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DETECTING THE MAXIMUM OF A SCALAR DIFFUSION WITH NEGATIVE DRIFT*

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Abstract. Let X be a scalar diffusion process with drift coefficient pointing towards the origin, i.e. X is mean-reverting. We denote by X^* the corresponding running maximum, T_0 the first time X hits the level zero. Given an increasing and convex loss function ℓ, we consider the following optimal stopping problem: $\inf_{0\le\theta\le T_0}\mathbb{E}[\ell(X^*_{T_0}-X_\theta)]$, over all stopping times θ with values in $[0,T_0]$. For the quadratic loss function and under mild conditions, we prove that an optimal stopping time exists and is defined by: $\theta^*=T_0\wedge\inf\{t\ge 0;\ X^*_t\ge\gamma(X_t)\}$, where the boundary γ is explicitly characterized as the concatenation of the solutions of two equations. We investigate some examples such as the Ornstein-Uhlenbeck process, the CIR–Feller process, as well as the standard and drifted Brownian motions.

Key words. maximum process, optimal stopping, free-boundary problem, smooth fit, verification argument, Markov process, ordinary differential equation

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1. Introduction. Motivated by applications in portfolio management, Graversen, Peskir, and Shiryaev [6] considered the problem of detecting the maximum of a Brownian motion W on a fixed time period. More precisely, [6] considers the optimal stopping problem

$$\inf_{0\le\theta\le 1}\mathbb{E}[(W^*_1-W_\theta)^p], \tag{1.1}$$

where $W^*_t:=\max_{s\le t}W_s$ is the running maximum of W, $p>0$ (and $p\neq 1$), and the infimum is taken over all stopping times θ taking values in $[0,1]$. Using properties of the Brownian motion and a relevant time change, [6] reduces the above problem to a one-dimensional infinite horizon optimal stopping problem and proves that the optimal stopping rule is given by

$$\hat\theta:=\inf\{t\le 1;\ W^*_t-W_t\ge b(t)\},$$

where the free boundary b is an explicit decreasing function.

A first extension of [6] was achieved by Pedersen [10], and later by Du Toit and Peskir [3], in the case of a Brownian motion with constant drift. A similar problem was solved by Shiryaev, Xu, and Zhou [13] in the context of exponential Brownian motion. See also Du Toit and Peskir [5], Dai, Yang, and Zhong [2] and Dai et al. [1].

We also mention a connection with the problem of detection of the last moment τ when W reaches its maximum before the terminal time $t=1$ (see Shiryaev in [12]):

$$\inf_{0\le\theta\le 1}\mathbb{E}|\theta-\tau|.$$

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This problem can indeed be related to the previous one by the observation of Urusov [14] that $\mathbb{E}(W_\tau - W_\theta)^2 = \mathbb{E}|\tau - \theta| + \frac{1}{2}$ for any stopping time θ. A similar problem formulated in the context of a drifted Brownian motion was solved by Du Toit and Peskir [4], although the latter identity stated by Urusov is no longer valid.

In the present paper, we consider a scalar Markov diffusion X, which "mean-reverts" toward the origin starting from a positive initial datum, and we consider the problem of optimal detection of the absolute maximum up to the first hitting time of the origin $T_0 := \inf\{t \geq 0 : X_t = 0\}$:

$$\inf_{0 \leq \theta \leq T_0} \mathbb{E}[\ell(X^*_{T_0} - X_\theta)].$$

Here, the infimum is taken over all stopping times with values in $[0, T_0]$, and ℓ is a nondecreasing and convex function, satisfying some additional technical conditions. We solve explicitly this problem as a free boundary problem and exhibit an optimal stopping time of the form:

$$\hat{\theta} = \inf\{t \geq 0;\ X^*_t \geq \gamma(X_t)\},$$

for some stopping boundary γ. Our analysis has some similarities with that of Peskir [11]; see also Obloj [9] and Hobson [7].

Notice that the formulation of the above optimal stopping problem involves the hitting time of the origin as the maturity for the problem. From the mathematical viewpoint, this is a crucial simplification, as the value function does not depend on the time variable. From the financial viewpoint, this formulation is also relevant, as it captures the practice of asset managers of trading at the extrema of excursions of some underlying asset. Namely, a popular strategy among portfolio managers is the following:

- Managers identify some mean-reverting asset or portfolio of assets; the portfolio composition may be estimated from historical data by minimizing empirical autocorrelations,

- Managers would then want to buy at the lowest price, along an excursion below the mean, and sell at the highest price, along an excursion above the mean; since trading decisions can occur only at stopping times, the only hope is to better approximate the extrema of the price process.

The above formulation corresponds exactly to a single-excursion problem of the asset managers. Clearly, a similar problem with fixed deterministic time horizon is not suitable for the present practical problem.

Using the dynamic programming approach, our problem leads to a two-dimensional elliptic variational inequality, in contrast with the finite horizon, where the problem can be reduced to a one-dimensional parabolic variational inequality. A major difficulty in the present context is that, in general, our solution exhibits a nonmonotonic free boundary γ made of two different parts and driven by two different equations. Except for [4], the latter feature does not appear in the literature mentioned above and has the following a posteriori interpretation. Because of the mean-reversion, we expect that stopping is optimal whenever the running maximum X^* is sufficiently larger than the level X, which corresponds to the intuitive increasing part of the boundary. On the other hand, for some specific dynamics, we may expect that when the process approaches the origin, the martingale part dominates the mean-reversion, implying that the process has equal chances to be pushed away from the origin, so that the investor may defer the stopping decision. This indeed turns out to be the case for the Ornstein–Uhlenbeck process and induces a decreasing part of the boundary near the origin.

The paper is organized as follows. Section 2 presents the general framework and provides some necessary and sufficient conditions for the problem to be well defined. In section 3, we derive the formulation as a free boundary problem, and we prove a verification result together with some preliminary properties. Sections 4–6 focus on the case of a quadratic loss function. In section 4, we study a certain set Γ^+ which plays an essential role in the construction of the solution. The candidate boundary is exhibited in section 5, and in section 6 the corresponding candidate value function is shown to satisfy the assumptions of the verification result of section 3. Section 7 is dedicated to some examples. In section 8, we provide sufficient conditions which guarantee that a similar solution is obtained for a general loss function.

2. Problem formulation. Let W be a scalar Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding augmented canonical filtration. Given two Lipschitz functions $\mu, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$, we consider the scalar diffusion defined by the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t \geq 0,$$

together with some initial datum $X_0 > 0$. We assume throughout that

$$\mu < 0 \text{ and } \sigma > 0 \ \text{ on } \ (0, \infty) \tag{2.1}$$

as well as the following stronger restrictions:

$$\text{the function } \alpha := \frac{-2\mu}{\sigma^2} : \ [0, \infty) \longrightarrow \mathbb{R} \text{ is } C^2 \text{ and concave.} \tag{2.2}$$

Remark 2.1. Conditions (2.2) are needed only for technical reasons. See, in particular, Remark 2.2 for some crucial implications of the concavity condition. In the context of our problem defined below, we shall consider only the process X up to the first hitting time of 0. Therefore the negative drift in condition (2.1) models the mean-reversion of X. Notice that we could formulate a symmetric problem on the negative real line under the condition of a positive drift on $(-\infty, 0)$.

The scale function S is defined by (see [8])

$$S(x) := \int_0^x e^{\int_0^u \alpha(r)dr} du, \quad x \geq 0. \tag{2.3}$$

By the mean-reversion condition (2.1),

$$S \text{ is convex and } \lim_{x \to \infty} S(x) = \infty. \tag{2.4}$$

Remark 2.2. For later use, we observe that the restriction (2.2) has the following useful consequences:

(i) The function α is nonnegative and nondecreasing. Consequently, $\int_0^u \alpha(r)dr < \infty$ and (2.3) is well defined.
(ii) $(1/\alpha)'(x) \to 0$ as $x \to \infty$, and therefore $\alpha' = \circ(\alpha^2)$.
(iii) The function $2S' - \alpha S - 2$ is nonnegative and increasing.

We denote by $T_y := \inf\{t > 0 : X_t = y\}$ the first hitting time of the barrier y. We recall that, for the above homogeneous scalar diffusion with positive diffusion coefficients, we have

$$\mathbb{P}_x[T_y < T_0] = \frac{S(x)}{S(y)} \quad \text{for } 0 \le x < y, \tag{2.5}$$

Our main objective is to solve the optimization problem

$$V_0 := \inf_{\theta \in \mathcal{T}_0} \mathbb{E}\left[\ell\left(X^*_{T_0} - X_\theta\right)\right], \tag{2.6}$$

where $X^*_t := \max_{s \le t} X_s,\ t \ge 0$, is the running maximum process of X; $\ell : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing, strictly convex function; and $\mathcal{T}_0$ is the collection of all $\mathbb{F}$ stopping times θ with $\theta \le T_0$ almost surely.

Remark 2.3. Our main results (sections 4–6) concern the quadratic loss function $\ell(x) = \frac{x^2}{2}$. However, a large part of the analysis is valid for general loss functions. In particular, we provide a natural extension of the quadratic case in section 8, but we have not succeeded in obtaining satisfactory conditions which guarantee that the extension holds true.

We shall approach this problem by the dynamic programming technique. We then introduce the dynamic version

$$V(x,z) := \inf_{\theta \in \mathcal{T}_0} \mathbb{E}_{x,z}\left[\ell\left(Z_{T_0} - X_\theta\right)\right] \quad \text{where } Z_t := z \vee X^*_t,\ t \ge 0, \tag{2.7}$$

and $\mathbb{E}_{x,z}$ denotes the expectation operator conditional on $(X_0, Z_0) = (x,z)$. Clearly, the process (X, Z) takes values in the state space,

$$\mathbf{\Delta} := \{(x,z);\ 0 \le x \le z\}, \tag{2.8}$$

and we may rewrite this problem in the standard form of an optimal stopping problem,

$$V(x,z) = \inf_{\theta \in \mathcal{T}_0} \mathbb{E}_{x,z}\left[g\left(X_\theta, Z_\theta\right)\right] \quad \text{with } g(x,z) := \mathbb{E}_{x,z}\left[\ell\left(Z_{T_0} - x\right)\right],\ (x,z) \in \mathbf{\Delta}. \tag{2.9}$$

Observing that $\mathbb{P}_{x,z}[Z_{T_0} \le u] = \mathbb{P}_x[T_u \ge T_0]\mathbf{1}_{u \ge z}$, we deduce from (2.5) that the reward function g is given by

$$\begin{aligned} g(x,z) &= \ell(z-x)\left(1 - \frac{S(x)}{S(z)}\right) + S(x)\int_z^\infty \ell(u-x)\frac{S'(u)}{S(u)^2}du \\ &= \ell(z-x) + S(x)\int_z^\infty \frac{\ell'(u-x)}{S(u)}du, \quad 0 < x \le z, \end{aligned} \tag{2.10}$$

where ℓ' is the generalized derivative of ℓ and the last expression in (2.10) is obtained by integration by parts together with the observation that for all $x \ge 0$,

$$\int^\infty \ell(u-x)\frac{S'(u)}{S(u)^2}du < \infty \quad \text{iff} \quad \int^\infty \frac{\ell'(u-x)}{S(u)}du < \infty. \tag{2.11}$$

Proof of (2.11). Denote $R := S^{-1}$, and assume $x = 0$, without loss of generality. Then

$$-\int_z^A \ell(u)R'(u)du = \ell(z)R(z) - \ell(A)R(A) + \int_z^A \ell'(u)R(u)du \text{ for } A \ge z > 0. \tag{2.12}$$

That $\int^\infty \ell(u)R'(u)du = -\infty$ implies $\int^\infty \ell'(u)R(u)du = \infty$ follows immediately from $\ell R \geq 0$. Conversely, since R is nonincreasing, $\int_z^A \ell'(u)R(u)du \geq \big(\ell(A) - \ell(z)\big)R(A)$. It follows from (2.12) that $-\int_z^A \ell(u)R'(u)du \geq \ell(z)R(z) - \ell(z)R(A) \longrightarrow \ell(z)R(z)$ when $A \to \infty$, by (2.4). Then $-\int^\infty \ell(u)R'(u)du < \infty$ implies that $\lim_{A\to\infty} \ell(A)$ $R(A) = 0$, and therefore $\int_z^\infty \ell'(u)R(u)du < \infty$ by (2.12). ∎

Remark 2.4. For the linear loss function $\ell(x) = x$, we have $V = g$. Indeed, $V(x,z) = \mathbb{E}_{x,z}[Z_{T_0}] - W(x)$ with $W(x) := \sup_{\theta\in\mathcal{T}_0} \mathbb{E}_x X_\theta$. Since $\alpha \geq 0$, $X_{t\wedge T_0}$ is a local supermartingale, bounded from below. By Fatou's lemma, this implies that $\mathbb{E}_x X_\theta \leq x$ for $\theta \leq T_0$.

We now provide necessary and sufficient conditions on the loss function ℓ which ensure that V is finite on $\mathbb{R}_+$. Recall that $V(0,z) = g(0,z) = \ell(z)$ is always finite.

PROPOSITION 2.1. *Assume that* $\alpha \geq 0$; *then* (iii) $\Longleftrightarrow$ (iii′) $\Longrightarrow$ (ii) $\Longrightarrow$ (i), *where*

(i) $V(x,z) < \infty$ *for every* $0 \leq x \leq z$,

(ii) $g(x,z) < \infty$ *for every* $0 \leq x \leq z$,

(iii) $\int^\infty \ell'(u-x)S(u)^{-1}du < \infty$ *for all* $x \geq 0$,

(iii)′ $\int^\infty \ell'(u)S(u)^{-1}du < \infty$.

If, in addition,

$$\sup_{u\geq z} \frac{\ell(u)}{\ell(u-x)} < \infty \quad \text{for every} \quad (x,z) \in \boldsymbol{\Delta}, \tag{2.13}$$

then all of the above items are equivalent to the following:

(i)′ $V(x_0,z_0) < \infty$ *for some* $0 < x_0 \leq z_0$,

(ii)′ $g(x_0,z_0) < \infty$ *for some* $0 < x_0 \leq z_0$.

The proof of this proposition, together with discussion of the conditions, is reported in section 9.1.

3. A verification result.

From now on, we assume

$$\int^\infty \ell'(u)S(u)^{-1}du < \infty, \tag{3.1}$$

so that, by Proposition 2.1, g and V are finite everywhere. Our general approach to solving the optimal detection problem is to exhibit a candidate solution for the corresponding dynamic programming equation,

$$\begin{aligned} &\max\{-Lv, v-g\} = 0 \text{ on } \mathrm{Int}(\boldsymbol{\Delta}), \text{ and} \\ &v(0,z) = \ell(z), \quad v_z(z,z) = 0 \quad \text{for} \quad z \geq 0, \end{aligned} \tag{3.2}$$

where L is the second order differential operator

$$Lv(x) = v''(x) - \alpha(x)v'(x), \tag{3.3}$$

and α is defined as in (2.2). Notice that $LS = 0$. We do not intend to prove directly that V satisfies this differential equation. Instead, we shall guess a candidate solution v of (3.2) and show that v indeed coincides with the value function V by a verification argument.

In order to exhibit a solution of (3.2), we guess that there should exist a free boundary $\gamma(x)$ so that stopping is optimal in the region $\{z \geq \gamma(x)\}$, while continuation is optimal in the remaining region $\{z < \gamma(x)\}$. If such a stopping boundary exists, then the above dynamic programming equation reduces to

$$Lv(x,z) = 0, \quad v(x,z) \le g(x,z) \quad \text{for} \quad 0 < z < \gamma(x), \tag{3.4}$$

$$v(x,z) = g(x,z), \quad Lg(x,z) \ge 0 \quad \text{for} \quad z \ge \gamma(x), \tag{3.5}$$

$$v(0,z) = \ell(z), \tag{3.6}$$

$$v_z(z,z) = 0. \tag{3.7}$$

The verification step requires that the value function be C^1 and piecewise C^2 in order to allow for the application of Itô's formula. We then complement the above system by the continuity and the smoothfit conditions

$$v(x,\gamma(x)) = g(x,\gamma(x)) \quad \text{and} \quad v_x(x,\gamma(x)) = g_x(x,\gamma(x)). \tag{3.8}$$

Our objective is to find a candidate v which satisfies (3.4)–(3.8) and an optimal stopping boundary γ so as to apply the following verification result.

THEOREM 3.1. *Assume that* (3.1) *holds true. Let γ be continuous and let v be a solution of* (3.4)–(3.8)*, which is $C^{1,0}$ and piecewise $C^{2,1}$ w.r.t. (x,z) on $\mathbf{\Delta}$, bounded from below, such that $v \le g$ on $\mathbf{\Delta}$ and $v < g$ on the continuation region $\{(x,z);\ 0 < x \le z$ and $z < \gamma(x)\}$.*

Then $v = V$ and $\theta^* = T_0 \wedge \inf\{t \ge 0;\ Z_t \ge \gamma(X_t)\}$ is an optimal stopping time.

Moreover if τ is another optimal stopping time, then $\theta^* \le \tau$ a.s.

Proof. (i) We first prove that $V \ge v$. Let $\theta \in \mathcal{T}_0$, and for $n \in \mathbb{N}$, define $\theta_n = n \wedge \theta \wedge \inf\{t \ge 0;\ |Z_t| \ge n\}$. Then from the assumed regularity of v, we may apply Itô's formula to obtain:

$$\begin{aligned} v(x,z) = v(X_{\theta_n}, Z_{\theta_n}) &- \int_0^{\theta_n} \frac{\sigma(X_t)^2}{2} Lv(X_t,Z_t)dt \\ &- \int_0^{\theta_n} v_x(X_t,Z_t)\sigma(X_t)dW_t - \int_0^{\theta_n} v_z(X_t,Z_t)dZ_t. \end{aligned}$$

Using the fact that $v_z(X_t,Z_t)dZ_t = v_z(Z_t,Z_t)dZ_t = 0$, $Lv \ge 0$, and $v \le g$, this implies

$$\begin{aligned} v(x,z) &\le \mathbb{E}_{x,z} v(X_{\theta_n}, Z_{\theta_n}) \\ &\le \mathbb{E}_{x,z} g(X_{\theta_n}, Z_{\theta_n}) = \mathbb{E}_{x,z}[\mathbb{E}_{X_{\theta_n},Z_{\theta_n}} \ell(Z_{T_0} - X_{\theta_n})] = \mathbb{E}_{x,z} \ell(Z_{T_0} - X_{\theta_n}). \end{aligned} \tag{3.9}$$

Clearly, as $n \to \infty$, $\theta_n \to \theta$ a.s. Notice that $0 \le \ell(Z_{T_0} - X_{\theta_n}) \le \ell(Z_{T_0}) \in \mathbb{L}^1(\mathbb{P})$ by (3.1). Then it follows from the dominated convergence theorem that

$$v(x,z) \le \mathbb{E}_{x,z} \ell(Z_{T_0} - X_\theta) \quad \text{for all} \quad \theta \in \mathcal{T}_0.$$

(ii) That $V \le v$ for $z \ge \gamma(x)$ is immediate. We now prove that $V \le v$ for $z < \gamma(x)$. Let $\theta^* = T_0 \wedge \inf\{t \ge 0;\ Z_t \ge \gamma(X_t)\}$. By the assumed regularity of v, we have $Lv(X_t,Z_t) = 0$ for $t \in [0,\theta^*)$, and by the same calculation as in (i), we see that

$$v(x,z) = \mathbb{E}_{x,z} v(X_{\theta_n}, Z_{\theta_n}) \quad \text{with} \quad \theta_n = n \wedge \theta^* \wedge \inf\{t \ge 0;\ |Z_t| \ge n\}. \tag{3.10}$$

Since v is bounded from below and $v \le g$, we have $|v| \le c + g$ for some constant c. Since $0 \le \ell(Z_{T_0} - X_{\theta_n}) \le \ell(Z_{T_0}) \in \mathbb{L}^1(\mathbb{P})$ by (3.1), the sequence $(\mathbb{E}[\ell(Z_{T_0})|X_{\theta_n}, Z_{\theta_n}])_n$ is uniformly integrable. This property is then inherited by the sequences $(g(X_{\theta_n}, Z_{\theta_n}))_n$ and $(v(X_{\theta_n}, Z_{\theta_n}))_n$. Then, sending $n \to \infty$ in (3.10), it follows from the continuity of γ that

$$\begin{aligned} v(x,z) \ = \ \mathbb{E}_{x,z} v(X_{\theta^*}, Z_{\theta^*}) &= \mathbb{E}_{x,z}\big[v(X_{\theta^*}, \gamma(X_{\theta^*}))\mathbf{1}_{\{\theta^* < T_0\}} + v(0, Z_{T_0})\mathbf{1}_{\{\theta^* = T_0\}}\big] \\ &= \mathbb{E}_{x,z}\big[g(X_{\theta^*}, \gamma(X_{\theta^*}))\mathbf{1}_{\{\theta^* < T_0\}} + \ell(Z_{T_0})\mathbf{1}_{\{\theta^* = T_0\}}\big] \\ &= \mathbb{E}_{x,z}\big[\ell(Z_{T_0} - X_{\theta^*})\big] \ \ge \ V(x,z). \end{aligned}$$

(iii) Finally we show the minimality of θ^*. Assume to the contrary that there exists τ satisfying $\mathbb{P}[\tau < \theta^*] > 0$ and $\mathbb{E}_{x,z}\ell(Z_{T_0} - X_\tau) = \inf_\theta \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta) = V(x,z)$.

On $\{\tau < \theta^*\}$, we have by assumption $V(X_\tau, Z_\tau) < g(X_\tau, Z_\tau)$, while we always have $V(X_\tau, Z_\tau) \le g(X_\tau, Z_\tau)$. This leads to the following contradiction:

$$V(x,z) = \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\tau) = \mathbb{E}_{x,z} g(X_\tau, Z_\tau) > \mathbb{E}_{x,z} V(X_\tau, Z_\tau) \ge V(x,z),$$

where the last inequality follows immediately from the definition of V. ∎

In the rest of this paper, our objective is to exhibit functions γ and v satisfying the assumptions of the previous theorem. For the quadratic loss function, this is the content of our main theorem, Theorem 6.1. In view of (3.5), the stopping region satisfies

$$\{(x,z): \; z \ge \gamma(x)\} \subset \Gamma^+ := \{(x,z): \; Lg(x,z) \ge 0\}. \tag{3.11}$$

We therefore need to study the structure of the set Γ^+.

In the subsequent sections we shall first focus on quadratic loss functions. For general loss functions, we shall provide some conditions which guarantee that the structure of the solution agrees with that of the quadratic case; see section 8.

4. The set Γ^+ for a quadratic loss function. Throughout this section as well as sections 5 and 6, we consider the quadratic loss function

$$\ell(x) := \frac{1}{2}\, x^2 \quad \text{for} \quad x \ge 0,$$

and we assume that the coefficient α satisfies the following additional condition:

$$\text{either } \exists K \ge 0, \text{ for } x \ge K, \; \alpha'(x) = 0, \text{ or, as } x \to \infty, \; \alpha''(x) = \circ\left([\alpha^2]'(x)\right). \tag{4.1}$$

Since α is positive on $(0,\infty)$ by (2.2), we immediately check that (3.1) holds true, so that g and V are finite on $\mathbf{\Delta}$. In order to study the set Γ^+ defined by (3.11), we compute that

$$Lg(x,z) = 1 + \alpha(x)(z-x) - \left(2S'(x) - \alpha(x)S(x)\right)\int_z^\infty \frac{du}{S(u)}, \quad (x,z) \in \mathbf{\Delta}, \tag{4.2}$$

which takes values in $\mathbb{R} \cup \{-\infty\}$. Since $\alpha \ge 0$ and S is increasing, $\frac{\partial}{\partial z} Lg(x,z) \ge 2\frac{S'(x)}{S(z)} > 0$, so it follows that for every fixed $x \ge 0$, the function $z \longmapsto Lg(x,z)$ is strictly increasing on $[x,\infty)$. Now since $\int_z^\infty \frac{du}{S(u)} \to 0$ when $z \to \infty$, we see that $\lim_{z\to\infty} Lg(x,z) > 0$ for any $x \ge 0$. This shows that $\Gamma^+ \neq \emptyset$ and that $\Gamma^+ = \mathrm{Epi}(\Gamma) := \{(x,z) \in \mathbf{\Delta};\; z \ge \Gamma(x)\}$, where

$$\Gamma(x) := \inf\{z \ge x: \; Lg(x,z) \ge 0\}. \tag{4.3}$$

Moreover, $\Gamma^+ \setminus graph(\Gamma) = \mathrm{Int}(\Gamma^+) \subset \{(x,z) \in \mathbf{\Delta};\; Lg(x,z) > 0\}$ and Γ is continuous. Let

$$\Gamma^0 := \Gamma(0) \text{ and } \Gamma^\infty := \sup\{x > 0, \; Lg(x,x) < 0\} \in (0, +\infty]. \tag{4.4}$$

By direct computation, we see that for $x > 0$,

$$\frac{\partial^2}{\partial x^2} Lg(x,z) = -2\alpha'(x) + \alpha''(x)(z-x) - (\alpha^2(x)S'(x) - \alpha''(x)S(x))\int_z^\infty \frac{du}{S(u)} \; < \; 0$$

by the concavity, the nondecrease, and the positivity of α on $(0,\infty)$. This implies that the function Γ is U-shaped in the sense of Proposition 4.2(i). We first isolate some asymptotic results that will be needed.

Proposition 4.1. *Under* (2.2)*, we have the following asymptotics, as $z \to \infty$:*

$$\text{(i)}\ \ S(z) \sim \frac{S'(z)}{\alpha(z)};$$

$$\text{(ii)}\ \ \int_z^\infty \frac{du}{S(u)} \sim \frac{1}{S'(z)}, \quad \int_z^\infty \frac{u}{S(u)}du \sim \frac{z}{S'(z)}, \quad \textit{and} \int_z^\infty \frac{u-z}{S(u)} \sim \frac{1}{\alpha(z)S'(z)}.$$

Proof. See section 9.2. ∎

Proposition 4.2. *Under conditions* (2.2)*, we have the following:*

(i) Γ *is decreasing on* $[0, \zeta]$ *and increasing on* $[\zeta, +\infty)$ *for some constant* $\zeta \geq 0$;

(ii) $\lim_{x\to+\infty} \Gamma(x) - x = 0$;

(iii) $0 < \Gamma^0 < \Gamma^\infty$*, where* Γ^0 *and* Γ^∞ *are as defined in* (4.4)*.*

Proof. (i) We first show that for $x_1 < x_3$, $\lambda \in (0,1)$, and $x_2 = \lambda x_1 + (1-\lambda)x_3$, we have $\Gamma(x_2) < \max(\Gamma(x_1), \Gamma(x_3))$. Assume to the contrary that $\Gamma(x_2) \geq \max(\Gamma(x_1), \Gamma(x_3))$; then from the strict concavity of Lg w.r.t. x and its nondecrease w.r.t. z, we see that

$$\begin{aligned} Lg(x_2, \Gamma(x_2)) &> \lambda Lg(x_1, \Gamma(x_2)) + (1-\lambda)Lg(x_3, \Gamma(x_2)) \\ &\geq \lambda Lg(x_1, \Gamma(x_1)) + (1-\lambda)Lg(x_3, \Gamma(x_3)) \geq 0, \end{aligned}$$

By the continuity of Lg, $Lg(x_2, \Gamma(x_2)) > 0$ implies that $\Gamma(x_2) = x_2$, which is in contradiction with $\Gamma(x_2) \geq \Gamma(x_3) \geq x_3 > x_2$. Since $\Gamma(x) \geq x$, this implies (i).

(ii) For an arbitrary $a > 0$, it follows from Proposition 4.1 that

$$Lg(z-a, z) = 1 + a\alpha(z-a) - e^{-\int_{z-a}^z \alpha(u)du} + \circ(1)$$

where $\circ(1) \to 0$ as $z \to \infty$. Notice that $\lim_{z\to\infty} e^{-\int_{z-a}^z \alpha(u)du} < 1$. Then $Lg(z-a, z) > 0$ for z large enough, and therefore $0 \leq \Gamma(z) - z < a$.

(iii) To see that $\Gamma^0 = \Gamma(0) > 0$, we first observe that $S(x) \sim x$ as $x \to 0$ implies that $\int_0^\infty \frac{du}{S(u)} = \infty$, and therefore $Lg(x, z) < 0$ on $\mathbf{\Delta}$ for z sufficiently small. In particular, for sufficiently small $z > 0$ we have $Lg(0, z) < 0$. Then $\Gamma^0 > 0$, and by continuity of Lg, $Lg(0, \Gamma^0) = 0$. Using Remark 2.2(ii) and the fact that $Lg(0, \Gamma^0) = 0$, we compute

$$\begin{aligned} Lg(\Gamma^0, \Gamma^0) &= 1 - (2S' - \alpha S)(\Gamma^0) \int_{\Gamma^0}^\infty \frac{du}{S(u)} \\ &< 1 - 2\int_{\Gamma^0}^\infty \frac{du}{S(u)} \ = \ Lg(0, \Gamma^0) - \alpha(0)\Gamma^0 \ = \ -\alpha(0)\Gamma^0 \leq 0. \end{aligned}$$

By continuity of Lg, this implies that $\Gamma^\infty > \Gamma^0$. ∎

Remark 4.1. If $z \leq \Gamma^0$, then $Lg(0, z) \leq 0$, and therefore by adapting the proof of Proposition 4.2(iii), we see that $Lg(z, z) < 0$ for $z \leq \Gamma^0$.

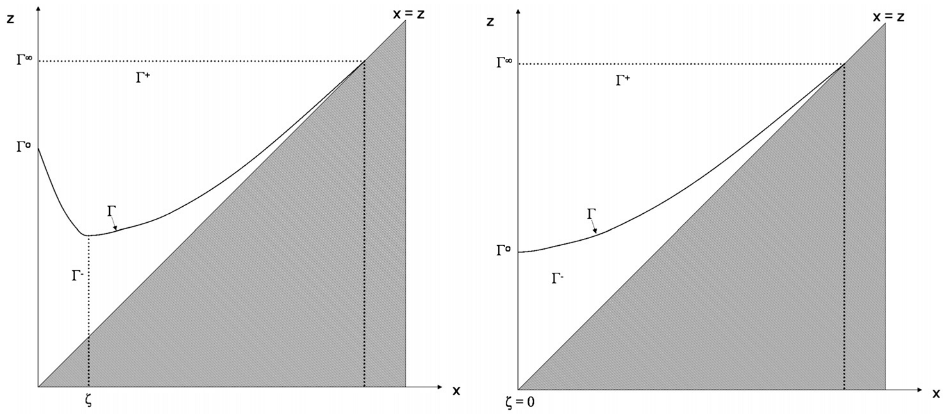
Remark 4.2. The fact that $\Gamma^0 < \Gamma^\infty$ implies, in the quadratic case, that the increasing part of Γ will never be reduced to a subset of the diagonal, or, in other words, that $\Gamma(\zeta) > \zeta$.

Figures 1(a) and 1(b) exhibit the two possible shapes of the function Γ and the location of Γ^+. Notice that in both cases, Γ^∞ can be finite or infinite. We refer the reader to section 7 for examples of both cases.

We now give a result, stronger than Proposition 4.2(ii) above, concerning the behavior of Γ at infinity. Recall that Γ^∞ was defined by (4.4).

Proposition 4.3. *Let the coefficient* α *satisfy conditions* (2.2) *and* (4.1)*. Then*

(i) *there exists* $\Gamma^{\max} > 0$ *such that*



(a) $\zeta > 0$ (Ornstein–Uhlenbeck process) (b) $\zeta = 0$ (Brownian motion with negative drift)

FIG. 1. *The two possible shapes of* Γ.

- *either for any* $x \geq \Gamma^{\max}$, $\Gamma(x) > x$,
- *or for any* $x \geq \Gamma^{\max}$, $\Gamma(x) = x$;

(ii) *if in addition* $\lim_{x\to\infty} \alpha(x) = \infty$, *then* $\Gamma^\infty < \infty$.

Proof. (i) By the definition of the scale function (2.3), for $x > 0$,

$$S(x) = A(x) + \frac{S'(x)}{\alpha(x)},$$
$$\text{where } A(x) := S(1) - \frac{S'(1)}{\alpha(1)} - \int_1^x \left(\frac{1}{\alpha}\right)'(u)S'(u)du. \tag{4.5}$$

Since A is nondecreasing, we may define $A^\infty := \lim_{x\to\infty} A(x) \in \mathbb{R} \cup \{+\infty\}$.

Case 1: $A^\infty < \infty$. We first observe that

$$\alpha = \alpha^\infty \quad \text{and} \quad A = A^\infty \quad \text{are constant on} \quad [K, \infty) \quad \text{for some} \quad K \geq 0. \tag{4.6}$$

By Condition (4.1), we need only verify this under the condition $\alpha'' = \circ\big((\alpha^2)'\big)$. By (2.2) and Remark 2.2(i), α is concave and nondecreasing. Then, if (4.6) does not hold, it follows that $\alpha'(x) > 0$ for all x, and we compute that $A'' = \big(\alpha\alpha'' + \alpha^2\alpha' - 2(\alpha')^2\big)S'/(\alpha^3) \sim \alpha'S'/\alpha > 0$ by the fact that $\alpha'' = \circ\big((\alpha^2)'\big)$ and by Remark 2.2(ii). This implies that A is nondecreasing and strictly convex for large x, which is in contradiction with $A^\infty < \infty$.

Since $S(x) \to \infty$ as $x \to \infty$, (4.5) implies that $\lim_{x\to\infty} \frac{\alpha(x)}{S'(x)} = 0$. Then, for $x \geq K$, we compute, from (4.5) and the fact that $S'' = \alpha S'$,

$$\int_x^\infty \frac{du}{S(u)} = \int_x^\infty \frac{du}{A^\infty + (S'/\alpha)(u)} = \int_x^\infty (S''/(S')^2)(u)\frac{du}{1 + A^\infty(S''/(S')^2)(u)}.$$

By a Taylor expansion, together with the boundedness of $\alpha/S' = S''/(S')^2$ on $[x, \infty)$, this provides

$$\int_x^\infty \frac{du}{S(u)} = \frac{1}{S'(x)^2}\left(S'(x) - \frac{1}{2}A^\infty\alpha^\infty + A^\infty\alpha^\infty \circ(1)\right).$$

By Proposition 4.1(i), this provides

$$Lg(x,x) = 1 - \big(2S'(x) - \alpha(x)S(x)\big)\int_x^\infty \frac{du}{S(u)} = \frac{A^\infty\alpha^\infty}{2S'(x)}\big(1 + \circ(1)\big).$$

By the definition of the function Γ, this implies that

$$\Gamma(x) = x \quad \text{whenever} \quad A^\infty \geq 0, \quad \text{and} \quad \Gamma(x) > x \quad \text{whenever} \quad A^\infty < 0.$$

Case 2: $A^\infty = \infty$. In this case, $\alpha'(x) > 0$ for all $x \geq 0$.

Set $\beta := 1/\alpha$. Since $S'' = \alpha S$, it follows from an integration by parts that

$$\begin{aligned}\int_1^x \beta'(u)S'(u)du &= \int_1^x (\beta\beta')(u)\alpha(u)S'(u)du \\ &= [(\beta\beta')(u)S'(u)]_1^x - \int_1^x \big(\beta\beta'\big)'(u)S'(u)du.\end{aligned}$$

By Remark 2.2 and (4.1), we observe that $(\beta\beta')' = \circ(\beta')$, and therefore

$$\int_1^x \beta'(u)S'(u)du = \beta(x)\beta'(x)S'(x) + \circ\left(\beta(x)\beta'(x)S'(x)\right).$$

Plugging this into (4.5), we see that

$$S(x) = \beta(x)S'(x)\Big(1 - \beta'(x) + \circ\big(\beta'(x)\big)\Big).$$

By a Taylor expansion, together with $\beta'(x) \to 0$ and $\alpha = S''/S'$, this implies that

$$\int_x^\infty \frac{du}{S(u)} = \int_x^\infty \frac{S''}{(S')^2}\big(1 + \beta' + \circ(\beta')\big) = \frac{1}{S'(x)} + \int_x^\infty \frac{S''}{(S')^2}\beta'\big(1 + \circ(1)\big).$$

Integrating by parts and using (4.1) together with the observation that $\beta'' = \circ(\alpha\beta')$, we also compute

$$\int_x^\infty \frac{S''}{(S')^2}\beta' = \frac{\beta'(x)}{S'(x)} + \int_x^\infty \frac{\beta''}{S'(u)} = \frac{\beta'(x)}{S'(x)} + \circ\left(\frac{\beta'(x)}{S'(x)}\right).$$

Hence,

$$\begin{aligned}Lg(x,x) &= 1 - \big(2S'(x) - \alpha(x)S(x)\big)\int_x^\infty \frac{du}{S(u)} \\ &= 1 - \Big(1 + \beta'(x) + \circ\big(\beta'(x)\big)\Big)\Big(1 + \beta'(x) + \circ\big(\beta'(x)\big)\Big) \; = \; -2\beta'(x) + \circ\big(\beta'(x)\big).\end{aligned}$$

Since $\beta' = (1/\alpha)' < 0$, this implies that for large x, $Lg(x,x) > 0$, and therefore $\Gamma(x) = x$.

(ii) We now assume that $\lim_{x\to\infty} \alpha(x) = \infty$, and we intend to prove that $A^\infty = \infty$, which would imply that $\Gamma^\infty < \infty$ by Case 2 above. Let $x \geq 1$. Since α is nondecreasing, we have

$$S'(x) = e^{\int_0^x \alpha(u)du} \geq e^{\int_{x-1}^x \alpha(u)du} \geq e^{\alpha(x-1)}.$$

Since α' is nonincreasing and nonnegative, α' is bounded on $[1,\infty)$. Therefore, there exists $K > 0$ such that $0 \leq \alpha(x) - \alpha(x-1) \leq K$, so that $S'(x) \geq e^{\alpha(x)-K}$ for $x \geq 1$.

On the other hand, $\lim_{x\to\infty} \alpha(x) = \infty$ implies that $\alpha(x)^2 = \circ(e^{\alpha(x)-K})$, which means that $\frac{S'(x)}{\alpha(x)^2} \to \infty$. Finally, as $x \to \infty$, we get

$$\alpha'(x) = \circ\big(-\beta'(x)S'(x)\big).$$

Since α is not bounded, the left-hand side is not integrable at infinity, so the right-hand side is also not integrable. In other words, $\int_1^\infty (1/\alpha)'(u)S'(u)du = -A^\infty = -\infty$. $\square$

5. The stopping boundary in the quadratic case. We now turn to the characterization of the stopping boundary γ. Following Proposition 4.2(i), we define

$$\Gamma_{\downarrow} = \Gamma_{|_{[0,\zeta]}} \quad \text{and} \quad \Gamma_{\uparrow} = \Gamma_{|_{[\zeta,\infty)}}$$

as the restrictions of Γ to the intervals $[0,\zeta]$ and $[\zeta,\infty)$.

5.1. The increasing part of the stopping boundary $\boldsymbol{\gamma}$. Our objective is to find a solution v of (3.4)–(3.8) on $\{(x,z) \in \mathbf{\Delta};\ x < z < \gamma(x)\}$. First, by (3.4), v is of the form

$$v(x,z) = A(z) + B(z)S(x).$$

We first guess that the free boundary γ is continuous and increasing near the diagonal. Then, denoting its inverse by γ^{-1}, the continuity and smoothfit conditions (3.8) imply that

$$v(x,z) = g(\gamma^{-1}(z),z) + \frac{g_x(\gamma^{-1}(z),z)}{S'(\gamma^{-1}(z))}[S(x) - S(\gamma^{-1}(z))].$$

Finally, the Neumann condition (3.7), together with (2.10) and the specific form of the loss function ℓ, implies that the boundary γ satisfies the following ODE:

$$\gamma' = \frac{Lg(x,\gamma)}{1 - \frac{S(x)}{S(\gamma)}}. \tag{5.1}$$

In what follows, we take this ODE (with no initial condition!) as a starting point to construct the boundary γ. Notice that this ODE has infinitely many solutions, as the Cauchy–Lipschitz condition is locally satisfied whenever (5.1) is complemented with the condition $\gamma(x_0) = z_0$ for any $0 < x_0 < z_0$. This feature is similar to that in Peskir [11]. The following result selects an appropriate solution of (5.1).

PROPOSITION 5.1. *Let the coefficient α satisfy conditions* (2.2) *and* (4.1). *Then, there exists a continuous function γ defined on $\mathbb{R}_+$ with graph $\{(x,\gamma(x)) : x \in \mathbb{R}_+\} \subset \mathbf{\Delta}$ such that*

(i) *on the set $\{x > 0:\ \gamma(x) > x\}$, γ is a C^1 solution of ODE* (5.1);

(ii) *$\{(x,\gamma(x)):\ x \geq \zeta\} \subset \Gamma^+$, and $\{(x,\gamma(x)):\ x > \zeta$ and $\gamma(x) > x\} \subset \mathrm{Int}(\Gamma^+)$;*

(iii) *if $\Gamma^\infty < \infty$, then $\gamma(x) = x$ for all $x \geq \Gamma^\infty$;*

(iv) *$\gamma(x) - x \longrightarrow 0$ as $x \to \infty$.*

The remaining part of this section is dedicated to the proof of this result. We first introduce some notation. We recall from Remark 4.2 that the graph of $\Gamma_\uparrow$ is not reduced to the diagonal, and therefore

$$b := \inf\{x \geq 0:\ \Gamma(x) = x\} \ \in \ (\zeta,\infty], \tag{5.2}$$

where b may take infinite value. We also introduce

$$\mathbf{D}^- := \{x > \zeta:\ Lg(x,x) < 0\} \supset (\zeta,b), \tag{5.3}$$

and for all $x_0 \in \mathbf{D}^-$,

$$d(x_0) := \sup\{x \leq x_0:\ Lg(x,x) \geq 0\} \quad \text{and} \qquad u(x_0) := \inf\{x \geq x_0;\ Lg(x,x) \geq 0\}, \tag{5.4}$$

with the convention that $d(x_0) = 0$ if $\{x \le x_0 :\ Lg(x, x) \ge 0\} = \emptyset$ and $u(x_0) = \infty$ if $\{x \ge x_0 :\ Lg(x, x) \ge 0\} = \emptyset$. Since Lg is continuous and $x_0 \in \mathbf{D}^-$, $d(x_0) < x_0 < u(x_0) \le \infty$. Notice that if $x_0 \in (\zeta, b)$, then $d(x_0) = 0$.

Let $x_0 \in \mathbf{D}^-$ be an arbitrary point. For all $z_0 > x_0$, we denote by $\gamma_{x_0}^{z_0}$ the maximal solution of the Cauchy problem complemented with the condition $\gamma(x_0) = z_0$, and we denote by $\big(\ell_{x_0}^{z_0}, r_{x_0}^{z_0}\big)$ the associated (open) interval. Notice that since the right-hand side of ODE (5.1) is locally Lipschitz on the set $\{(x, \gamma),\ 0 < x < \gamma\}$, the maximal solution will be defined as long as $0 < x < \gamma$.

The following result provides more properties on the maximal solutions.

LEMMA 5.2. *Assume that α satisfies conditions* (2.2) *and let $x_0 \in \mathbf{D}^-$ be fixed.*

(i) *For all $z > x_0$, $\ell_{x_0}^z \le d(x_0)$, and if $\ell_{x_0}^z > 0$, then $Lg(\ell_{x_0}^z, \ell_{x_0}^z) \ge 0$;*

(ii) *for all $z \in (x_0, \Gamma(x_0)]$, $Lg(x, \gamma_{x_0}^z(x)) < 0$ for any $x \in (x_0, r_{x_0}^z)$;*

(iii) *for z sufficiently large, we have $r_{x_0}^z = +\infty$.*

Before proceeding to the proof of this result, we turn to the main construction of the stopping boundary γ. Let

(5.5)
$$\mathbf{Z}(x_0) := \big\{z > x_0 :\ Lg\big(x, \gamma_{x_0}^z(x)\big) < 0 \text{ for some } x \in \big[x_0, r_{x_0}^z\big)\big\},\ z^*(x_0) := \sup \mathbf{Z}(x_0).$$

Moreover, whenever $z^*(x_0) < \infty$, we denote

$$\gamma_{x_0}^* := \gamma_{x_0}^{z^*(x_0)}, \quad \ell_{x_0}^* := \ell_{x_0}^{z^*(x_0)}, \quad r_{x_0}^* := r_{x_0}^{z^*(x_0)}, \quad \text{and} \quad I_{x_0}^* := \big(\ell_{x_0}^*, r_{x_0}^*\big). \tag{5.6}$$

LEMMA 5.3. *Assume that α satisfies conditions* (2.2)*, and let x_0 be arbitrary in $\mathbf{D}^-$. Then*

(i) *$z^*(x_0) \in (\Gamma(x_0), \infty)$, $\big(d(x_0), u(x_0)\big) \subset I_{x_0}^*$, and the corresponding maximal solution $\gamma_{x_0}^*$ has a positive derivative on the interval $I_{x_0}^* \cap (\zeta, \infty)$.*

(ii) *For $x_0, x_1 \in \mathbf{D}^-$, we have*
- *either $I_{x_0}^* \cap I_{x_1}^* = \emptyset$,*
- *or $I_{x_0}^* = I_{x_1}^*$ and $\gamma_{x_0}^* = \gamma_{x_1}^*$.*

Proof. (i) By Lemma 5.2(iii), there exists $a = a(x_0)$ such that for any $z \ge a$, $r_{x_0}^z = \infty$. If $Lg\big(x_1, \gamma_{x_0}^z(x_1)\big) < 0$ for some $x_1 \ge x_0$, then by (5.1), $\gamma_{x_0}^z$ is decreasing in a neighborhood of x_1 and as long as $\big(x, \gamma_{x_0}^z(x)\big) \in \mathrm{Int}(\Gamma^-)$. Since $x_1 \ge x_0 > \zeta$, Γ is increasing on $[x_1, \infty)$ so that $\gamma_{x_0}^z$ is decreasing on $[x_1, r_{x_0}^z)$, which implies that $r_{x_0}^z < \infty$. Therefore $\mathbf{Z}(x_0)$ is bounded by a, and $z^*(x_0) < \infty$. Since $x_0 \in \mathbf{D}^-$, we have $\Gamma(x_0) \in \mathbf{Z}(x_0)$, and therefore $z^*(x_0) \ge \Gamma(x_0)$. We next assume to the contrary that $z^*(x_0) = \Gamma(x_0)$ and work toward a contradiction.

Notice that $\mathbf{D}^-$ is an open set as a consequence of the continuity of the function Lg. Then there exists $\varepsilon > 0$ such that $(x_0, x_0 + 2\varepsilon) \subset \mathbf{D}^- \cap (x_0, r_{x_0}^*)$ and $d(x) = d(x_0)$ for any $x \in (x_0, x_0 + \varepsilon)$. Let $x_\varepsilon := x_0 + \varepsilon$ and $z_\varepsilon := \Gamma(x_\varepsilon) > \Gamma(x_0)$. By Lemma 5.2(ii), $\gamma_{x_0}^*$ is decreasing on $(x_0, r_{x_0}^*)$ so that $\gamma_{x_0}^*(x_\varepsilon) < \gamma_{x_0}^*(x_0) = \Gamma(x_0) < \Gamma(x_\varepsilon) = \gamma_{x_\varepsilon}^{z_\varepsilon}(x_\varepsilon)$. Since by Lemma 5.2(i) we have $\ell_{x_\varepsilon}^{z^\varepsilon} \le d(x_0) < x_0$, together with the one-to-one property of the flow, it follows that $z^* = \gamma_{x_0}^*(x_0) < \gamma_{x_\varepsilon}^{z_\varepsilon}(x_0)$. Therefore, by Lemma 5.2(ii), $\gamma_{x_\varepsilon}^{z_\varepsilon}(x_0) \in \mathbf{Z}(x_0)$, which contradicts the maximality of z^*.

A similar argument proves that $(x, \gamma_{x_0}^*(x)) \in \mathrm{Int}(\Gamma^+)$ for $x \in I_{x_0}^* \cap [\zeta, \infty)$, which implies $r_{x_0}^* \ge u(x_0)$ (possibly infinite). Using (5.1), we see that $\gamma_{x_0}^*$ has a positive derivative on the same interval. Finally, Lemma 5.2(i) implies that $\ell_{x_0}^* \le d(x_0)$.

(ii) Let $x_0 < x_1$ in $\mathbf{D}^-$, and assume that there exists $x_2 \in I_{x_0}^* \cap I_{x_1}^*$. We prove below that $\gamma_{x_0}^*(x_2) = \gamma_{x_1}^*(x_2)$. Then the one-to-one property of the flow and the maximality of I^* imply that $I_{x_0}^* = I_{x_1}^*$ and $\gamma_{x_0}^* = \gamma_{x_1}^*$.

To see that $\gamma_{x_0}^*(x_2) = \gamma_{x_1}^*(x_2)$, first assume to the contrary that $\gamma_{x_0}^*(x_2) < \gamma_{x_1}^*(x_2)$. Then, the one-to-one property of the flow implies that $\gamma_{x_0}^* < \gamma_{x_1}^*$ on $I_{x_0}^* \cap I_{x_1}^*$, and therefore the maximality of $I^*_{x_1}$ implies that $I^*_{x_0} \subset I^*_{x_1}$. By the definition of $z^*(x_1)$ and the continuity of the flow with respect to initial data, there exists $z < z^*(x_1)$ such that $\gamma^*_{x_0}(x_2) < \gamma^z_{x_1}(x_2) < \gamma^*_{x_1}(x_2)$ and $z \in \mathbf{Z}(x_1)$. For the same reasons as before, we have $I^*_{x_0} \subset (\ell^z_{x_1}, r^z_{x_1})$ and $\gamma^*_{x_0} < \gamma^z_{x_1} < \gamma^*_{x_1}$ on $I^*_{x_0}$. Therefore $\gamma^z_{x_1}(x_0) \in \mathbf{Z}(x_0)$, while $\gamma^z_{x_1}(x_0) > z^*(x_0) = \gamma^*_{x_0}(x_0)$, which is impossible. A similar argument can be used if $\gamma^*_{x_0}(x_2) > \gamma^*_{x_1}(x_2)$. ∎

We are now ready for the following proof.

Proof of Proposition 5.1. We first define γ and then prove the announced properties.

1. Let

$$\mathcal{D} := \bigcup_{x_0 \in \mathbf{D}^-} I^*_{x_0} \supset \mathbf{D}^-. \tag{5.7}$$

By Lemma 5.3, for any x and y in $\mathbf{D}^-$, we have either $I^*_x = I^*_y$ or $I^*_x \cap I^*_y = \emptyset$. Therefore, there exists a subset $\mathbf{D}^-_0$ of $\mathbf{D}^-$ such that $\mathcal{D} = \bigcup_{x_0 \in \mathbf{D}^-_0} I^*_{x_0}$, and for any $x, y \in \mathbf{D}^-_0$, $x \neq y$ implies that $I^*_x \cap I^*_y = \emptyset$.

We now define the function γ on $\mathbb{R}_+ \setminus \{0\}$ by

$$\gamma(x) := \begin{cases} \gamma^*_{x_0}(x) \text{ if } x \in I^*_{x_0}, \text{ for some } x_0 \in \mathbf{D}^-_0, \\ x \text{ otherwise.} \end{cases} \tag{5.8}$$

By Lemma 5.3, this definition does not depend on the choice of $\mathbf{D}^-_0$.

2. We first prove that γ is continuous on $\mathbb{R}_+$. This is nontrivial only at the endpoints $\ell^*_{x_0}$ and $r^*_{x_0}$, $x_0 \in \mathbf{D}^-$. Recalling that γ is increasing on $I^*_{x_0}$, we see that both limits exist. By the maximality of $I^*(x_0)$, it is immediate that $\lim_{r^*_{x_0}} \gamma = r^*_{x_0}$ and, whenever $\ell^*_{x_0} > 0$, $\lim_{\ell^*_{x_0}} \gamma = \ell^*_{x_0}$. If $\ell^*_{x_0} = 0$, which is the case for $x_0 \in (\zeta, b)$, then the limit also exists and is in fact positive since $Lg(x, \gamma(x)) < 0$ for any $x > 0$ such that $\gamma(x) < \zeta$. Setting $\gamma(0) := \lim_{x \to 0} \gamma(x)$, we obtain a continuous function γ on $\mathbb{R}_+$.

3. Proposition 5.1(i) follows immediately from Lemma 5.3. To prove (ii), we first notice that $\{x \geq \zeta : \ \gamma(x) = x\} = \mathbb{R}_+ \setminus \mathcal{D} \subset \mathbb{R}_+ \setminus \mathbf{D}^-$ so that $Lg(x,x) \geq 0$ on the set $\{x \geq \zeta : \ \gamma(x) = x\}$. On the set $\{x > \zeta : \ \gamma(x) > x\}$, since γ has a positive derivative and satisfies (5.1), we have $Lg(x, \gamma(x)) > 0$. Finally, since for $x_0 \in (\zeta, b)$, where b was defined by (5.2), $d(x_0) = 0$, Lemma 5.3 and the continuity of Lg imply that $Lg(\zeta, \gamma(\zeta)) \geq 0$.

4. We next prove (iii). Assume $\Gamma^\infty < \infty$ and let $x_0 \in \mathbf{D}^-$ be arbitrary. Then by the continuity of Lg, $Lg(\Gamma^\infty, \Gamma^\infty) = 0$, and therefore $x_0 < \Gamma^\infty$. Assume that $r^*_{x_0} > \Gamma^\infty$, and let us work toward a contradiction. Then by continuity of the flow with respect to the initial data, there exists $\varepsilon > 0$ such that for any $z \in \big(z^*(x_0) - \varepsilon, z^*(x_0)\big)$, the function $\gamma^z_{x_0}$ is defined on $[x_0, \frac{\Gamma^\infty + r^*_{x_0}}{2}]$. By Lemma 5.2(ii), we deduce that $(x, \gamma^z_{x_0}(x)) \in \Gamma^+$ on the same interval. By the definition of Γ^∞ and recalling that $\frac{\partial}{\partial z} Lg > 0$, we get that $z \notin \mathbf{Z}(x_0)$. By the arbitrariness of z in $(z^*(x_0) - \varepsilon, z^*(x_0))$, this contradicts the definition of $z^*(x_0)$.

5. We finally prove (iv). First, the claim is obvious when $\mathcal{D}$ is bounded, as $\gamma(x) = x$ for $x \geq \sup \mathcal{D}$. We then concentrate on the case where $\mathcal{D}$ is not bounded. From Proposition 4.3, either $\mathbf{D}^-$ is bounded or $Lg(x,x) < 0$ for any $x \in [\Gamma^{\max}, \infty)$, and by Lemma 5.3, $r^*_{x_0} \geq u(x_0)$. In both cases, there exists $x_0 \in \mathbf{D}^-$ such that $r^*_{x_0} = \infty$. To complete the proof, we now intend to show that, for $a > 0$ and $x > x_0$ large enough, $\gamma(x) \leq x + a$.

Using Proposition 4.1, we compute

$$Lg(x, x+a) = 1 + a\alpha(x) - e^{-\int_x^{x+a} \alpha(u)du} + \circ(1). \tag{5.9}$$

- If $\lim_{x\to\infty} \alpha(x) = \infty$, then, for any $\varepsilon > 0$, we get that $Lg(x, x+a) > 1+\varepsilon$ for x large enough.

- If $\lim_{x\to\infty} \alpha(x) = M > 0$, then $\frac{Lg(x,x+a)}{1-\frac{S(x)}{S(x+a)}} = \frac{1-e^{-aM}+aM}{1-e^{-aM}} + \circ(1)$ so that for any $\varepsilon \in \left(0, \frac{aM}{1-e^{-aM}}\right)$, we get that $\frac{Lg(x,x+a)}{1-\frac{S(x)}{S(x+a)}} > 1+\varepsilon$ for x large enough.

In both cases, we can find a sufficiently small $\varepsilon > 0$ such that $\frac{Lg(x,x+a)}{1-\frac{S(x)}{S(x+a)}} > 1+\varepsilon$ for any sufficiently large x, say $x \geq x_1$. We now assume that $\gamma(x_1) > x_1 + a$ and work toward a contradiction. Since $\gamma(x) > x$ on $[x_0, +\infty)$, using the continuity of the flow with respect to the initial data, we can find $z < z^*(x_0)$ such that $\gamma^z_{x_0}(x) > x$ on $[x_0, x_1]$ and $\gamma^z_{x_0}(x_1) > x_1 + a$. Using (5.9) together with (5.1), we therefore have for $x \in [x_1, +\infty)$

$$\begin{aligned} &\gamma^z_{x_0}(x) - \gamma^z_{x_0}(x_1) \geq (1+\varepsilon)(x - x_1) \\ &\quad \text{and so } \gamma^z_{x_0}(x) > (1+\varepsilon)(x-x_1) + x_1 + a \geq x + a \end{aligned}$$

so that $r^z = \infty$, and the same holds for any $y \in [z, z^*(x_0)]$, which contradicts the definition of $z^*(x_0)$ as $\sup \mathbf{Z}(x_0)$. $\square$

We finally turn to the proof of Lemma 5.2. Let

$$\Gamma^- := \{(x,z) \in \mathbf{\Delta} : \; Lg(x,z) \leq 0\}. \tag{5.10}$$

Proof of Lemma 5.2(i). The right-hand side of (5.1) is locally Lipschitz as long as $0 < x < \gamma^z_{x_0}(x)$. Now $\gamma^z_{x_0}$ is nonincreasing if $(x, \gamma^z_{x_0}(x)) \in \Gamma^-$. Therefore, since $d(x_0) < x_0 < u(x_0)$ and $\Gamma(x) > x$ for any $x \in \mathbf{D}^- \supset \big(d(x_0), u(x_0)\big)$, the minimality of $\ell^z_{x_0}$ implies that $\ell^z_{x_0} \leq d(x_0)$ and that $\ell^z_{x_0} \notin \mathbf{D}^-$.

(ii) Since $x_0 > \zeta$, the function Γ is increasing on $[x_0, +\infty)$, while by (5.1), for any $z > x_0$, $\gamma^z_{x_0}$ is nonincreasing as long as $(x, \gamma_{x_0}(x)) \in \Gamma^-$. Therefore for any $z \in (x_0, \Gamma(x_0))$, $(x, \gamma^z_{x_0}(x))$ remains in $\mathrm{Int}(\Gamma^-)$ on $\big[x_0, r^z_{x_0}\big)$.

Assume now that $z = \Gamma(x_0)$. Since $\Gamma(x_0) > x_0$, Γ satisfies $Lg(x, \Gamma(x)) = 0$ in a neighborhood of x_0. Since $\frac{\partial}{\partial z} Lg > 0$ on $\mathbf{\Delta}$, while $\frac{\partial}{\partial x} Lg(x, \Gamma(x)) < 0$ as soon as $\Gamma(x) > x$, the implicit functions theorem implies that Γ is C^1 with positive derivative in a neighborhood of x_0. If $z = \Gamma(x_0)$, $(\gamma^z_{x_0})'(x_0) = 0$ by (5.1), and therefore $\gamma' - \Gamma'$ is negative in a neighborhood of x_0, and we can conclude as in the case $z < \Gamma(x_0)$ that $\big(x, \gamma^z_{x_0}(x)\big) \in \mathrm{Int}(\Gamma^-)$ on $\big(x_0, r^z_{x_0}\big)$.

(iii) Let $\varepsilon > 0$ be given. From Proposition 4.1(ii), we see that

$$\begin{aligned} Lg(x, (1+\varepsilon)x) &= 1 + \varepsilon x \alpha(x) - \frac{S'(x)}{S'((1+\varepsilon)x)} + \circ(1) \\ &= 1 + \varepsilon x \alpha(x) - e^{-\int_x^{x+\varepsilon x} \alpha(v)dv} + \circ(1) \quad \text{as } x \to \infty. \end{aligned}$$

Since $x\alpha(x) \to +\infty$, this implies that

$$Lg(., (1+\varepsilon).) \geq 1 + 3\varepsilon \text{ on } [A, \infty) \quad \text{for some } \; A \geq 0. \tag{5.11}$$

In particular, $(A, (1+\varepsilon)A) \in \mathrm{Int}(\Gamma^+)$. Let $D := \max((1+\varepsilon)A, \Gamma^0)$. Since Γ is U-shaped, it follows that $[0, A] \times [D, \infty) \subset \Gamma^+$. Since $\gamma^z_{x_0}$ is nondecreasing as long as

$(x, \gamma_{x_0}^z(x)) \in \mathrm{Int}(\Gamma^+)$, by (5.1) it follows that $r_{x_0}^z > A$ and $\gamma_{x_0}^z(A) > (1+\varepsilon)A$ for all $z \geq D$.

In order to complete the proof, we now show that

$$\gamma_{x_0}^z(x) \geq (1+\varepsilon)x \quad \text{for all } x \geq A \text{ and } z \geq D.$$

To see this, assume to the contrary that $\gamma_{x_0}^z(\xi) \leq (1+\varepsilon)\xi$ for some $\xi > A$ and define

$$x_1 := \inf\{x > A;\ \gamma_{x_0}^z(x) = (1+\varepsilon)x)\}.$$

By continuity of $\gamma_{x_0}^z$, we have $A < x_1 \leq \xi$, and therefore $Lg(x_1, (1+\varepsilon)x_1) \geq 1 + 3\varepsilon$ by (5.11). Since Lg is also continuous, there is a neighborhood $\mathcal{O}$ of $(x_1, (1+\varepsilon)x_1)$ such that for $(x, z) \in \mathcal{O}$, $Lg(x, z) \geq 1 + 2\varepsilon$. We then deduce that there exists $\eta > 0$ such that

$$(\gamma_{x_0}^z)'(x) \geq Lg(x, \gamma^z(x)) \geq 1 + 2\varepsilon \quad \text{for any} \;\; x \in [x_1 - \eta, x_1 + \eta],$$

and then, for $x \in (x_1 - \eta, x_1) \cap [A, \infty)$,

$$\gamma_{x_0}^z(x) \leq \gamma_{x_0}^z(x_1) - (1+2\varepsilon)(x_1 - x) = (1+\varepsilon)x_1 - (1+2\varepsilon)(x_1 - x) < (1+\varepsilon)x.$$

Since $\gamma_{x_0}^z(A) > (1+\varepsilon)A$, this contradicts the definition of x_1. $\square$

5.2. The decreasing part. The problem now is that there is no reason for the function γ constructed in the previous section to be entirely in Γ^+ since it can cross graph($\Gamma_\downarrow$). In section 7, numerical computations suggest that this is indeed the case in the context of an Ornstein–Uhlenbeck process. In fact, in general the boundary is made of two parts, as shown in Figure 2. Therefore we need to consider the area that lies between the axis $\{x = 0\}$ and graph(γ). While the previous part of γ is characterized by the ODE (5.1) because of the Neumann condition, here we must take into account the Dirichlet condition (3.6).

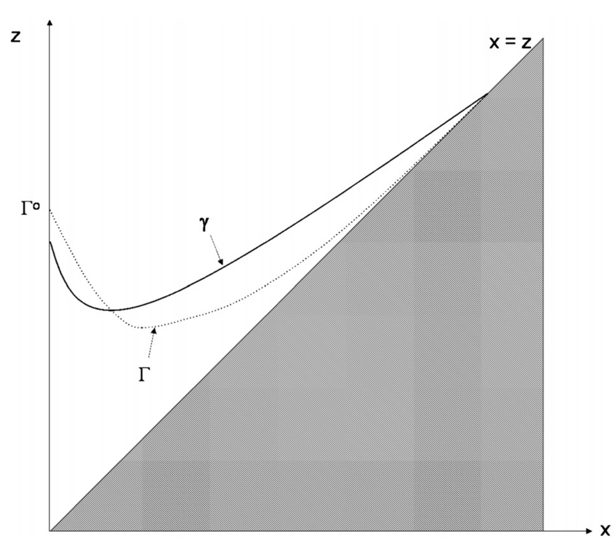


FIG. 2. *On the left part, the graph of γ is inside* $\mathrm{Int}(\Gamma^-)$ *and γ is decreasing.*

Therefore, we consider the following problem for a fixed $z > 0$: Find $x(z)$ such that $f\big(x(z), z\big) = 0$, where

$$f(x,z) := g(x,z) - g_x(x,z)\frac{S(x)}{S'(x)} - \frac{z^2}{2}. \tag{5.12}$$

Proposition 5.4. *Assume that α satisfies conditions* (2.2) *and that $\Gamma_\downarrow$ is not degenerate (i.e., $\zeta > 0$). Then there exists $x^* > 0$ and a function $\gamma_\downarrow$ defined on $[0, x^*]$, which is C^0 on $[0, x^*]$, C^1 with negative derivative on $(0, x^*)$, and such that*

(i) for all $x \in [0, x^*]$, $f(x, \gamma_\downarrow(x)) = 0$;
(ii) for all $x \in (0, x^*)$, $(x, \gamma_\downarrow(x)) \in \mathrm{Int}(\Gamma^+)$;
(iii) $\gamma_\downarrow(0) = \Gamma^0$;
(iv) $(x^*, \gamma_\downarrow(x^*)) \in graph(\Gamma_\uparrow)$.

Proof. By direct calculation and the fact that $LS = 0$, we have

$$\text{for all } (x,z) \in \mathbf{\Delta}, \quad \frac{\partial}{\partial x}\left(\frac{g_x(x,z)}{S'(x)}\right) = \frac{Lg(x,z)}{S'(x)}. \tag{5.13}$$

Then by direct differentiation of (5.12), we get

$$f_x(x,z) = g_x(x,z) - S(x)\frac{Lg(x,z)}{S'(x)} - g_x(x,z) = -S(x)\frac{Lg(x,z)}{S'(x)}.$$

Therefore $f_x(x, \Gamma^0) < 0$. Since for all z, $f(0,z) = 0$, we deduce that $f(x, \Gamma^0) < 0$ for all $x \in \big(0, \Gamma_\uparrow^{-1}(\Gamma^0)\big]$. On the other hand, if $z < \Gamma^0$, then $f(x,z) > 0$ for any $x \in (0, \Gamma_\downarrow^{-1}(z)]$, where $\Gamma_\downarrow^{-1}(z) > 0$.

By continuity of f, there exists $\varepsilon > 0$ and $x > 0$ such that for any $z \in \big(\Gamma^0 - \varepsilon, \Gamma^0\big]$, $f(x,z) < 0$. Therefore there exists $x \in \big(\Gamma_\downarrow^{-1}(z), \Gamma_\uparrow^{-1}(z)\big]$ satisfying $f(x,z) = 0$. Let z_0 be in such a neighborhood and let $x_0 \in \big(\Gamma_\downarrow^{-1}(z_0), \Gamma_\uparrow^{-1}(z_0)\big]$ satisfying $f(x_0, z_0) = 0$. By definition, $(x_0, z_0) \in \mathrm{Int}(\Gamma^+)$.

We consider now the following Cauchy problem:

$$\gamma_\downarrow'(x) = \frac{Lg(x, \gamma_\downarrow(x))S(x)}{S(x) - xS'(x) - \frac{(S(x))^2}{S(\gamma_\downarrow)}}, \tag{5.14}$$

with the additional condition $\gamma_\downarrow(x_0) = z_0$. ODE (5.14) is obtained by a formal derivation of the equation $f(x, \gamma(x)) = 0$. Indeed, assuming that γ is C^1, we see that

$$f_x(x, \gamma(x)) + \gamma'(x) f_z(x, \gamma(x)) = 0.$$

We compute

$$\begin{aligned}
f_z(x,z) &= g_z(x,z) - g_{xz}(x,z)\frac{S(x)}{S'(x)} - z \\
&= z - x - \frac{S(x)}{S(z)}(z-x) + \frac{S(x)}{S'(x)}\left(1 + \frac{S'(x)(z-x)}{S(z)} - \frac{S(x)}{S(z)}\right) - z \\
&= -x + \frac{S(x)}{S'(x)} - \frac{(S(x))^2}{S'(x)S(z)}.
\end{aligned}$$

Thus we get

$$\gamma'\left[-xS'(x) + S(x) - \frac{(S(x))^2}{S(\gamma)}\right] = S(x)Lg(x, \gamma).$$

As long as $x > 0$, $S(x) - xS'(x) - \frac{(S(x))^2}{S(\gamma_\downarrow)} \le S(x) - xS'(x) < 0$, so the Cauchy problem is well defined (since $0 < x_0 \le z_0$). The maximal solution will be defined on an interval (x_-, x_+), with $x_0 \in (x_-, x_+)$. We also have $\gamma'_\downarrow < 0$ as long as $(x, \gamma_\downarrow(x)) \in \mathrm{Int}(\Gamma^+)$ and $(x_0, z_0) \in \mathrm{Int}(\Gamma^+)$, so we have $graph(\gamma_\downarrow) \cap \Gamma \neq \emptyset$.

Since $\frac{\partial}{\partial z} Lg > 0$, $\frac{\partial^2}{\partial x^2} Lg < 0$ and $Lg(x, \Gamma(x)) = 0$ on $[0, \zeta]$, the implicit functions theorem implies that $\Gamma_\downarrow$ is C^1 with negative derivative. We also have that if $(x_\Gamma, z_\Gamma) \in graph(\gamma_\downarrow) \cap \Gamma$, then $\gamma'_\downarrow(x_\Gamma) = 0$. Therefore $(x_\Gamma, z_\Gamma) \in graph(\Gamma_\uparrow)$. This implies that $x_- = 0$, and we can define $x^* = \inf\{x \ge x_0, (x, \gamma_\downarrow(x)) \in \Gamma\}$. $\gamma_\downarrow$ is defined on $(0, x^* + \varepsilon)$ for a certain $\varepsilon > 0$ and on $(0, x^*)$, $(x, \gamma_\downarrow(x)) \in \mathrm{Int}(\Gamma^+)$. Using (5.14), we see that $\gamma'_\downarrow$ is negative on $(0, x^*)$.

By construction $f(x, \gamma_\downarrow(x)) = \text{constant} = f(x_0, z_0) = 0$, $(x, \gamma_\downarrow(x)) \in \Gamma^+$ and $(x^*, \gamma_\downarrow(x^*)) \in graph(\Gamma_\uparrow)$.

Finally, since $\gamma_\downarrow$ is decreasing it has a limit at 0. The fact that $(x, \gamma_\downarrow(x)) \in \Gamma^+$ implies that $\gamma_\downarrow(0) \ge \Gamma^0$, and if we had $\gamma_\downarrow(0) > \Gamma^0$, then by continuity of $\gamma_\downarrow$, there would exist $x \in (0, \Gamma_\uparrow^{-1}(\Gamma^0)]$ such that $f(x, \Gamma^0) = 0$, which is impossible. So we have the result. ∎

The function $\gamma_\downarrow$ defined in the previous proposition will be the second part of our boundary. We denote by $\gamma_\uparrow$ the boundary constructed in the previous paragraph. We now check that the two boundaries $\gamma_\uparrow$ and $\gamma_\downarrow$ do intersect. This is provided in the following proposition.

PROPOSITION 5.5. *We have that either $\gamma_\uparrow$ is increasing on $[0, +\infty)$, or $|graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)| = 1$. In the first case, we write $\bar{x} = 0$ and $\bar{z} = \gamma_\uparrow(0)$. In the second case, we write $(\bar{x}, \bar{z}) = graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)$. In both cases, we have $(\bar{x}, \bar{z}) \in \Gamma^+$ and $\{(x, \gamma_\uparrow(x));\ x > \bar{x}$ and $\gamma_\uparrow(x) > x\} \subset \mathrm{Int}(\Gamma^+)$.*

Proof. $\gamma_\uparrow$ is increasing as long as $Lg(x, \gamma_\uparrow(x)) > 0$. By Proposition 5.1, if we do not have $\gamma_\uparrow$ increasing on $[0, +\infty)$, then there exists $x_0 \le \zeta$ such that $Lg(x_0, \gamma_\uparrow(x_0)) = 0$ while $\gamma_\uparrow$ is increasing on $(x_0, +\infty)$. Since $\Gamma_\downarrow$ is decreasing on $(0, \zeta)$ while $\gamma_\uparrow$ is increasing as long as $(x, \gamma_\uparrow(x)) \in \mathrm{Int}(\Gamma^+)$, $(x, \gamma_\uparrow(x)) \in \Gamma^-$ on $(0, x_0)$.

On the other hand, $\gamma_\downarrow$ is defined on $[0, x^*]$, decreasing, continuous, and $(x, \gamma_\downarrow(x)) \in \mathrm{Int}(\Gamma^+)$ on $(0, x^*)$. Therefore we have $|graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)| = 1$; this intersection is in Γ^+, and by construction the last property is immediate.

If $\gamma_\uparrow$ is increasing on $[0, +\infty)$, then $(x, \gamma_\uparrow(x)) \in \Gamma^+$ for all $x > 0$, so by continuity of $\gamma_\uparrow$ and since Γ^+, is a closed set, it is still true for $x = 0$. ∎

From now on, we denote by γ the concatenation of $\gamma_\downarrow$ and $\gamma_\uparrow$, which is continuous and piecewise C^1:

$$\gamma(x) = \begin{cases} \gamma_\downarrow(x) \text{ if } x < \bar{x}, \\ \gamma_\uparrow(x) \text{ if } x \ge \bar{x}. \end{cases}$$

We also introduce

$$\phi_\downarrow = \gamma_\downarrow^{-1} \quad \text{and} \quad \phi_\uparrow = \gamma_\uparrow^{-1}. \tag{5.15}$$

Notice that Proposition 5.1 (respectively, Proposition 5.4) implies that $\phi_\uparrow$ (resp., $\phi_\downarrow$) is C^1 on $\{z > \bar{z},\ \phi_\uparrow(z) < z\}$ (resp., on $(\bar{z}, \Gamma^0)$), with positive (resp., negative) derivative.

Notice also that if $\gamma_\downarrow$ is degenerate, then $\gamma = \gamma_\uparrow$.

Remark 5.1. Notice that, if $\bar{x} > 0$, γ is not differentiable at the point $\bar{x}$. Indeed, assuming to the contrary that $\bar{x} > 0$ and γ is differentiable at $\bar{x}$, it follows from the increase of $\gamma_\uparrow$ and the decrease of $\gamma_\downarrow$ that $\gamma'(\bar{x}) = 0$. By ODE (5.1) satisfied by $\gamma_\uparrow$, we see that $Lg(\bar{x}, \bar{z}) = 0$, so that $\bar{z} = \Gamma(\bar{x})$. Following the proof of Proposition

5.5, this also implies that $\bar{x} = \zeta$, the point where the minimum of Γ is attained. By differentiating (5.1) and using $\gamma'(\bar{x}) = 0$, we compute that the second derivative of γ at the right of $\bar{x}$ is given by $\gamma''(\bar{x}+) = \frac{\frac{\partial}{\partial x} Lg(\zeta,\gamma(\zeta))}{S(\gamma(\zeta))-S(\zeta)}$. However, it follows from Proposition 4.2 that $\frac{\partial}{\partial x} Lg(\zeta, \gamma(\zeta)) < 0$, implying that $\gamma''(\bar{x}+) < 0$. This is in contradiction with the nondecrease of γ at the right of $\bar{x}$.

6. Definition of v and verification result. We now have all the ingredients to define our candidate function v and to prove that it coincides with the value function V defined by (2.7).

We first decompose $\mathbf{\Delta}$ into four disjoint sets. We define

$$\begin{aligned}
A_1 &= \{(x,z),\ x \in [0,\bar{x}],\ \text{and}\ \bar{z} < z < \gamma(x)\},\\
A_2 &= \{(x,z),\ x \geq \bar{x},\ \text{and}\ \bar{z} < z < \gamma(x)\},\\
A_3 &= \{(x,z),\ 0 \leq x \leq z \leq \bar{z}\},\\
A_4 &= \{(x,z),\ x \geq 0,\ \text{and}\ z \geq \gamma(x)\}.
\end{aligned}$$

(A_1, A_2, A_3, A_4) is a partition of $\mathbf{\Delta}$. Notice that if $(x,z) \in A_2$, then by Proposition 5.1(iii), $x \leq \Gamma^\infty$, and recall that $\bar{x} < \bar{z}$ were defined in Proposition 5.5, while $\phi_\downarrow$ and $\phi_\uparrow$ were defined by (5.15). Notice also that A_2 is not necessarily connected.

We refer to Figure 3 for a better understanding of the different areas. Let

$$K := \int_{\bar{z}}^{\infty} \frac{u}{S(u)} du - \frac{g_x(\bar{x},\bar{z})}{S'(\bar{x})}; \tag{6.1}$$

we define v in the following way:

$$v(x,z) = \frac{z^2}{2} + g_x(\phi_\downarrow(z),z)\frac{S(x)}{S'(\phi_\downarrow(z))} \quad \text{if}(x,z) \in A_1, \tag{6.2}$$

$$v(x,z) = g(\phi_\uparrow(z),z) + g_x(\phi_\uparrow(z),z)\frac{S(x) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \quad \text{if}(x,z) \in A_2, \tag{6.3}$$

$$v(x,z) = \frac{z^2}{2} + S(x)\left[\int_z^\infty \frac{u}{S(u)} du - K\right] \quad \text{if}(x,z) \in A_3, \tag{6.4}$$

$$v(x,z) = g(x,z) \quad \text{if}(x,z) \in A_4. \tag{6.5}$$

The main result of this section is the following.

THEOREM 6.1. *Let the coefficient α satisfy conditions* (2.2) *and* (4.1). *Let γ be given by Proposition* 5.1 *and v be defined by* (6.2)–(6.5). *Then $v = V$, and $\theta^* = \inf\{t \geq 0; Z_t \geq \gamma(X_t)\}$ is an optimal stopping time.*

Moreover, if τ is another optimal stopping time, then $\theta^* \leq \tau$ a.s.

Proof. From Proposition 5.1, Lemmas 6.2 and 6.3, and Propositions 6.4 and 6.5, v and γ satisfy the assumptions of Theorem 3.1. ∎

We first prove that v has the required regularity.

LEMMA 6.2. *v is C^0 w.r.t. (x,z), C^1 w.r.t. x, and piecewise $C^{2,1}$ w.r.t. (x,z). More precisely, except on $\cup_{i\neq j}\big(\mathrm{Cl}(A_i) \cap \mathrm{Cl}(A_j)\big)$, it is $C^{2,1}$.*

Proof. From the definition of v, $\phi_\downarrow$, and $\phi_\uparrow$, it is immediate that v can be extended as a $C^{2,1}$ function on any $\mathrm{Cl}(A_i)$.

Let us denote by v_i the expression of v on $\mathrm{Cl}(A_i)$. Since $\phi_\downarrow$ satisfies (5.12), it is immediate that v is C^0 w.r.t. (x,z) and C^1 w.r.t. x on the boundary (v_1 with v_4 and v_2 with v_4). On $z = \bar{z}$, it is easy to check that the expressions of v_2 and v_3 coincide. It

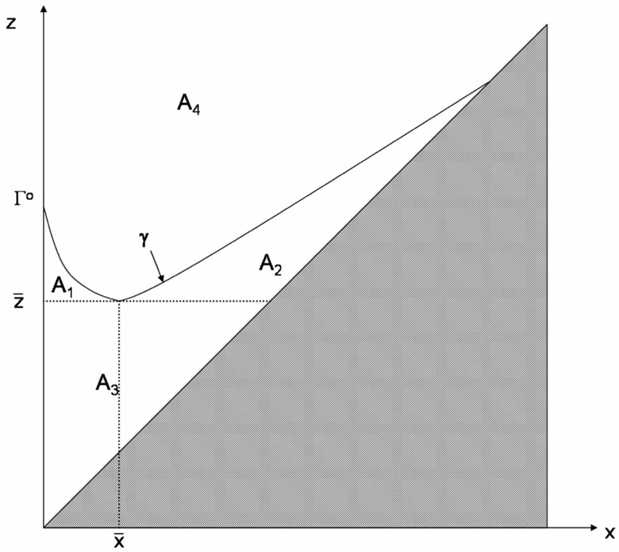


FIG. 3. *The different areas.*

is also true for v_1 and v_3 since $\phi_\downarrow$ satisfies (5.12) and $\bar{x} = \phi_\downarrow(\bar{z})$. It is straightforward that it is also C^1 and even C^2 w.r.t. x. ∎

We now show that v satisfies the boundary conditions.

LEMMA 6.3. *For all* $z \geq 0$, $v(0,z) = \frac{z^2}{2}$ *and* $v_z(z,z) = 0$.

Proof. Since $S(0) = 0$, $v(0,z) = \frac{z^2}{2}$ is immediate.

For $(z,z) \in \mathrm{Int}(A_4)$, since $g_z(z,z) = 0$, we have $v_z(z,z) = 0$. For $(z,z) \in \mathrm{Int}(A_3)$ it is immediate that $v_z(z,z) = 0$. For $(z,z) \in \mathrm{Int}(A_2)$, since $\gamma_\uparrow$ satisfies ODE (5.1), $\phi'_\uparrow(z) Lg(\phi_\uparrow(z), z) = 1 - \frac{S(\phi_\uparrow(z))}{S(z)}$. We then compute

$$\begin{aligned} v_z(z,z) &= g_z(\phi_\uparrow(z), z) + g_{xz}\frac{S(z) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} + \phi'_\uparrow(z) Lg(\phi_\uparrow(z), z)\frac{S(z) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \\ &= -\left(1 - \frac{S(\phi_\uparrow(z))}{S(z)}\right)\frac{S(z) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} + \left(1 - \frac{S(\phi_\uparrow(z))}{S(z)}\right) \\ &\quad \frac{S(z) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} = 0. \end{aligned}$$

To complete the proof, we need to show that $v_z(\bar{z},\bar{z}) = 0$ and $v_z(\Gamma^\infty, \Gamma^\infty) = 0$ if $\Gamma^\infty < \infty$. The previous computations and the definition of v on A_3 and A_4 show that at those points, $v_z(z,z)$ has right and left limits that are both equal to 0, so we have the result. ∎

PROPOSITION 6.4. *Let the coefficient* α *satisfy conditions* (2.2) *and* (4.1). *Then the function* v *is bounded from below and* $\lim_{z\to\infty} v(z,z) - g(z,z) = 0$.

Proof. If $\Gamma^\infty < \infty$, it is immediate since in this case, by Proposition 5.1(iii), $v = g$ outside a compact set, v is continuous and g is nonnegative. So let us focus on the case $\Gamma^\infty = \infty$. If (4.1) is satisfied, by Proposition 4.3, we know that α is bounded. We write $\alpha \leq M$.

We first prove that v is bounded from below and that $v(z,z) - g(\phi_\uparrow(z), z) \to 0$ as $z \to \infty$. A_1 is bounded because of the definition of $\gamma_\downarrow$, and A_3 is bounded by definition. Since $v = g$ on A_4 and $g \geq 0$, we need only check that v is bounded from below on A_2.

On the set $\{(x, \gamma_\uparrow(x));\ x \in [\bar{x}, \infty)\}$, $v = g$, and for $(x, z) \in A_2$, we have

$$v(x,z) = g(\phi_\uparrow(z), z) + g_x(\phi_\uparrow(z), z)\frac{S(x) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))}. \tag{6.6}$$

In particular, we see that for each z, $v(.,z)$ is monotonic on $[\phi_\uparrow(z), z]$. Therefore, since $v(\phi_\uparrow(z), z) \geq g(\phi_\uparrow(z), z) \geq 0$, it is sufficient to check that v is bounded from below on the diagonal $\{(z,z);\ z \in [\bar{x}, \infty)\}$.

We compute

$$g_x(\phi_\uparrow(z), z) = -(z - \phi_\uparrow(z)) + S'(\phi_\uparrow(z)) \int_z^\infty \frac{u - \phi_\uparrow(z)}{S(u)} du - S(\phi_\uparrow(z)) \int_z^\infty \frac{du}{S(u)}.$$

From Proposition 5.1 we know that $\lim_{x\to\infty} \gamma_\uparrow(x) - x = 0$, so that $\lim_{z\to\infty} z - \phi_\uparrow(z) = 0$. Using Proposition 4.1, the fact that $\phi_\uparrow(z) < z$ since $\Gamma^\infty = \infty$, and the increase of S', we have, as $z \to \infty$,

$$\begin{aligned}
&S'(\phi_\uparrow(z)) \int_z^\infty \frac{u - \phi_\uparrow(z)}{S(u)} du \\
&\quad = S'(\phi_\uparrow(z)) \left(\int_z^\infty \frac{u - z}{S(u)} du + \int_z^\infty \frac{z - \phi_\uparrow(z)}{S(u)} du \right) \\
&\quad = S'(\phi_\uparrow(z)) \left(\frac{1}{\alpha(z) S'(z)} + \frac{z - \phi_\uparrow(z)}{S'(z)} + \circ\left(\frac{1}{S'(z)}\right) \right) = O(1), \\
&S(\phi_\uparrow(z)) \int_z^\infty \frac{du}{S(u)} \sim \frac{S'(\phi_\uparrow(z))}{\alpha(\phi_\uparrow(z)) S'(z)} = O(1)
\end{aligned}$$

so that $g_x(\phi_\uparrow(z), z) = O(1)$.

Since $\alpha \leq M$ and S' is increasing,

$$S'(z) = S'(\phi_\uparrow(z)) e^{\int_{\phi_\uparrow(z)}^z \alpha(u) du} \leq S'(\phi_\uparrow(z)) e^{M(z - \phi_\uparrow(z))},$$

so that

$$\begin{aligned}
S(z) - S(\phi_\uparrow(z)) &\leq (z - \phi_\uparrow(z)) S'(z) \\
&\leq (z - \phi_\uparrow(z)) S'(\phi_\uparrow(z)) e^{M(z - \phi_\uparrow(z))},
\end{aligned} \tag{6.7}$$

and therefore $0 \leq \frac{S(z) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \leq (z - \phi_\uparrow(z)) e^{M(z - \phi_\uparrow(z))} = \circ(1)$.

Since v is continuous and $g \geq 0$, by (6.6) we see that v is bounded from below and $v(z,z) - g(\phi_\uparrow(z), z) \to 0$.

Finally, we show that $g(z,z) - g(\phi_\uparrow(z), z) \to 0$. Indeed, we compute

$$\begin{aligned}
g(z,z) - g(\phi_\uparrow(z), z) = -\frac{(z - \phi_\uparrow(z))^2}{2} &+ \Big(S(z) - S(\phi_\uparrow(z))\Big) \int_z^\infty \frac{u - z}{S(u)} du \\
&- S(\phi_\uparrow(z)) \int_z^\infty \frac{z - \phi_\uparrow(z)}{S(u)} du.
\end{aligned}$$

Using Proposition 4.1(ii) and (6.7), we get

$$\begin{aligned}
\Big(S(z) - S(\phi_\uparrow(z))\Big) \int_z^\infty \frac{u - z}{S(u)} du &\sim \frac{S(z) - S(\phi_\uparrow(z))}{\alpha(z) S'(z)} \\
&\leq \frac{z - \phi_\uparrow(z)}{\alpha(z)} = \circ(1).
\end{aligned}$$

Using again Proposition 4.1, we also get

$$S(\phi_\uparrow(z))\int_z^\infty \frac{z-\phi_\uparrow(z)}{S(u)}du \sim (z-\phi_\uparrow(z))\frac{S'(\phi_\uparrow(z))}{\alpha(\phi_\uparrow(z))S'(z)} = \circ(1),$$

and as a consequence,

$$g(z,z) - g(\phi_\uparrow(z),z) = \circ(1).$$

Therefore we finally have $\lim_{z\to\infty} v(z,z) - g(z,z) = 0$. ∎

The final property of v required by the verification Theorem 3.1 is the following.

PROPOSITION 6.5. *Let the coefficient α satisfy conditions* (2.2) *and* (4.1). *Then $v \le g$ on $\mathbf{\Delta}$ and $v < g$ on the continuation region $\{(x,z) \in \mathbf{\Delta};\ x > 0$ and $z < \gamma(x)\}$.*

Proof. We analyze separately the different subsets A_1, A_2, A_3.

On A_1: For $\bar z < z < \Gamma^0$ and $0 \le x < \phi_\downarrow(z)$, we have

$$\begin{aligned} v(x,z) - g(x,z) &= \frac{z^2}{2} + g_x(\phi_\downarrow(z),z)\frac{S(x)}{S'(\phi_\downarrow(z))} - g(x,z),\\ v_x(x,z) - g_x(x,z) &= g_x(\phi_\downarrow(z),z)\frac{S'(x)}{S'(\phi_\downarrow(z))} - g_x(x,z)\\ &= S'(x)\int_x^{\phi_\downarrow(z)} \frac{Lg(u,z)}{S'(u)}du, \end{aligned}$$

where we used (5.13) for the last equality.

For $\bar z \le z < \Gamma^0$, $(0,z) \in \Gamma^-$ while $(\phi_\downarrow(z),z) \in \Gamma^+$, so we can a priori have three behaviors for $v(.,z) - g(.,z)$:

- it is increasing on $[0,\phi_\downarrow(z)]$,
- or it is decreasing on $[0,\phi_\downarrow(z)]$,
- or it is decreasing on $[0,\delta)$ and increasing on $(\delta,\phi_\downarrow(z)]$ for a certain $\delta \in (0,\phi_\downarrow(z))$.

Since $v(0,z) = g(0,z)$ and $v(\phi_\downarrow(z),z) = g(\phi_\downarrow(z),z)$, only the last behavior can occur and $v \le g$ on A_1. Moreover, $v < g$, except if $x = 0$ or $x = \phi_\downarrow(z)$.

On A_2: For $x > \phi_\uparrow(z)$ and $\bar z < z < \Gamma^\infty$, we compute

$$v(x,z) - g(x,z) = g(\phi_\uparrow(z),z) + g_x(\phi_\uparrow(z),z)\frac{S(x) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} - g(x,z).$$

So, similarly,

$$v_x(x,z) - g_x(x,z) = -S'(x)\int_{\phi_\uparrow(z)}^x \frac{Lg(u,z)}{S'(u)}du. \tag{6.8}$$

Here again only three behaviors are a priori possible, for $(v-g)(.,z)$:

- increasing on $[\phi_\uparrow(z),z]$,
- decreasing on $[\phi_\uparrow(z),z]$,
- decreasing on $[\phi_\uparrow(z),\delta)$ and increasing on $(\delta,z]$ for a certain $\delta \in (\phi_\downarrow(z),z)$.

Since $v(\phi_\uparrow(z),z) = g(\phi_\uparrow(z),z)$, we need only consider $v(z,z) - g(z,z)$.

We write $n(z) = v(z,z) - g(z,z)$. Since $v_z(z,z) = g_z(z,z) = 0$,

$$\begin{aligned} \frac{\partial}{\partial z}(v(z,z) - g(z,z)) = n'(z) &= v_x(z,z) - g_x(z,z)\\ &= -S'(z)\int_{\phi_\uparrow(z)}^z \frac{Lg(u,z)}{S'(u)}du. \end{aligned}$$

We find the same expression as before, with $x = z$. First, if $n'(z) \leq 0$, we have $\int_{\phi_\uparrow(z)}^{z} \frac{Lg(u,z)}{S'(u)} du \geq 0$, which implies that for any $x \in (\phi_\uparrow(z), z)$, $\int_{\phi_\uparrow(z)}^{x} \frac{Lg(u,z)}{S'(u)} du > 0$. Therefore from (6.8), $(v-g)(.,z)$ is decreasing on $[\phi_\uparrow(z), z]$, and since $(v-g)(\phi_\uparrow(z), z) = 0$, we get $n(z) < 0$ if $\phi_\uparrow(z) < z$.

Assume now that there exists $z \in [\bar{z}, \Gamma^\infty)$ such that $n(z) \geq 0$ and $\phi_\uparrow(z) < z$. Then, from the previous argument, $n'(z) > 0$. Since n is continuous, this implies that n is increasing on any connected subset of $\{z' \geq z,\ \phi_\uparrow(z') < z'\}$ that contains z. Let $a := \inf\{z' > z;\ \phi_\uparrow(z') = z'\}$. If $a < \infty$, the definition of $\phi_\uparrow$ implies $v(a,a) = g(a,a)$, which contradicts $n(a) > 0$, and if $a = \infty$, Proposition 6.4 gives $\lim_{z\to\infty} n(z) = 0$, so again this is impossible. As a consequence, $n(z) < 0$ if $\phi_\uparrow(z) < z$. Therefore $v \leq g$ on A_2 and $v < g$, except if $x = \phi_\uparrow(z)$.

On A_3: Recall the definition of K given by (6.1). For $x \leq z \leq \bar{z}$, we have

$$v(x,z) - g(x,z) = \frac{z^2}{2} - KS(x) - \frac{(z-x)^2}{2} + xS(x) \int_z^{+\infty} \frac{du}{S(u)},$$

$$\text{so} \quad v_z(x,z) - g_z(x,z) = x\left(1 - \frac{S(x)}{S(z)}\right).$$

The latter expression is nonnegative and positive if $x \neq 0$. Since v and g are continuous, the result for A_1 and A_2 tells us that $v(.,\bar{z}) \leq g(.,\bar{z})$ so that $v \leq g$ on A_3 and $v < g$ if $x \neq 0$. ∎

7. Examples.

7.1. Brownian motion with negative drift. We first observe that the problem is degenerate for a standard Brownian motion. Indeed, in this case, $\alpha(x) = 0$ and $S(x) = x$. Since (2.11) will never be satisfied for a nondecreasing and convex function ℓ, Proposition 2.1 tells us that V and g will be infinite if ℓ satisfies (2.13). Moreover, for any $0 < x \leq z$ and any convex and nondecreasing function ℓ, we have the following:

(i) $\mathbb{E}_{x,z} T_0 = +\infty$,

(ii) $\mathbb{E}_{x,z} Z_{T_0} = \mathbb{E}_{x,z}(Z_{T_0})^2 = +\infty$,

(iii) V and g are infinite everywhere except for $x = 0$.

Point (i) is a classical result, (ii) comes directly from (2.10), and (iii) comes from (ii) and arguments similar to the proof of Proposition 2.1.

We now consider the following diffusion for constant $\mu < 0$ and $\sigma > 0$:

$$dX_t = \mu dt + \sigma dW_t.$$

Therefore $\alpha(x) = -\frac{2\mu}{\sigma^2} = \alpha > 0$, $S(x) = \frac{e^{\alpha x} - 1}{\alpha}$, and $S'(x) = e^{\alpha x}$.

We have an interesting homogeneity result for this process, which allows us to assume that $\alpha = 1$. In the following statement, we denote by γ_α the corresponding boundary, given by Theorem 6.1.

PROPOSITION 7.1. *Let $\alpha > 0$ be given, and consider the quadratic loss function $\ell(x) = \frac{x^2}{2}$. Then*

$$\gamma_\alpha(z) = \frac{\gamma_1(\alpha z)}{\alpha}.$$

Proof. Let X be a drifted Brownian motion with parameter $\alpha_X = \alpha$, and define $\bar{X} = \alpha X$. The dynamics of $\bar{X}$ is

$$d\bar{X}_t = \alpha dX_t = \alpha\mu dt + \alpha\sigma dW_t$$

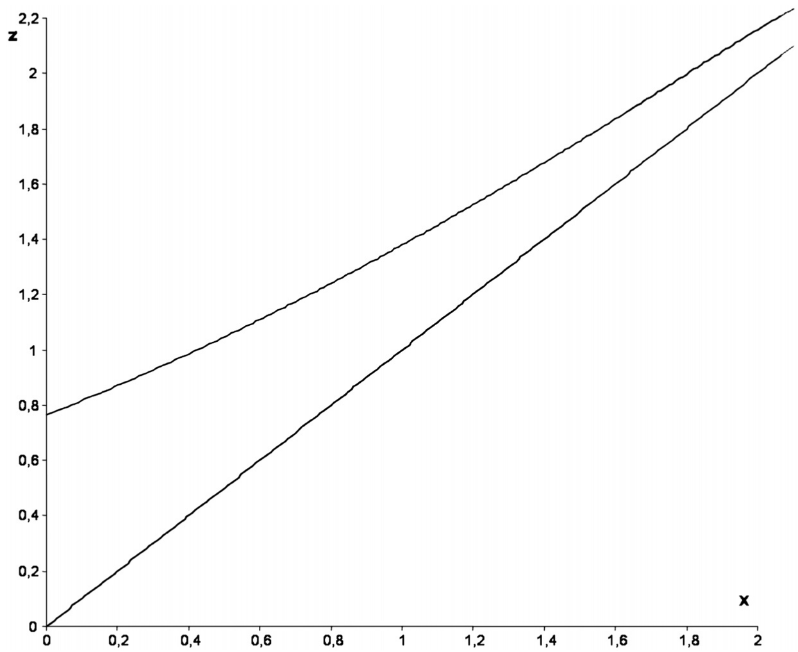


FIG. 4. *γ for a Brownian motion with negative drift and $\ell(x) = \frac{x^2}{2}$.*

so that $\alpha_{\bar{X}} = \frac{-2\mu\alpha}{\sigma^2\alpha^2} = 1$. Let $\bar{Z}$ be the corresponding running maximum, started from αz. Then $\bar{Z} = \alpha Z$, $T_0(X) = T_0(\bar{X}) = T_0$, and for any θ,

$$\mathbb{E}_{kx,kz}(\bar{Z}_{T_0} - \bar{X}_\theta)^2 = \alpha^2 \mathbb{E}_{x,z}(Z_{T_0} - X_\theta)^2.$$

This equality implies that if τ is optimal for one problem, it is also optimal for the other one. Together with the minimality of θ^*, it means that

$$Z_t = \gamma_\alpha(X_t) \Leftrightarrow \bar{Z}_t = \gamma_1(\bar{X}_t) \Leftrightarrow \alpha Z_t = \gamma_1(kX_t),$$

which completes the proof. $\square$

In the quadratic case $\ell(x) = \frac{x^2}{2}$, we have $Lg(x,z) = 1 + \alpha(z-x) + (1+e^{\alpha x})\ln(1-e^{-\alpha z})$.

We can see that $\frac{\partial}{\partial x} Lg < 0$, so that Γ is increasing (i.e. $\zeta = 0$). Moreover, for any $x \in (0,1)$, $\ln(1-x) < -x$ so that for $z > 0$, $Lg(z,z) < -e^{-\alpha z} < 0$, which implies $\Gamma^\infty = +\infty$.

Figure 4 is a numerical computation of γ for $\ell(x) = \frac{x^2}{2}$. Since Γ is increasing, γ is necessarily increasing too ($\gamma_\downarrow$ is degenerate). Even though it does not affect the shape because of Proposition 7.1, this plot was computed for $\alpha = 1$.

7.2. The CIR–Feller process. Let $b \geq 0$, $\mu < 0$, and $\sigma > 0$; then the dynamics of X is

$$dX_t = \mu X_t dt + \sigma\sqrt{b + X_t} dW_t.$$

Here, $\alpha(x) = \alpha\frac{x}{x+b}$ with $\alpha > 0$. In the degenerate case $b = 0$, we are reduced to the context of the Brownian motion with negative drift. We then focus on the case $b > 0$ with a quadratic loss function $\ell(x) = \frac{x^2}{2}$. Proceeding as in the proof of Proposition 4.3, we can see that $\Gamma^\infty < \infty$, unlike in the case $b = 0$.

Moreover, as $x \to 0$, $\alpha(x) \sim \frac{\alpha x}{b}$, $\alpha'(x) \sim \frac{\alpha}{b}$ so that we can see that for any $z > 0$, $\frac{\partial}{\partial x} Lg > 0$ for x small enough, which means that $\Gamma_\downarrow$ is not degenerate, or equivalently, that $\zeta > 0$.

7.3. Ornstein–Uhlenbeck process. The dynamics of X is now given by

$$dX_t = \mu X_t dt + \sigma dW_t$$

so that $\alpha(x) = \alpha x$, $S'(x) = e^{\alpha \frac{x^2}{2}}$.

This case and the Brownian motion with negative drift case can be seen as the extreme cases of our framework. Indeed, here $\alpha(x) = \alpha x$ is the "most increasing" concave function, while $\alpha(x) = \alpha$ is the "least nondecreasing" function.

As for the Brownian motion with negative drift, we have a homogeneity result for this process, for $\ell(x) = \frac{x^2}{2}$, which allows us to assume that $\alpha(x) = x$.

PROPOSITION 7.2. *Let $\alpha(x) = \alpha x$ with $\alpha > 0$ and $\ell(x) = \frac{x^2}{2}$. Then the corresponding boundary γ_α satisfies*

$$\gamma_\alpha(z) = \frac{\gamma_1(z\sqrt{\alpha})}{\sqrt{\alpha}}.$$

Proof. We follow the proof in the case of a Brownian motion with negative drift. Let X be a process with $\alpha_X(x) = \alpha x$. Then the process $\bar{X} = \sqrt{\alpha} X$ is such that $\alpha_{\bar{X}} = 1$. Denote by $\bar{Z}$ the corresponding running maximum process. Then $\bar{Z} = \sqrt{\alpha} Z$, $T_0(X) = T_0(\bar{X}) = T_0$, and for any θ,

$$\mathbb{E}_{\sqrt{\alpha}x, \sqrt{\alpha}z}(\bar{Z}_{T_0} - \bar{X}_\theta)^2 = \alpha \mathbb{E}_{x,z}(Z_{T_0} - X_\theta)^2.$$

Then by the minimality of θ^* we have

$$X_t = \gamma_\alpha(Z_t) \Leftrightarrow \bar{X}_t = \gamma_1(\bar{Z}_t) \Leftrightarrow \sqrt{\alpha} X_t = \gamma_1(\sqrt{\alpha} Z_t),$$

which provides the required result. ∎

Then, again in the case $\ell(x) = \frac{x^2}{2}$, we show that Γ is decreasing in a neighborhood of 0 so that $\zeta > 0$ and that $\Gamma^\infty < +\infty$.

PROPOSITION 7.3. *For an Ornstein–Uhlenbeck process;*

- *$Lg(x, \Gamma^0) > 0$ for $x > 0$ in a neighborhood of 0, and therefore $\Gamma_\downarrow$ is not degenerate;*
- *$Lg(z, z) > 0$ in a neighborhood of $+\infty$, and therefore $\Gamma^\infty < +\infty$.*

Proof. Since $\alpha(x) \to \infty$ as $x \to \infty$, Proposition 4.3 implies that $\Gamma^\infty < \infty$.

If x is small, we have $S(x) \sim x$, $S'(x) = 1 + S''(0)x + \circ(x) = 1 + \circ(x)$, and by definition of Γ^0, $\int_{\Gamma^0}^\infty \frac{du}{S(u)} = \frac{1}{2}$. Therefore, as $x \to 0$, we can write

$$Lg(x, \Gamma^0) = 1 + \alpha x \Gamma^0 - 1 + \circ(x).$$

Since $\alpha > 0$ and $\Gamma^0 > 0$ by Proposition 4.2, $Lg(x, \Gamma^0) > 0$ for $x > 0$ and sufficiently small. ∎

Finally, Figure 5 is a numerical computation of the boundary γ for $\ell(x) = \frac{x^2}{2}$. While we do not prove it, we can see that γ is, in this case, decreasing first and then increasing. Although it does not affect the shape because of Proposition 7.2, it was computed for $\alpha = 1$.

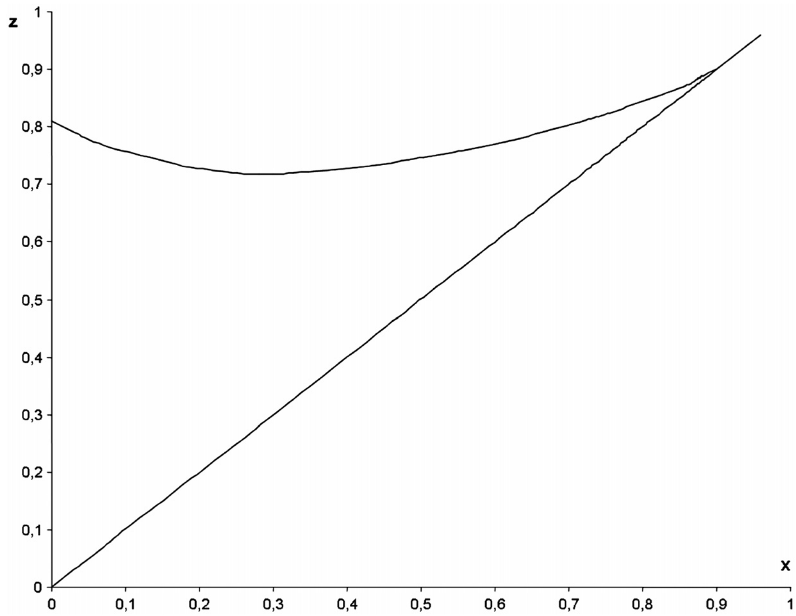


FIG. 5. γ *for an Ornstein–Uhlenbeck process with* $\ell(x) = \frac{x^2}{2}$.

8. Extension to general loss functions. Except for sections 2 and 3, the previous analysis considered only the case of the quadratic loss function $\ell(x) = \frac{x^2}{2}$. In fact, as the reader has probably noticed, the quadratic loss function plays a special role here, since $\ell^{(3)} = 0$, inducing a substantial simplification of the analysis of the set Γ^+ and the asymptotic behavior of Lg.

Unfortunately, we were not able to extend some crucial properties established in the quadratic case. Therefore, this section can be seen as a first attempt for the present more general framework. In particular, the case of a general loss function introduces the possibility that the free boundary γ is decreasing until it reaches the diagonal, a case which was not possible for a quadratic loss function.

8.1. Additional assumptions and shape of Γ. Recall from section 3 that we assume (3.1) holds true. Moreover, if ℓ is not the quadratic loss function, we require the following technical assumptions:

$$\ell \text{ is } C^3,\ \ell' > 0,\ \ell'' > 0,\ \ell^{(3)} \geq 0 \text{ and } \ell,\ \ell',\ \ell'' \text{ satisfy (3.1)}, \tag{8.1}$$

$$K_1 := \sup_{y \geq 0} \frac{\ell^{(3)}(y)}{\ell''(y)} < \infty \text{ and } \lim_{x\to\infty} \alpha(x) > K_1, \tag{8.2}$$

$$K_2 := \sup_{y \geq 0} \frac{\ell''(y)}{\ell'(y)} < \infty \text{ and } \lim_{x\to\infty} \alpha(x) > K_2. \tag{8.3}$$

Notice that (8.1)–(8.3) are satisfied for exponential loss functions $\ell(x) = \lambda e^x$ with $\lambda > 0$ or for power loss functions of the form $\lambda(x + \varepsilon)^p$ with $\varepsilon > 0$ and $p \geq 2$. They are mainly needed in order to derive asymptotic expansions similar to Proposition 4.1.

Let us now compute

$$Lg(x,z) = \ell''(z-x) + \alpha(x)\ell'(z-x) - (2S'(x) - \alpha(x)S(x)) \int_z^\infty \frac{\ell''(u-x)}{S(u)} du + S(x) \int_z^\infty \frac{\ell^{(3)}(u-x)}{S(u)} du.$$

Since $\ell''(x) > 0$ for $x > 0$ and $\ell^{(3)} \geq 0$, for any $x \geq 0$, $z \mapsto \frac{\partial}{\partial z} Lg(x,z)$ is increasing. Moreover, we have $\ell'(x) \to \infty$ as $x \to \infty$ so that for any $x \geq 0$, $\lim_{z\to\infty} Lg(x,z) > 0$. As a consequence, $\Gamma^+ \neq \emptyset$, and the definition of Γ in (4.3) can be extended.

The main problem is that Lg is no longer concave w.r.t. x, and it is not clear how to show that Γ is U-shaped. In fact Propositions 4.2(i) and 4.3 are crucial, but we are unable to prove them in general. Therefore we assume the following conditions:

$$\exists \zeta \geq 0 \text{ such that } \Gamma \text{ is decreasing on } [0,\zeta] \text{ and increasing on } [\zeta,+\infty), \tag{8.4}$$

$$\text{if } \lim_{x\to\infty} \alpha(x) = \infty, \text{ then } \Gamma^\infty < \infty. \tag{8.5}$$

Unfortunately, we failed to derive conditions directly on ℓ and α that guarantee that these conditions hold true.

In the present context, notice that in contrast to Proposition 4.2(iii), Γ^0 may be larger than Γ^∞. This means that we have a new possibility for the shape of γ: $\gamma_\uparrow(x) = x$ for every $x \geq \bar{x}$.

8.2. The increasing part of the boundary. In order to determine the increasing part of the free boundary, ODE (5.1) is replaced by

$$\gamma' = \frac{Lg(x,\gamma)}{\ell''(\gamma - x)\left(1 - \frac{S(x)}{S(\gamma)}\right)}. \tag{8.6}$$

Since $\ell'' > 0$, the Cauchy problem is well defined for any $x_0 > 0$ and $\gamma(x_0) > x_0$, and the maximal solution is defined as long as $\gamma(x) > x$.

In order to extend Proposition 5.1, the asymptotic results of Proposition 4.1 must be adapted; see section 9.3. Using Proposition 9.1, we can easily adapt the proofs of Lemmas 5.2 and 5.3 and show that they still hold true. However, in order to adapt the proof of Proposition 5.1, we make the following assumption:

$$\text{either } \alpha(x) \to \infty \text{ as } x \to \infty, \tag{8.7}$$

$$\text{or in Proposition 9.1(ii), for any } a > 0 \text{ and } \varphi(z) = z - a,\ \delta \equiv 1. \tag{8.8}$$

This additional assumption is made in order to prove that for sufficiently large x,

$$\frac{Lg(x,x+a)}{\ell''(a)\left(1 - \frac{S(x)}{S(x+a)}\right)} > 1 + \varepsilon,$$

while the other arguments of the proof remain exactly the same.

8.3. The decreasing part and the definition of v. Now we examine the decreasing part of γ. Equation (5.12) is replaced by

$$g(x(z),z) - g_x(x(z),z)\frac{S(x(z))}{S'(x(z))} - \ell(z) = 0. \tag{8.9}$$

The proof of Proposition 5.4 can be extended almost immediately by replacing ODE (5.14) with

$$\gamma'(x) = \frac{Lg(x,\gamma)S(x)}{(\ell'(\gamma - x) - \ell'(\gamma))S'(x) + \ell''(\gamma - x)S(x)\left(1 - \frac{S(x)}{S(\gamma)}\right)},$$

and noticing that, since $\ell^{(3)} \geq 0$, for any x and γ, there exists $y \in (\gamma - x, \gamma)$ such that

$$\begin{aligned}(\ell'(\gamma - x) - \ell'(\gamma))S'(x) + \ell''(\gamma - x)S(x)\left(1 - \frac{S(x)}{S(\gamma)}\right)&\\ &= -x\ell''(y)S'(x) + \ell''(\gamma - x)S(x)\left(1 - \frac{S(x)}{S(\gamma)}\right)\\ &\leq \ell''(\gamma - x)(S(x) - xS'(x)).\end{aligned}$$

Then, Proposition 5.5 still holds, except how a new case can occur; that is, $\gamma_\downarrow(x^*) = x^*$ and $x^* \geq \Gamma^\infty$, which implies $\Gamma^0 > \Gamma^\infty$.

Remark 8.1. In the new case stated above, the condition $x^* \geq \Gamma^\infty$ is not a priori a consequence of $\gamma_\downarrow(x^*) = x^*$, since there is no reason in general for the set $\mathrm{Int}(\Gamma^-)$ to be connected.

Finally, Theorem 6.1 can be proved in the same way for a general loss function, using the asymptotic expansions of Proposition 9.1, where v is defined by formulas generalizing (6.2) to (6.5).

9. Appendix.

9.1. Proof of Proposition 2.1.

Proof. The implications (iii) $\Longrightarrow$ (iii)$'$, (ii) $\Longrightarrow$ (i), (i) $\Longrightarrow$ (i)$'$ and (ii) $\Longrightarrow$ (ii)$'$ are immediate. Since ℓ is nondecreasing and nonnegative, we also have (iii)$'$ $\Longrightarrow$ (iii). Using (2.10) and (2.11), we get (iii) $\Longrightarrow$ (ii).

Assume now that Condition (2.13) holds true. The implications (ii)$'$ $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) follow immediately from the definition of g in (2.10) together with condition (2.13) and the nondecrease of ℓ.

We conclude the proof by showing that (i)$'$ $\Longrightarrow$ (iii). Let (i)$'$ hold true and assume to the contrary that $\int^\infty \ell'(u-x)S(u)^{-1}du = \infty$ for all $x \geq 0$. For arbitrary $0 < x \leq z$ and $\theta \in \mathcal{T}_0$, we have from (2.10) that

$$\mathbb{E}[\ell(Z_{T_0} - X_\theta)|X_\theta, Z_\theta] = g(X_\theta, Z_\theta) = \ell(Z_\theta)\mathbf{1}_{\{X_\theta = 0\}} + \infty\mathbf{1}_{\{X_\theta > 0\}}.$$

- If $\mathbb{P}(\{\theta \neq T_0\}) > 0$, then

$$\begin{aligned}J(\theta, x, z) :=&\mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta) = \mathbb{E}_{x,z}\mathbb{E}[\ell(Z_{T_0} - X_\theta)|X_\theta, Z_\theta]\\ \geq&\mathbb{E}_{x,z}\mathbf{1}_{\{\theta\neq T_0\}}\mathbb{E}[\ell(Z_{T_0} - X_\theta)|X_\theta, Z_\theta] = +\infty.\end{aligned}$$

- Alternatively, if $\theta = T_0$ a.s., then $J(\theta, x, z) = J(T_0, x, z) = \ell(z) + S(x)\int_z^\infty \ell'(u)S(u)^{-1}du = +\infty$.

By arbitrariness of $0 < x \leq z$ and $\theta \in \mathcal{T}_0$, this shows that $V = +\infty$ everywhere. $\square$

Notice that if (2.11) holds for $x = 0$, then (2.10) is also valid for $x = 0$.

Remark 9.1. Without assuming (2.13), (i) and (ii) can hold true while (iii) does not. Indeed, consider for example a process with scale function $S(x) = e^{x^2}$ and the loss function $\ell(x) = \int_0^x e^{u^2}du$. Then $\int_z^\infty \ell'(u)S(u)^{-1}du = +\infty$, while for

$x > 0$, $\int_z^\infty \ell'(u-x)S(u)^{-1}du = \frac{e^{x^2+2xz}}{2x}$ so that (i) and (ii) are satisfied (recall that $V(0,z) = g(0,z) = \ell(z)$).

Remark 9.2. Condition (2.13) is satisfied by power and exponential loss functions $\ell(x) = x^p$ for some $p \geq 1$, or $e^{\eta x}$ for some $\eta > 0$. Without condition (2.13), (i)$' \Longrightarrow$ (i) or (ii)$' \Longrightarrow$ (ii) are not true in general. Consider for instance the process with scale function $S(x) = e^{x^2}$ and, for $\varepsilon > 0$, the loss function $\ell(x) = \int_0^x e^{(u+\varepsilon)^2}du$. Then if $x \leq \varepsilon$, $\int_z^\infty \ell'(u-x)S(u)^{-1}du = \infty$, while if $x > \varepsilon$, $\int_z^\infty \ell'(u-x)S(u)^{-1}du = \frac{e^{(x-\varepsilon)^2+2(x-\varepsilon)z}}{2(x-\varepsilon)}$. So $g(x,z) < \infty$ iff $x > \varepsilon$ or $x = 0$. In other words (ii)$'$ is true while (ii) is false. Adapting the proof of (i)$'\Longrightarrow$(iii) by considering the set $\{X_\theta \in (0,\varepsilon)\}$, which has a nonzero probability if $x \in (0,\varepsilon)$ and θ is not a.s. equal to T_0, we see that we also have (i)$'$ but not (i) (so that $V(x,z) < \infty$ iff $x \geq \varepsilon$ or $x = 0$).

Remark 9.3. From the previous proof, we also observe that we have ($g = +\infty$ everywhere except for $x = 0$) implies ($V = +\infty$ everywhere except for $x = 0$). This statement does not require condition (2.13).

9.2. Proof of Proposition 4.1.

Proof. Recall that $\left(\frac{1}{\alpha}\right)' \to 0$ at infinity as stated in Remark 2.2(ii). The below limits and equivalents are considered when $z \to +\infty$.

(i) As $S(z) \to +\infty$, $S(z) = \int_0^z e^{\int_0^u \alpha(v)dv} \sim \int_1^z e^{\int_0^u \alpha(v)dv}$. Integrating by parts, we get

$$\int_1^z e^{\int_0^u \alpha(v)dv} = \left[\frac{e^{\int_0^u \alpha(v)dv}}{\alpha(u)}\right]_1^z - \int_1^z \left(\frac{1}{\alpha}\right)'(u)e^{\int_0^u \alpha(v)dv}du.$$

Since $\left(\frac{1}{\alpha}\right)' \to 0$, $\int_1^z \left(\frac{1}{\alpha}\right)'(u)e^{\int_0^u \alpha(v)dv}du = \circ\left(\int_1^z e^{\int_0^u \alpha(v)dv}\right)$ so that $S(z) \sim \frac{S'(z)}{\alpha(z)}$.

(ii) Using (i) and integrating by parts, we get

$$\int_z^\infty \frac{du}{S(u)} \sim \int_z^\infty \frac{\alpha(u)}{S'(u)}du = \int_z^\infty \alpha(u)e^{-\int_0^u \alpha(v)dv}du \quad = \frac{1}{S'(z)};$$

$$\int_z^\infty \frac{udu}{S(u)} \sim \int_z^\infty \frac{u\alpha(u)}{S'(u)}du = \frac{z}{S'(z)} + \int_z^\infty \frac{1}{S'(u)}du.$$

But $u\alpha(u) \to \infty$ as $u \to \infty$ so that

$$\int_z^\infty \frac{1}{S'(u)}du = \circ\left(\int_z^\infty \frac{u\alpha(u)}{S'(u)}du\right),$$

and therefore

$$\int_z^\infty \frac{udu}{S(u)} \sim \frac{z}{S'(z)}.$$

Finally, by integrating by parts twice, we get

$$\begin{aligned}\int_z^\infty \frac{u-z}{S(u)}du &\sim \int_z^\infty \frac{(u-z)\alpha(u)}{S'(u)}du = \int_z^\infty \frac{1}{S'(u)}du \\ &= \int_z^\infty \frac{\alpha(u)}{\alpha(u)S'(u)}du = \frac{1}{\alpha(z)S'(z)} + \int_z^\infty \left(\frac{1}{\alpha}\right)'(u)\frac{1}{S'(u)}du.\end{aligned}$$

As $\left(\frac{1}{\alpha}\right)'(u) \to 0$ as $u \to \infty$, we get the result. $\square$

9.3. Asymptotic results for a general loss function.

PROPOSITION 9.1. *Assume* (8.1)–(8.3). *Let φ be a measurable function such that $0 \leq \varphi(z) \leq z$ for all z (large enough). Then we have the following asymptotic behaviors as $z \to \infty$:*

(i) *There exists a bounded function δ (depending on φ) satisfying $\delta(z) \geq 1$ for z large enough and such that*

$$\int_z^\infty \frac{\ell''(u-\varphi(z))}{S(u)} du \sim \delta(z) \frac{\ell''(z-\varphi(z))}{S'(z)};$$

(ii) *there exists a bounded function ν satisfying $\nu(z) \geq 1$ for z large enough and such that*

$$\int_z^\infty \frac{\ell'(u-\varphi(z))}{S(u)} du \sim \nu(z) \frac{\ell'(z-\varphi(z))}{S'(z)}.$$

Moreover if $\lim_{x\to\infty} \alpha(x) = \infty$, then for any function φ, δ and ν are constant and equal to 1.

Proof. (i) The proof is close to the proof of Proposition 4.1(ii). First, as φ is measurable and satisfies $0 \leq \varphi(z) \leq z$, the expressions make sense and the integrals exist. Then, using Proposition 4.1(i) and integrating by parts, we have

$$\begin{aligned}\int_z^\infty \frac{\ell''(u-\varphi(z))}{S(u)} du &\sim \int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)} du = \int_z^\infty \alpha(u) e^{-\int_0^u \alpha(v)dv} \ell''(u-\varphi(z)) du \\ &= \frac{\ell''(z-\varphi(z))}{S'(z)} + \int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)} du.\end{aligned}$$

According to assumption (8.1), all of the terms above are nonnegative. Moreover, using (8.2) we get

$$\int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)} du \leq K_1 \int_z^\infty \frac{\ell''(u-\varphi(z))}{S'(u)} du,$$

$$\text{while} \int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)} du \geq \alpha(z) \int_z^\infty \frac{\ell''(u-\varphi(z))}{S'(u)} du \ (> 0),$$

so that

$$A := \limsup_{z\to\infty} \frac{\int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)} du}{\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)} du} < 1,$$

which means that, for z large enough, there exists a certain $k(z) \in \left[0, \frac{1+A}{2}\right)$ such that

$$\begin{aligned}&\int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)} du \\ &\quad = k(z) \int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)} du + \circ\left(\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)} du\right).\end{aligned}$$

As $\varphi(z) < z$ if $z > 0$, $\ell''(z-\varphi(z)) > 0$, and this implies that

$$(1-k(z)) \int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)} du \sim \frac{\ell''(z-\varphi(z))}{S'(z)}.$$

Setting $\delta(z) = \frac{1}{1-k(z)} \in \left[1, \frac{2}{1-A}\right]$, we have the result. We also see that if $\alpha(x) \to \infty$ as $x \to \infty$, then $k(z) = 0$, so $\delta(z) = 1$.

(ii) This follows along the lines of (i), replacing ℓ'' by ℓ' and using (8.3) instead of (8.2). $\square$

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