)dr - \int_t^T Z_r \cdot \sigma_r(X, \beta_r^o) dW_r^{\mathbb{P}^{\beta^o}},$$

is well-posed, because we also have that $\mathbb{P}^{\beta^o}$ satisfies the predictable martingale representation property. Moreover, we then have automatically $(Z,0) \in \mathcal{V}(t,x)$. This implies that ξ can indeed be represented by a process Y which is of the form (4.2). □

Remark 4.2. *Let us comment on the additional assumptions of Theorem 4.1. We assume that the Blumenthal* 0-1 *law holds only to simplify the proof in this section, and we provide the proof for the general case in the next section, without this assumption. The predictable martingale representation property holds if, for instance,* $\sigma_t(x, \beta_t^o(x))(\sigma_t(x, \beta_t^o(x)))^\top$ *is always invertible and if the solutions to the SDE* (2.1) *are strong solutions instead of weak solutions. For example, it holds if both* σ *and* λ *have linear growth in* x (*with respect to the uniform topology on the space of continuous functions*) *and are Lipschitz continuous in* x*, uniformly in* t *and* α*, which is the case in the typical applications.*

[5] For a semimartingale probability measure $\mathbb{P}$, we denote by $\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$ its right-continuous limit, and by $\mathcal{F}_{t+}^{\mathbb{P}}$ the corresponding completion under $\mathbb{P}$. The completed right-continuous filtration is denoted by $\mathbb{F}_+^{\mathbb{P}}$.

4.2 The general case

The purpose of this section is to extend Theorem 4.1 to the case in which Agent controls both the drift and the volatility of the output process X. Similarly to the previous section, the critical tool is the theory of Backward SDEs, but the control of volatility requires to invoke the recent extension of Backward SDE's to the second order case. This needs additional notation, as follows. Let $\mathbb{M}$ denote the collection of all probability measures on $(\Omega, \mathcal{F}_T)$. The universal filtration $\mathbb{F}^U = \left(\mathcal{F}_t^U\right)_{0\le t\le T}$ is defined by

$$\mathcal{F}_t^U := \bigcap_{\mathbb{P}\in\mathbb{M}} \mathcal{F}_t^{\mathbb{P}}, t\in[0,T],$$

and we denote by $\mathbb{F}_+^U$, the corresponding right-continuous limit. Moreover, for a subset $\mathcal{P}\subset\mathbb{M}$, we introduce the set of $\mathcal{P}-$polar sets $\mathcal{N}^{\mathcal{P}} := \big\{N\subset\Omega : N\subset A$ for some $A\in\mathcal{F}_T$ with $\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}(A)=0\big\}$, and we introduce the $\mathcal{P}-$completion of $\mathbb{F}$

$$\mathbb{F}^{\mathcal{P}} := \left(\mathcal{F}_t^{\mathcal{P}}\right)_{t\in[0,T]}, \text{ with } \mathcal{F}_t^{\mathcal{P}} := \mathcal{F}_t^U \vee \sigma\left(\mathcal{N}^{\mathcal{P}}\right),\ t\in[0,T],$$

together with the corresponding right-continuous limit $\mathbb{F}_+^{\mathcal{P}}$.

Finally, for technical reasons, we work under the classical ZFC set-theoretic axioms, as well as the continuum hypothesis[6].

4.2.1 2BSDE characterization of Agent's problem

We now provide a representation of Agent's value function by means of the so-called second order BSDEs, or 2BSDEs as introduced by Soner, Touzi and Zhang (2011), (see also Cheridito, Soner, Touzi and Victoir (2007)). Furthermore, we use crucially recent results of Possamaï, Tan and Zhou (2015), PTZ (2015) to bypass the regularity conditions in Soner, Touzi and Zhang (2011).

We first re-write mapping H in (3.1) as:

$$\begin{aligned} H_t(x,y,z,\gamma) &= \sup_{\beta\in B}\Big\{F_t\big(x,y,z,\Sigma_t(x,\beta)\big)+\frac{1}{2}\Sigma_t(x,\beta):\gamma\Big\},\\ F_t(x,y,z,\Sigma) &:= \sup_{(\alpha,\beta)\in A\times B_t(x,\Sigma)}\big\{-c_t(x,\alpha,\beta)-k_t(x,\alpha,\beta)y+\sigma_t(x,\beta)\lambda_t(x,\alpha)\cdot z\big\}.\end{aligned}$$

We reformulate Assumption 3.2 in this setting:

Assumption 4.3. *Functional F has at least one measurable maximizer $u^\star=(\alpha^\star,\beta^\star):[0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}_d^+\longrightarrow A\times B$, i.e. $F(\cdot,y,z,\Sigma)=-c(\cdot,\alpha^\star,\beta^\star)-k(\cdot,\alpha^\star,\beta^\star)y+\sigma(\cdot,\beta^\star)\lambda(\cdot,\alpha^\star)\cdot z$. Moreover, for all $(t,x)\in[0,T]\times\Omega$, and for all admissible controls $\beta\in\overline{\mathcal{U}}(t,x)$, the control process*

$$\nu_s^{\star,Y,Z,\beta} := (\alpha_s^\star,\beta_s^*)\big(Y_s,Z_s,\Sigma_s(\beta_s)\big),\quad s\in[t,T],$$

is admissible, that is $\nu^{\star,Y,Z,\beta}\in\mathcal{U}(t,x)$.

[6] Actually, we do not need the continuum hypothesis, *per se*. Indeed, we want to be able to use the main result of Nutz (2012), which only requires axioms ensuring the existence of medial limits in the sense of Mokobodzki. We make this choice here for ease of presentation.

We also need the following condition.

Assumption 4.4. *For any $(t,x,\beta) \in [0,T] \times \Omega \times B$, the matrix $(\sigma_t \sigma_t^\top)(x,\beta)$ is invertible, with a bounded inverse.*

The following lemma shows that sets $\overline{\mathcal{P}}(t,x)$ satisfy natural properties.

Lemma 4.5. *The family $\{\overline{\mathcal{P}}(t,x), (t,x) \in [0,T] \times \Omega\}$ is saturated, and satisfies the dynamic programming requirements of Assumption 2.1 in PTZ (2015).*

Proof. Consider some $\mathbb{P} \in \overline{\mathcal{P}}(t,x)$ and some $\mathbb{P}^{'}$ under which X is a martingale, and which is equivalent to $\mathbb{P}$. Then, the quadratic variation of X under $\mathbb{P}^{'}$ is the same as its quadratic variation under $\mathbb{P}$, that is $\int_t^{\cdot} (\sigma_s \sigma_s^\top)(X, \beta_s) ds$. By definition, $\mathbb{P}^{'}$ is therefore a weak solution to (2.4) and belongs to $\overline{\mathcal{P}}(t,x)$.

The dynamic programming requirements of Assumption 2.1 in PTZ (2015) follow from the more general results given in El Karoui and Tan (2013a) and (2013b). □

Given an admissible contract ξ, we consider the following saturated 2BSDE (in the sense of Section 5 of PTZ (2015)):

$$Y_t \;=\; \xi + \int_t^T F_s(X, Y_s, Z_s, \widehat{\sigma}_s^2) ds - \int_t^T Z_s \cdot dX_s + \int_t^T dK_s, \tag{4.4}$$

where Y is $\mathbb{F}_+^{\overline{\mathcal{P}}_0}-$progressively measurable process, Z is an $\mathbb{F}^{\overline{\mathcal{P}}_0}-$predictable process, with appropriate integrability conditions, and K is an $\mathbb{F}^{\overline{\mathcal{P}}_0}-$optional non-decreasing process with $K_0 = 0$, and satisfying the minimality condition

$$K_t \;=\; \underset{\mathbb{P}^{'} \in \overline{\mathcal{P}}_0(t,\mathbb{P},\mathbb{F}^+)}{\operatorname{essinf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}^{'}} \big[K_T \big| \mathcal{F}_t^{\mathbb{P}+} \big], \quad 0 \le t \le T, \quad \mathbb{P}-\text{a.s. for all} \quad \mathbb{P} \in \overline{\mathcal{P}}_0. \tag{4.5}$$

Notice that, in contrast with the 2BSDE definition in Soner, Touzi and Zhang (2011) and PTZ (2015), we are using here an aggregated non-decreasing process K. This is possible because of the general aggregation result of stochastic integrals in Nutz (2012).

Since k, σ, λ are bounded, and $\sigma\sigma^\top$ is invertible with a bounded inverse, it follows from the definition of admissible controls that F satisfies the integrability and Lipschitz continuity assumptions required in PTZ (2015), that is for some $\kappa \in [1,p)$ and for any $(s,x,y,y',z,z',a) \in [0,T] \times \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2d} \times \mathcal{S}_d^+$

$$|F_s(x,y,z,a) - F_s(x,y',z',a)| \le C \Big(|y-y'| + |a^{1/2}(z-z')| \Big),$$

$$\sup_{\mathbb{P} \in \overline{\mathcal{P}}_0} \mathbb{E}^{\mathbb{P}} \left[\underset{0 \le t \le T}{\operatorname{essup}^{\mathbb{P}}} \left(\mathbb{E}^{\mathbb{P}} \left[\int_0^T |F_s(X,0,0,\widehat{\sigma}_s^2)|^{\kappa} \middle| \mathcal{F}_{t+} \right] \right)^{\frac{p}{\kappa}} \right] < +\infty.$$

Then, in view of Lemma 4.5, the well-posedness of the saturated 2BSDE (4.4) is a direct consequence of Theorems 4.1 and 5.1 in PTZ (2015).

We use 2BSDE's (4.4) because of the following representation result.

Proposition 4.6. *Let Assumptions 4.3 and 4.4 hold. Then, we have*

$$V^A(t,x,\xi) \quad = \quad \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}\left[Y_t\right].$$

Moreover, $\xi \in \Xi(t,x)$ if and only if there is an $\mathbb{F}-$adapted process $\beta^\star$ with values in B, such that $\nu^\star := \big(a^\star_\cdot(X, Y_\cdot, Z_\cdot, \Sigma_\cdot(X,\beta^\star_\cdot)), \beta^\star_\cdot\big) \in \mathcal{U}(t,x)$, and

$$K_T \quad = \quad 0, \quad \mathbb{P}^{\beta^\star} - \text{a.s.}$$

for any associated weak solution $\mathbb{P}^{\beta^\star}$ of (2.4).

Proof. By Theorem 4.2 of PTZ (2015), we know that we can write the solution of the 2BSDE (4.4) as a supremum of solutions of BSDEs, that is

$$Y_t \;=\; \underset{\mathbb{P}'\in\overline{\mathcal{P}}_0(t,\mathbb{P},\mathbb{F}^+)}{\operatorname{essup}^{\mathbb{P}}} \; \mathcal{Y}_t^{\mathbb{P}'}, \quad \mathbb{P}-\text{a.s. for all} \quad \mathbb{P}\in\overline{\mathcal{P}}_0,$$

where for any $\mathbb{P}\in\overline{\mathcal{P}}_0$ and any $s\in[0,T]$,

$$\mathcal{Y}_s^{\mathbb{P}} = \xi + \int_s^T F_s(X,\mathcal{Y}_r,\mathcal{Z}_r,\widehat{\sigma}_r^2)dr - \int_s^T \mathcal{Z}_r^{\mathbb{P}}\cdot dX_r - \int_s^T d\mathcal{M}_r^{\mathbb{P}}, \ \mathbb{P}-\text{a.s.}$$

with a càdlàg $(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})-$martingale $\mathcal{M}^{\mathbb{P}}$ orthogonal to $W^{\mathbb{P}}$.

For any $\mathbb{P}\in\overline{\mathcal{P}}_0$, let $\mathcal{B}(\widehat{\sigma}^2,\mathbb{P})$ denote the collection of all control processes β with $\beta\in B_t(X,\widehat{\sigma}_t^2)$, $dt\otimes\mathbb{P}-$a.e. For all $(\mathbb{P},\alpha)\in\overline{\mathcal{P}}_0\times\mathcal{A}$, and $\beta\in\mathcal{B}(X,\widehat{\sigma}^2,\mathbb{P})$, we next introduce the backward SDE

$$\begin{aligned}\mathcal{Y}_s^{\mathbb{P},\alpha,\beta} = \xi + &\int_s^T \big(-c_r(X,\alpha_r,\beta_r) - k_r(X,\alpha_r,\beta_r)\mathcal{Y}_r^{\mathbb{P},\alpha,\beta} + \sigma_r(X,\beta_r)\lambda_r(X,\alpha_r)\cdot\mathcal{Z}_r^{\mathbb{P},\alpha,\beta}\big)\,dr\\ &- \int_s^T \mathcal{Z}_r^{\mathbb{P},\alpha,\beta}\cdot dX_r - \int_s^T d\mathcal{M}_r^{\mathbb{P},\alpha,\beta}, \ \mathbb{P}-\text{a.s.}\end{aligned}$$

Let $\mathbb{P}^{\alpha,\beta}$ be the probability measure, equivalent to $\mathbb{P}$, defined by

$$\frac{d\mathbb{P}^{\alpha,\beta}}{d\mathbb{P}} := \mathcal{E}\Big(\int_t^T R_s(X,\beta_s)\lambda_s(X,\alpha_s)\cdot dW_s^{\mathbb{P}}\Big).$$

Then, the solution of the last linear backward SDE is given by:

$$\mathcal{Y}_t^{\mathbb{P},\alpha,\beta} = \mathbb{E}^{\mathbb{P}^{\alpha,\beta}}\left[\mathcal{K}_{t,T}^{(\alpha,\beta)}(X)\xi - \int_t^T \mathcal{K}_{t,s}^{(\alpha,\beta)}(X)c_s\big(X,\alpha 1_s,\beta_s\big)ds\,\middle|\,\mathcal{F}_t^+\right], \quad \mathbb{P}-\text{a.s.}$$

By Assumption 4.3, from El Karoui, Peng & Quenez (1997) it follows that the processes $\mathcal{Y}^{\mathbb{P},\alpha,\beta}$ induce then following stochastic control representation for $\mathcal{Y}^{\mathbb{P}}$ (see also Lemma A.3 in PTZ (2015)):

$$\mathcal{Y}_t^{\mathbb{P}} = \underset{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}(\widehat{\sigma}^2,\mathbb{P})}{\operatorname{essup}^{\mathbb{P}}} \; \mathcal{Y}_t^{\mathbb{P},\alpha,\beta}, \ \mathbb{P}-\text{a.s., for any } \mathbb{P}\in\overline{\mathcal{P}}_0.$$

This implies that

$$\mathcal{Y}_t^{\mathbb{P}} = \underset{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}(\widehat{\sigma}^2,\mathbb{P})}{\operatorname{essup}^{\mathbb{P}}} \; \mathbb{E}^{\mathbb{P}^{\alpha,\beta}}\left[\mathcal{K}_{t,T}^{(\alpha,\beta)}(X)\xi - \int_t^T \mathcal{K}_{t,s}^{(\alpha,\beta)}(X)c_s\big(X,\alpha_s,\beta_s\big)ds\,\middle|\,\mathcal{F}_t^+\right],$$

and therefore for any $\mathbb{P} \in \overline{\mathcal{P}}_0$, we have $\mathbb{P} -$ a.s.

$$Y_t = \underset{(\mathbb{P}',\alpha,\beta)\in\overline{\mathcal{P}}_0(t,\mathbb{P},\mathbb{F}^+)\times\mathcal{A}\times\mathcal{B}(\widehat{\sigma}^2,\mathbb{P}')}{\operatorname{essup}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'^{\alpha,\beta}}\left[\mathcal{K}_{t,T}^{(\alpha,\beta)}(X)\xi - \int_t^T \mathcal{K}_{t,s}^{(\alpha,\beta)}(X)c_s\big(X,\alpha_s,\beta_s\big)ds\middle|\mathcal{F}_t^+\right]$$
$$= \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}^+)}{\operatorname{essup}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[\mathcal{K}_{t,T}^{\nu^{\mathbb{P}'}}(X)\xi - \int_t^T \mathcal{K}_{t,s}^{\nu^{\mathbb{P}'}}(X)c_s\big(X,\alpha_s^{\mathbb{P}'},\beta_s^{\mathbb{P}'}\big)ds\middle|\mathcal{F}_t^+\right],$$

where we have used the connection between $\mathcal{P}_0$ and $\overline{\mathcal{P}}_0$ recalled at the end of Section 2.1. The desired result follows by classical arguments similar to the ones used in the proofs of Lemma 3.5 and Theorem 5.2 of PTZ (2015).

By the above equalities, together with Assumption 4.3, it is clear that a probability measure $\mathbb{P} \in \mathcal{P}(t,x)$ is in $\mathcal{P}^\star(t,x,\xi)$ if and only if

$$\nu^\star_\cdot = (a^\star_\cdot,\beta^\star_\cdot)(X,Y_\cdot,Z_\cdot,\Sigma^\star_\cdot),$$

where $\Sigma^\star$ is such that for any associated weak solution $\mathbb{P}^{\beta^\star}$ to (2.4), we have

$$K_T^{\mathbb{P}^{\beta^\star}} = 0, \quad \mathbb{P}^{\beta^\star} - \text{a.s.}$$

□

4.3 The main result

Theorem 4.7. *Let Assumptions 3.2, 4.3, and 4.4 hold true. Then*

$$V^P(t,x) \;=\; \sup_{y\geq R} \underline{V}(t,x,y) \quad \textit{for all} \quad (t,x)\in[0,T]\times\Omega.$$

Proof. The inequality $V^P(t,x) \leq \sup_{y\geq R}\underline{V}(t,x,y)$ was already stated in Proposition 3.4. To prove the converse inequality we consider an arbitrary $\xi \in \Xi(t,x)$ and we intend to prove that Principal's objective function $J^P(t,x,\xi)$ can be approximated by $J^P(t,x,\xi^\varepsilon)$, where $\xi^\varepsilon = Y_T^{Z^\varepsilon,\Gamma^\varepsilon}$ for some $(Z^\varepsilon,\Gamma^\varepsilon)\in\mathcal{V}(t,x)$.

Step 1: Let (Y,Z,K) be the solution of the 2BSDE (4.4)

$$Y_t \;=\; \xi + \int_t^T F(s,X_\cdot,Y_s,Z_s,\widehat{\sigma}_s^2)ds - \int_t^T Z_s\cdot dX_s + \int_t^T dK_s,$$

where we recall again that the aggregated process K exists as a consequence of the aggregation result of Nutz (2012); see Remark 4.1 in PTZ (2015). By Proposition 4.6, we know that for every $\mathbb{P}^\star \in \mathcal{P}(t,x,\xi)$, we have

$$K_T \;=\; 0, \;\; \mathbb{P}^\star - \text{a.s.}$$

For all $\varepsilon > 0$, define the absolutely continuous approximation of K:

$$K_t^\varepsilon \;:=\; \frac{1}{\varepsilon}\int_{(t-\varepsilon)\wedge 0}^t K_s ds, \quad t\in[0,T].$$

Clearly, K^ε is $\mathbb{F}^{\overline{\mathcal{P}}_0}-$predictable, non-decreasing $\overline{\mathcal{P}}_0-$q.s. and

$$K_T^\varepsilon = 0, \quad \mathbb{P}^\star - \text{a.s. for all} \quad \mathbb{P}^\star \in \mathcal{P}(t,x,\xi). \tag{4.6}$$

We next define for any $t \in [0,T]$ the process

$$Y_t^\varepsilon \quad := \quad Y_0 - \int_0^t F_s(X, Y_s^\varepsilon, Z_s, \widehat{\sigma}_s^2)ds + \int_0^t Z_s \cdot dX_s - \int_0^t dK_s^\varepsilon, \tag{4.7}$$

and verify that $(Y^\varepsilon, Z, K^\varepsilon)$ solves the 2BSDE (4.4) with terminal condition $\xi^\varepsilon := Y_T^\varepsilon$ and generator F. This requires to check that K^ε satisfies the required minimality condition, which is obvious by (4.6).

Step 2: For $(t,x,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$, notice that the map $\gamma \longmapsto H_t(x,y,z,\gamma) - F_t(x,y,z,\widehat{\sigma}_t^2(x)) - \frac{1}{2}\widehat{\sigma}_t^2(x) : \gamma$ is valued in $\mathbb{R}_+$, convex, continuous on the interior of its domain, attains the value 0 by Assumption 3.2, and is coercive by the boundedness of λ, σ, k. Then, this map is surjective on $\mathbb{R}_+$. Let $\dot{K}^\varepsilon$ denote the density of the absolutely continuous process K^ε with respect to the Lebesgue measure. Applying a classical measurable selection argument, we may deduce the existence of an $\mathbb{F}-$predictable process Γ^ε such that

$$\dot{K}_s^\varepsilon \quad = \quad H_s(X, \bar{Y}_s^\varepsilon, \bar{Z}_s, \bar{\Gamma}_s^\varepsilon) - F_s(X, \bar{Y}_s^\varepsilon, \bar{Z}_s, \widehat{\sigma}_s^2) - \frac{1}{2}\widehat{\sigma}_s^2 : \bar{\Gamma}_s^\varepsilon.$$

Substituting in (4.7), it follows that the following representation of Y_t^ε holds:

$$Y_t^\varepsilon \quad = \quad Y_0 - \int_0^t H_s(X, Y_s^\varepsilon, Z_s, \Gamma_s^\varepsilon)ds + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\int_0^t \Gamma_s^\varepsilon : d\langle X \rangle_s.$$

Step 3: The contract $\xi^\varepsilon := Y_T^\varepsilon$ takes the required form (3.3), for which we know how to solve Agent's problem, *i.e.* $V^A(t,x,\xi^\varepsilon) = Y_t$, by Proposition 3.3. Moreover, it follows from (4.6) that

$$\xi \quad = \quad \xi^\varepsilon, \quad \mathbb{P}^\star - \text{a.s.}$$

Consequently, for any $\mathbb{P}^\star \in \mathcal{P}^\star(t,x,\xi)$, we have

$$\mathbb{E}^{\mathbb{P}^\star}\big[\mathcal{K}_{t,T}^P U(\ell(X_T) - \xi^\varepsilon)\big] \quad = \quad \mathbb{E}^{\mathbb{P}^\star}\big[\mathcal{K}_{t,T}^P U(\ell(X_T) - \xi)\big],$$

which implies that $J^P(t,x,\xi) = J^P(t,x,\xi^\varepsilon)$. □

5 Special cases and examples

5.1 Coefficients independent of X

In Theorem 3.5 we saw that Principal's problem involves both x and y as state variables. We now identify conditions under which Principal's problem can be somewhat simplified, for example by reducing the number of state variables. We first provide conditions under which Agent's participation constraint is tight.

We assume that

$$\sigma, \ \lambda, \ c, \ k, \ \text{and } k^P \text{ are independent of } x. \tag{5.1}$$

In this case, the Hamiltonian H introduced in (3.1) is also independent of x, and we re-write the dynamics of the controlled process $Y^{Z,\Gamma}$ as:

$$Y_s^{Z,\Gamma} \;:=\; Y_t - \int_t^s H_r\left(Y_r^{Z,\Gamma}, Z_r, \Gamma_r\right) dr + \int_t^s Z_r \cdot dX_r + \frac{1}{2}\int_t^s \Gamma_r : d\langle X\rangle_r, \; s \in [t,T].$$

By classical comparison result of stochastic differential equation, this implies that the flow $Y_s^{Z,\Gamma}$ is increasing in terms of the corresponding initial condition Y_t. Thus, optimally, Principal will provide Agent with the minimum reservation utility R he requires. In other words, we have the following simplification of Principal's problem, as a direct consequence of Theorem 4.7.

Proposition 5.1. *Let Assumptions 3.2, 4.3, and 4.4 hold true. Then, assuming* (5.1)*, we have:*

$$V^P(t,x) \;=\; \underline{V}(t,x,R) \quad \text{for all} \quad (t,x) \in [0,T] \times \Omega.$$

We now consider cases in which the number of state variables is reduced.

Example 5.2 (Exponential utility)**.**

(i) *Let $U(y) := -e^{-\eta y}$, and assume $k \equiv 0$. Then, under the conditions of Proposition 5.1, it follows that*

$$V^P(t,x) \;=\; e^{\eta R}\underline{V}(t,x,0) \quad \text{for all} \quad (t,x) \in [0,T] \times \Omega.$$

Consequently, the HJB equation of Theorem 3.5, corresponding to $\underline{V}$, may be reduced to a two-dimensional problem on $[0,T] \times \mathbb{R}^d$, by applying the change of variables $v(t,x,y) = e^{\eta y} f(t,x)$.

(ii) *Assume in addition that, for some $h \in \mathbb{R}^d$, the liquidation function is linear, $\ell(x) = h \cdot x$ is linear. Then, it follows that*

$$V^P(t,x) \;=\; e^{-\eta(h\cdot x - R)}\underline{V}(t,0,0) \quad \text{for all} \quad (t,x) \in [0,T] \times \Omega.$$

Consequently, the HJB equation of Theorem 3.5 corresponding to $\underline{V}$ can be reduced to an ODE on $[0,T]$ by applying the change of variables $v(t,x,y) = e^{-\eta(h\cdot x - R)} f(t)$.

Example 5.3 (Risk-neutral Principal)**.** *Let $U(x) := x$, and assume $k \equiv 0$. Then, under the conditions of Proposition 5.1, it follows that*

$$V^P(t,x) \;=\; -R + \underline{V}(t,x,0) \quad \text{for all} \quad (t,x) \in [0,T] \times \Omega.$$

Consequently, the HJB equation of Theorem 3.5 corresponding to $\underline{V}$ can be reduced to $[0,T] \times \mathbb{R}^d$ by applying the change of variables $v(t,x,y) = -y + f(t,x)$.

5.2 Drift control with quadratic cost: Cvitanić, Wan and Zhang (2009)

We now consider the only tractable case from Cvitanić, Wan and Zhang (2009), from now on CWZ (2009).

Suppose $\xi = U_A(C_T)$ where U_A is Agent's utility function, and C_T is the contract payment. Then, we need to replace ξ by $U_A^{-1}(\xi)$, where the inverse function is assumed to exist. Assume that $d = n = 1$ and, for some constants $c > 0$, $\sigma > 0$,

$$\sigma(x,\beta) \equiv \sigma, \; \lambda = \lambda(\alpha) = \alpha, \; k = k^P \equiv 0, \; \ell(x) = x, \; c(t,\alpha) = -\frac{1}{2}c\alpha^2.$$

That is, the volatility is uncontrolled (as in Section 4.1) and the output is of the form

$$dX_t = \sigma\alpha_t dt + \sigma dW_t^\alpha,$$

and Agent and Principal are respectively maximizing

$$\mathbb{E}^{\mathbb{P}}\left[U_A(C_T) - \frac{c}{2}\int_0^T \alpha_t^2 dt\right] \text{ and } \mathbb{E}^{\mathbb{P}}\left[U_P(X_T - C_T)\right],$$

denoting Principal utility U_P instead of U. In particular, and this is important for tractability, the cost of drift effort α is quadratic.

We recover the following result from CWZ (2009), using our approach, and under a different set of technical conditions.

Proposition 5.4. *Assume that Principal's value function $v(t,x,y)$ is the solution of its corresponding HJB equation, in which the supremum over (z,γ) is attained at the solution (z^*,γ^*) to the first order conditions, and that v is in class $C^{2,3,3}$ on its domain, including at $t = T$. Then, we have, for some constant L,*

$$v_y(t, X_t, Y_t) = -\frac{1}{c}v(t, X_t, Y_t) - L.$$

In particular, the optimal contract C_T satisfies the following equation, almost surely,

$$\frac{\tilde{U}_P'(X_T - C_T)}{U_A'(C_T)} = \frac{1}{c}U_P(X_T - C_T) + L. \tag{5.2}$$

Moreover, if this equation has a unique solution $C_T = C(X_T)$, if the Backward SDE under the Wiener measure $\mathbb{P}_0$

$$P_t = e^{U_A(C(X_T))/c} - \int_t^T \frac{1}{c}P_s Z_s dX_s,\ t \in [0,T],$$

has a unique solution (P,Z), and if Agent's value function is the solution of its corresponding HJB equation in which the supremum over α is attained at the solution α^* to the first order condition, then the contract $C(X_T)$ is optimal.

Thus, the optimal contract C_T is a function of the terminal value X_T only. This can be considered as a moral hazard modification of the Borch rule valid in the first best case: the ratio of Principal's and Agent's marginal utilities is constant under first best risk-sharing, but here, it is a linear function of the Principal's utility.

Proof. Agent's Hamiltonian is maximized by $\alpha^*(z) = \frac{1}{c}\sigma z$. The HJB equation for Principal's value function $v = v^P$ of Theorem 3.5 becomes then, with $U = U_P$,

$$\begin{cases} \partial_t v + \sup\limits_{z\in\mathbb{R}}\left\{\frac{1}{c}\sigma^2 z v_x + \frac{1}{2c}\sigma^2 z^2 v_y + \frac{1}{2}\sigma^2\left(v_{\mathrm{xx}} + z^2 v_{\mathrm{yy}}\right) + \sigma^2 z v_{\mathrm{xy}}\right\} = 0, \\ v(T, \mathrm{x}, y) = U_P(\mathrm{x} - U_A^{-1}(y)). \end{cases}$$

Optimizing over z gives

$$z^* = -\frac{v_x + cv_{xy}}{v_y + cv_{yy}}.$$

We have that $v(t, X_t, Y_t)$ is a martingale under the optimal measure P, satisfying

$$dv_t = \sigma(v_x + z^* v_y)dW_t.$$

Thus, the volatility of v is σ times

$$v_x + z^* v_y = \frac{c(v_x v_{yy} - v_y v_{xy})}{v_y + c v_{yy}}.$$

We also have, by Ito's rule,

$$\begin{aligned} dv_y = & \left(\partial_t v_y + \frac{1}{c}\sigma^2 z^* v_{xy} + \frac{1}{2c}\sigma^2 (z^*)^2 v_{yy} + \frac{1}{2}\sigma^2 \left(v_{\mathrm{xxy}} + (z^*)^2 v_{\mathrm{yyy}} \right) + \sigma^2 z^* v_{\mathrm{xyy}} \right) dt \\ & + \sigma(v_{xy} + z^* v_{yy})dW_t, \end{aligned}$$

$$v_y(T, \mathrm{x}, y) = -\frac{U_P'(\mathrm{x} - U_A^{-1}(y))}{U_A'(U_A^{-1}(y))}.$$

Thus, the volatility of v_y is σ times

$$v_{xy} + z^* v_{yy} = \frac{v_{xy} v_y - v_{yy} v_x}{v_y + c v_{yy}},$$

that is, equal to the minus volatility of v divided by c. For the first statement, it only remains to prove that the drift of $v_y(t, X_t, Y_t)$ is zero. This drift is equal to

$$\partial_t v_y - \sigma^2 \frac{v_x/c + v_{xy}}{v_y/c + v_{yy}}(v_{xy}/c + v_{xyy}) + \frac{1}{2}\sigma^2 \frac{(v_x/c + v_{xy})^2}{(v_y/c + v_{yy})^2}(v_{yy}/c + v_{\mathrm{yyy}}) + \frac{1}{2}\sigma^2 v_{\mathrm{xxy}}.$$

However, note that the HJB equation can be written as

$$\partial_t v = \frac{\sigma^2}{2}\left(\frac{(v_x/c + v_{xy})^2}{v_y/c + v_{yy}} - v_{xx} \right),$$

and that differentiating it with respect to y gives

$$\partial_t v_y = \frac{\sigma^2}{2}\left(\frac{2(v_x/c + v_{xy})(v_{xy}/c + v_{xyy})(v_y/c + v_{yy}) - (v_x/c + v_{xy})^2 (v_{yy}/c + v_{yyy})}{(v_y/c + v_{yy})^2} - v_{xxy} \right).$$

Using this, it is readily seen that the above expression for the drift is equal to zero.

Next, denoting by W^0 the Brownian motion for which $dX = \sigma dW^0$, from (3.3) we have

$$dY = -\frac{1}{2c}\sigma^2 (Z^*)^2 dt + \sigma Z^* dW^0$$

and thus, by Ito's rule

$$de^{Y/c} = \frac{1}{c} e^{Y/c} \sigma Z^* dW^0$$

Suppose now the offered contract $C_T = C(X_T)$ is the one determined by equation (5.2). Agent's optimal effort is $\hat{\alpha} = \sigma V_x^A / c$, where Agent's value function V^A satisfies

$$\partial_t V^A + \frac{1}{2c}\sigma^2 (V_x^A)^2 + \frac{1}{2}\sigma^2 V_{xx}^A = 0.$$

Using Ito's rule, this implies that the $\mathbb{P}_0-$martingale processes $e^{V^A(t,X_t)/c}$ and $e^{Y(t)/c}$ satisfy the same stochastic differential equation. Moreover, they are equal almost surely at $t = T$ because $V^A(T, X_T) = Y_T = U_A(C(X_T))$, hence, by the uniqueness of the solution of the Backward SDE, they are equal for all t, and, furthermore, $V_x^A(t, X_t) = Z^*(t)$. This implies that Agent's effort $\hat{\alpha}$ induced by $C(X_T)$ is the same as the effort α^* optimal for Principal, and both Agent and Principal get their optimal expected utilities. $\square$

We now present a completely solvable example of the above model from CWZ (2009), solved here using our approach.

Example 5.5. Risk-neutral principal and logarithmic agent; CWZ (2009). In addition to the above assumptions, suppose, for notational simplicity, that $c = 1$. Assume also that Principal is risk-neutral while Agent is risk averse with

$$U_P(C_T) = X_T - C_T \ , \quad U_A(C_T) = \log C_T.$$

We also assume that the model for X is, with $\sigma > 0$ being a positive constant,

$$dX_t = \sigma\alpha_t X_t dt + \sigma X_t dW_t^\alpha.$$

Thus, $X_t > 0$ for all t. We will show that the optimal contract payoff C_T satisfies

$$C_T = \frac{1}{2}X_T + const.$$

This can be seen directly from (5.2), or as follows. Similarly as in the proof above (replacing σ with σx), the HJB equation of Theorem 3.5 is

$$\partial_t v = \frac{\sigma^2 x^2}{2}\left(\frac{(v_x + v_{xy})^2}{v_y + v_{yy}} - v_{xx}\right), \ v(T, \mathrm{x}, y) = \mathrm{x} - e^y.$$

It is straightforward to verify that the solution is given by

$$v(t, x, y) = x - e^y + \frac{1}{4}e^{-y}x^2\left(e^{\sigma^2(T-t)} - 1\right).$$

We have, denoting $E(t) := e^{\sigma^2(T-t)} - 1$,

$$v_x = 1 + \frac{1}{2}E(t)e^{-y}x, v_{xy} = -v_x - 1, v_y = -e^y - \frac{1}{4}E(t)e^{-y}x^2, v_{yy} = -e^y + \frac{1}{4}E(t)e^{-y}x^2,$$

and therefore

$$z^* = \frac{1}{2}e^{-y}, \alpha^* = \frac{1}{2}\sigma e^{-y}.$$

Hence, from (3.3),

$$dY = -\frac{1}{8}\sigma^2 e^{-2Y}dt + \frac{1}{2}e^{-Y}dX,$$

and

$$d(e^Y) = \frac{1}{2}dX.$$

Since $e^{Y_T} = C_T$, we get $C_T = \frac{1}{2}X_T + const.$

5.3 Volatility control with no cost; Cadenillas, Cvitanić and Zapatero (2007)

We now apply our method to the main model of interest in Cadenillas, Cvitanić and Zapatero (2007), CCZ (2007). That paper considered the risk-sharing problem between Agent and Principal, when choosing the first best choice of volatility β_t, with no moral hazard, with general utility functions. In that case, it is possible to apply convex duality methods to solve the problem. Those methods do not work for the general setup of the current paper, which is the first paper that provides a method for Principal-Agent problems with volatility choice that enables us to solve both the special, first best case of CCZ (2007), and the second best, moral hazard case [7].

Suppose again that $\xi = U_A(C_T)$ where U_A is Agent's utility function, and C_T is the contract payment. Assume also for some constants $c > 0$, $\sigma > 0$ that the output is of the form, for a one-dimensional Brownian motion W,[8] and a fixed constant λ,

$$dX_t = \lambda \beta_t dt + \beta_t dW_t.$$

We assume that Agent is maximizing $E[U_A(C_T)]$ and Principal is maximizing $E[U_P(X_T - C_T)]$. In particular, there is zero cost of volatility effort β. This is a standard model for portfolio management, in which case β has the interpretation of the vector of positions in risky assets.

Since there is no cost of effort, first best is attained - Principal can offer a constant payoff C such that $U_A(C) = R$, and Agent will be indifferent with respect to which action β to apply. Nevertheless, we look for a possibly different contract, which would provide Agent with strict incentives. We recover the following result from CCZ (2007) using our approach, and under a different set of technical conditions.

Proposition 5.6. *Given constants κ and λ, consider the following ODE*

$$\frac{U_P'(x - F(x))}{U_A'(F(x))} = \kappa F'(x), \tag{5.3}$$

and boundary condition $F(0) = \lambda$, with a solution (if exists) denoted $F(x) = F(x; \kappa, \lambda)$. Consider the set $\mathcal{S}$ of (κ, λ) such that a solution F exists, and if Agent is offered the contract $C_T = F(X_T)$, his value function $V(t, x) = V(t, x; \kappa, \lambda)$ solves the corresponding HJB equation, in which the supremum over β is attained at the solution β^* to the first order conditions, and V is a $C^{2,3}$ function on its domain, including at $t = T$. With W_T denoting a normally distributed random variable with mean zero and variance T, suppose there exists m_0 such that

$$\mathbb{E}\left[U_P\left((U_P')^{-1}\left(m_0 \exp\{-\frac{1}{2}\lambda^2 T + \lambda W_T\}\right)\right)\right],$$

is equal to Principal's expected utility in the first best risk-sharing, for the given Agent's expected utility R. Assume also that there exists $(\kappa_0, \lambda_0) \in \mathcal{S}$ such that $\kappa_0 = m_0 / V_x(0, X_0; \kappa_0, \lambda_0)$, and that Agent's optimal expected utility under the contract $C_T = F(X_T; \kappa_0, \lambda_0)$ is equal to his reservation utility R. Then, under that contract, Agent will choose actions that will result in Principal attaining her corresponding first best expected utility.

[7] The special case of moral hazard with CARA utility functions and linear output dynamics is solved using the method of this paper in Cvitanić, Possamaï and Touzi (2015).

[8] The $n-$dimensional case with $n > 1$ is similar.

Note that the action process β chosen by Agent is not necessarily the same as the action process Principal would dictate as the first best when paying Agent with cash. However, the expected utilities are the same as the first best. We also mention that CCZ (2007) present a number of examples for which the assumptions of the proposition are satisfied, and in which, indeed, (5.3) provides the optimal contract.

Proof. Suppose the offered contract is of the form $C_T = F(X_T)$ for some function F for which Agent's value function $V(t,x)$ satisfies $V_{xx} < 0$ and the corresponding HJB equation, given by

$$\partial_t V + \sup_\beta \left\{ \lambda\beta V_x + \frac{1}{2}\beta^2 V_{xx} \right\} = 0.$$

We get that Agent's optimal action is $\beta^* = -\lambda \frac{V_x}{V_{xx}}$ and the HJB equation becomes

$$\partial_t V - \frac{1}{2}\lambda^2 \frac{V_x^2}{V_{xx}} = 0, \; V(T,x) = U_A(F(x)).$$

On the other hand, using Ito's rule, we get

$$dV_x = \left(\partial_t V_x - \lambda^2 V_x + \frac{1}{2}\lambda^2 \frac{V_x^2}{V_{xx}^2} V_{xxx} \right) dt - \lambda V_x dW.$$

Differentiating the HJB equation for V with respect to x, we see that the drift term is zero, and we have

$$dV_x = -\lambda V_x dW, \; V_x(T,x) = U_A'(F(x))F'(x).$$

The solution $V_x(t, X_t)$ to the SDE is a martingale given by

$$V_x(t, X_t) = V_x(0, X_0) M_t,$$

where

$$M_t := e^{-\frac{1}{2}\lambda^2 t + \lambda W_t}.$$

From the boundary condition we get

$$U_A'(F(X_T))F'(X_T) = V_x(0, X_0) M_T.$$

On the other hand, it is known from CCZ (2007) that the first best utility for Principal is attained if

$$U_P'(X_T - C_T) = m_0 M_T, \tag{5.4}$$

where m_0 is chosen so that Agent's participation constraint is satisfied. If we choose F that satisfies the ODE (5.3), with κ_0 satisfying $\kappa_0 = m_0/V_x(0.X_0; \kappa_0, \lambda_0)$, then (5.4) is satisfied and we are done. □

We now present a way to arrive at condition (5.4) using our approach. For a given (z, γ), Agent maximizes $\lambda\beta z + \frac{1}{2}\gamma\beta^2$, thus the optimal β is, assuming $\gamma < 0$,

$$\beta^*(z, \gamma) = -\lambda \frac{z}{\gamma}.$$

The HJB equation of Theorem 3.5 becomes then, with $U = U_P$, and $w = z/\gamma$,

$$\begin{cases} \partial_t v + \sup_{z,w \in \mathbb{R}^2} \left\{ -\lambda^2 w v_x + \frac{1}{2}\lambda^2 w^2 \left(v_{\mathrm{xx}} + z^2 v_{\mathrm{yy}} \right) + \lambda^2 z w^2 v_{\mathrm{xy}} \right\} = 0, \\ v(T, \mathrm{x}, y) = U_P(\mathrm{x} - U_A^{-1}(y)). \end{cases}$$

First order conditions are

$$z^* = -\frac{v_{xy}}{v_{yy}}, \;\; w^* = \frac{v_x}{v_{\mathrm{xx}} - \frac{v_{xy}^2}{v_{yy}}}.$$

The HJB equation becomes

$$\begin{cases} \partial_t v - \frac{1}{2}\lambda^2 \frac{v_x^2}{v_{\mathrm{xx}} - \frac{v_{xy}^2}{v_{yy}}} = 0, \\ v(T, \mathrm{x}, y) = U_P(\mathrm{x} - U_A^{-1}(y)). \end{cases}$$

We also have, by Ito's rule,

$$dv_x = \left(\partial_t v_x - \lambda^2 \frac{v_x v_{xx}}{v_{\mathrm{xx}} - \frac{v_{xy}^2}{v_{yy}}} + \frac{1}{2}\lambda^2 \frac{v_x^2}{\left(v_{\mathrm{xx}} - \frac{v_{xy}^2}{v_{yy}^2} \right)^2} \left[v_{\mathrm{xxx}} + \frac{v_{xy}^2}{v_{yy}^2} v_{xyy} - 2\frac{v_{xy}}{v_{yy}} v_{xxy} \right] \right) dt - \lambda v_x dW,$$

$$v_x(T, \mathrm{x}, y) = U_P'(\mathrm{x} - U_A^{-1}(y)).$$

Differentiating the HJB equation for v with respect to x, we see that the drift term is zero, and we have

$$dv_x = -\lambda v_x dW,$$

with the solution

$$v_x(t, X_t, Y_t) = m_0 e^{-\frac{1}{2}\lambda^2 t + \lambda W_t}.$$

From the boundary condition we get that the optimal contract payoff satisfies

$$U_P'(X_T - C_T) = m_0 M_T.$$

6 Conclusions

We consider a very general Principal Agent problem, with a lump-sum payment at the end of the contracting period. While we develop a simple to use approach, our proofs rely on deep results from the recent theory of Backward Stochastic Differential equations of the second order. The method consists of considering only the contracts that allow a dynamic programming representation of the agent's value function, for which it is straightforward to identify the agent's incentive compatible effort, and then showing that this leads to no loss of generality. While our method encompasses all the existing continuous-time Brownian motion models with only the final lump-sum payment, it remains to be extended to the model with possibly continuous payments. While that might involve technical difficulties, the road map we suggest is clear - identify the generic dynamic programming representation of the agent's value process, express the contract payments in terms of the value process, and optimize the principal's objective over such payments.

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