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OPTIMAL INVESTMENT UNDER RELATIVE PERFORMANCE CONCERNS

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We consider the problem of optimal investment when agents take into account their relative performance by comparison to their peers. Given N interacting agents, we consider the following optimization problem for agent i, $1 \leq i \leq N$:

$$\sup_{\pi^i \in \mathcal{A}_i} \mathbb{E} U_i \left((1 - \lambda_i) X_T^{\pi^i} + \lambda_i \left(X_T^{\pi^i} - \bar{X}_T^{i,\pi} \right) \right),$$

where U_i is the utility function of agent i, π^i his portfolio, X^{π^i} his wealth, $\bar{X}^{i,\pi}$ the average wealth of his peers, and λ_i is the parameter of relative interest for agent i. Together with some mild technical conditions, we assume that the portfolio of each agent i is restricted in some subset A_i. We show existence and uniqueness of a Nash equilibrium in the following situations:

- unconstrained agents,
- constrained agents with exponential utilities and Black–Scholes financial market.

We also investigate the limit when the number of agents N goes to infinity. Finally, when the constraints sets are vector spaces, we study the impact of the λ_is on the risk of the market.

KEY WORDS: portfolio optimization, relative concerns, Nash equilibrium, differential game, backward stochastic differential equations.

1. INTRODUCTION

The seminal papers of Merton (1969, 1971) generated a huge literature extending the optimal investment problem in various directions and using different techniques. We refer to Pliska (1986), Cox and Huang (1989), or Karatzas, Lehoczky, and Shreve (1987) for the complete market situation, to Cvitanic and Karatzas (1992) or Zariphopoulou (1994) for constrained portfolios; to Constantinides and Magill (1976), Davis and Norman (1990), Shreve and Soner (1994), Duffie and Sun (1990), or Akian, Menaldi, and Sulem (1995) for transactions costs; to Constantinides (1983), Jouini, Koehl, and Touzi (1997, 1999), Damon, Spatt, and Zhang (2001), or Ben Tahar, Soner, and Touzi (2008a, 2008b) for

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taxes; and to He and Pearson (1991a,b), Karatzas, Lehoczky, Shreve, and Xu (1991), Kramkov and Schachermayer (1999, 2003), or Kramkov and Sirbu (2006) for general incomplete markets.

However, in all of these works, no interaction between agents is taken into account. The most natural framework to model such interaction would be a general equilibrium model where the behavior of the investors is coupled through the market equilibrium conditions. But this typically leads to untractable calculations. Instead, we shall model the interactions based on some simplified context of comparison of the performance to that of the competitors or to some benchmark. A return of 5% during a crisis is not equivalent to the same return during a financial bubble. Moreover, human beings tend to compare themselves to their peers. In fact, economic and sociological studies have emphasized the importance of relative concerns in human behaviors, see Veblen (1899) for the sociological part, and Abel (1990), Gali (1994), Gomez, Priestley, and Zapatero (2007), or DeMarzo, Kaniel, and Kremer (2008) for economic works, considering simple models in discrete time frameworks.

In this paper, we study the optimal investment problem under relative performance concerns, in a continuous-time framework. More precisely, there are N particular investors that compare themselves to each other. Agents are heterogeneous (different utility functions and different constraints sets) and instead of considering only his absolute wealth, each agent takes into account a convex combination of his wealth (with weight $1 - \lambda$, $\lambda \in [0, 1]$) and the difference between his wealth and the average wealth of the other investors (with weight λ). This creates interactions between agents and therefore leads to a differential game with N players. We also consider that each agent's portfolio must stay in a set of constraints.

In the context of a complete market situation where all agents have access to the entire financial market, we prove existence and uniqueness of a Nash equilibrium for general utility functions. The optimal performances at equilibrium are explicit, and therefore allow for many interesting qualitative results.

We next turn to the case where the agents have different access to the financial market, i.e., their portfolio constraints sets are different. Our solution approach requires to restrict the utility functions to the exponential framework. Then, assuming mainly that the agents positions are constrained to lie in closed convex subsets, and that the drift and volatility of the log prices are deterministic, we show the existence and uniqueness of a Nash equilibrium, using the backward stochastic differential equation (BSDE) techniques introduced by El Karoui and Rouge (2000) and further developed by Hu, Imkeller, and Müller (2005). The Nash equilibrium optimal positions are more explicit in the case of constraints defined by linear subspaces. In this setting, we analyze the limit when the number of players N goes to infinity where the situation considerably simplifies in the spirit of mean field games, see Lasry and Lions (2007). Notice that our problem does not fit in the framework of Lasry and Lions (2007) for the two following reasons. First, in Lasry and Lions (2007), the authors consider similar agents, which is not the case in the present paper, as the utility functions, the parameters λ_is, and the sets of constraint can be specific. More importantly, in Lasry and Lions (2007), the sources of randomness of two different agents are independent.

We finally investigate the impact of the interaction coefficient λ. Under some additional assumptions, which are satisfied in many examples, we show that the local volatility of the wealth of each agent is nondecreasing with respect to λ. In other words, the more investors are concerned about each other (λ large), the more risky is the (equilibrium) portfolio of each investor. However, in general, this can fail to hold. But in the limit,

N goes to infinity, the same phenomenon holds for the average portfolio of the market, without any additional assumption. Roughly speaking, this means that the global risk of the market increases with λ, although it can fail for the portfolio of some specific agent.

This paper is organized as follows. Section 2 introduces the problem. In Section 3, we solve the complete market situation, for general utility functions. In Section 4, we deal with the general case with exponential utility functions and portfolios that are constrained to remain inside closed convex sets. In Section 5, we restrict the sets of constraints to linear spaces that allow us in particular to derive some interesting economic implications.

REMARK 1.1. This paper is the submitted version from the content of the PhD thesis of the first author Espinosa (2010). The delayed date of submission is due to our wish to provide more relevant examples. Based on the content of Espinosa (2010), Frei and dos Reis developed an interesting article, Frei and dos Reis (2011), during their Post-Doc at Ecole Polytechnique. Their article, in particular, highlights the difficulty in the existence and uniqueness of the quadratic multidimensional backward SDE of the present paper, and establishes the existence of a *sequentially delayed* Nash equilibrium in the general case. Frei and dos Reis (2011) also provide some extensions of our results, and some shorter proofs. But they crucially make use of many results of this paper.

Notations. $\mathbb{H}^2(\mathbb{R}^m)$ denotes the space of all predictable processes φ, with values in $\mathbb{R}^m$, and satisfying $\mathbb{E}\int_0^T |\varphi_t|^2 dt < \infty$. The corresponding localized space is denoted by $\mathbb{H}^2_{loc}(\mathbb{R}^m)$. When there is no risk of confusion, we simply write $\mathbb{H}^2$ and $\mathbb{H}^2_{loc}$.

2. PROBLEM FORMULATION

Let W be a d-dimensional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding completed canonical filtration. We assume that $\mathcal{F}$ is generated by W. Let $T > 0$ be the investment horizon so that $t \in [0, T]$. Given two $\mathbb{F}$-predictable processes θ taking values in $\mathbb{R}^d$ and σ taking values in $\mathbb{R}^{d\times d}$, satisfying:

$$\sigma \text{ symmetric, uniformly definite positive, } \int_0^T |\sigma_t|^2 dt < +\infty \text{ a.s.,} \tag{2.1}$$

$$\text{and } \theta \text{ is bounded, } dt \otimes d\mathbb{P}-\text{a.e,} \tag{2.2}$$

we consider a market with a nonrisky asset with interest rate $r = 0$ and a d-dimensional risky asset $S = (S^1, \ldots, S^d)$ given by the following dynamics:

$$dS_t = \operatorname{diag}(S_t)\sigma_t(\theta_t dt + dW_t), \tag{2.3}$$

where for $x \in \mathbb{R}^d$, $\operatorname{diag}(x)$ is the diagonal matrix with ith diagonal term equal to x^i.

A portfolio is an $\mathbb{F}$-predictable process $\{\pi_t, t \in [0, T]\}$ taking values in $\mathbb{R}^d$. Here, π_t^j is the amount invested in the jth risky asset at time t. Under the self-financing condition, the associated wealth process X_t^π is defined by

$$X_t^\pi = X_0 + \int_0^t \pi_r \cdot \operatorname{diag}(S_r)^{-1} dS_r, \; t \in [0, T].$$

Given an integer $N \geq 2$, we consider N portfolio managers whose preferences are characterized by a utility function $U_i : \mathbb{R} \to \mathbb{R}$, for each $i = 1, \ldots, N$. We assume that U_i is

C^1, increasing, strictly concave and satisfies Inada conditions:

$$U_i'(-\infty) = +\infty, \; U_i'(+\infty) = 0. \tag{2.4}$$

In addition, we assume that each investor is concerned about the average performance of his peers. Given the portfolio strategies π^i, $i = 1, \dots, N$, of the managers, we introduce the average performance viewed by agent i as

$$\bar{X}^{i,(\pi^j)_{j\neq i}} := \frac{1}{N-1} \sum_{j \neq i} X^{\pi^j}. \tag{2.5}$$

The portfolio optimization problem of the ith agent is then defined by

$$\begin{aligned} V_0^i\left((\pi^j)_{j\neq i}\right) := V_0^i &:= \sup_{\pi^i \in \mathcal{A}^i} \mathbb{E}\left[U_i\left((1-\lambda_i)X_T^{\pi^i} + \lambda_i\left(X_T^{\pi^i} - \bar{X}_T^{i,(\pi^j)_{j\neq i}}\right)\right)\right] \\ &= \sup_{\pi^i \in \mathcal{A}^i} \mathbb{E}\left[U_i\left(X_T^{\pi^i} - \lambda_i \bar{X}_T^{i,(\pi^j)_{j\neq i}}\right)\right], \; 1 \le i \le N, \end{aligned} \tag{2.6}$$

where $\lambda_i \in [0, 1]$ measures the sensitivity of agent i to the performance of his peers, and the set of admissible portfolios $\mathcal{A}^i$ will be defined later. Roughly speaking, we impose integrability conditions as well as the constraints π^i take values in A_i, a given closed convex subset of $\mathbb{R}^d$.

Our main interest is to find a Nash equilibrium in the context where each agent is "small" in the sense that his actions do not impact the market prices S.

DEFINITION 2.1. A Nash equilibrium for the N portfolio managers is an N-tuple $(\hat{\pi}^1, \dots, \hat{\pi}^N) \in \mathcal{A}^1 \times \dots \mathcal{A}^N$ such that, for every $i = 1, \dots, N$, given $(\hat{\pi}^j)_{j\neq i}$, the portfolio strategy $\hat{\pi}^i$ is a solution of the portfolio optimization problem $V_0^i((\hat{\pi}^j)_{j\neq i})$.

If in addition, for each $i = 1, \dots, N$, $\hat{\pi}^i$ is a deterministic and continuous function of $t \in [0, T]$, we say that $(\hat{\pi}^1, \dots, \hat{\pi}^N)$ is a deterministic Nash equilibrium.

Our main results are the following:

MAIN THEOREM 2.2. *Assume that for each $i = 1, \dots, N$, $U_i(x) = -e^{-\frac{x}{\eta_i}}$ for some constant $\eta_i > 0$, the portfolio constraints sets A_i are closed convex, $\prod_{i=1}^N \lambda_i < 1$, and that there exists a Nash equilibrium $(\tilde{\pi}^1, \dots, \tilde{\pi}^N)$. Then, there exists a solution to the N-dimensional quadratic BSDE* (4.20) *below, and $(\tilde{\pi}^1, \dots, \tilde{\pi}^N)$ can be expressed in terms of this solution.*

Unfortunately, the well-posedness of BSDE (4.20) below is an open problem in the present literature, thus preventing Main Theorem 2.2 from providing a characterization of Nash equilibria. We refer in particular to Frei and dos Reis (2011) who further explored this question following a previous version of this paper based on Espinosa (2010). In particular, Frei and dos Reis (2011) highlight some examples of nonexistence by allowing the agents to have some final reward.

In the context of deterministic coefficients, we obtain a complete characterization.

MAIN THEOREM 2.3. *Assume that θ and σ are deterministic and continuous functions of $t \in [0, T]$, and that for each $i = 1, \dots, N$, $U_i(x) = -e^{-\frac{x}{\eta_i}}$ for some constant $\eta_i > 0$, the portfolio constraints sets A_i are closed convex, and $\prod_{i=1}^N \lambda_i < 1$. Then, there exists a unique deterministic Nash equilibrium.*

We also observe that the unique Nash equilibrium of Main Theorem 2.3 can be computed explicitly in various interesting situations that will be described throughout the paper.

In order to simplify notations, from now on, we will write

$$X^i_t := X^{\pi^i}_t \quad \text{and} \quad \bar{X}^i_t := \bar{X}^{i,(\pi^j)_{j\neq i}}_t,\ t \in [0, T].$$

In Section 3, we shall consider the complete market situation in which the portfolios will be free of constraints (in other words, $A_i = \mathbb{R}^d$ for each i). This will be solved for general utility functions. In the next sections, we will derive results for more general types of constraints, but we will focus on the case of exponential utility functions: $U_i(x) = -e^{-\frac{x}{\eta_i}}$. We will first consider the general case in Section 4, and then in Section 5, we will focus on the case of linear constraints, where the A_is are (vector) subspaces of $\mathbb{R}^d$.

3. THE COMPLETE MARKET SITUATION

In this section, we consider the case where there are no constraints on the portfolios:

$$A_i = \mathbb{R}^d, \quad \text{for all } i = 1, \ldots, N.$$

In the present situation, the density of the unique equivalent martingale measure is

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^T \theta(u)\cdot dW_u - \frac{1}{2}\int_0^T |\theta(u)|^2 du}. \tag{3.1}$$

We shall denote by $\mathbb{E}^{\mathbb{Q}}$ the expectation under $\mathbb{Q}$.

In contrast with the general results in the subsequent sections, the complete market situation can be solved for general utility functions. In this case, the set of admissible strategies $\mathcal{A} = \mathcal{A}_i$ is the set of predictable processes π such that

$$\begin{aligned} &\sigma\pi \in \mathbb{H}^2_{loc}(\mathbb{R}^d),\ X^\pi \text{ is a} \mathbb{Q}\text{-martingale}, \\ &\text{and } U_j\big(-2^k\big(X^\pi_T\big)^+\big), U_j\big(-2^k\big(X^\pi_T\big)^-\big) \in \mathbb{L}^1(\mathbb{P}) \quad \text{for all } j \le N \quad \text{and } k \in \mathbb{N}. \end{aligned} \tag{3.2}$$

To simplify the notations and presentation in this introductory example, we also assume that all agents have the same relative performance coefficient λ:

$$\lambda_i = \lambda \in [0, 1), \quad \text{for all } i = 1, \ldots, N, \tag{3.3}$$

see, however, Remark 3.4. Nevertheless, we allow the investors to have different utility functions U_i and different initial endowments $x^i \in \mathbb{R}$. We denote

$$\bar{x}^i := \frac{1}{N-1}\sum_{j\neq i} x^j,\ i = 1, \ldots, N.$$

3.1. Single-Agent Optimization

We first check that the single-agent optimization problem is well-posed under the following additional conditions:

$$\text{for all } y > 0, \mathbb{E}\left|U_i \circ I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right| < \infty, \mathbb{E}^{\mathbb{Q}}\left|I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right| < \infty, \text{ and} \tag{3.4}$$

$$U_j\left(-2^k I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^+\right), \quad U_j\left(-2^k I_i\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^-\right) \in \mathbb{L}^1(\mathbb{P}) \quad \text{for all } j \le N \quad \text{and } k \in \mathbb{N}.$$

LEMMA 3.1. *Under condition (3.4), we have* $U_i(X_T^i - \lambda \bar{X}_T^i) \in \mathbb{L}^1(\mathbb{P})$ *for all* $(\pi^1, \dots, \pi^N) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_N$ *and* $i = 1, \dots, N$.

Proof. Using the convex dual of $-U_i(-x)$, we have for any $y > 0$:

$$U_i\big(X_T^i - \lambda \bar{X}_T^i\big) \le \left| U_i \circ I_i\left(y \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right| + y \frac{d\mathbb{Q}}{d\mathbb{P}} \left(\big| X_T^i - \lambda \bar{X}_T^i \big| + \left| I_i\left(y \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right| \right).$$

The right-hand side is integrable under $\mathbb{P}$ by the admissibility conditions (3.2) and the integrability assumptions (3.4). Then, $U_i(X_T^i - \lambda \bar{X}_T^i)^+$ is integrable. On the other hand, by the increase of U_i, we have $U_i(x - y) \ge U_i(-x^- - y^+) \ge -|U_i(-2x^-)| - |U_i(-2y^+)|$. Then, it follows from the concavity of U_i that

$$\begin{aligned}
U_i\big(X_T^i - \lambda \bar{X}_T^i\big) &\ge (1-\lambda) U_i\big(X_T^i\big) + \frac{\lambda}{N-1} \sum_{j \ne i} U_i\big(X_T^i - X_T^j\big) \\
&\ge (1-\lambda) U_i\big(-2\big(X_T^i\big)^-\big) - \frac{\lambda}{N-1} \sum_{j \ne i} \big\{ \big| U_i\big(-2\big(X_T^i\big)^-\big)\big| + \big| U_i\big(-2\big(X_T^j\big)^+\big)\big| \big\} \\
&\ge -\big| U_i\big(-2\big(X_T^i\big)^-\big)\big| - \frac{\lambda}{N-1} \sum_{j \ne i} \big| U_i\big(-2\big(X_T^j\big)^+\big)\big|.
\end{aligned}$$

Hence, $U_i(X_T^i - \lambda \bar{X}_T^i) \in \mathbb{L}^1(\mathbb{P})$ by our definition of $\mathcal{A}_i$. □

We now characterize the optimal portfolio and wealth of each agent, given the strategies of his peers. In other words, we try to find the best response of agent i to the strategies of his peers. As in the classical case of optimal investment in complete market, we will use the convex dual of $-U_i(-x)$. Since U_i is strictly concave and C^1, we can define $I_i := (U_i')^{-1}$, which is a bijection from $\mathbb{R}_+^*$ onto $\mathbb{R}$ because of (2.4). The main result of this section requires the following integrability conditions:

LEMMA 3.2. *For any* $i = 1, \dots, N$, *let the strategies* $\pi^j \in \mathcal{A}$ *for* $j \ne i$ *be given. Then, under (3.4), there exists a unique optimal portfolio for the optimization problem (2.6) of agent* i *with optimal final wealth:*

$$X_T^{i*} = I_i\left(y^i \frac{d\mathbb{Q}}{d\mathbb{P}}\right) + \lambda \bar{X}_T^i, \quad \text{where } y^i \text{ is defined by } \mathbb{E}^{\mathbb{Q}} I_i\left(y^i \frac{d\mathbb{Q}}{d\mathbb{P}}\right) = x^i - \lambda \bar{x}^i. \tag{3.5}$$

Proof. Since the market is complete, and under conditions (3.2) and (3.4), there exist portfolio strategies π_i^* such that $X_T^{i*} = X_T^{\pi_i^*}$ is a $\mathbb{Q}$-martingale. We only verify that $\pi_i^* \in \mathcal{A}_i$ for all i, the rest of the proof is omitted as it follows the classical martingale approach in the simple complete market framework. Writing $I_i := I_i(y^i \frac{d\mathbb{Q}}{d\mathbb{P}})$, we compute that

$$\begin{aligned}
U_j(0) \ge U_j\big(-2^k\big(X_T^{i*}\big)^{\pm}\big) &\ge U_j\big(-2^k I_i^{\pm} - 2^k\big(\bar{X}_T^i\big)^{\pm}\big) \\
&\ge \frac{1}{N-1} \sum_{\ell \ne i} U_j\big(-2^k I_i^{\pm} - 2^k\big(X_T^{\ell}\big)^{\pm}\big) \\
&\ge \frac{-1}{N-1} \sum_{\ell \ne i} \big\{ \big| U_j\big(-2^{k+1} I_i^{\pm}\big)\big| + \big| U_j\big(2^{k+1}\big(X_T^{\ell}\big)^{\pm}\big)\big| \big\}.
\end{aligned}$$

Then, the required integrability follows from (3.4) and the definition (3.2) of the sets of admissible portfolios $\mathcal{A}_i$, $i = 1, \dots, N$. □

3.2. Partial Nash Equilibrium

The second step is to search for a Nash equilibrium between the N agents. Let $\mathbf{X_N} := (X_T^{i*})_{1\le i\le N}$ be the vector of terminal wealth of the investors associated to $(\pi^1, \dots, \pi^N)$. From Lemma 3.2, $(\pi^1, \dots, \pi^N)$ is a Nash equilibrium if and only if we have

$$A_N \mathbf{X_N} = J_N,$$

$$\text{where } A_N = \begin{pmatrix} 1 & & -\frac{\lambda}{N-1} \\ & \ddots & \\ -\frac{\lambda}{N-1} & & 1 \end{pmatrix} \in M_N(\mathbb{R}); \quad J_N = \left(I_i\left(y^i \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right)_{1\le i\le N}.$$

Under the condition $\lambda \neq 1$ in (3.3), it follows that A_N is invertible and we can compute explicitly that

$$A_N^{-1} = \begin{pmatrix} 1 + \frac{\lambda^2}{(1-\lambda)(N+\lambda-1)} & & \frac{\lambda}{(1-\lambda)(N+\lambda-1)} \\ & \ddots & \\ \frac{\lambda}{(1-\lambda)(N+\lambda-1)} & & 1 + \frac{\lambda^2}{(1-\lambda)(N+\lambda-1)} \end{pmatrix},$$

thus providing the existence of a unique Nash equilibrium:

THEOREM 3.3. *There exists a unique Nash equilibrium, and the equilibrium terminal wealth for each $i = 1, \dots, N$ is given by*

$$\hat{X}_T^i = \left(1 + \frac{\lambda^2}{(1-\lambda)(N+\lambda-1)}\right) I_i\left(y^i \frac{d\mathbb{Q}}{d\mathbb{P}}\right) + \frac{\lambda}{(1-\lambda)(N+\lambda-1)} \sum_{j\neq i} I_j\left(y^j \frac{d\mathbb{Q}}{d\mathbb{P}}\right).$$

REMARK 3.4. In the case of specific λ_is, the previous arguments can be adapted. In the expression of A_N, λ_i appears on the ith line instead of λ, A_N is invertible if and only if $\prod_{i=1}^N \lambda_i < 1$ (for more details, see the proof of Lemma 4.6 below), and then its inverse is given by

$$\left(A_N^{-1}\right)_{ii} = 1 + \frac{\lambda_i^N \sum_{k\neq i} \frac{\lambda_k^N}{1+\lambda_k^N}}{1 - \sum_{k\neq i} \frac{\lambda_k^N(1+\lambda_i^N)}{1+\lambda_k^N}}, \quad \text{and} \quad \left(A_N^{-1}\right)_{ij} = \frac{\frac{\lambda_i^N}{1+\lambda_j^N}}{1 - \sum_{k\neq i} \frac{\lambda_k^N(1+\lambda_i^N)}{1+\lambda_k^N}} \quad \text{for } i \neq j,$$

where we denoted $\lambda_i^N := \lambda_i/(N-1)$. The equilibrium performances are given by

$$\hat{X}_T^i = \sum_{j=1}^N \left(A_N^{-1}\right)_{ij} I_j\left(y^j \frac{d\mathbb{Q}}{d\mathbb{P}}\right), \quad i = 1, \dots, N.$$

REMARK 3.5. In the case $\lambda = 1$, it turns out that there exist either an infinity of Nash equilibria or no Nash equilibrium. Indeed, in this case, A_N is of rank $N-1$. Therefore, if J_N belongs to the image of A_N, then there is an affine space of dimension 1 of Nash equilibria, while if J_N is not in the image of A_N, then there is no Nash equilibrium.

In particular, in the exponential utility context (further developed below), we directly compute that $J_N = A_N x + \eta \int_0^T \theta(t) \cdot (\theta(t)dt + dW_t)$, where x is the vector of initial data

x^i and η is the vector of risk tolerances η_i of each agent. Therefore, J_N belongs to the image of A_N if and only if η belongs to it.

3.3. The Exponential Utility Case

In order to push further the analysis of the complete market situation, we now consider the exponential utility case:

$$U_i(x) = -e^{-\frac{x}{\eta_i}}, x \in \mathbb{R}, \tag{3.6}$$

where $\eta_i > 0$ is the risk tolerance parameter for agent i, i.e., the inverse of his absolute risk aversion coefficient. We denote the average risk tolerance by

$$\bar{\eta}_N := \frac{1}{N}\sum_{j=1}^{N} \eta_j. \tag{3.7}$$

In the present context, $I_i(y) = -\eta_i \ln(\eta_i y)$, so that the equilibrium wealth process is

$$\hat{X}^i_T = x^i - \frac{\eta_i}{1-\lambda}\left[\left(1 - \frac{\lambda N}{N+\lambda-1}\right) + \frac{\lambda N}{N+\lambda-1}\frac{\bar{\eta}_N}{\eta_i}\right]\left(\ln\frac{d\mathbb{Q}}{d\mathbb{P}} - \mathbb{E}^{\mathbb{Q}}\ln\frac{d\mathbb{Q}}{d\mathbb{P}}\right).$$

We denote by $\hat{\pi}^{i,N,\lambda}$ the corresponding equilibrium portfolio strategy of agent i, where we emphasize its dependence on the parameters N and λ.

In order to have explicit formulas, we assume that the risk premium θ is a (deterministic) continuous function of t. Then, it is well known that the classical portfolio optimization problem with no interaction between managers leads to the optimal portfolios

$$\hat{\pi}^{0,i}_t := \eta_i \sigma_t^{-1}\theta(t), \quad t \in [0, T].$$

PROPOSITION 3.6. *In the above setting, the equilibrium portfolio for agent i is given by*

$$\hat{\pi}^{i,N,\lambda}_t = k^{i,N}_\lambda \hat{\pi}^{0,i}_t, \quad \textit{where } k^{i,N}_\lambda := \frac{1}{1-\lambda}\left[\left(1 - \frac{\lambda N}{N+\lambda-1}\right) + \frac{\lambda N}{N+\lambda-1}\frac{\bar{\eta}_N}{\eta_i}\right].$$

REMARK 3.7. Assume further that $\bar{\eta}_N \longrightarrow \eta > 0$ as $N \to \infty$. Then, $k^{i,N}_\lambda \longrightarrow 1 + \frac{\lambda}{1-\lambda}\frac{\eta}{\eta_i}$. In particular, if all agents have the same risk aversion coefficient $\eta_i = \eta > 0$, then

$$\hat{\pi}^{i,N,\lambda} = \hat{\pi}^\lambda := \frac{1}{1-\lambda}\hat{\pi}^0_t. \quad \text{for all } i.$$

REMARK 3.8. In the case of similar agents, i.e., for any $i = 1, \dots, N$, $\eta_i = \eta$ and $\lambda_i = \lambda$, we can find the equilibrium portfolio very easily. Indeed, by symmetry considerations, all the X^is must be equal, $X^i = \bar{X}^i$, and the optimization problem reduces to

$$\sup_\pi -\mathbb{E}e^{-\frac{1-\lambda}{\eta}X^i_T}.$$

This is the classical case with η replaced by $\frac{\eta}{1-\lambda}$ so that the optimal portfolio is given by $\hat{\pi}_t = \eta\sigma_t^{-1}\theta(t)/(1-\lambda)$, in agreement with our results.

In the general case of the following sections, we will not always be able to conclude anything on the behavior of every agent; therefore, we introduce the following definition:

DEFINITION 3.9. The market index and the corresponding market portfolio are defined by

$$\bar{X}_t := \frac{1}{N}\sum_{i=1}^{N} X_t^i \quad \text{and} \quad \bar{\pi}_t := \frac{1}{N}\sum_{i=1}^{N} \pi_t^i,\ t \in [0, T].$$

We recall the definition of the Sharpe ratio SR and introduce the variance risk ratio VRR:

$$\text{(3.8)} \qquad \mathrm{SR} = \frac{\text{expected excess return}}{\text{volatility}}, \quad \mathrm{VRR} := \frac{\text{expected excess return}}{\text{variance}}.$$

For practical purposes, the VRR is a better criterion for the two following reasons:

- VRR is robust to the investment duration, while SR is not: for a time period L and a scalar $k > 0$, we have $\mathrm{SR}(kL) = k\,\mathrm{SR}(L)$, while $\mathrm{VRR}(kL) = \mathrm{VRR}(L)$.
- VRR accounts for the illiquidity risk related to the size of the position, while SR does not: for a portfolio X and a scalar $k > 0$, we have $\mathrm{SR}(kX) = \mathrm{SR}(X)$ and $\mathrm{VRR}(kX) = \mathrm{VRR}(X)/k$.

We have the following results for the impact of λ:

PROPOSITION 3.10.

(i) For any linear form φ, $|\varphi(\hat{\pi}_t^{i,N,\lambda})|$ is increasing w.r.t. λ.
(ii) The dynamics of the market index and the corresponding market portfolio are given by

$$d\bar{X}_t = \frac{\bar{\eta}_N}{1-\lambda}\theta(t)\cdot[\theta(t)dt + dW_t] \quad \text{and} \quad \bar{\pi}_t = \frac{\bar{\eta}_N}{1-\lambda}\sigma_t^{-1}\theta(t).$$

In particular, for any linear form φ, $|\varphi(\bar{\pi}_t)|$ is increasing w.r.t. λ.

Proof. (ii) is immediate, so we only prove (i). By Proposition 3.6, $\hat{\pi}^{i,N,\lambda} = k_\lambda^{i,N}\hat{\pi}_t^{0,i}$, and we directly compute that

$$\frac{\partial k_\lambda^{i,N}}{\partial\lambda} = \frac{1}{(1-\lambda)^2(N+\lambda-1)^2}\left[N+\lambda-1+N\big(\lambda+(N-1)(1-\lambda)\big)\left(\frac{\bar{\eta}_N}{\eta_i}-1\right)\right].$$

By definition of $\bar{\eta}_N$ in (3.7) and the fact that $\eta_j \geq 0$ for all j, we have $\frac{\bar{\eta}_N}{\eta_i} - 1 \geq \frac{1-N}{N}$. Therefore,

$$\begin{aligned}(1-\lambda)^2(N+\lambda-1)^2\frac{\partial k_\lambda}{\partial\lambda} &\geq N(N+\lambda-1) - \lambda(N-1) - (N-1)^2(1-\lambda)\\ &\geq (N-1)(1-\lambda) + \lambda(N^2-N+1) > 0.\end{aligned}$$

□

In words, Proposition 3.10 states that the more investors are concerned about each other, the more risk they will undertake. In each investment direction, the global position of agents, described by $|\varphi(\bar{\pi}_t)|$, will increase with λ and in the limit $\lambda \to 1$, we even have a limit of infinite positions $|\varphi(\bar{\pi}_t)| \to \infty$ a.s. Furthermore, the drift and volatility of the market index are both increasing w.r.t. λ. The corresponding Sharpe ratio is $\mathrm{SR} = |\theta(t)|$, independent of λ, while the variance risk ratio is $\mathrm{VRR} = \frac{1-\lambda}{\bar{\eta}_N}$, a decreasing function of λ. This is a perverse aspect of the present financial markets that may provide an explanation of the emergence of financial bubbles, when managers use the Sharpe ratio as a reliable indicator.

3.4. General Equilibrium

In the previous sections, the price process S was given exogeneously. We now analyze the effect of the relative performance coefficient λ when the price process S is determined at the equilibrium.

For each fixed price process S, defined as in Section 2, there exists a unique Nash equilibrium in the sense of Definition 2.1. Similar to Karatzas and Shreve (1998), our objective is to search for a market equilibrium price S that is consistent with market equilibrium conditions:

$$\sum_{i=1}^{N} \pi_t^{i,j} = K^j S_t^j \quad \text{for all } j = 1, \dots, d \quad \text{and } t \in [0, T], \tag{3.9}$$

$$\sum_{i=1}^{N} x^i = \sum_{j=1}^{d} K^j S_0^j, \tag{3.10}$$

where K^j is a constant such that $K^j S_t^j$ is the market capitalization of the jth firm. Equation (3.9) says that the total amount invested in the stocks of the jth firm is equal to the market capitalization of this firm. Equation (3.10) says that the initial endowment of the investors equals the initial market capitalizations. With $\mathbf{1} := (1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^d$, we observe that (3.9) and (3.10) imply that

$$\begin{aligned}
\sum_{i=1}^{N} X_t^i &= \sum_{i=1}^{N} \left(x^i + \int_0^t \pi_u^i \cdot \mathrm{diag}(S_u)^{-1} d S_u \right) \\
&= \sum_{i=1}^{N} x^i + \sum_{j=1}^{d} \int_0^t K^j d S_u^j \\
&= \sum_{i=1}^{N} x^i + \sum_{j=1}^{d} K^j \left(S_t^j - S_0^j \right) = \sum_{j=1}^{d} K^j S_t^j = \sum_{i=1}^{N} \pi_t^i \cdot \mathbf{1},
\end{aligned}$$

i.e., the total amount invested in the nonrisky asset is zero at any time $t \in [0, T]$.

DEFINITION 3.11. We say that a process S is an equilibrium market if there exists a Nash equilibrium $\hat{\pi} = (\hat{\pi}^1, \dots, \hat{\pi}^N)$ associated with the price dynamics S, in the sense of Definition 2.1, such that S and $\hat{\pi}$ satisfy (3.9) and (3.10).

In order to simplify notations, we set $K^j = k^j N$ and $k := (k^1, \dots, k^d)$.

PROPOSITION 3.12. *Let θ be a deterministic and continuous function of $t \in [0, T]$. Then, there exists an equilibrium market whose risk premium is θ. Moreover, in this equilibrium market, the market index is given by*

$$\bar{X}_t = \bar{x} + \frac{\bar{\eta}_N}{1 - \lambda} \int_0^t \theta(u) \cdot (\theta(u)\, du + d W_u), \; t \in [0, T].$$

Proof. By Proposition 3.10 (ii), it follows that

$$S_t = \frac{\bar{\eta}_N}{1-\lambda}\operatorname{diag}(k)\sigma(t)^{-1}\theta(t).$$

Notice that the previous equation does not define σ uniquely for $d > 1$.

Conversely, let θ be some given continuous function. Then, we can choose a diagonal matrix $\sigma_t = \sigma(t, S_t)$, with diagonal elements

$$\sigma^{ii}(t, S_t) = \frac{\bar{\eta}_N}{(1-\lambda)k^i S_t^i}\theta^i(t).$$

Notice that σ satisfies the conditions for S to be a strong solution of (2.3). Then, it follows from Proposition 3.6 that:

$$d\bar{X}_t = \frac{1}{N}\sum_{i=1}^{N} dX_t^i = \frac{1}{N}\sum_{i=1}^{N}\hat{\pi}_t^i \cdot \operatorname{diag}(S_t)^{-1} dS_t = \frac{\bar{\eta}_N}{1-\lambda}\theta(t)\cdot(\theta(t)dt + dW_t).$$

□

We next analyze the impact of λ on the drift and the volatility of the market index. Despite the multiplicity of market equilibria, they all lead to similar conclusions. Let us, for example, assume that the risk premium is independent of λ. Then, the drift of the market index is $\eta|\theta(t)|^2/(1-\lambda)$ and the volatility is $\eta|\theta(t)|/(1-\lambda)$; thus, both are increasing w.r.t. λ and with the same order. We may interpret this equilibrium as a financial bubble, where the return and the volatility are both increased by the agents' interactions. An alternative interpretation for a fund manager is that for the same given return, the agents' interaction coefficient increases the volatility of the optimal portfolio.

Notice that in the present setting, the variance risk ratio $\mathrm{VRR} = (1-\lambda)/\eta$ is decreasing in λ and tends to zero as $\lambda \to 1$. This indicates that according to this criterion, the agents' interactions lead to market inefficiency.

4. GENERAL CONSTRAINTS WITH EXPONENTIAL UTILITY

In the rest of this paper, we consider a general case with constrained portfolios. We assume

$$A_i \text{ is a closed convex set of } \mathbb{R}^d, \quad \text{for all } i = 1, \ldots, N. \tag{4.1}$$

We denote by P_t^i the orthogonal projection on $\sigma_t A_i$, which is well defined by (4.1). For $x \in \mathbb{R}^d$, we denote $\operatorname{dist}(x, \sigma_t A_i) := |x - P_t^i x|$ the Euclidean distance from x to the closed convex subset $\sigma_t A_i$.

REMARK 4.1. Recall that for a closed convex set A in a Euclidean space, the orthogonal projection on A, denoted by P, is well defined, is a contraction, and satisfies for any $x, y \in \mathbb{R}^d$:

$$|P(x) - P(y)|^2 \le (x-y)\cdot(P(x) - P(y)) \le |x-y|^2.$$

Moreover, $P(x)$ is the only point satisfying $(x - P(x))\cdot(a - P(x)) \le 0$ for all $a \in A$.

For technical reasons, we restrict our analysis to exponential utility functions (3.6).

DEFINITION 4.2. The set of admissible strategies $\mathcal{A}_i$ is the collection of all predictable processes π with values in A_i, $dt \otimes d\mathbb{P}$−a.e., such that $\sigma\pi \in \mathbb{H}^2_{loc}(\mathbb{R}^d)$ and such that the family

$$\{e^{\pm(X^{\pi}_{\tau}-X^{\pi}_{\nu})};\ \nu,\tau \text{ stopping times on } [0,T] \text{ with } \nu \le \tau \text{ a.s.}\} \tag{4.2}$$

is uniformly bounded in $\mathbb{L}^p(\mathbb{P})$ for all $p > 0$.

In comparison with the admissibility conditions of Section 3, the previous definition requires the uniform boundedness condition of the above family, which is needed in order to prove a dynamic programming principle similar to Lim and Quenez (2009).

REMARK 4.3. The set of admissible strategies introduced in Definition 4.2 is strictly smaller than the corresponding set in the one-agent framework of Hu, Imkeller, and Müller (2005). This deficiency was further overcome by Frei and dos Reis (2011) who used the results of the present paper (which are contained in Espinosa 2010).

4.1. A Formal Argument

In this section, we provide a formal argument that helps to understand the construction of Nash equilibrium of the subsequent section. For fixed $i = 1, \dots, N$, we rewrite (2.6) as

$$V_0^i := \sup_{\pi^i \in \mathcal{A}_i} \mathbb{E}\left[U_i\left(X_T^{\pi^i} - \tilde{\xi}^i\right)\right], \quad \text{where } \tilde{\xi}^i := \lambda_i \bar{X}_T^i =: \lambda_i \bar{x}^i + \tilde{\xi}_0^i. \tag{4.3}$$

Then, following El Karoui and Rouge (2000) or Hu, Imkeller, and Müller (2005), we expect that the value function V_0^i and the corresponding optimal solution be given by

$$V_0^i = -e^{-(x^i - \lambda_i \bar{x}^i - \tilde{Y}_0^i)/\eta_i}, \quad \sigma_t \hat{\pi}_t^i = P_t^i(\tilde{\zeta}_t^i + \eta_i \theta_t) \quad \text{for all } t \in [0,T],$$

and $(\tilde{Y}^i, \tilde{\zeta}^i)$ is the solution of the quadratic BSDE:

$$\tilde{Y}_t^i = \tilde{\xi}_0^i + \int_t^T \left(-\tilde{\zeta}_u^i \cdot \theta_u - \frac{\eta_i}{2}|\theta_u|^2 + \tilde{f}_u^i\left(\tilde{\zeta}_u^i + \eta_i\theta_u\right)\right) du - \int_t^T \tilde{\zeta}_u^i \cdot dW_u,\ t \le T, \tag{4.4}$$

where the generator $\tilde{f}^i$ is given by

$$\tilde{f}_t^i(z^i) := \frac{1}{2\eta_i} \operatorname{dist}(z^i, \sigma_t A_i)^2, \quad z^i \in \mathbb{R}^d. \tag{4.5}$$

This suggests that one can search for a Nash equilibrium by solving the BSDEs (4.4) for all $i = 1, \dots, N$. However, this raises the following difficulties:

- the final data $\tilde{\xi}_0^i$ do not have to be bounded as it is defined in (4.3) through the performance of the other portfolio managers;
- the situation is even worse because the final data $\tilde{\xi}_0^i$ induce a coupling of the BSDEs (4.4) for $i = 1, \dots, N$. To express this coupling in a more transparent way, we substitute the expressions of $\tilde{\xi}_0^i$ and rewrite (4.4) for $t = 0$ into:

$$\tilde{Y}_0^i = \eta_i \xi + \int_0^T \tilde{f}_u^i(\zeta_u^i)\, du - \int_0^T \left(\zeta_u^i - \lambda_i^N \sum_{j \ne i} P_u^j\left(\zeta_u^j\right)\right) \cdot dB_u,$$

where $B := W + \int_0^\cdot \theta_r dr$ is the Brownian motion under the equivalent martingale measure

$$\lambda_i^N := \frac{\lambda_i}{N-1}, \quad \zeta_t^i := \tilde{\zeta}_t^i + \eta_i \theta_t, \quad t \in [0, T], \tag{4.6}$$

and the final data are expressed in terms of the unbounded r.v.

$$\xi := \int_0^T \theta_u \cdot dB_u - \frac{1}{2}\int_0^T |\theta_u|^2 du. \tag{4.7}$$

Then, $\tilde{Y}_0 = Y_0$, where (Y, ζ) is defined by the BSDE

$$Y_t^i = \eta_i \xi + \int_t^T \tilde{f}_u^i(\zeta_u^i)\, du - \int_t^T \left(\zeta_u^i - \lambda_i^N \sum_{j \neq i} P_u^j(\zeta_u^j)\right) \cdot dB_u. \tag{4.8}$$

In order to sketch (4.8) into the BSDEs framework, we further introduce the mapping $\varphi_t : \mathbb{R}^{Nd} \longrightarrow \mathbb{R}^{Nd}$ defined by the components:

$$\varphi_t^i(\zeta^1, \dots, \zeta^N) := \zeta^i - \lambda_i^N \sum_{j \neq i} P_t^j(\zeta^j) \quad \text{for all } \zeta^1, \dots, \zeta^N \in \mathbb{R}^d. \tag{4.9}$$

It turns out that the mapping φ_t is invertible under fairly general conditions. We shall prove this result in Lemma 4.6 for general convex constraints and in Lemma 5.1 in the case of linear constraints. We denote $\psi_t := \varphi_t^{-1}$ and $\psi_t^i(z)$ the ith block component of size d of $\varphi_t^{-1}(z)$. Then, one can rewrite (4.8) as

$$Y_t^i = \eta_i \xi + \int_t^T f_u^i(Z_u)\, du - \int_t^T Z_u^i \cdot dB_u, \tag{4.10}$$

where the generator f^i is now given by

$$f_t^i(z) := \tilde{f}_t^i\big(\psi_t^i(z)\big) \quad \text{for all } z = (z^1, \dots, z^N) \in \mathbb{R}^{Nd}. \tag{4.11}$$

A Nash equilibrium should then satisfy for each i:

$$\hat{\pi}_t^i = \sigma_t^{-1} P_t^i\big(\psi_t^i(Z_t)\big), \quad i = 1, \dots, N. \tag{4.12}$$

4.2. Auxiliary Results

Our first objective is to verify that the map φ introduced in (4.9) is invertible. The crucial condition for the rest of this section is

$$\prod_{1 \le i \le N} \lambda_i < 1. \tag{4.13}$$

Recall the notation λ_i^N from (4.6).

LEMMA 4.4. *Under (4.1) and (4.13), for any $t \in [0, T]$, the map $I + \lambda_j^N P_t^j$ is a bijection on $\mathbb{R}^d$ and its inverse is a contraction, for all $j = 1, \dots, N$.*

Proof. Let $t \in [0, T]$ be fixed, for ease of notation, we omit all t subscripts. Since $\sigma_t A_j$ is a closed convex set, from Remark 4.3, $(x - y) \cdot (P^j(x) - P^j(y)) \ge |P^j(x) - P^j(y)|^2 \ge 0$, for any $x, y \in \mathbb{R}^d$. Notice that $I + \lambda_j^N P^j$ is a bijection if and only if, for all $y \in \mathbb{R}^d$, the map

$$f_y(x) := y - \lambda_j^N P^j(x) \tag{4.14}$$

has a unique fixed point. Since P^j is a contraction, we compute, for any x, x' in $\mathbb{R}^d$:

$$|f_y(x) - f_y(x')| = \lambda_j^N |P^j(x) - P^j(x')| \le \lambda_j^N |x - x'| = \frac{\lambda_j}{N-1}|x - x'|.$$

Case 1: If $N \ge 3$ or $\lambda_j < 1$, then f_y is a strict contraction of $\mathbb{R}^d$. We prove now that the inverse of $I + \lambda_j^N P^j$ is a contraction. Indeed, if $x \neq y$, we have

$$\begin{aligned} \left|x - y + \lambda_j^N(P^j(x) - P^j(y))\right|^2 &= |x-y|^2 + \left(\lambda_j^N\right)^2 |P^j(x) - P^j(y)|^2 \\ &\quad + 2\lambda_j^N (x-y)\cdot(P^j(x) - P^j(y)) \\ &\ge |x-y|^2 > 0, \end{aligned} \tag{4.15}$$

where we used the fact that $(x - y) \cdot (P^j(x) - P^j(y)) \ge 0$, see Remark 4.15.

Case 2: If $N = 2$ and $\lambda_j = 1$, f_y fails to be a strict contraction. However, (4.15) still holds and implies that $I + P^j$ is one-to-one. Using Lemma 4.5 below, we get the bijection property of $I + P^j$ and the contraction property of the inverse function follows from (4.15). $\square$

LEMMA 4.5. *Let A be a closed convex set of $\mathbb{R}^d$. Then $(I + P_A)(\mathbb{R}^d) = \mathbb{R}^d$.*

Proof. Let $B := 2A = \{y \in \mathbb{R}^d; \exists x \in A, y = 2x\}$, and let us prove that

$$P_A\left(y - \frac{1}{2}P_B(y)\right) = \frac{1}{2}P_B(y) \quad \text{for all } y \in \mathbb{R}^d. \tag{4.16}$$

This implies that $y = (I + P_A)(y - \frac{1}{2}P_B(y)) \in (I + P_A)(\mathbb{R}^d)$ for all $y \in \mathbb{R}^d$, which gives the required result.

To prove (4.16), define $x := \frac{1}{2}P_B(y)$ and $z := y - 2x$. By Remark 4.3, $P_B(y)$ is the only point in B satisfying $(y - P_B(y)) \cdot (b - P_B(y)) \le 0$ for all $b \in B$. In other words, we have for any $b \in B$, $z \cdot (b - 2x) \le 0$, or by definition of B, for any $a \in A$, $z \cdot (2a - 2x) \le 0$; hence,

$$(x + z - x) \cdot (a - x) \le 0 \quad \text{for all } a \in A,$$

which means that $x = P_A(x + z)$ and therefore $(I + P_A)(x + z) = x + z + x = y$. $\square$

Recall the definition of φ in (4.9).

LEMMA 4.6. *Under (4.1) and (4.13), we have for $t \in [0, T]$:*

(i) φ_t is a bijection of $\mathbb{R}^{Nd}$, and we write $\psi_t := \varphi_t^{-1}$.
(ii) ψ_t is Lipschitz continuous with a constant depending only on N and the λ_is.

Proof. For ease of notation, we omit all t subscripts. For arbitrary $z = (z^1, \dots, z^N)$ in $\mathbb{R}^{Nd}$, we want to find a solution $\zeta \in \mathbb{R}^{Nd}$ to the following system:

$$\varphi^i(\zeta) = \zeta^i - \lambda_i^N \sum_{j \neq i} P^j(\zeta^j) = z^i, \quad 1 \le i \le N. \tag{4.17}$$

Subtracting λ_j times equation i from λ_i times equation j in (4.17), we see that

$$\lambda_i \left(I+\lambda_j^N P^j\right)(\zeta^j) = \lambda_j \left(I+\lambda_i^N P^i\right)(\zeta^i) + \lambda_i z^j - \lambda_j z^i, \quad i,j=1,\dots,N. \tag{4.18}$$

1. From Lemma 4.4, we know that $I+\lambda_j^N P^j$ is a bijection; thus, from (4.18), we compute

$$\sum_{j\neq i} P^j(\zeta^j) = \sum_{j\neq i} P^j \circ \left(I+\lambda_j^N P^j\right)^{-1}\left(\frac{\lambda_j}{\lambda_i}(I+\lambda_i^N P^i)(\zeta^i) + z^j - \frac{\lambda_j}{\lambda_i} z^i\right),$$

so that from (4.17)

$$\zeta^i = z^i + \lambda_i^N \sum_{j\neq i} P^j \circ \left(I+\lambda_j^N P^j\right)^{-1}\left(\frac{\lambda_j}{\lambda_i}(I+\lambda_i^N P^i)(\zeta^i) + z^j - \frac{\lambda_j}{\lambda_i} z^i\right) =: g^{i,z}(\zeta^i). \tag{4.19}$$

2. We next show that under Condition (4.13), $g^{i,z}$ has a unique fixed point. We have

$$\begin{aligned}\left|\left(I+\lambda_j^N P^j\right)(x) - \left(I+\lambda_j^N P^j\right)(y)\right|^2 &= |x-y|^2 + 2\lambda_j^N (x-y)\cdot(P^j(x)-P^j(y)) \\ &\quad + \left(\lambda_j^N\right)^2 |P^j(x)-P^j(y)|^2 \\ &\geq \left(1+2\lambda_j^N + \left(\lambda_j^N\right)^2\right)|P^j(x)-P^j(y)|^2 \\ &\geq \left(1+\lambda_j^N\right)^2 |P^j(x)-P^j(y)|^2.\end{aligned}$$

Therefore, $P^j \circ (I+\lambda_j^N P^j)^{-1}$ is $\frac{1}{1+\lambda_j^N}-$Lipschitz. Then, since $(I+\lambda_i^N P^i)$ is $1+\lambda_i^N-$Lipschitz:

$$|g^{i,z}(x) - g^{i,z}(y)| \leq \frac{1}{N-1}\sum_{j\neq i}\frac{\lambda_j}{1+\lambda_j^N}\left(1+\lambda_i^N\right)|x-y|.$$

Notice that $\frac{\lambda_j}{1+\lambda_j^N}(1+\lambda_i^N) \leq \max(\lambda_i,\lambda_j)$, with equality if and only if $\lambda_i = \lambda_j$. Therefore, condition (4.13) implies that $K^i := \frac{1}{N-1}\sum_{j\neq i}\frac{\lambda_j}{1+\lambda_j^N}(1+\lambda_i^N) < 1$, where K^i depends only on N and the λ_js. Then, $g^{i,z}$ is a strict contraction and admits a unique fixed point that we write $\{\psi(z)\}^i$. It is then immediate that $\zeta = \psi(z)$ is the unique solution of (4.17).

3. Finally, we prove that ψ is Lipschitz with a constant depending only on N and the λ_js. Let $z_1, z_2 \in \mathbb{R}^{Nd}$, from (4.19), we compute

$$|\psi(z_1)^i - \psi(z_2)^i| \leq |z_1^i - z_2^i| + K^i|\psi(z_1)^i - \psi(z_2)^i| + 2\sup_{1\leq j\leq N}|z_1^j - z_2^j|.$$

Since $K := \sup_{1\leq j\leq N} K^j < 1$, we get $\sup_{1\leq j\leq N}|\psi(z_1)^j - \psi(z_2)^j| \leq \frac{3}{1-K}\sup_{1\leq j\leq N}|z_1^j - z_2^j|$, which completes the proof since K depends only on N and the λ_js. □

4.3. The Main Results

Similar to the classical literature on portfolio optimization with exponential utility (El Karoui and Rouge 2000; Hu, Imkeller, and Müller 2005; Mania and Schweizer 2005), we first establish a connection between Nash equilibria and a quadratic multidimensional BSDE.

THEOREM 4.7. *Under (4.1) and (4.13), assume that* $(\tilde{\pi}^1, \dots, \tilde{\pi}^N)$ *is a Nash equilibrium. Then, there exists a solution* $(Y, Z) \in \mathbb{H}^2(\mathbb{R}^N) \times \mathbb{H}^2_{loc}(\mathbb{R}^{Nd})$ *of the following N-dimensional BSDE:*

$$Y_t^i = \eta_i \xi + \frac{1}{2\eta_i} \int_t^T \left| \left(I - P_u^i\right) \circ \psi_u^i(Z_u) \right|^2 du - \int_t^T Z_u^i \cdot dB_u, \tag{4.20}$$

where ξ *is defined by (4.7), and we have*

$$\tilde{\pi}_t^i = \sigma_t^{-1} P_t^i\left(\psi_t^i(Z_t)\right) \quad \text{and} \quad V_i = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - Y_0^i)}.$$

Proof. See Section 4.5. □

Our second main result focuses on the multidimensional Black–Scholes financial market, where we can guess an explicit solution to the BSDE (4.20). Although no uniqueness result is available for the BSDE (4.20) in this context, the following complete characterization is obtained by means of a PDE verification argument.

In view of Lemma 4.6, under Condition (4.13), the maps

$$\bar{\psi}_t^i(x) := \psi_t^i(\eta_1 x, \dots, \eta_N x) \quad \text{for all } x \in \mathbb{R}^d, i = 1, \dots, N, t \in [0, T], \tag{4.21}$$

are well defined and Lipschitz continuous on $\mathbb{R}^d$.

THEOREM 4.8. *Under (4.1) and (4.13), assume that* σ *and* θ *are deterministic continuous functions. Then, there exists a unique deterministic Nash equilibrium:*

$$\hat{\pi}_t^i = \sigma(t)^{-1} P_t^i \circ \bar{\psi}_t^i(\theta(t)) \quad \text{for all } t \in [0, T]. \tag{4.22}$$

Moreover, the value function for agent i at equilibrium is given by

$$V_i = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - Y_0^i)}, \quad Y_0^i = -\frac{\eta_i}{2} \int_0^T |\theta(t)|^2 dt + \frac{1}{2\eta_i} \int_0^T \left| \left(I - P_t^i\right) \circ \bar{\psi}_t^i(\theta(t)) \right|^2 dt.$$

Proof. See Section 4.6. □

We conclude this section by two simple examples. More interesting situations will be obtained later under the additional condition that the constraints sets are linear.

EXAMPLE 4.9 (Common investment). Let $\sigma = I_d, \lambda_i = \lambda, \eta_i = \eta$, and $A_i = \bar{B}(x, r)$ for some $x \in \mathbb{R}^d$ and $r > 0$, $i = 1, \dots, N$. Here, $\bar{B}(x, r)$ is the closed ball centered at x with radius $r > 0$ for the canonical Euclidean norm of $\mathbb{R}^d$. Using Theorem 4.8, we compute the following equilibrium portfolio:

$$\hat{\pi}_t^i = P\left(\frac{\eta\theta(t)}{1-\lambda}\right) = \begin{cases} \dfrac{\eta\theta(t)}{1-\lambda} & \text{if } \dfrac{\eta\theta(t)}{1-\lambda} \in \bar{B}(x, r) \\ x + \dfrac{r}{\left|\frac{\eta\theta(t)}{1-\lambda} - x\right|}\left(\dfrac{\eta\theta(t)}{1-\lambda} - x\right) & \text{otherwise.} \end{cases}$$

Notice in particular that as one could expect, $\hat{\pi}_t^i - x$ is collinear to $\frac{\eta\theta(t)}{1-\lambda} - x$ and that $\hat{\pi}_t^i$ is in the boundary of $\bar{B}(x,r)$ whenever $\frac{\eta\theta(t)}{1-\lambda} \notin \bar{B}(x,r)$. One can prove that $|\hat{\pi}|$ is nondecreasing w.r.t. λ and η. Notice also that this expression is independent of N.

EXAMPLE 4.10 (Specific independent investments). Let $\sigma = I_d$, $\lambda_i = \lambda$, $\eta_i = \eta$, and $A_i = [a_i, b_i]e_i$, for some $a_i \le b_i$, $i = 1, \dots, N$. Here, $(e_j, 1 \le j \le d)$ is the canonical basis of $\mathbb{R}^d$. Using Theorem 4.8, we compute the following equilibrium portfolio for agent i:

$$\hat{\pi}_t^i = P^i\big(\eta\theta(t)\big) = a_i \vee \big(\eta\theta(t)\big) \wedge b_i.$$

This is exactly the same expression as in the classical case with no interaction between managers. Hence, the equilibrium portfolio is not affected by λ and N.

REMARK 4.11. Suppose that the portfolio constraints sets A_i are not convex. Then, we have to face two major problems. First, the projection operators A_i are not well defined. Second, and more importantly, the map φ may fail to be one-to-one or surjective onto $\mathbb{R}^{Nd}$. The failure of the one-to-one property means that there could exist more than one Nash equilibrium. However, the failure of the surjectivity onto $\mathbb{R}^{Nd}$, as illustrated by Examples A.1 and A.2 in the Appendix section, would lead to a constrained (N-dimensional) BSDE with no additional nondecreasing penalization process. Such BSDEs do not have solutions even in the case of Lipschitz generators, meaning that there is no Nash equilibrium in this context.

4.4. Infinite Managers Asymptotics

In the spirit of the theory of mean-field games, see Lasry and Lions (2007), we examine the situation when the number of mangers N increases to infinity with the hope of getting some more explicit qualitative results with behavioral implications. In this section, we assume that the number of assets d is not affected by the increase of the number of managers, see, however, the examples of Section 5.3. We also specialize the discussion to the case where the agents have similar preferences and only differ by their specific access to market.

The following result is similar to Proposition 5.1 in Espinosa (2010). Therefore, the proof is omitted.

PROPOSITION 4.12. *Let* $\lambda_j = \lambda \in [0, 1)$ *and* $\eta_j = \eta > 0$ *for all* $j \ge 1$. *Assume* $\frac{1}{N}\sum_{i=1}^N P_t^i \longrightarrow U_t^1$ *uniformly on any compact subsets, for all* $t \in [0, T]$ *(respectively, uniformly on* $[0, T] \times K$*, for any compact subset* K *of* $\mathbb{R}^d$*). Then,*

$$\hat{\pi}_t^{i,N} \longrightarrow \hat{\pi}_t^{i,\infty} := \sigma(t)^{-1} \circ P_t^i \circ \big(I - U_t^1 \circ (\lambda I)\big)^{-1}\big(\eta_i\theta(t) + U_t^1(0)\big)$$

for all $t \in [0, T]$ *(respectively, uniformly in* $t \in [0, T]$*).*

4.5. Proof of Theorem 4.7

Assume that $(\tilde{\pi}^1, \dots, \tilde{\pi}^N)$ is a Nash equilibrium for our problem. First, by Hölder's inequality, the admissibility conditions for all $i = 1, \dots, N$ imply that $e^{-\frac{1}{\eta_i}(X_T^i - \lambda_i \bar{X}_T^i)}$ belongs to $\mathbb{L}^p$, for any $p > 0$. Let $\mathcal{T}$ be the set of all stopping times with values in $[0, T]$, we define the following family of random variables:

$$J^{i,\pi}(\tau) := \mathbb{E}\big[-e^{-\frac{1}{\eta_i}(\int_\tau^T \sigma_u\pi_u\cdot dB_u-\lambda_i(\bar{X}^i_T-\bar{x}^i))}|\mathcal{F}_\tau\big], \tag{4.23}$$

$$\mathcal{V}^i(\tau) := \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_i} J^{i,\pi}(\tau) \quad \text{for all } \tau\in\mathcal{T} \quad \text{so that } \mathcal{V}^i(0)=e^{\frac{1}{\eta_i}(x^i-\lambda_i\bar{x}^i)}V_0^i. \tag{4.24}$$

1. By Lemma 4.13 below, the family $\{\mathcal{V}^i(\tau);\tau\in\mathcal{T}\}$ satisfies a supermartingale property. Indeed, let $\beta_t^{i,\pi} := e^{-\frac{1}{\eta_i}\int_0^t\sigma(u)\pi_u\cdot dB_u}$ for all $\pi\in\mathcal{A}_i$, we have

$$\beta_\tau^{i,\pi}\mathcal{V}_\tau^i \geq \mathbb{E}(\beta_\theta^{i,\pi}\mathcal{V}_\theta^i|\mathcal{F}_\tau) \quad \text{for all stopping times } \tau\leq\theta.$$

Then, we can extract a process $(\mathcal{V}_t^i)$ that is càdlàg and consistent with the family defined previously in the sense that $\mathcal{V}_\tau^i=\mathcal{V}^i(\tau)$ a.s. (see Karatzas and Shreve 1991, proposition I.3.14 p. 16, for more details). Moreover, this process also satisfies the dynamic programming principle stated in Lemma 4.13 so that for any $\pi\in\mathcal{A}_i$, the process $\beta^{i,\pi}\mathcal{V}^i$ is a $\mathbb{P}$-supermartingale.

The definition of a Nash equilibrium implies that $\tilde{\pi}^i$ is optimal for agent i, i.e.,

$$\begin{aligned}\mathcal{V}_0^i &= \sup_{\pi\in\mathcal{A}_i}\mathbb{E}-e^{-\frac{1}{\eta_i}(X_T^\pi-x^i-\lambda_i(\bar{X}_T^i-\bar{x}^i))}\\ &= \mathbb{E}-e^{-\frac{1}{\eta_i}(X_T^{\tilde{\pi}^i}-x^i-\lambda_i(\bar{X}_T^i-\bar{x}^i))},\end{aligned} \tag{4.25}$$

which implies that the process $\beta^{i,\tilde{\pi}^i}\mathcal{V}^i$ is a square integrable martingale, as the conditional expectation of a r.v. in $\mathbb{L}^2$.

2. We now show that the adapted and continuous process:

$$\gamma_t^i := X_t^{\tilde{\pi}^i}-x^i+\eta_i\ln\big(-\beta_t^{i,\tilde{\pi}^i}\mathcal{V}_t^i\big),\ t\in[0,T], \tag{4.26}$$

solves the required BSDE.

(a) First, by Jensen's inequality, and the fact that $\ln x\leq x$ for any $x>0$, we have

$$\begin{aligned}-\frac{1}{\eta_i}\mathbb{E}\big[X_T^{\tilde{\pi}^i}-x^i-\lambda_i\big(\bar{X}_T^i-\bar{x}^i\big)|\mathcal{F}_t\big] &\leq \ln\big(-\beta_t^{i,\tilde{\pi}^i}\mathcal{V}_t^i\big)\\ &\leq \mathbb{E}\big[-e^{-\frac{1}{\eta_i}(X_T^{\tilde{\pi}^i}-x^i-\lambda_i(\bar{X}_T^i-\bar{x}^i))}|\mathcal{F}_t\big].\end{aligned} \tag{4.27}$$

By the admissibility conditions, both sides of (4.27) belong to $\mathbb{H}^2$, as conditional expectations of random variables in $\mathbb{L}^2$. Since $X^{\tilde{\pi}^i}$ is also in $\mathbb{H}^2$, we see that γ^i is in $\mathbb{H}^2$. Then, for all $\pi\in\mathcal{A}_i$, we have that

$$M_t^{i,\pi} := -e^{-\frac{1}{\eta_i}(X_t^\pi-x^i-\gamma_t^i)} = \tilde{M}_t^i e^{-\frac{1}{\eta_i}(X_t^\pi-X_t^{\tilde{\pi}^i})}, \quad t\in[0,T],$$

where $\tilde{M}^i=\beta^{i,\tilde{\pi}^i}\mathcal{V}^i$ is a square integrable martingale. By Hölder's inequality, it follows that $e^{-\frac{1}{\eta_i}(X_t^\pi-X_t^{\tilde{\pi}^i})}\in\mathbb{L}^p$ for all $p>0$. Then, $M^{i,\pi}$ is integrable.

(b) In this step, we prove that $M^{i,\pi}$ is a supermartingale for all $\pi\in\mathcal{A}_i$. Assume, to the contrary, that there exists $\pi\in\mathcal{A}_i$, $t\geq s$ and $A\in\mathbb{F}_s$, with $\mathbb{P}(A)>0$ and such that

$$\mathbb{E}\big(-e^{-\frac{1}{\eta_i}(X_t^\pi-x^i-\gamma_t^i)}|\mathcal{F}_s\big) > -e^{-\frac{1}{\eta_i}(X_s^\pi-x^i-\gamma_s^i)} \text{ on } A,$$

and let us work toward a contradiction. Define:

$$\hat{\pi}_u(\omega) := \pi_u(\omega)\mathbf{1}_{\{[s,t]\times A\}}(u,\omega) + \tilde{\pi}_u(\omega)\mathbf{1}_{\{([s,t]\times A)^c\}}(u,\omega).$$

Since $A \in \mathcal{F}_s$, using Hölder's inequality, we see that $\hat{\pi} \in \mathcal{A}_i$ and we have

$$\begin{aligned}\mathcal{V}_0^i &\geq \mathbb{E} - e^{-\frac{1}{\eta_i}(X_T^{\hat{\pi}} - x^i - \gamma_T^i)} \\ &= \mathbb{E}\big[\mathbb{E}\big\{- e^{-\frac{1}{\eta_i}(X_T^{\hat{\pi}} - x^i - \gamma_T^i)}|\mathcal{F}_t\big\}\big] = \mathbb{E} - e^{-\frac{1}{\eta_i}(X_t^{\hat{\pi}} - x^i - \gamma_t^i)}\end{aligned}$$

by the fact that $\hat{\pi} = \tilde{\pi}$ on $[t, T]$ and $\tilde{M}^i$ is a martingale. Since $\mathbb{P}[A] > 0$, $A \in \mathcal{F}_s$, and recalling the definition of $\hat{\pi}$, this implies that

$$\begin{aligned}\mathcal{V}_0^i &\geq \mathbb{E}\big[\mathbb{E}\big\{- e^{-\frac{1}{\eta_i}(X_t^{\hat{\pi}} - x^i - \gamma_t^i)}|\mathcal{F}_s\big\}\big] \\ &= \mathbb{E}\big[\mathbb{E}\big\{- e^{-\frac{1}{\eta_i}(X_t^{\pi} - x^i - \gamma_t^i)}|\mathcal{F}_s\big\}1_A + \mathbb{E}\big\{- e^{-\frac{1}{\eta_i}(X_t^{\tilde{\pi}} - x^i - \gamma_t^i)}|\mathcal{F}_s\big\}1_{A^c})\big] \\ &> \mathbb{E} - e^{-\frac{1}{\eta_i}(X_s^{\tilde{\pi}} - x^i - \gamma_s^i)} = -e^{\frac{1}{\eta_i}\gamma_0^i} = \mathcal{V}_0^i,\end{aligned}$$

which provides the required contradiction.

(c) Since $\tilde{M}^i = \beta^{i,\tilde{\pi}^i}\mathcal{V}^i$ is a martingale, it follows from the martingale representation theorem that $\tilde{M}^i$ is an Itô process. Therefore, (4.26) implies that γ^i is also an Itô process defined by some coefficients b^i and ζ^i:

$$d\gamma_t^i = -b_t^i dt + \zeta_t^i \cdot dW_t \quad \text{with } (\gamma^i, \zeta^i) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_{loc}(\mathbb{R}^d). \tag{4.28}$$

Moreover, by Jensen's inequality, $\ln(-M^{i,\tilde{\pi}^i})$ is a supermartingale, and by (4.27), it is bounded in $\mathbb{L}^2$. Therefore, it admits a Doob–Meyer decomposition $\ln(-M^{i,\tilde{\pi}^i}) = N + A$, where N is a (uniformly integrable) martingale and A a decreasing process. The martingale representation theorem then implies that there exists a process $\delta \in \mathbb{H}^2_{loc}(\mathbb{R}^d)$ such that $N_t = \int_0^t \delta_u \cdot dW_u$. Using (4.27) and (4.28), we get $\zeta_t^i = \sigma_t\tilde{\pi}_t^i + \eta_i\delta_t$.

(d) We next compute the drift of $M^{i,\pi}$. From the previous supermartingale and martingale properties of $M^{i,\pi}$ and $\tilde{M}^i$, respectively, together with (4.28), we get

$$\begin{aligned}b_t^i &\leq \frac{1}{2\eta_i}\left|\sigma_t\pi_t - \left(\zeta_t^i + \eta_i\theta_t\right)\right|^2 - \frac{\eta_i}{2}|\theta_t|^2 - \zeta_t^i \cdot \theta_t \quad \text{for all } \pi \in \mathcal{A}_i, \\ \text{and } \mathrm{b}_t^i &= \frac{1}{2\eta_i}\left|\sigma_t\tilde{\pi}_t^i - \left(\zeta_t^i + \eta_i\theta_t\right)\right|^2 - \frac{\eta_i}{2}|\theta_t|^2 - \zeta_t^i \cdot \theta_t.\end{aligned}$$

This implies that

$$\begin{aligned}\tilde{\pi}_t^i &= \sigma_t^{-1} P_t^i\left(\zeta_t^i + \eta_i\theta_t\right), \\ b_t^i &= f^i(t, \zeta_t^i) = \frac{1}{2\eta_i} d\left(\zeta_t^i + \eta_i\theta_t, \sigma_t A_i\right)^2 - \frac{\eta_i}{2}|\theta_t|^2 - \zeta_t^i \cdot \theta_t,\end{aligned} \tag{4.29}$$

and therefore $(\gamma^i, \zeta^i) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_{loc}(\mathbb{R}^d)$ is a solution of the BSDE:

$$\begin{aligned}d\gamma_t^i &= \left(\zeta_t^i \cdot \theta_t + \frac{\eta_i|\theta_t|^2}{2} - \frac{1}{2\eta_i}\left|\left(I - P_t^i\right)\left(\zeta_t^i + \eta_i\theta_t\right)\right|^2\right)dt + \zeta_t^i \cdot dW_t, \\ \gamma_T^i &= \lambda_i\left(\bar{X}_T^i - \bar{x}_i\right) = \lambda_i^N \sum_{j\neq i}\int_0^T \tilde{\pi}_u^j \cdot \sigma_u(dW_u + \theta_u du).\end{aligned}$$

Recalling that $dB_t = dW_t + \theta_t dt$, we can write it:

$$(4.30)\qquad d\gamma_t^i = \left(\frac{\eta_i|\theta_t|^2}{2} - \frac{\eta_i}{2}\big|(I - P_t^i)(\zeta_t^i + \eta_i\theta_t)\big|^2\right)dt + \zeta_t^i \cdot dB_t,$$
$$\gamma_T^i = \lambda_i(\bar{X}_T^i - \bar{x}_i) = \lambda_i^N \sum_{j\neq i} \int_0^T \tilde{\pi}_u^j \cdot \sigma_u dB_u.$$

3. We finally put together the N BSDEs obtained in step 2. Since $(\tilde{\pi}^1, \dots, \tilde{\pi}^N)$ is a Nash equilibrium, equation (4.30) holds for each $i = 1, \dots, N$. Replacing the value of $\tilde{\pi}^j$ by (4.29) in the expression of γ^i and writing $\Gamma^i := \zeta^i + \eta_i\theta$, we see that (γ^i, Γ^i) must satisfy for each $t \in [0, T]$:

$$\gamma_t^i = \lambda_i^N \sum_{j\neq i} \int_0^T P_u^j(\Gamma_u^j) \cdot dB_u - \frac{\eta_i}{2}\int_t^T |\theta_u|^2 du + \frac{1}{2\eta_i}\int_t^T \big|(I - P_u^i)(\Gamma_u^i)\big|^2 du$$
$$- \int_t^T (\Gamma_u^i - \eta_i\theta_u) \cdot dB_u,$$

so that the adapted process $Y_t^i := \gamma_t^i - \frac{\eta_i}{2}\int_0^t |\theta_u|^2 du + \frac{\eta_i}{2}\int_0^t \theta_u \cdot dB_u - \lambda_i^N \sum_{j\neq i} \int_0^t P_u^j(\Gamma_u^j) \cdot dB_u$, $t \in [0, T]$, satisfies:

$$Y_t^i = \eta_i\xi + \frac{1}{2\eta_i}\int_t^T \big|(I - P_u^i)(\Gamma_u^i)\big|^2 du - \int_t^T \left(\Gamma_u^i - \lambda_i^N \sum_{j\neq i} P_u^j(\Gamma_u^j)\right) \cdot dB_u,$$

with $(Y^i, \Gamma^i) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_{loc}(\mathbb{R}^d)$. We finally define

$$Z_t^i := \varphi_t^i(\Gamma_t) = \Gamma_t^i - \lambda_i^N \sum_{j\neq i} P_t^j(\Gamma_t^j).$$

Under (4.13), using Lemma 4.6, we know that φ_t is invertible. As a consequence, $(Y, Z) \in \mathbb{H}^2(\mathbb{R}^N) \times \mathbb{H}^2_{loc}(\mathbb{R}^{Nd})$ is a solution of the following system of BSDEs:

$$Y_0^i = \eta_i\xi + \frac{1}{2\eta_i}\int_0^T \big|(I - P_t^i)(\psi_t^i(Z_t))\big|^2 dt - \int_0^T Z_t^i \cdot dB_t.$$

Moreover, for each i, the equilibrium portfolio is given by

$$\sigma(t)\tilde{\pi}_t^i = P_t^i[\psi_t(Z_t)^i], \quad t \in [0, T].$$

The following dynamic programming principle was used in step 1 of the previous proof.

LEMMA 4.13 (Dynamic Programming). *For any stopping times $\tau \le \nu$ in $\mathcal{T}$, we have*

$$\mathcal{V}^i(\tau) = ess\sup_{\pi\in\mathcal{A}_i} \mathbb{E}\big[e^{-\frac{1}{\eta_i}\int_\tau^\nu \sigma_u\pi_u\cdot, dB_u}\mathcal{V}^i(\nu)|\mathcal{F}_\tau\big], \quad i = 1, \dots, N.$$

Proof. Let $\tau \le \nu \le T$ a.s. We first obtain by the tower property that

$$\begin{aligned}\mathcal{V}^i(\tau) &= \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_i} \mathbb{E}\big[\mathbb{E}\big[-e^{-\frac{1}{\eta_i}(\int_\nu^T \sigma_u\pi_u\cdot dB_u-\lambda_i(\bar{X}^i_T-\bar{x}^i))}|\mathcal{F}_\nu\big]e^{-\frac{1}{\eta_i}\int_\tau^\nu \sigma(u)\pi_u\cdot dB_u}|\mathcal{F}_\tau\big]\\ &\le \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_i} \mathbb{E}\big[e^{-\frac{1}{\eta_i}\int_\tau^\nu \sigma_u\pi_u\cdot dB_u}\mathcal{V}^i(\nu)|\mathcal{F}_\tau\big].\end{aligned}$$

To prove the converse inequality, we fix $\pi^0 \in \mathcal{A}_i$ and we observe that $J^{i,\pi}(\nu)$ defined by (4.23) depends on π only through its values on $[\nu, T]$. Therefore, we have the identity:

$$\mathcal{V}^i(\nu) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_i(\nu)} J^{i,\pi}(\nu), \quad \text{where } \mathcal{A}_i(\nu) := \{\pi \in \mathcal{A}_i; \pi = \pi^0 \quad \text{on } [0,\nu], dt\otimes d\mathbb{P}\text{-a.e}\}.$$

We next observe that the family $\{J^{i,\pi}(\nu), \pi \in \mathcal{A}_i(\nu)\}$ is closed under pairwise maximization. Indeed, let π_1, π_2 in $\mathcal{A}_i(\nu)$, $A := \{\omega \in \Omega; J^{i,\pi_1}(\nu)(\omega) \ge J^{i,\pi_2}(\nu)(\omega)\}$ and define the process $\pi := 1_A\pi_1 + 1_{\Omega\setminus A}\pi_2$. Since $\pi^1 = \pi^2 = \pi^0$ on $[\![0,\nu]\!]$, and $A \in \mathcal{F}_\nu$, it is immediate that π is predictable. We compute $\mathbb{E}e^{\pm\frac{p}{\eta_i}(X^\pi_\tau - X^\pi_\nu)} = \mathbb{E}e^{\pm\frac{p}{\eta_i}\int_\vartheta^\tau \sigma_t\pi^1_t\cdot dB_t}1_A + \mathbb{E}e^{\pm\frac{p}{\eta_i}\int_\vartheta^\tau \sigma_t\pi^2_t\cdot dB_t}1_{\Omega\setminus A}$ so that since $\pi^1, \pi^2 \in \mathcal{A}_i(\nu)$, the family $\{e^{\pm\frac{1}{\eta_i}(X^\pi_\tau - X^\pi_\vartheta)}; \vartheta \le \tau \in \mathcal{T}\}$ is uniformly bounded in any $\mathbb{L}^p$, $p > 1$. Therefore, $\pi \in \mathcal{A}_i(\nu)$ and it is immediate that $J^{i,\pi}(\nu) = \max(J^{i,\pi_1}(\nu), J^{i,\pi_2}(\nu))$. Then, it follows from Theorem A.3 (p. 324 in Karatzas and Shreve 1998) that there exists a sequence $(\hat{\pi}_n)$ satisfying:

- $\forall n, \hat{\pi}_n = \pi^0$ on $[\![0,\nu]\!]$
- $(J^{i,\hat{\pi}_n}(\nu))$ is nondecreasing and converges to $\mathcal{V}^i(\nu)$.

Then, we have

$$J^{i,\hat{\pi}_n}(\tau) = \mathbb{E}[J^{i,\hat{\pi}_n}(\nu)e^{-\frac{1}{\eta_i}\int_\tau^\nu \sigma_u\pi^0_u\cdot dB_u}|\mathcal{F}_\tau].$$

Since $J^{i,\hat{\pi}_n}(\nu)$ is nondecreasing and converges to $\mathcal{V}^i(\nu)$, it follows from the monotone convergence theorem that

$$\mathcal{V}^i(\tau) \ge \mathbb{E}\big[e^{-\frac{1}{\eta_i}\int_\tau^\nu \sigma_u\pi^0_u\cdot dB_u}\mathcal{V}^i(\nu)|\mathcal{F}_\tau\big],$$

and the required inequality follows from the arbitrariness of π^0. □

4.6. Proof of Theorem 4.8

1. We first prove that the portfolio (4.22) is indeed a Nash equilibrium. The idea is to show that we can make the formal computations of Section 4.1 in the reverse sense.

(a) Let

$$\xi_t := \int_0^t \theta(u).dB_u - \frac{1}{2}\int_0^T |\theta(u)|^2 du, \quad t \in [0,T], \tag{4.31}$$

Since θ and σ are deterministic and continuous functions, the functions P^is are also deterministic and continuous w.r.t. $(t,z) \in [0,T]\times\mathbb{R}^d$. Let us prove that the same holds for ψ, and therefore for that $\hat{\pi}^i_t := \sigma(t)^{-1}P^i_t \circ \psi^i_t(Z_t)$ is deterministic and continuous w.r.t. $t \in [0,T]$. Indeed, it is immediate that φ is deterministic and continuous w.r.t. (t,ζ) so that ψ is a deterministic function of (t,z). Then, from Lemma 4.6, under condition (4.13), ψ_t is Lipschitz in z, uniformly in t, so that there exists a constant $K > 0$ such that for all $t \in [0,T]$, and all $z, z' \in \mathbb{R}^d$, $|\psi_t(z) - \psi_t(z')| \le |z - z'|$. Let $t_n \to t$, $z \in \mathbb{R}^{Nd}$, and $\zeta := \varphi_t(z)$.

We define $z_n := \varphi_{t_n}(\zeta)$ for each n. Since φ is continuous w.r.t. t, $z_n \to z$, and we have, for all n, $\psi_{t_n}(z_n) = \zeta$, so that $|\psi_{t_n}(z) - \zeta| = |\psi_{t_n}(z) - \psi_{t_n}(z_n)| \le K|z - z_n| \to 0$. Therefore, ψ is continuous w.r.t. t. Then, if $z_n \to z$ and $t_n \to t$, we compute $|\psi_{t_n}(z_n) - \psi_t(z)| \le |\psi_{t_n}(z_n) - \psi_{t_n}(z)| + |\psi_{t_n}(z) - \psi_t(z)| \to 0$, since ψ is continuous w.r.t. t and Lipschitz in z uniformly in t. As a consequence, we can define the following adapted and continuous processes:

$$Z_t^i := \eta_i \theta(t) \quad \text{and } Y_t^{\,i} := \eta_i \xi_t + \frac{1}{2\eta_i} \int_t^T \left|\left(I - P_u^i\right) \circ \psi_u^i(Z_u)\right|^2 du, \quad t \in [0, T].$$

Then, we directly verify that (Y, Z) satisfies the following N-dimensional BSDE:

$$Y_t^{\,i} = \eta_i \xi + \frac{1}{2\eta_i} \int_t^T \left|\left(I - P_u^i\right)\left(\psi_u^i(Z_u)\right)\right|^2 du - \int_t^T Z_u^{\,i} \cdot dB_u.$$

Set

$$\begin{aligned}
\gamma_t^i &= Y_t^{\,i} + \frac{\eta_i}{2} \int_0^t |\theta(u)|^2 du - \eta_i \int_0^t \theta(u) \cdot dB_u + \lambda_i^N \sum_{j \neq i} \int_0^t P_u^j\left(\psi_u^j(Z_u)\right) \cdot dB_u,\\
\zeta_t^i &= \psi_t(Z_t)^i - \eta_i \theta(t) = \left(\bar{\psi}_t^i - \eta_i I\right)(\theta(t)).
\end{aligned}$$

By the same computations as in Section 4.1, we see that for all $i = 1, \dots, N$, (γ^i, ζ^i) is a solution of the one-dimensional BSDE:

$$\begin{aligned}
d\gamma_t^i &= \left(\zeta_t^i \cdot \theta(t) + \frac{\eta_i |\theta(t)|^2}{2} - \frac{1}{2\eta_i}\left|\left(I - P_t^i\right)\left(\zeta_t^i + \eta_i \theta(t)\right)\right|^2\right) dt + \zeta_t^i \cdot dW_t,\\
\gamma_T^i &= \lambda_i^N \sum_{j \neq i} \int_0^T \hat{\pi}_u^j \cdot \sigma(u)(dW_u + \theta(u)\, du).
\end{aligned}$$

Then, using the definition of φ and ψ, we can rewrite γ^i as

$$\begin{aligned}
\gamma_t^i &= -\frac{\eta_i}{2} \int_0^t |\theta(u)|^2 du + \frac{1}{2\eta_i} \int_t^T \left|\left(I - P_u^i\right) \circ \bar{\psi}_u^i(\theta(u))\right|^2 du + \lambda_i^N \sum_{j \neq i} \int_0^t P_u^j[\psi_u(Z_u)]^j \cdot dB_u\\
&= -\frac{\eta_i}{2} \int_0^t |\theta(u)|^2 du + \frac{1}{2\eta_i} \int_t^T \left|\left(I - P_u^i\right) \circ \bar{\psi}_u^i(\theta(u))\right|^2 du + \int_0^t \zeta_u^i \cdot dB_u\\
&= -\frac{\eta_i}{2} \int_0^t |\theta(u)|^2 du + \frac{1}{2\eta_i} \int_t^T \left|\left(I - P_u^i\right) \circ \bar{\psi}_u^i(\theta(u))\right|^2 du + \int_0^t \left(\bar{\psi}_u^i - \eta_i I\right)(\theta(u)) \cdot dB_u.
\end{aligned}$$

(b) Throughout this step, we fix an integer $i \in \{1, \dots, N\}$, and we define

$$M_t^{\pi} := -e^{-\frac{1}{\eta_i}\left(X_t^{i,\pi} - x^i - \gamma_t^i\right)} \quad \text{for all } \pi \in \mathcal{A}_i.$$

By Itô's formula, it follows that M^{π} is a local supermartingale for each $\pi \in \mathcal{A}_i$, and $M^{\hat{\pi}^i}$ is a local martingale. Then, there exist increasing sequences of stopping times (τ_n^{π}) in $\mathcal{T}$ such that for each π, $\tau_n^{\pi} \to T$ a.s. and for each n and any $s \le t$:

$$\mathbb{E}\left[M_{t \wedge \tau_n^{\pi}}^{\pi} | \mathcal{F}_s\right] \le M_{s \wedge \tau_n^{\pi}}^{\pi} \quad \text{for all} \quad \pi \in \mathcal{A}_i \quad \text{and} \quad \mathbb{E}\left[M_{t \wedge \tau_n^{\hat{\pi}}}^{\hat{\pi}^i} | \mathcal{F}_s\right] = M_{s \wedge \tau_n^{\hat{\pi}}}^{\hat{\pi}^i}. \tag{4.32}$$

We next introduce the measure $\mathbb{Q}^i$, equivalent to $\mathbb{P}$, defined by its Radon–Nikodym density:

$$L_t^i = \frac{d\mathbb{Q}^i}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = e^{\int_0^t (\frac{1}{\eta_i}\bar{\psi}_u^i - I)(\theta(u))\cdot dW_u - \frac{1}{2}\int_0^t |(\frac{1}{\eta_i}\bar{\psi}_u^i - I)(\theta(u))|^2 du}. \tag{4.33}$$

We denote by $\mathbb{E}^i$ the expectation operator under $\mathbb{Q}^i$. Since θ is a deterministic and continuous function on $[0, T]$, $-\frac{\eta_i}{2}\int_0^t |\theta(u)|^2 du + \frac{1}{2\eta_i}\int_t^T |(I - P_u^i) \circ \bar{\psi}_u^i(\theta(u))|^2 du$ is bounded. Then, for any $\pi \in \mathcal{A}_i$:

$$\begin{aligned} \mathbb{E} M_{t\wedge\tau_n}^{\pi} = \frac{1}{L_s^i}\mathbb{E}^i\big[- e^{-\frac{1}{\eta_i}(X_{t\wedge\tau_n}^{\pi} - x^i) - \frac{1}{2}\int_0^{t\wedge\tau_n} |\theta(u)|^2 du + \frac{1}{2\eta_i^2}\int_{t\wedge\tau_n}^T |(I - P_u^i)\circ f_u^i(\theta(u))|^2 du} \\ \times e^{\int_0^{t\wedge\tau_n} (\frac{1}{\eta_i}\bar{\psi}_u^i - I)(\theta(u))\cdot\theta(u) du + \frac{1}{2}\int_0^{t\wedge\tau_n} |(\frac{1}{\eta_i}\bar{\psi}_u^i - I)(\theta(u))|^2 du}\big], \end{aligned} \tag{4.34}$$

where we simply denoted $\tau_n := \tau_n^{\pi}$. In (4.34), all the terms inside the expectation other than $e^{-\frac{1}{\eta_i}X_{t\wedge\tau_n}^{\pi}}$ are bounded. We shall prove in step 1(c) below that the family $\{e^{-\frac{1}{\eta_i}X_\tau^{\pi}}; \tau \in \mathcal{T}\}$ is uniformly integrable under $\mathbb{Q}^i$. Hence, the sequence of processes inside the expectation in (4.34) is uniformly integrable under $\mathbb{Q}^i$, and we may apply the dominated convergence theorem to pass to the limit $n \to \infty$, and we obtain $\lim_{n\to\infty} \mathbb{E} M_{t\wedge\tau_n}^{\pi} = \mathbb{E} M_t^{\pi}$. Together with (4.32), this implies that

$$\begin{aligned} &\mathbb{E} - e^{-\frac{1}{\eta_i}(X_t^{\pi} - x^i - \gamma_t^i)} \le -e^{\frac{1}{\eta_i}\gamma_0^i} \quad \text{for all } \pi \in \mathcal{A}_i \text{ and} \\ &\mathbb{E} - e^{-\frac{1}{\eta_i}(X_t^{\hat{\pi}^i} - x^i - \gamma_t^i)} = -e^{\frac{1}{\eta_i}\gamma_0^i}. \end{aligned}$$

Multiplying by $e^{-\frac{1}{\eta_i}(x^i - \lambda_i\bar{x}^i)}$, we finally get $V_i = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i\bar{x}^i - Y_0^i)}$, since $Y_0^i = \gamma_0^i$, and $\hat{\pi}^i$ is optimal for agent i. Hence, $(\hat{\pi}^1, \dots, \hat{\pi}^N)$ is a Nash equilibrium.

(c) In this step, we prove that the family $\{Y_\tau := e^{-\frac{1}{\eta_i}X_\tau^{i,\pi}} : \tau \in \mathcal{T}\}$ is $\mathbb{Q}^i$ uniformly integrable for all $\pi \in \mathcal{A}_i$. Fix some $p > 1$. Then, by the admissibility condition, the family $\{Y_\tau : \tau \in \mathcal{T}\}$ is uniformly bounded in $\mathbb{L}^p(\mathbb{P})$. With $r := (1 + p)/2$, it follows that the family $\{Y_\tau^r : \tau \in \mathcal{T}\}$ is uniformly integrable. Then, for all $c > 0$ and $\tau \in \mathcal{T}$, it follows from Hölder's inequality:

$$\mathbb{E}^{\mathbb{Q}^i}[Y_\tau 1_{Y_\tau \ge c}] = \mathbb{E}\big[L_T^i Y_\tau 1_{Y_\tau \ge c}\big] \le \big\|L_T^i\big\|_{\mathbb{L}^q(\mathbb{P})} \|Y_\tau^r 1_{Y_\tau \ge c}\|_{\mathbb{L}^r(\mathbb{P})},$$

where q is defined by $(1/q) + (1/r) = 1$. Since $\{Y^r : \tau \in \mathcal{T}\}$ is uniformly integrable, the last term uniformly goes to 0 as $c \to \infty$.

REMARK 4.14. We mention again that based on a first version of this paper contained in the PhD thesis Espinosa (2010), Frei and dos Reis developed many interesting extensions in Frei and dos Reis (2011). In particular, their argument for the existence part of Theorem 4.8 is shorter than our previous step 1.

2. We now prove uniqueness by using a verification argument.

(a) Let $(\pi^1, \dots, \pi^N)$ be a deterministic Nash equilibrium, and define for all $i = 1, \dots, N$:

$$u^i(t, x, y) := -e^{-\frac{1}{\eta_i}(x - \lambda_i y) - \frac{1}{2}\int_t^T |\theta(u)|^2 du + \frac{1}{2\eta_i^2}\int_t^T |(I - P_u^i)(\eta_i\theta(u) + \lambda_i\sigma(u)\bar{\pi}_N^i(u))|^2 du}, \tag{4.35}$$

where

$$\bar{\pi}_N^i(u) := \frac{1}{N-1}\sum_{j\neq i}\pi^j(u).$$

Since π^j is a continuous function for all $j = 1, \dots, N$, the functions u^i are C^1 in the t variable. Direct calculation reveals that u^i is a classical solution of the equation:

$$-\partial_t u^i - \sup_{p\in A_i} L^p u^i = 0 \quad \text{and} \quad u^i(T, x, y) = -e^{-(x-\lambda_i y)/\eta_i},$$

where for all $p \in A_i$, L^p is the linear second-order differential operator:

$$\begin{aligned} L^p &:= \sigma(t)\bar{\pi}_N^i(t)\cdot\theta(t)\partial_y + \frac{1}{2}\left|\sigma(t)\bar{\pi}_N^i(t)\right|^2\partial_{yy}^2 \\ &\quad + \sigma(t)p.\theta(t)\partial_x + \sigma(t)p\cdot\sigma(t)\bar{\pi}_N^i(t)\partial_{xy}^2 + \frac{1}{2}|\sigma(t)p|^2\partial_{xx}^2, \end{aligned}$$

and the supremum is attained at a unique point

$$\pi_t^* := \sigma(t)^{-1}P_t^i\big(\eta_i\theta(t) + \lambda_i\sigma(t)\bar{\pi}_N^i(t)\big). \tag{4.36}$$

(b) In this step, we prove that $u^i(0, X_0^i, \bar{X}_0^i) = V_0^i$. First, by Itô's formula, we have for all $\pi \in \mathcal{A}_i$:

$$u^i(t, x, y) = u^i\big(\tau_n, X_{\tau_n}^\pi, \bar{X}_{\tau_n}^i\big) - \int_t^{\tau_n} L^\pi u^i\big(r, X_r^\pi, \bar{X}_r^i\big)\,dr - \int_t^{\tau_n}\big(\pi - \bar{\pi}_N^i\big)(r)\cdot\sigma(r)\,dW_r, \tag{4.37}$$

where $\tau_n := \inf\{r \ge t, |X_r^\pi - x| \ge n \text{ or } |\bar{X}_r^i - \bar{x}^i| \ge n\}$. Taking conditional expectations in (4.37), and using the fact that $L^{\pi_t}u^i \le 0$ for any $\pi \in \mathcal{A}_i$, we get

$$u^i(t, x, y) \ge \mathbb{E}_{t,x,y}u^i(\tau_n, X_{\tau_n}^\pi, \bar{X}_{\tau_n}^i) \quad \text{for all } \pi \in \mathcal{A}_i. \tag{4.38}$$

Since the π_js, σ, and θ are continuous deterministic functions and $\pi^i \in \mathcal{A}_i$, it follows from Hölder's inequality that $\{e^{-\frac{1}{\eta_i}(X_\tau^i - \lambda_i\bar{X}_\tau^i)}, \tau \in \mathcal{T}\}$ is uniformly bounded in any $\mathbb{L}^p$. By the definition of U^i, this property is immediately inherited by the family $\{u^i(\tau, X_\tau^\pi, \bar{X}_\tau^i), \tau \in \mathcal{T}\}$. Therefore, taking the limit $n \to \infty$ in (4.38), we get $u^i(t, x, y) \ge \mathbb{E}_{t,x,y}e^{-\frac{1}{\eta_i}(X_T^\pi - \lambda_i\bar{X}_T^i)}$. By the arbitrariness of $\pi \in \mathcal{A}_i$, this implies that $u^i(0, X_0^i, \bar{X}_0^i) \ge V_0^i$.

We next observe that $\pi^* \in \mathcal{A}_i$ and the inequality in (4.38) is turned into an equality if π^* is substituted to π. By the dominated convergence theorem, this provides

$$u^i(t, x, y) = \mathbb{E}_{t,x,y}e^{-\frac{1}{\eta_i}(X_T^{\pi^*} - \lambda_i\bar{X}_T^i)},$$

which, in view of (4.38), shows that $u^i(0, X_0^i, \bar{X}_0^i) = V_0^i$.

(c) To see that the continuous deterministic Nash equilibrium is unique, consider another continuous deterministic Nash equilibrium $(\hat{\pi}^1, \dots, \hat{\pi}^N)$, and denote by $\hat{u}^i$ the corresponding value functions as in (4.35). It suffices to observe that $L^\pi\hat{u}^i < 0$ on any nonempty open subset B of $[0, T]$ such that $\pi \neq \pi^*$ on B, and the inequality (4.38) is strict. Therefore, any Nash equilibrium must satisfy (4.36) for every $i = 1, \dots, N$.

Set $\hat{\gamma}_t^i := \sigma(t)\hat{\pi}_t^i$, and let $\hat{\gamma}$ be the matrix whose ith line is $\hat{\gamma}^i$. From the previous argument, $(\hat{\pi}^1, \ldots, \hat{\pi}^N)$ is a Nash equilibrium if and only:

$$\Gamma_t^i(\hat{\gamma}_t) := P_t^i\left(\eta_i\theta(t) + \lambda_i^N \sum_{j\neq i} \hat{\gamma}_t^j\right) = \hat{\gamma}_t^i, \qquad i = 1, \ldots, N, t \in [0, T], \tag{4.39}$$

i.e., $\hat{\gamma}_t$ is a fixed point of Γ_t for all $t \in [0, T]$. Using Lemma 4.15 below, we have the uniqueness of a Nash equilibrium. Finally, the expression for V^i at equilibrium follows from the last statement of Lemma 4.15 together with (4.35).

Recall the function $\bar{\psi}^i$ defined in (4.21).

LEMMA 4.15. *Under (4.13), the function Γ_t defined in (4.39) has a unique fixed point $\hat{\gamma}_t$ for all $t \in [0, T]$, given by*

$$\hat{\gamma}_t^i = P_t^i \circ \bar{\psi}_t^i(\theta) \quad \textit{and satisfying } \bar{\psi}_t^i(\theta) = \eta_i\theta + \lambda_i^N \sum_{j\neq i} \hat{\gamma}_t^j.$$

Proof.

1. Since P_t^i is a contraction, we compute

$$|\Gamma_t(x_1) - \Gamma_t(x_2)|_1 := \sum_{i=1}^N \left|\Gamma_t^i(x_1) - \Gamma_t^i(x_2)\right| \leq \sum_{i=1}^N \frac{1}{N-1} \sum_{j\neq i} \left|x_1^j - x_2^j\right| = |x_1 - x_2|_1,$$

proving that Γ_t is a contraction.

2. We next show that $(\Gamma_t)^2 := \Gamma_t \circ \Gamma_t$ is a strict contraction. Indeed, under (4.13), we may assume without loss of generality that $\lambda_1 < 1$. Then,

$$\left|\Gamma_t^i \circ \Gamma_t(x_1) - \Gamma_t^i \circ \Gamma_t(x_2)\right| \leq \lambda_i^N \sum_{j\neq i} \left|\Gamma_t^j(x_1) - \Gamma_t^j(x_2)\right| \leq \lambda_i^N \sum_{j\neq i} \lambda_j^N \sum_{k\neq j} \left|x_1^k - x_2^k\right|,$$

so that

$$\begin{aligned}
|(\Gamma_t)^2(x_1) - (\Gamma_t)^2(x_2)|_1 &\leq \sum_{i=1}^N \sum_{j\neq i} \sum_{k\neq j} \lambda_i^N \lambda_j^N \left|x_1^k - x_2^k\right| \\
&\leq \frac{\lambda_1}{(N-1)^2}\left(\sum_{k=1}^N (N-2)\left|x_1^k - x_2^k\right| + \sum_{k\neq 1}\left|x_1^k - x_2^k\right|\right) \\
&\quad + \frac{N-2}{N-1}\left|x_1^1 - x_2^1\right| + \frac{N-1+(N-2)^2}{(N-1)^2} \sum_{k\neq 1} \left|x_1^k - x_2^k\right| \\
&\leq \left(\frac{\lambda_1(N-2)}{(N-1)^2} + \frac{(N-2)^2 + N - 1}{(N-1)^2}\right)|x_1 - x_2|_1.
\end{aligned}$$

Observe that $N - 2 + N - 1 + (N-2)^2 = (N-1)^2$. Then, $\lambda_1 < 1$ implies that $(\Gamma_t)^2$ is a strict contraction.

3. Therefore, $(\Gamma_t)^n$ is a strict contraction as well for any $n \geq 2$. As a consequence, $(\Gamma_t)^2$, $(\Gamma_t)^3$, and $(\Gamma_t)^6$, respectively, admit a unique fixed point x_2, x_3, and x_6, respectively. It is immediate that x_2 and x_3 are also fixed points for $(\Gamma_t)^6$; therefore, $x_2 = x_3 = x_6$, and finally, $x_2 = (\Gamma_t)^3(x_2) = \Gamma_t \circ (\Gamma_t)^2(x_2) = \Gamma_t(x_2)$ so that x_2 is a

fixed point of Γ_t. The uniqueness is immediate since a fixed point of Γ_t is also a fixed point of $(\Gamma_t)^2$.

4. Let $\Theta \in \mathbb{R}^{Nd}$ be defined by $\Theta^i = \eta_i\theta$. By definition of ψ_t in Lemma 4.6, $\varphi_t^i \circ \psi_t(\Theta) = \eta_i\theta$ for all $i = 1, \dots, N$. Using the definition of φ_t in (4.9), this implies that

$$\psi_t^i(\Theta) = \eta_i\theta + \lambda_i^N \sum_{j\neq i} P_t^j \circ \psi_t^j(\Theta). \tag{4.40}$$

Applying P_t^i and setting $\hat{\gamma}_t^i = P_t^i \circ \psi_t^i(\Theta)$, this provides $\hat{\gamma}_t^i = \Gamma_t^i(\hat{\gamma}_t)$, for each $i = 1, \dots, N$. By the definition of $\bar{\psi}^i$ together with the expression of ψ, we have $\bar{\psi}_t^i(\theta) = \psi_t^i(\Theta)$ so that $\hat{\gamma}_t^i = P_t^i \circ \bar{\psi}_t^i(\theta)$. Plugging it into (4.40) provides the last statement of the lemma. □

5. LINEAR PORTFOLIO CONSTRAINTS

We now focus on the case where the sets of constraints are such that

$$A_i \text{ is a vector subspace of } \mathbb{R}^d, \quad \text{for all } i = 1, \dots, N. \tag{5.1}$$

Our main objective in this section is to exploit the linearity of the projection operators P^i in order to derive more explicit results.

5.1. Nash Equilibrium Under Linear Portfolio Constraints

In the present context, we show that condition (4.13) in Theorem 4.8 can be weakened to

$$\prod_{i=1}^{N} \lambda_i < 1 \quad \text{or} \quad \bigcap_{i=1}^{N} A_i = \{0\}. \tag{5.2}$$

In view of Lemma 4.4 (which is obvious in the present linear case), the map

$$R_t^i := \frac{1}{N-1} \sum_{j\neq i} \lambda_j P_t^j \left(I + \lambda_j^N P_t^j\right)^{-1} \left(I + \lambda_i^N P_t^i\right) \tag{5.3}$$

is well defined. Moreover, for any $j = 1, \dots, N$, since P_t^j is a projection, we compute that $(I + \lambda_j^N P_t^j)^{-1} = I - \frac{\lambda_j^N}{1+\lambda_j^N} P_t^j$ so that

$$R_t^i = \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P_t^j \left(I + \lambda_i^N P_t^i\right).$$

The following statement is more precise than Lemma 4.6.

LEMMA 5.1. *Let $(A_i)_{1\leq i\leq N}$ be vector subspaces of $\mathbb{R}^d$. Then, for all $t \in [0, T]$:*

(i) the linear operator φ_t is invertible if and only if (5.2) is satisfied,
(ii) this condition is equivalent to the invertibility of the linear operators $I - R_t^i$, $i = 1, \dots, N$,

(iii) under (5.2), the ith component of $\psi_t = \varphi_t^{-1}$ is given by

$$\psi_t^i(z) = (I - R_t^i)^{-1} \left(z^i + \sum_{j \neq i} \frac{1}{1 + \lambda_j^N} P_t^j \left(\lambda_i^N z^j - \lambda_j^N z^i \right) \right).$$

The proof of this lemma is reported in Section 5.4. We now proceed to the characterization of Nash equilibria in the context of the multivariate Black–Scholes financial market. From Lemma 5.1, if condition (5.2) is satisfied, $\bar{\psi}^i$ defined by (4.21) is well defined, is a linear operator, and has the following expression:

(5.4)
$$\bar{\psi}_t^i = M_t^i := \left(I - \sum_{j \neq i} \frac{\lambda_j^N}{1 + \lambda_j^N} P_t^j \left(I + \lambda_i^N P_t^i \right) \right)^{-1} \left(\eta_i I + \sum_{j \neq i} \frac{\lambda_i^N \eta_j - \lambda_j^N \eta_i}{1 + \lambda_j^N} P_t^j \right).$$

THEOREM 5.2. *Assume that σ and θ are deterministic, and (5.2) is satisfied. Then, there exists a unique deterministic Nash equilibrium given by*

$$\hat{\pi}_t^i = \sigma(t)^{-1} P_t^i M_t^i \theta(t) \quad \text{for } i = 1, \ldots, N, t \in [0, T].$$

Moreover, the value function for agent i at equilibrium is given by

$$V_i = -e^{-\frac{1}{\eta_i}(x^i - \lambda_i \bar{x}^i - Y_0^i)}$$

where

$$Y_0^i = -\frac{\eta_i}{2} \int_0^T |\theta(t)|^2 dt + \frac{1}{2\eta_i} \int_0^T \left| \left(I - P_t^i \right) M_t^i \theta(t) \right|^2 dt.$$

Proof. Follow the lines of the proof of Theorem 4.8, replacing Lemma 4.6 by Lemma 5.1 and Lemma 4.15 by the following Lemma 5.3. □

LEMMA 5.3. *Let $\theta \in \mathbb{R}^d$ be arbitrary and $\Gamma : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ be defined for any $\gamma \in \mathbb{R}^{Nd}$ by*

$$\Gamma^i(\gamma) = P^i \left(\eta_i \theta + \lambda_i^N \sum_{j \neq i} \gamma^j \right).$$

Then, under (5.2), Γ admits a unique fixed point $\hat{\gamma}$ given by $\hat{\gamma}^i = P^i \bar{\psi}^i(\theta)$.

The proof of this lemma is reported in Section 5.4. We illustrate the previous Nash equilibrium in the context of symmetric managers with different access to the financial market.

EXAMPLE 5.4 (Similar agents with different investment constraints). Assume that σ and θ are deterministic, and let $\lambda_j = \lambda \in [0, 1)$ and $\eta_j = \eta > 0$, $j = 1, \ldots, N$. Then, there exists a unique deterministic Nash equilibrium given by

$$\hat{\pi}_t^i = \eta \sigma(t)^{-1} P_t^i \left(I - \frac{\lambda^N}{1 + \lambda^N} \sum_{j \neq i} P_t^j \left(I + \lambda^N P_t^i \right) \right)^{-1} \theta(t), \quad i = 1, \ldots, N.$$

We conclude this section with the following qualitative result that shows, in particular, that the managers' interactions induce an overinvestment on the risky assets, and imply that the market portfolio $\bar{\pi}$ of Definition 3.9 is nondecreasing in the interaction coefficients λ_i, in agreement with Proposition 3.10. This result requires a quite restrictive condition that, however, covers many examples, see also Remark 5.6 below.

PROPOSITION 5.5. *Assume that the projection operators P^i commute, i.e., $P^i P^j = P^j P^i$ for all $i, j = 1, \dots, N$. Then, under the conditions of Theorem 5.2, Agent i's equilibrium portfolio is such that $|\sigma(t)\hat{\pi}_t^i|$ is nondecreasing w.r.t. λ_j and η_j, for all $i, j = 1, \dots, N$ and $t \in [0, T]$.*

Proof. We fix an agent $i = 1, \dots, N$, and omit all t-dependence. The assumption that the P^i's commute is equivalent to the existence of an orthonormal basis $\{u_k, k = 1, \dots, d\}$ such that for all i, u_k is an eigenvector of P_i for all k. We write $P^i u_k = \varepsilon_{i,k} u_k$, and we observe that $\varepsilon_{i,k} \in \{0, 1\}$ by the fact that P^i is a projection. Then, by the explicit expression of $\hat{\pi}^i$ in Theorem 5.2, writing $\theta = \sum_{k=1}^d \theta^k u_k$, we directly compute that $|\sigma\hat{\pi}^i|^2 = \sum_{k=1}^d (\theta^k)^2 (\ell_{i,k})^2$, where

$$\ell_{i,k} = \varepsilon_{i,k}\left(1 - \sum_{m\neq i} \frac{\lambda_m^N \varepsilon_{m,k}}{1+\lambda_m^N}\left(1 + \lambda_i^N \varepsilon_{i,k}\right)\right)^{-1}\left(\eta_i + \sum_{m\neq i} \frac{\lambda_i^N \eta_m - \lambda_m^N \eta_i}{1+\lambda_m^N}\varepsilon_{m,k}\right). \tag{5.5}$$

We now verify that $\ell_{i,k}$ is nondecreasing w.r.t. λ_j and η_j, for all $j = 1, \dots, N$ and $k = 1, \dots, d$, which implies the required result by the orthogonality of the basis $\{u_k, k = 1, \dots, d\}$.

- That $\ell_{i,k}$ is nondecreasing in η_j is obvious from (5.5).
- That $\ell_{i,k}$ is nondecreasing in λ_i is also obvious from (5.5).
- Finally, for $j \neq i$, we directly differentiate (5.5), and see that the sign of $\partial \ell_{i,k}/\partial \lambda_j^N$ is given by the sign of

$$\begin{aligned}
&\varepsilon_{i,k}\varepsilon_{j,k}\left(\left(1+\lambda_i^N \varepsilon_{i,k}\right)\left(\eta_i + \sum_{m\neq i}\frac{\lambda_i^N \eta_m - \lambda_m^N \eta_i}{1+\lambda_m^N}\varepsilon_{m,k}\right)\right.\\
&\qquad \left. - \eta_i\left(1 - \sum_{m\neq i}\frac{\lambda_m^N \varepsilon_{m,k}}{1+\lambda_m^N}\left(1+\lambda_i^N \varepsilon_{i,k}\right)\right)\right)\\
&\quad = \varepsilon_{i,k}\varepsilon_{j,k}\left(\lambda_i^N \eta_i + \lambda_i^N\left(1 + \lambda_i^N \sum_{m\neq i}\frac{\eta_m}{1+\lambda_m^N}\varepsilon_{m,k}\right)\right) \geq 0.
\end{aligned}$$

□

REMARK 5.6. The statement of Proposition 5.5 is not valid for general portfolio constraints, as illustrated by the following example. Let $N = d = 2$, $A_1 = \mathbb{R}e_1$, $A_2 = \mathbb{R}(e_1 + e_2)$, and $\sigma = I$. Then, the projection operators P^1 and P^2 are defined by the following matrices in the basis (e_1, e_2):

$$P^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

respectively. By direct calculation, $|\hat{\pi}^1| = \frac{1}{2-\lambda_1\lambda_2}|(2\eta_1 + \lambda_1\eta_2)\theta_1 + \lambda_1\eta_2\theta_2|$, which can be increasing or decreasing in η_i and λ_i, $i = 1, 2$ for appropriate choices of the risk premium θ.

5.2. Infinite Managers Asymptotics

We now investigate the limiting behavior when the number of agents N goes to infinity with fixed number of assets d.

Recall that $|.|$ denotes the canonical Euclidean norm on $\mathbb{R}^d$, and $\mathcal{L}(\mathbb{R}^d)$ is the space of linear mappings on $\mathbb{R}^d$ endowed with operator norm $\|U\| = \sup_{|x|=1} |U(x)|$ for all $U \in \mathcal{L}(\mathbb{R}^d)$.

PROPOSITION 5.7. *Let d be fixed and the sequence $(\eta_i)_{i\in\mathbb{N}}$ bounded in $\mathbb{R}$. Assume that*

$$\frac{1}{N}\sum_{i=1}^{N} \lambda_i P_t^i \longrightarrow U_t^\lambda in \mathcal{L}(\mathbb{R}^d) \quad and \quad \frac{1}{N}\sum_{i=1}^{N} \eta_i P_t^i \to U_t^\eta \quad in \quad \mathcal{L}(\mathbb{R}^d), \tag{5.6}$$

for all (respectively, uniformly in) $t \in [0, T]$. Assume further that $\|U_t^\lambda\| < 1$, $t \in [0, 1]$. Then,

$$\hat{\pi}_t^{i,N} \longrightarrow \hat{\pi}_t^{i,\infty} := \sigma(t)^{-1} P_t^i (I - U_t^\lambda)^{-1}\big(\eta_i\big(I - U_t^\lambda\big) + \lambda_i U_t^\eta\big)\theta(t)$$

for all (respectively, uniformly in) $t \in [0, T]$.

Proof. By Theorem 5.2, we have $\hat{\pi}_t^{i,N} = \sigma(t)^{-1} P_t^i A_t^i B_t^i \theta(t)$, where

$$A_t^i := \left(I - \sum_{j\neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P_t^j (I + \lambda_i^N P_t^i)\right)^{-1} \quad \text{and} \quad B_t^i := \eta_i I + \sum_{j\neq i} \frac{\lambda_i^N \eta_j - \lambda_j^N \eta_i}{1+\lambda_j^N} P_t^j.$$

Since $\|P_t^j\| \leq 1$, we have

$$\begin{aligned}
&\left\| \frac{1}{N-1}\sum_{j\neq i} \frac{\lambda_j}{1+\lambda_j^N} P_t^j - \frac{1}{N}\sum_{j=1}^{N} \lambda_j P_t^j \right\| \\
&\quad \leq \frac{1}{N-1}\left\| \sum_{j\neq i} \frac{\lambda_j}{1+\lambda_j^N} P_t^j - \lambda_j P_t^j \right\| + \left\| \frac{1}{N-1}\sum_{j\neq i} \lambda_j P_t^j - \frac{1}{N}\sum_{j=1}^{N} \lambda_j P_t^j \right\| \\
&\quad \leq \left\| \frac{1}{(N-1)^2}\sum_{j\neq i} \frac{\lambda_j^2}{1+\lambda_j^N} P_t^j \right\| + \left\| \frac{1}{N}\lambda_i P_t^i + \frac{1}{N(N-1)}\sum_{j\neq i} \lambda_j P_t^j \right\| \leq \frac{3}{N}.
\end{aligned}$$

Similarly, by the boundedness of the sequence $(\eta_i)_{i\geq 1}$:

$$\left\| \frac{1}{N-1}\sum_{j\neq i} \frac{\eta_j}{1+\lambda_j^N} P_t^j - \frac{1}{N}\sum_{j=1}^{N} \eta_j P_t^j \right\| \leq \frac{3|\eta|_\infty}{N}.$$

Then, as $N \to \infty$, we have in $\mathcal{L}(\mathbb{R}^d)$

$$I + \lambda_i^N P_t^i \to I, \quad \frac{1}{N-1} \sum_{j \neq i} \frac{\lambda_j}{1+\lambda_j^N} P_t^j \longrightarrow U_t^\lambda, \quad \frac{1}{N-1} \sum_{j \neq i} \frac{\eta_j}{1+\lambda_j^N} P_t^j \longrightarrow U_t^\eta,$$

and $A_t^i \to (I - U_t^\lambda)^{-1}$, $B_t^i \to \eta_i I + \lambda_i U_t^\eta - \eta_i U_t^\lambda$. Under the condition $\|U_t^\lambda\| < 1$, the limit is finite. Moreover, the convergence is uniform in t whenever the convergence (5.6) holds uniformly in t. □

EXAMPLE 5.8 (Symmetric agents with different access to the financial market). Let $\lambda_i = \lambda \in [0, 1)$ and $\eta_i = \eta > 0$, $i \geq 1$. Then, the limiting Nash equilibrium portfolio reduces to

$$\hat{\pi}_t^{i,\infty} = \eta\sigma(t)^{-1} P_t^i \big(I - \lambda U_t^1\big)^{-1} \theta(t), \quad t \in [0, T], i \geq 1.$$

EXAMPLE 5.9 (Symmetric agents with finite market access possibilities). In the context of the previous example, suppose further that $\{A_i, i \geq 1\} = \{A_j, j = 1, \dots, p\}$ for some integer $p > 1$. We denote by k_j^N the number of agents with portfolio constraint A_j, and we assume that $k_j^N / N \longrightarrow \kappa_j \in [0, 1]$ for all $j = 1, \dots, p$. Then, an immediate application of Proposition 5.7 provides the limit Nash equilibrium portfolio:

$$\hat{\pi}_t^{i,\infty} = \eta\sigma(t)^{-1} P_t^i \left(I - \lambda \sum_{j=1}^{p} \kappa_j P_t^j \right)^{-1} \theta(t).$$

REMARK 5.10. We may also adopt the following probabilistic point of view to reformulate Proposition 5.7. Assume that there is a continuum of independent players modeled through a probability space $(\Delta, \mathcal{D}, \mu)$ independent from the space $(\Omega, \mathcal{F}, \mathbb{P})$ describing the financial market uncertainty. In such a setting, the market interactions, the risk tolerance, and the projection operators are defined by the random variables λ and η and the process $P = \{P_t, t \in [0, T]\}$ taking values, respectively, in $[0, 1]$, $(0, +\infty)$ and $\mathcal{L}(\mathbb{R}^d)$. The limiting Nash equilibrium portfolio is then given by

$$\hat{\pi}_t^{i,\infty} := \sigma(t)^{-1} P_t (I - \mu(\lambda P_t))^{-1} (\eta(I - \mu(\lambda P_t)) + \lambda\mu(\eta P_t))\theta(t),$$

provided that $\mu(\lambda \|P_t\|) + \mu(\eta \|P_t\|) < \infty$ and $\|\mu(\lambda P_t)\| < 1$.

Our next comment concerns the asymptotics of the market index $\bar{X}^N$ and the market portfolio $\bar{\pi}^N$ of Definition 3.9.

REMARK 5.11. In the context of Remark 5.10, we further assume that the random variables λ, η, and P_t are independent, and we denote $\bar{P}_t := \mu(P_t)$, $\bar{\lambda} := \mu(\lambda)$, $\bar{\eta} := \mu(\eta)$. Then, under the condition $\bar{\lambda}\bar{P}_t < 1$, the limit market portfolio and market index are given by

$$\bar{\pi}_t^\infty = \sigma(t)^{-1} \bar{v}_t^\infty, \ \bar{X}_t^\infty = \bar{x} + \int_0^t \bar{v}_t^\infty \cdot \big(dW_t + \theta(t)dt\big), \text{ where } v_t^\infty := \bar{\eta}\bar{U}_t \big(I - \bar{\lambda}\bar{U}_t\big)^{-1} \theta(t).$$

In particular, we have the following observations that are consistent with Proposition 3.10:

- the drift of the market index is nonnegative,
- the drift and the volatility of the market index are nondecreasing in $\bar{\eta}$ and $\bar{\lambda}$,

- the VRR index of the market portfolio is given by

$$\overline{\mathrm{VRR}}_t^{\infty} = \frac{\theta(t) \cdot \bar{P}_t(I - \bar{\lambda}\bar{P}_t)^{-1}\theta(t)}{\theta(t) \cdot (\bar{P}_t(I - \bar{\lambda}\bar{P}_t))^2\theta(t)},$$

and is nonincreasing in $\bar{\eta}$ and $\bar{\lambda}$.

5.3. Examples with Linear Constraints

For simplicity, except for Example 5.16, we assume that the agents are symmetric $\lambda_i = \lambda$ and $\eta_i = \eta$ for $i = 1, \ldots, N$ and only differ by their access to the financial market.

Except for the last Example 5.17, we shall consider a diagonal multidimensional Black–Scholes model with volatility matrix $\sigma = I_d$, i.e., the risky assets price processes are independent.

Under the conditions of Theorem 5.2, the optimal Nash equilibrium is given by

$$(5.7)\quad \hat{\pi}_t^i = \eta P^i \left(I - \frac{\frac{\lambda}{N-1}}{1 + \frac{\lambda}{N-1}} \sum_{j \neq i} P^j \left(I + \frac{\lambda}{N-1} P^i \right) \right)^{-1} \theta(t) \quad \text{for } i = 1, \ldots, N,$$

see Example 5.4. Let $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{R}^d$.

EXAMPLE 5.12. Let $d = N$ and $A_i = \mathbb{R}e_i$, $i = 1, \ldots, N$. Notice that $\cap_{i=1}^n A_i = \{0\}$. Then, Theorem 5.2 applies for all $\lambda \in [0, 1]$. The projection matrices P^i are all diagonal with unique nonzero diagonal entry $P_{i,i}^i = 1$. The calculation of the Nash equilibrium is then easy and provides

$$\hat{\pi}_t^i = \eta\sigma(t)^{-1}\theta_i(t)e_i, \quad i = 1, \ldots, N.$$

Hence, in agreement with the economic intuition, the interaction has no impact in this example, and the optimal Nash equilibrium portfolio coincides with the classical case with no interactions ($\lambda = 0$).

EXAMPLE 5.13. Let $d = 3$, $N = 2$, and $A_1 = \mathbb{R}e_1 + \mathbb{R}e_2$, $A_2 = \mathbb{R}e_2 + \mathbb{R}e_3$. Since $A_1 \cap A_2 \neq \{0\}$, Theorem 5.2 requires that $\lambda \in [0, 1)$. In the present context, the projection matrices are diagonal with $P_{1,1}^1 = P_{2,2}^1 = 1$, $P_{3,3}^1 = 0$, and $P_{1,1}^2 = 0$, $P_{2,2}^2 = P_{3,3}^2 = 1$. An easy calculation provides the optimal Nash equilibrium:

$$\hat{\pi}_t^1 = \eta\theta^1(t)e_1 + \frac{\eta}{1-\lambda}\theta^2(t)e_2 \quad \text{and} \quad \hat{\pi}_t^2 = \frac{\eta}{1-\lambda}\theta^2(t)e_2 + \eta\theta^3(t)e_3.$$

Notice that the optimal investment in the first and the third stock for agent 1 and agent 2, respectively, is the same as in the classical case ($\lambda = 0$). However, the investment in stock 2, which both agents can trade, is dilated by the factor $(1 - \lambda)^{-1} \in [1, +\infty)$.

EXAMPLE 5.14. Let $d = N = 3$ and $A_1 = \mathbb{R}e_1 + \mathbb{R}e_2$, $A_2 = \mathbb{R}e_2 + \mathbb{R}e_3$, $A_3 = \mathbb{R}e_3$. Since $A_1 \cap A_2 \cap A_3 = \{0\}$, Theorem 5.2 applies for $\lambda \in [0, 1]$. The projection matrices P^1 and P^2 are the same as in the previous example, and we similarly see that P^3 is diagonal with $P_{1,1}^3 = P_{2,2}^3 = 0$, $P_{3,3}^3 = 1$. Direct calculation provides the optimal Nash equilibrium:

$$\hat{\pi}_t^1 = \eta\theta^1(t)e_1 + \frac{\eta}{1-\frac{\lambda}{2}}\theta^2(t)e_2, \quad \hat{\pi}_t^2 = \frac{\eta}{1-\frac{\lambda}{2}}\theta^2(t)e_2 + \frac{\eta}{1-\frac{\lambda}{2}}\theta^3(t)e_3,$$
$$\hat{\pi}_t^3 = \frac{\eta}{1-\frac{\lambda}{2}}\theta^3(t)e_3.$$

Similar to the previous example, we see that the optimal investment in the first stock for agent 1 and agent 2, respectively, is the same as in the classical case ($\lambda = 0$), while the investment in stocks 2 and 3, which can both be traded by two agents, is dilated by the factor $(1 - \frac{\lambda}{2})^{-1} \in [1, +\infty)$. Notice that the dilation factor in the present example is smaller than that of the previous one.

EXAMPLE 5.15 (Investment with respect to hyperplanes). Let $d = N$ and $A_i = (\mathbb{R}e_i)^{\perp}$. In words, each manager has access to the whole market except for its own stock or those of the firms for which some private information is available to the manager. Direct calculation from the expression of Theorem 5.2 provides the following unique Nash equilibrium:

$$\hat{\pi}^{i,N} = \frac{\eta}{1 - \lambda + \frac{\lambda}{N-1}} \sum_{j \neq i} \theta_j e_j, \quad i = 1, \dots, N.$$

EXAMPLE 5.16 (Groups of managers investing in independent sectors). We assume that there are d groups of managers. The jth group consists of k_j symmetric agents with risk tolerance coefficient η_j, interaction coefficient λ_j, and market access defined by the constraints set $A_j = \mathbb{R}e_j$. The total number of managers is $N = \sum_{j=1}^{d} k_j$. Then, it follows from Theorem 5.2 that the Nash equilibrium portfolio for an agent of the jth group is

$$\begin{aligned}
\hat{\pi}^j &= P^j \left(I - \sum_{m \neq j} k_m \frac{\lambda_m^N}{1 + \lambda_m^N} P^m \big(I + \lambda_j^N P^j\big) - (k_j - 1) \frac{\lambda_j^N}{1 + \lambda_j^N} P^j \big(I + \lambda_j^N P^j\big) \right)^{-1} \\
&\quad \times \left(\eta_j I + \sum_{m \neq j} \frac{\lambda_j^N \eta_m - \lambda_m^N \eta_j}{1 + \lambda_m^N} P^m \right) \theta \\
&= P^j \left(I - \sum_{m \neq j} k_m \frac{\lambda_m^N}{1 + \lambda_m^N} P^m - (k_j - 1) \lambda_j^N P^j \right)^{-1} \left(\eta_j I + \sum_{m \neq j} \frac{\lambda_j^N \eta_m - \lambda_m^N \eta_j}{1 + \lambda_m^N} P^m \right) \theta,
\end{aligned}$$

where we used the fact that $P^j P^m = 0$ for $m \neq j$. The inverse matrix in the previous expression can be computed explicitly, and we get

$$\hat{\pi}^j = P^j \left(\frac{1}{1 - (k_j - 1)\lambda_j^N} P^j + \sum_{m \neq j} \frac{1}{1 - k_m \frac{\lambda_m^N}{1 + \lambda_m^N}} P^m \right) \left(\eta_j I + \sum_{m \neq j} \frac{\lambda_j^N \eta_m - \lambda_m^N \eta_j}{1 + \lambda_m^N} P^m \right) \theta.$$

Using again the fact that $P^j P^m = 0$ for $m \neq j$, we see that

$$\hat{\pi}^j = \frac{\eta_j}{1 - \frac{k_j - 1}{N} \lambda_j} \theta_j e_j \quad \text{for each agent of group } j, \quad j = 1, \dots, d.$$

EXAMPLE 5.17 (Correlated investments). Let $d = N$, $A_i = \mathbb{R}e_i$, $i = 1, \dots, N$, and

$$\theta = \theta_N \sigma \sum_{i=1}^{d} e_i, \qquad \sigma^2 = \sigma_N^2 \begin{pmatrix} 1 & & \rho^2 \\ & \ddots & \\ \rho^2 & & 1 \end{pmatrix}, \tag{5.8}$$

for some $\theta_N \in \mathbb{R}$, $\rho \in (-1, 1)$ and $\sigma_N > 0$.

Since σ is invertible, $(u_i := \sigma e_i)_{1\le i\le d}$ forms a basis of $\mathbb{R}^d$. We directly verify that for $j \neq i$ and $x = \sum_{i=1}^d x_i u_i$,

$$P^j(x) = \left(x_j + \rho^2 \sum_{k\neq j} x_k\right) u_j, \quad P^j P^i(x) = \rho^2 \left(x_i + \rho^2 \sum_{k\neq i} x_k\right) u_j. \tag{5.9}$$

By (5.7), the Nash equilibrium portfolio for the ith manager is given by $\hat{\pi}^i_t = \eta P^i x$, where x satisfies

$$\left(I - \frac{\frac{\lambda}{N-1}}{1+\frac{\lambda}{N-1}} \sum_{j\neq i} P^j \left(I + \frac{\lambda}{N-1} P^i\right)\right) x = \theta \quad \text{for } i = 1, \dots, N.$$

Given the particular structure of the risk premium in (5.8), we search for a solution of this linear system of the form $x = x_i u_i + x_0 \sum_{k\neq i} u_k$. By (5.9), this reduces the previous linear system to

$$\theta = x_i u_i + \sum_{j\neq i} \left(x_0 - \frac{\lambda}{N-1}\rho^2 x_i - \frac{\frac{\lambda}{N-1}}{1+\frac{\lambda}{N-1}}(1 + (N-2)\rho^2 + \lambda\rho^4)x_0\right) u_j,$$

and provides the solution of the system

$$x_i = \theta_N \quad \text{and} \quad x_0 = \frac{(1 + \frac{\lambda}{N-1}\rho^2)\theta_N}{1 - \frac{\frac{\lambda}{N-1}}{1+\frac{\lambda}{N-1}}(1 + (N-2)\rho^2 + \lambda\rho^4)},$$

and therefore, using again (5.9), the Nash equilibrium $\hat{\pi}^i = \eta P^i x$ is given by

$$\hat{\pi}^i = \eta\theta_N \left(1 + \frac{(N-1)\rho^2(1 + \frac{\lambda}{N-1}\rho^2)}{1 - \frac{\frac{\lambda}{N-1}}{1+\frac{\lambda}{N-1}}(1 + (N-2)\rho^2 + \lambda\rho^4)}\right) u_i, \quad i = 1, \dots, N.$$

We finally observe that $\hat{\pi}^i \sim \eta\theta_N \frac{1+(N-1-\lambda)\rho^2}{1-\lambda\rho^2}$ as $N \to \infty$. Then,

$$\hat{\pi}^i \sim \frac{\eta\theta_0\rho^2}{1-\lambda\rho^2} u_i \quad \text{whenever } \theta_N \equiv \frac{\theta_0}{N} \text{ as } N \to \infty.$$

This shows that the Nash equilibrium portfolio consists again of a dilation of the no-interaction optimal portfolio. However, in the present context, in addition to the dilation due to the interaction coefficient λ, there is an additional dilation caused by the correlation coefficient ρ. The dilation factor is increasing both in λ and ρ.

5.4. Proof of Technical Lemmas

Proof of Lemma 5.1. We omit all t subscripts. For arbitrary $z^1, \dots, z^N$ in $\mathbb{R}^d$, we want to find a unique solution to the system:

$$z^i - \lambda_i^N \sum_{j\neq i} P^j(z^j) = \zeta^i, \quad 1 \le i \le N. \tag{5.10}$$

1. We reduce (5.10) to a simpler form. Subtracting λ_i times equation j from λ_j times equation i in (5.10), we get for any i, j

$$\lambda_i \left(I + \lambda_j^N P^j\right) z^j = \lambda_j \left(I + \lambda_i^N P^i\right) z^i + \lambda_i \zeta^j - \lambda_j \zeta^i.$$

Since $(I + \lambda_j^N P^j)^{-1} = I - \frac{\lambda_j^N}{1+\lambda_j^N} P^j$, we have

$$\lambda_i P^j z^j = \frac{1}{1+\lambda_j^N} P^j \big(\lambda_j \left(I + \lambda_i^N P^i\right) z^i + \lambda_i \zeta^j - \lambda_j \zeta^i\big).$$

Thus, using (5.10) it follows that

$$\zeta^i = z^i - \frac{1}{N-1} \sum_{j \neq i} \frac{1}{1+\lambda_j^N} P^j \left[\lambda_j \left(I + \lambda_i^N P^i\right) z^i + \lambda_i \zeta^j - \lambda_j \zeta^i\right],$$

and we can rewrite (5.10) equivalently as

(5.11)
$$\left(I - \sum_{j \neq i} \frac{\lambda_j^N}{1+\lambda_j^N} P^j \left(I + \lambda_i^N P^i\right)\right) z^i = \zeta^i + \frac{1}{N-1} \sum_{j \neq i} \frac{1}{1+\lambda_j^N} P^j \left(\lambda_i \zeta^j - \lambda_j \zeta^i\right),$$

so that the invertibility of φ is equivalent to the invertibility of the linear operators $I - R^i$, for $i = 1, \dots, N$, where the R^is are introduced in the statement of the lemma.

2. We prove that the $I - R^i$s are all invertible iff (5.2) holds true.

(a) First, assume that $\lambda_j = 1$ for all j and that $x \in \bigcap_{j=1}^N A_j \neq \{0\}$ satisfies $x \neq 0$. Then, we have for any j, $P^j x = x$ and so:

$$R^i x = \frac{1}{N-1} \sum_{j \neq i} \frac{1}{1 + \frac{1}{N-1}} P^j \left(I + \frac{1}{N-1} P^i\right) x = x.$$

Therefore, $I - R^i$ is not invertible.

(b) Conversely, assume that (5.2) holds true. We consider two separate cases.

- If $\lambda_{i_0} < 1$, for some $i_0 \in \{1, \dots, N\}$, then we estimate that

$$\frac{\lambda_{i_0}^N}{1+\lambda_{i_0}^N} < \frac{\frac{1}{N-1}}{1+\frac{1}{N-1}} \qquad \text{and} \qquad \frac{\lambda_j^N}{1+\lambda_j^N} \leq \frac{\frac{1}{N-1}}{1+\frac{1}{N-1}} \quad \text{for any } j \neq i_0.$$

  Then, since $1 + \lambda_{i_0}^N < 1 + \frac{1}{N-1}$, for any i and any $x \neq 0$, $|R^i x| < |x|$, proving that $I - R^i$ is invertible.
- If $\lambda_i = 1$, for all $i = 1, \dots, N$ and $\bigcap_{i=1}^N A_i = \{0\}$. Let $x \in Ker(I - R^i)$ for some i, using the fact that the P^js are contractions, we have

$$|x| = |R^i x| = \left| \frac{1}{N-1} \sum_{j \neq i} \frac{1}{1+\frac{1}{N-1}} P^j \left(I + \frac{1}{N-1} P^i\right) x \right|$$
$$\leq \frac{1}{N-1} \sum_{j \neq i} |x| = |x|,$$

so that equality holds in the above inequality, which can only happen if $P^j x = x$ for all $j = 1, \dots, N$, which implies that $x \in \bigcap_{j=1}^N A_j$, and therefore $x = 0$, which completes the proof. □

Proof of Lemma 5.3. We want to show that the system $\eta_i P^i \theta + \lambda_i^N \sum_{j \neq i} P^i \gamma^j = \gamma^i$, for all $i = 1, \dots, N$, has a unique solution, or equivalently that $\lambda_i^N \sum_{j \neq i} P^i \gamma^j - \gamma^i = 0$ is satisfied for all $i = 1, \dots, N$ if and only if $\gamma = 0$. Writing this linear system $A\gamma = 0$, we have

$$\left| \lambda_i^N \sum_{j \neq i} P^i \gamma^j - \gamma^i \right| \geq |\gamma^i| - \lambda_i^N \sum_{j \neq i} |P^i \gamma^j|,$$

so that $\gamma \in \operatorname{Ker} A$ implies that $|\gamma^i| = |\gamma^j|$ for any i, j. Having equality for i implies that $P^i \gamma^j = \gamma^j$ for all j, the γ^js are all collinear (i included) and $\lambda_i = 1$. Therefore, if $\prod_i \lambda_i < 1$ or $\cap_i A_i = \{0\}$, the previous inequality becomes strict if $\gamma \in \operatorname{Ker} A \neq 0$.

Then, as in the proof of Lemma 4.15, we have $\hat{\gamma}^i = \Gamma^i(\hat{\gamma})$, for each $i = 1, \dots, N$. □

APPENDIX

EXAMPLE A.1. Let $N = 2$, $\sigma = I_d$, $\lambda_i = \lambda$, and $A_i = A := \{x \in \mathbb{R}^d; |x_1| \geq 1\}$, $i = 1, 2$. The projection is uniquely determined for $x_1 \neq 0$, and we can take, for example, the following:

$$P(x) = \begin{cases} x, & \text{if } x \in A \\ (1, x_2, \dots, x_d)^t, & \text{if } x_1 \in [0, 1) \\ (-1, x_2, \dots, x_d)^t, & \text{if } x_1 \in (-1, 0). \end{cases}$$

If φ was surjective onto $\mathbb{R}^{2d}$, then subtracting the expressions of φ^1 and φ^2, we see that $I + \lambda P$ would be surjective onto $\mathbb{R}^d$. Let $y \in \mathbb{R}^d$, we want to find x such that $x + \lambda P(x) = y$.

- If $x_1 \geq 1$, then $(1 + \lambda)x_1 = y_1$ so that $y_1 \geq 1 + \lambda$;
- if $x_1 \in [0, 1)$, then $x_1 + \lambda = y_1$ so that $y_1 \in [\lambda, 1 + \lambda)$;
- if $x_1 \in (-1, 0)$, then $x_1 - \lambda = y_1$ so that $y_1 \in (-1 - \lambda, -\lambda)$;
- if $x_1 \leq -1$, then $(1 + \lambda)x_1 = y_1$ so that $y_1 \leq -1 - \lambda$.

Therefore, $\{x \in \mathbb{R}^d; x_1 \in [-\lambda, \lambda)\}$ is not attained by $I + \lambda P$ so that as soon as $\lambda > 0$, φ is not surjective. Moreover, the interior of the complementary of its image is nonempty.

EXAMPLE A.2. Let $A_i = B := \{x \in \mathbb{R}^d; |x| \geq 1\}$, the complement of the unit (open) ball. The projection is uniquely determined for $x \neq 0$, and we can for example take:

$$P(x) = \begin{cases} x, & \text{if } x \in B, \\ \dfrac{1}{|x|} x, & \text{if } |x| \in (0, 1), \\ 1_d, & \text{if } x = 0. \end{cases}$$

Similar to the previous example, in order to have φ surjective, we need $I + \lambda P$ surjective onto $\mathbb{R}^d$. If $y \in \mathbb{R}^d$, and $x + \lambda P(x) = y$, we compute

- if $|x| \geq 1$, then $(1+\lambda)x = y$ so that $|y| \geq 1+\lambda$;
- if $|x| \in (0,1)$, then $(1+\frac{\lambda}{|x|})x = y$ so that $|y| \in (\lambda, 1+\lambda)$;
- if $x = 0$, then $y = \lambda 1_d$.

Therefore, $\{x \in \mathbb{R}^d; |x| < \lambda\}$ is not attained by $I + \lambda P$, so again as soon as $\lambda > 0$, φ is not surjective. Moreover, the interior of the complementary of its image is nonempty.

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