A STOCHASTIC CONTROL APPROACH TO NO-ARBITRAGE BOUNDS GIVEN MARGINALS, WITH AN APPLICATION TO LOOKBACK OPTIONS

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We consider the problem of superhedging under volatility uncertainty for an investor allowed to dynamically trade the underlying asset, and statically trade European call options for all possible strikes with some given maturity. This problem is classically approached by means of the Skorohod Embedding Problem (SEP). Instead, we provide a dual formulation which converts the superhedging problem into a continuous martingale optimal transportation problem. We then show that this formulation allows us to recover previously known results about lookback options. In particular, our methodology induces a new proof of the optimality of Azéma–Yor solution of the SEP for a certain class of lookback options. Unlike the SEP technique, our approach applies to a large class of exotics and is suitable for numerical approximation techniques.

1. Introduction. In a financial market allowing for the dynamic trading of a riskless asset, normalized to unity, and some given underlying assets without restrictions, the fundamental theorem of asset pricing essentially states that the absence of arbitrage opportunities is equivalent to the existence of a probability measure under which the underlying asset process is a martingale. See Kreps [24], Harrison and Pliska [18] and Delbaen and Schachermayer [14]. Then, for the purpose of hedging, the only relevant information is the quadratic variation of the assets price process under such a martingale measure. Without any further assumption on the quadratic variation, the robust superhedging cost reduces to an obvious bound which can be realized by static trading on the underlying assets; see Cvitanić, Pham and Touzi [12] and Frey [17].

In this paper, we examine the problem of superhedging, under the condition of no-arbitrage, when the financial market allows for the static trading of European call options in addition to the dynamic trading of the underlying asset with

price process $\{X_t, t \geq 0\}$. For simplicity, we consider the case where all available European call options have the same maturity T. However, we idealize the financial market assuming that such European call options are available for all possible strikes. Under the linearity (and continuity) assumption, this means that although the joint distribution $\mathbb{P}$ of the process X is unknown, the modeler has access to the function $K \in \mathbb{R}_+ \longmapsto \mathbb{E}[(X_T - K)^+]$, and therefore to the marginal distribution μ of the random variable X_T, as observed by Breeden and Litzenberger [6]. In particular, any T-maturity vanilla derivative, with payoff $g(X_T)$, can be perfectly replicated by a portfolio of European calls, and has an unambiguous no-arbitrage price $\int g \, d\mu$, as long as $g \in \mathbb{L}^1(\mu)$, which can be expressed as a linear combination of the given prices of the underlying calls.

This problem is classically approached in the literature by means of the Skorohod embedding problem (SEP) which shows up naturally due to the Dambis–Dubins–Schwarz time change result. However, only some special cases of derivatives are eligible for this approach, namely those defined by a payoff which is invariant under rime change. The use of the SEP techniques to solve the robust superhedging problem can be traced back to the seminal paper by Hobson [20]. The survey paper by Hobson [21] is very informative and contains the relevant references on the subject. For a derivative security $g(X_s, s \leq t)$ written on an underlying asset X, the idea is to search for a function λ and a martingale M such that $g(X_s, s \leq t) \leq \lambda(X_t) + M_t$ so that $\mathbb{E}[g(X_s, s \leq \tau)] \leq \int \lambda \, d\mu$ for all stopping times τ such that X_τ has the distribution μ determined from European call options, as outlined above. The bound is then obtained by designing λ, M and τ such that $g(X_s, s \leq \tau) = \lambda(X_\tau) + M_\tau$.

In this paper, we develop an alternative approach which relates the robust superhedging problem to the literature on stochastic control; see Fleming and Soner [16]. Our approach consists of solving directly the robust superhedging problem whose solution provides the above function λ and martingale M. As a consequence, unlike the SEP approach which requires that the payoff of the exotic to be invariant under time change, there is no restriction on the class of derivative securities as the time change step is avoided. Moreover, our methodology is related to the optimal transportation theory, and in fact opens the door to an original new ramification in this theory by imposing naturally that the transportation be performed along a continuous martingale.

Our first main result, reported in Sections 2.3 and 2.4, provides a formulation of the robust superhedging problem based on the Kantorovich duality in the spirit of Benamou and Brenier [4]; see Villani [39]. A finite dimensional version of this dual formulation was already used by Davis, Obłój and Raval [13]. Also, this duality is implicitly proved for special cases of derivatives in the previous literature based on the SEP. So the importance of our duality result lies in its generality. Moreover, similar to the SEP approach, the solution of the dual problem provides the corresponding optimal hedging strategy and worst case model. Finally, the dual

formulation is suitable for numerical approximation techniques as shown by Bonnans and Tan [5] and Tan and Touzi [38].

Our next concern is to show that this duality result is not merely a theoretical result, but may be applied to compute things in practice. In this paper, this is demonstrated in the context of lookback derivatives, where the robust superhedging problem is known to be induced by the Azéma–Yor solution of the SEP [1, 2]. A semi-static hedging strategy corresponding to this bound was produced by Hobson [20]. Our second main result, reported in Section 3, reproduces this bound by means of our dual formulation. In particular, this provides a new presentation of the fact that the Azéma–Yor solution of the SEP realizes the upper bound for a certain class of lookback options. We also recover in Section 4 the robust superhedging cost for the forward lookback option which was also derived in Hobson [20].

The results of Sections 3 and 4 are not new, and are reported here in order to show that the optimal transportation approach is suitable to recover these known results. Beyond rediscovering the formerly derived results in the literature, we would like to insist on the fact that the present optimal transportation approach complements the SEP approach by putting the emphasis on the superhedging problem whose solution provides the optimal semi-static superhedging strategy. This feature is perfectly illustrated in the related paper [19], where:

(i) this approach is used in order to provide the optimal semi-static superhedging strategy;

(ii) by using the optimal semi-static superhedging strategy from (i), Oblój and Spoida [31] prove an extension of the Azéma–Yor solution of the SEP to the case of multiple intermediate marginals.

The final outcome from [19, 31] is an extension of the previous results of Brown, Hobson and Rogers [7] and Madan and Yor [25].

We also observe that one should not expect to solve explicitly the problem of robust superhedging, in general. Therefore, an important advantage of the optimal transportation approach is that the corresponding dual formulation is suitable for numerical approximation techniques; see Bonnans and Tan [5] for the case of variance options, and Tan and Touzi [38] for a class of optimal transportation problems motivated by this paper.

Finally, we would like to emphasize that by expressing the robust bound as the value function of a superhedging problem, we are implicitly addressing the important issue of no-arbitrage under model uncertainty, as discussed in the previous literature; see, for example, Cox and Oblój [11] and Davis, Oblòj and Raval [13]. Indeed, if an arbitrage does exist, under some conveniently defined weak form, the value function would be infinite. Conversely, whenever the value function is finite, the financial market would either admit a genuine model-free arbitrage, if existence holds for the dual problem, or some convenient notion of weak arbitrage otherwise.

2. Model-free bounds of derivatives securities.

2.1. *The probabilistic framework.* Let $\Omega := \{\omega \in C([0, T], \mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0 \le t \le T} |\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0 \le t \le T}$ the filtration generated by B and $\mathbb{F}^+ := \{\mathcal{F}_t^+, 0 \le t \le T\}$ the right limit of $\mathbb{F}$, where $\mathcal{F}_t^+ := \bigcap_{s > t} \mathcal{F}_s$.

Throughout the paper, X_0 is some given initial value in $\mathbb{R}^d_+$, and we denote

$$X_t := X_0 + B_t \qquad \text{for } t \in [0, T].$$

For any $\mathbb{F}$-progressively measurable process α with values in $\mathcal{S}_d^{>0}$ (space of definite positive symmetric matrices) and satisfying $\int_0^T |\alpha_s| \, ds < \infty$, $\mathbb{P}_0$-a.s., we define the probability measure on $(\Omega, \mathcal{F})$,

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1},$$

where

$$X_t^\alpha := X_0 + \int_0^t \alpha_r^{1/2} \, dB_r, \qquad t \in [0, T], \mathbb{P}_0\text{-a.s.}$$

Then X is a $\mathbb{P}^\alpha$-local martingale. Following [34], we denote by $\overline{\mathcal{P}}_S$ the collection of all such probability measures on $(\Omega, \mathcal{F})$. The quadratic variation process $\langle X \rangle = \langle B \rangle$ is universally defined under any $\mathbb{P} \in \overline{\mathcal{P}}_S$, and takes values in the set of all nondecreasing continuous functions from $\mathbb{R}_+$ to $\mathcal{S}_d^{>0}$. Finally, we recall from [34] that

(2.1)
$$\begin{aligned}&\text{every } \mathbb{P} \in \overline{\mathcal{P}}_S \text{ satisfies the Blumenthal zero–one law}\\&\qquad \text{and the martingale representation property.}\end{aligned}$$

In this paper, we shall focus on the subset $\mathcal{P}_\infty$ of $\overline{\mathcal{P}}_S$ consisting of all measures $\mathbb{P}$ such that

$$X \text{ is a } \mathbb{P}\text{-uniformly integrable martingale with values in } \mathbb{R}^d.$$

The restriction of the probability measures in $\mathcal{P}_\infty$ to those induced by uniformly integrable martingales X is motivated by our subsequent interpretation of the entries X^i as price processes of financial securities.

2.2. *Model-free super-hedging problem.* For all $\mathbb{P} \in \mathcal{P}_\infty$, we denote

$$\mathbb{H}_{\text{loc}}^2(\mathbb{P}) := \left\{ H \in \mathbb{H}^0(\mathbb{P}) : \int_0^T \text{Tr}\big[H_t^{\mathrm{T}} H_t \, d\langle B \rangle_t\big] < \infty, \mathbb{P}\text{-a.s.} \right\},$$

where Tr denotes the trace operator. We assume that

the interest rate is zero.

Under the self-financing condition, for any portfolio process H, the portfolio value process

$$
(2.2) \qquad Y_t^H := Y_0 + \int_0^t H_s \cdot dB_s, \qquad t \in [0, T],
$$

is well defined $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}_\infty$, whenever $H \in \mathbb{H}_{\mathrm{loc}}^2$. This stochastic integral should be rather denoted $Y_t^{H\,\mathbb{P}}$ to emphasize its dependence on $\mathbb{P}$; see, however, Nutz [28].

Let ξ be an $\mathcal{F}_T$-measurable random variable. We introduce the subset of martingale measures

$$
\mathcal{P}_\infty(\xi) := \{ \mathbb{P} \in \mathcal{P}_\infty : \mathbb{E}^{\mathbb{P}}[\xi^-] < \infty \}.
$$

The reason for restricting to this class of models is that, under the condition that $\mathbb{E}^{\mathbb{P}}[\xi^+] < \infty$, the hedging cost of ξ under $\mathbb{P}$ is expected to be $-\infty$ whenever $\mathbb{E}^{\mathbb{P}}[\xi^-] = \infty$. As usual, in order to avoid doubling strategies, we introduce the set of admissible portfolios,

$$
\mathcal{H}(\xi) := \{ H : H \in \mathbb{H}_{\mathrm{loc}}^2 \text{ and } Y^H \text{ is a } \mathbb{P}\text{-supermartingale for all } \mathbb{P} \in \mathcal{P}_\infty(\xi) \}.
$$

The model-free superhedging problem is defined by

$$
(2.3) \qquad U^0(\xi) := \inf \{ Y_0 : \exists H \in \mathcal{H}(\xi), Y_1^H \geq \xi, \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_\infty(\xi) \}.
$$

We call U^0 the model-free superhedging bound, and we recall its interpretation as the no-arbitrage upper bound on the market price of the derivative security ξ, for an investor who has access to continuous-time trading the underlying securities with price process X.

2.3. *Dual formulation of the super-hedging bound.* We denote by $\mathrm{UC}(\Omega_{X_0})$ the collection of all uniformly continuous maps from Ω_{X_0} to $\mathbb{R}$, where $X_0 \in \mathbb{R}^d$ is a fixed initial value, and $\Omega_{X_0} := \{ \omega \in C([0, T], \mathbb{R}_+^d) : \omega_0 = X_0 \}$. The following result is a direct adaptation from Soner, Touzi and Zhang [35].

THEOREM 2.1.   *Let $\xi \in \mathrm{UC}(\Omega_{X_0})$ be such that $\sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi^+] < \infty$. Then*

$$
U^0(\xi) = \sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi].
$$

Assume further that $U^0(\xi)$ is finite. Then there exists a process $H \in \mathcal{H}(\xi)$ and a family of nondecreasing predictable processes $\{ K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_\infty(\xi) \}$, with $K_0^{\mathbb{P}} = 0$ for all $\mathbb{P} \in \mathcal{P}_\infty(\xi)$, s.t.

$$
(2.4) \qquad \xi = U^0(\xi) + \int_0^1 H_t \cdot dB_t - K_1^{\mathbb{P}}, \qquad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_\infty(\xi).
$$

The proof is reported in Section 5.

REMARK 2.1. A similar dual representation as in Theorem 2.1 was first obtained by Denis and Martini [15] in the bounded volatility case. Notice, however, that the family of nondominated singular measures in [15] is not included in our set $\overline{\mathcal{P}}_S$, and does not allow for the existence of an optimal super-hedging strategy.

While revising this paper, a new approach for the robust superhedging problem, initiated by Nutz and von Handel [29] and Neufeld and Nutz [27], allowed for an extension of Theorem 2.1 by Possamaï, Royer and Touzi [33].

2.4. *Calibration adjusted no-arbitrage bound.* In this section we specialize the discussion to the one-dimensional case. This is consistent with the one-dimensional practical treatment of vanilla options on real financial markets.

We assume that, in addition to the continuous-time trading of the primitive securities, the investor can take static positions on T-maturity European call or put options with all possible strikes $K \geq 0$. Then, from Breeden and Litzenberger [6], the investor can identify that the T-marginal distribution of the underlying asset under the pricing measure is some probability measure $\mu \in M(\mathbb{R})$, the set of all probability measures on $\mathbb{R}$.

REMARK 2.2. For the purpose of the present financial application, the measure μ has a support in $\mathbb{R}_+$. We consider, however, the general case $\mu \in M(\mathbb{R})$ in order to compare our results to the Azéma–Yor solution of the SEP.

For any scalar function $\lambda \in \mathbb{L}^1(\mu)$, the T-maturity European derivative defined by the payoff $\lambda(X_T)$ has an un-ambiguous no-arbitrage price

$$\mu(\lambda) = \int \lambda \, d\mu$$

and can be perfectly replicated by buying and holding a portfolio of European calls and puts of all strikes, with the density $\lambda''(K)$ at strike K (with λ'' understood in the sense of distributions). See Carr and Chou [8]. In particular, given the spot price $X_0 > 0$ of the underlying assets, the probability measure μ must satisfy

$$\int x \mu(dx) = X_0.$$

We now define an improvement of the no-arbitrage upper-bound by accounting for the additional possibility of statically trading the European call options. Let

(2.5)
$$\Lambda^\mu := \left\{ \lambda \in \mathbb{L}^1(\mu) : \sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}\big[\lambda(X_T)^-\big] < \infty \right\} \quad \text{and}$$
$$\Lambda^\mu_{\mathrm{UC}} := \Lambda^\mu \cap \mathrm{UC}(\mathbb{R}),$$

where $\mathrm{UC}(\mathbb{R})$ is the collection of all uniformly continuous maps from $\mathbb{R}$ to $\mathbb{R}$. For all $\lambda \in \Lambda^\mu$, we denote $\xi^\lambda := \xi - \lambda(X_1)$. The improved no-arbitrage upper bound

is defined by

$$
\begin{aligned}
U^{\mu}(\xi) := \inf\{Y_0 : \exists \lambda \in \Lambda_{\mathrm{UC}}^{\mu} \text{ and } H \in \mathcal{H}(\xi^{\lambda}), \\
\overline{Y}_1^{H,\lambda} \geq \xi, \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_{\infty}(\xi^{\lambda})\},
\end{aligned}
$$

(2.6)

where $\overline{Y}^{H,\lambda}$ denotes the portfolio value of a self-financing strategy with continuous trading H in the primitive securities, and static trading λ in the T-maturity European calls with all strikes

$$
(2.7) \qquad \overline{Y}_1^{H,\lambda} := Y_1^H - \mu(\lambda) + \lambda(X_T),
$$

indicating that the investor has the possibility of buying at time 0 any derivative security with payoff $\lambda(X_T)$ for the price $\mu(\lambda)$.

The next result is a direct application of Theorem 2.1.

PROPOSITION 2.1. *Let* $\mu \in M(\mathbb{R})$, *and* $\xi \in \mathrm{UC}(\Omega_{X_0})$ *with* $\sup_{\mathbb{P} \in \mathcal{P}_{\infty}} \mathbb{E}^{\mathbb{P}}[\xi^+] < \infty$. *Then*

$$
U^{\mu}(\xi) = \inf_{\lambda \in \Lambda_{\mathrm{UC}}^{\mu}} \sup_{\mathbb{P} \in \mathcal{P}_{\infty}} \{\mu(\lambda) + \mathbb{E}^{\mathbb{P}}[\xi - \lambda(X_T)]\}.
$$

PROOF. Observe that

$$
U^{\mu}(\xi) = \inf_{\lambda \in \Lambda_{\mathrm{UC}}^{\mu}} U^0\big(\xi + \mu(\lambda) - \lambda(X_T)\big).
$$

For every fixed λ, if $V(0) := \sup_{\mathbb{P} \in \mathcal{P}_{\infty}} \mathbb{E}^{\mathbb{P}}[\xi + \mu(\lambda) - \lambda(X_T)] < \infty$, then the proof of Theorem 2.1, reported in Section 5, applies and we get $U^0(\xi + \mu(\lambda) - \lambda(X_T)) = V(0)$. On the other hand, if $V(0) = \infty$, then notice from the proof of Theorem 2.1 that the inequality $U^0(\xi + \mu(\lambda) - \lambda(X_T)) \geq V(0)$ is still valid in this case, and therefore $U^0(\xi + \mu(\lambda) - \lambda(X_T)) = V(0)$. □

REMARK 2.3. As a sanity check, let us consider the case $\xi = g(X_T)$, for some uniformly continuous function g with $\mu(|g|) < \infty$ and $\sup_{\mathbb{P} \in \mathcal{P}_{\infty}} \mathbb{E}^{\mathbb{P}}[|g(X_T)|] < \infty$, and let us verify that $U^{\mu}(\xi) = \mu(g)$.

First, since $g \in \Lambda_{\mathrm{UC}}^{\mu}$, we may take $\lambda = g$, and it follows from the dual formulation of Proposition 2.1 that $U^{\mu}(\xi) \leq \mu(g)$. On the other hand, it is easily seen that

$$
\sup_{\mathbb{P} \in \mathcal{P}_{\infty}} \mathbb{E}^{\mathbb{P}}[g(X_T)] = g^{\mathrm{conc}}(X_0),
$$

where g^{conc} is the smallest concave majorant of g. Then, it follows from the dual formulation of Proposition 2.1 that $U^{\mu}(\xi) = \inf_{\lambda \in \Lambda_{\mathrm{UC}}^{\mu}} \mu(\lambda) + (g - \lambda)^{\mathrm{conc}}(X_0) \geq \inf_{\lambda \in \Lambda_{\mathrm{UC}}^{\mu}} \mu(\lambda) + \mu(g - \lambda) = \mu(g)$ as expected.

REMARK 2.4.   Similar to the SEP approach, the dual formulation of Proposition 2.1 gives access to the optimal hedging strategy and the worst case model. This requires that we prove an additional existence result of a solution to the inf-sup problem $(\lambda^*, \mathbb{P}^*)$. Then λ^* is the optimal T-maturity vanilla profile, and $\mathbb{P}^*$ is the worst case model corresponding to the upper bound. The optimal dynamic hedging strategy in the underlying asset is, as usual, obtained by representation of the residual security $\xi - \lambda^*(X_T)$; see the proof of Theorem 2.1 in Section 5.

REMARK 2.5.   The dual formulation of Proposition 2.1 is suitable for numerical approximation. Indeed, for each fixed multiplier λ, the maximization problem is a (singular) stochastic control problem which may be approximated by finite differences or Monte Carlo methods. Then optimization stage with respect to λ requires an additional iteration. This issue is addressed in Tan and Bonnans [5] and Tan and Touzi [38].

2.5. *Connection with optimal transportation theory.*   As an alternative point of view, one may directly imbed in the no-arbitrage bounds the calibration constraint that the risk neutral marginal distribution of B_T is given by μ.

For convenience of comparison with the optimal transportation theory, the discussion of this subsection will be focused on the no-arbitrage lower bound. A natural formulation of the calibration adjusted no-arbitrage lower bound is

$$(2.8) \qquad \ell(\xi, \mu) := \inf\{\mathbb{E}^{\mathbb{P}}[\xi] : \mathbb{P} \in \mathcal{P}_\infty, X_0 \sim_{\mathbb{P}} \delta_{X_0} \text{ and } X_T \sim_{\mathbb{P}} \mu\},$$

where δ_{X_0} denotes the Dirac mass at the point X_0. We observe that a direct proof that $\ell(\xi, \mu)$ coincides with the corresponding sub-hedging cost is not obvious in the present context.

Under this form, the problem appears as minimizing the coupling criterion $E^{\mathbb{P}}[\xi]$ which involves the law of the process X under $\mathbb{P}$, over all those probability measures $\mathbb{P} \in \mathcal{P}_\infty$ such that the marginal distributions of X at times 0 and T are fixed. This is the general scope of optimal transportation problems as introduced by Monge and Kantorovich; see, for example, Villani [39] and Mikami and Thieulen [26]. Motivated by the present financial application, Tan and Touzi [38] extended the Kantorovich duality as described below. However, the above problem $\ell(\xi, \mu)$ does not satisfy the assumptions in [38] so that none of the results contained in this literature apply to our context.

The classical approach in optimal transportation consists of deriving a dual formulation for problem (2.8) by means of the classical convex duality theory. Recall that $M(\mathbb{R}_+)$ denotes the collection of all probability measures on $\mathbb{R}_+$. Then, the Legendre dual with respect to μ is defined by

$$\ell^*(\xi, \lambda) := \sup_{\mu \in M(\mathbb{R}_+)} \{\mu(\lambda) - \ell(\xi, \mu)\} \qquad \text{for all } \lambda \in C_b^0(\mathbb{R}_+)$$

the set of all bounded continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. Direct calculation shows that

$$\ell^*(\xi, \lambda) = \sup\{\mathbb{E}^{\mathbb{P}}[\lambda(X_T) - \xi] : \mu \in M(\mathbb{R}_+), \mathbb{P} \in \mathcal{P}_\infty, X_0 \sim_{\mathbb{P}} \delta_{X_0} \text{ and } X_T \sim_{\mathbb{P}} \mu\}$$
$$= \sup\{\mathbb{E}^{\mathbb{P}}[\lambda(X_T) - \xi] : \mathbb{P} \in \mathcal{P}_\infty, X_0 \sim_{\mathbb{P}} \delta_{X_0}\}.$$

Observe that the latter problem is a standard (singular) diffusion control problem.

It is easily checked that ℓ is convex in μ. However, due to the absence of a uniform bound on the quadratic variation of X under $\mathbb{P} \in \mathcal{P}_\infty$, it is not obvious whether it is lower semicontinuous with respect to μ. If the latter property were true, then the equality $\ell^{**} = \ell$ provides

$$\ell(\xi, \mu) = \sup_{\lambda \in C_b^0} \{\mu(\lambda) - \ell^*(\xi, \lambda)\},$$

which is formally (up to the spaces choices) the lower bound analogue of the dual formulation of Proposition 2.1. A discrete-time analysis of this duality is contained in the parallel work to the present one by Beiglböck, Henry-Labordère and Penkner [3]. We also observe that, for special cases of payoffs ξ, this duality was implicitly proved in the previous literature based on the SEP approach; see, for example, Cox, Hobson and Oblój [10].

3. Application to lookback derivatives. Throughout this section, we consider the one dimensional case $d = 1$. The derivative security is defined by the lookback payoff

(3.1) $\xi = g(X_T^*)$ where $X_T^* := \max_{t \leq T} X_t$

and

(3.2) $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ is a C^1 nondecreasing function.

Our main interest is to show that the optimal upper bound given by Proposition 2.1,

$$U^\mu(\xi) = \inf_{\lambda \in \Lambda_{\mathrm{UC}}^\mu} \{\mu(\lambda) + u^\lambda(0, X_0, X_0)\}$$

reproduces the already known bound corresponding to the Azéma–Yor solution to the Skorohod embedding problem. Here, u^λ is the value function of the dynamic version of stochastic control problem

(3.3) $u^\lambda(t, x, m) := \sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[g(M_T^{t,x,m}) - \lambda(X_T^{t,x})],$ $t \leq T, (x, m) \in \Delta,$

where $\Delta := \{(x, m) \in \mathbb{R}^2 : x \leq m\}$, and

$X_u^{t,x} := x + (B_u - B_t),$ $M_u^{t,x,m} := m \vee \max_{t \leq r \leq u} X_r^{t,x},$ $0 \leq t \leq u \leq T.$

When the time origin is zero, we shall simply write $X_u^x := X_u^{0,x}$ and $M_u^{x,m} := M_u^{0,x,m}$.

For the subsequent analysis, we also claim that

$$(3.4) \qquad U^\mu(\xi) = \inf_{\lambda \in \Lambda^\mu} \{\mu(\lambda) + u^\lambda(0, X_0, X_0)\},$$

where Λ^μ is defined in (2.5). This follows from the recent extension by [33], which appeared during the revision of this paper (see Remark 2.1), and avoids placing further conditions on μ to ensure that the function λ^* defined in (3.14) below is uniformly continuous.

3.1. *Formulation in terms of optimal stopping.* We first convert the optimization problem u^λ into an infinite horizon optimal stopping problem.

PROPOSITION 3.1. *For any $\lambda \in \Lambda^\mu$, the functions u^λ is independent of t and*

$$(3.5) \qquad u^\lambda(x, m) = \sup_{\tau \in \mathcal{T}_\infty} \mathbb{E}^{\mathbb{P}_0}[g(M_\tau^{x,m}) - \lambda(X_\tau^x)] \qquad \text{for all } (x, m) \in \boldsymbol{\Delta},$$

where $\mathcal{T}_\infty$ is the collection of all stopping times τ such that the stopped process $\{X_{t \wedge \tau}, t \geq 0\}$ is a $\mathbb{P}_0$-uniformly integrable martingale.

PROOF. The present argument is classical, but we could not find a clear reference. We therefore report it for completeness.

By the definition of $\mathcal{P}_\infty$, we may write the stochastic control problem (3.3) in its strong formulation

$$u^\lambda(t, x, m) := \sup_{\sigma \in \Sigma^+} \mathbb{E}^{\mathbb{P}_0}[g(M_T^{\sigma,t,x,m}) - \lambda(X_T^{\sigma,t,x}) \mid (X_t^{\sigma,t,x}, M_t^{\sigma,t,x,m}) = (x, m)],$$

where

$$X_s^{\sigma,t,x} = x + \int_t^s \sigma_r \, dB_r, \qquad M_s^{\sigma,t,x,m} := m \vee \max_{t \leq r \leq s} X_r^{\sigma,t,x}, \qquad 0 \leq t \leq s \leq T$$

and Σ^+ is the set of all nonnegative progressively measurable processes, with $\int_0^T \sigma_s^2 \, ds < \infty$, and such that the process $\{X_s^{\sigma,t,x}, t \leq s \leq T\}$ is a uniformly integrable martingale.

We shall denote $\phi(x, m) := g(m) - \lambda(x)$.

(1) For a stopping time $\tau \in \mathcal{T}_\infty$, we define the processes $\sigma_t^\tau := \mathbf{1}_{\{\tau \wedge (t/(T-t))\}}$ and $X_t^{\sigma^\tau} := \int_0^t \sigma_s^\tau \, dB_s$, $t \in [t, T]$. Then the corresponding measure $\mathbb{P}^{\sigma^\tau} := \mathbb{P}_0 \circ (X^{\sigma^\tau})^{-1} \in \mathcal{P}_\infty$, and

$$(3.6) \qquad \sup_{\tau \in \mathcal{T}_\infty} \mathbb{E}^{\mathbb{P}_0}[\phi(X_\tau, M_\tau)] \leq \sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\phi(X_\tau, M_\tau)].$$

(2) To obtain the reverse inequality, we observe by the Dambis–Dubins–Schwarz theorem (see, e.g., Karatzas and Shreve [23], Theorem 3.4.6) that the law of (X, M) under $\mathbb{P} = \mathbb{P}^\sigma \in \mathcal{P}_\infty$ is the same as the law of (X_τ, M_τ) under $\mathbb{P}_0$ where $\tau := \int_0^T \sigma_t^2 \, dt$ is a stopping time with respected to the time-changed filtration $\{\mathcal{F}_{T_t}^B, t \geq 0\}$, $T_t := \inf\{s : \langle B \rangle_s > t\}$. In order to convert to the context of the canonical filtration of B, we use the result of Szpirglas and Mazziotto [37]. This allows us to conclude that equality holds in (3.6). $\quad \square$

In view of the previous results, we are reduced to the problem

$$(3.7) \qquad U^\mu(\xi) := \inf_{\lambda \in \Lambda_0^\mu} \{\mu(\lambda) + u^\lambda(X_0, X_0)\},$$

where

$$u^\lambda(X_0, X_0) := \sup_{\tau \in \mathcal{T}_\infty} J(\lambda, \tau), \qquad J(\lambda, \tau) := \mathbb{E}^{\mathbb{P}_0}\big[g(X_\tau^*) - \lambda(X_\tau)\big]$$

and the set Λ^μ of (2.5) translates in the present context to

$$(3.8) \qquad \Lambda_0^\mu = \Big\{\lambda \in \mathbb{L}^1(\mu) : \sup_{\tau \in \mathcal{T}_\infty} \mathbb{E}[\lambda(X_\tau)^-] < \infty\Big\}.$$

3.2. *The main result.* The endpoints of the support of the distribution μ are denoted by

$$\ell^\mu := \sup\{x : \mu([x, \infty)) = 1\} \quad \text{and} \quad r^\mu := \inf\{x : \mu((x, \infty)) = 0\}.$$

The Azéma–Yor solution of the *Skorohod embedding problem* is defined by means of the so-called barycenter function,

$$(3.9) \qquad b(x) := \frac{\int_{[x,\infty)} y\mu(dy)}{\mu([x, \infty))} \mathbf{1}_{\{x < r^\mu\}} + x\mathbf{1}_{\{x \geq r^\mu\}}, \qquad x \geq 0.$$

REMARK 3.1.   Hobson [20] observed that the barycenter function can be alternatively defined as the left-continuous inverse to the following function β. Given the European calls prices $c(x) := \int(y - x)^+\mu(dy)$ and $X_0 = \int y\mu(dy)$, define the function

$$(3.10) \quad \beta(x) := \max\Big\{\operatorname*{arg\,min}_{y < x} \frac{c(y)}{x - y}\Big\} \qquad \text{for } x \in [X_0, r^\mu),$$

$$(3.11) \quad \beta(x) = \ell^\mu \qquad \text{for } x \in [0, X_0) \quad \text{and} \quad \beta(x) = x \qquad \text{for } x \in [r^\mu, \infty).$$

On $[X_0, r^\mu)$, $\beta(x)$ is the largest minimizer of the function $y \longmapsto c(y)/(x - y)$ on $(-\infty, x)$. Then, β is nondecreasing, right-continuous, and $\beta(x) < x$ for all $x \in [X_0, r^\mu)$. Notice that $\beta(X_0) = \ell^\mu := \sup\{x : \mu((0, x]) > 0\}$.

Finally, we introduce the Hardy–Littlewood transform μ^{HL} of μ,

$$(3.12) \qquad \mu^{\mathrm{HL}}\big([y, \infty)\big) := \inf_{\xi < y} \frac{c(y)}{y - \xi} = \frac{c(\beta(y))}{y - \beta(y)} \qquad \text{for all } y \geq 0,$$

where the functions c and β are defined in the previous remark; see Proposition 4.10(c) in Carraro, El Karoui and Obłój [9].

The following result is a combination of [20, 32]. Our objective is to derive it directly from the dual formulation of Proposition 2.1. Let

$$(3.13) \qquad \tau^* := \inf\{t > 0 : X_t^* \geq b(X_t)\}$$

and

$$(3.14) \qquad \lambda^*(x) := \int_{\ell^\mu}^x \int_{\ell^\mu}^y g'\big(b(\xi)\big) \frac{b(d\xi)}{b(\xi) - \xi}\, dy; \qquad x < r^\mu.$$

Notice that $\lambda^* \in [0, \infty]$ as the integral of a nonnegative function. To see that $\lambda^* < \infty$, we compute by the Fubini theorem that

$$\lambda^*(x) = \int_{\ell^\mu}^x g'\big(b(\xi)\big) \frac{x - \xi}{b(\xi) - \xi} b(d\xi)$$

and we observe that $(x - \xi)/(b(\xi) - \xi)$ is bounded near ℓ^μ. Then $\lambda^*(x) \leq C(x)[g(b(x)) - g(X_0)] < \infty$ for some constant $C(x)$ depending on x.

THEOREM 3.1. *Let $\mu \in M(\mathbb{R})$, and $\xi = g(X_T^*)$ for some C^1 nondecreasing function g satisfying $\sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi^+] < \infty$, and $\mu^{\mathrm{HL}}(g) < \infty$. Then*

$$U^\mu(\xi) = \mu(\lambda^*) + J(\lambda^*, \tau^*) = \mu^{\mathrm{HL}}(g).$$

The proof is reported in the subsequent section.

3.3. *An upper bound for the optimal upper bound.* In this section, we prove that

$$(3.15) \qquad U^\mu(\xi) \leq \mu(\lambda^*) + J(\lambda^*, \tau^*).$$

Our first step is to use the following construction due to Peskir [32] which provides a guess of the value function u^λ for functions λ in the subset

$$(3.16) \qquad \hat{\Lambda}_0^\mu := \big\{\lambda \in \Lambda_0^\mu : \lambda \text{ is convex}\big\}.$$

By classical tools from stochastic control theory, the value function $u^\lambda(x, m)$ is expected to solve the dynamic programming equation

$$(3.17) \qquad \begin{aligned} \min\{u^\lambda - g + \lambda, -u_{xx}^\lambda\} &= 0 \qquad \text{on } \Delta \quad \text{and} \\ u_m^\lambda(m, m) &= 0 \qquad \text{for } m \in \mathbb{R}. \end{aligned}$$

The first part of the above DPE is an ODE for which m appears only as a parameter involved in the domain on which the ODE must hold. Since we are restricting to convex λ, one can guess a solution of the form

$$(3.18) \quad v^\psi(x,m) := g(m) - \lambda\big(x \wedge \psi(m)\big) - \lambda'(\psi(m))\big(x - x \wedge \psi(m)\big),$$

that is, $v^\psi(x,m) = g(m) - \lambda(x)$ for $x \le \psi(m)$ and is given by the tangent at the point $\psi(m)$ for $x \in [\psi(m), m]$. For later use, we observe that for $x \in [\psi(m), m]$,

$$
\begin{aligned}
v^\psi(x,m) &= g(m) - \lambda(\psi(m)) + \int_{\psi(m)}^x \frac{\partial}{\partial y}\big\{\lambda'(y)(x-y)\big\}\, dy \\
&= g(m) - \lambda(x) + \int_{\psi(m)}^x (x-y)\lambda''(dy) \qquad \text{for } x \in [\psi(m), m],
\end{aligned}
$$
(3.19)

where λ'' is the second derivative measure of the convex function λ.

We next choose the function ψ in order to satisfy the Neumann condition in (3.17). Assuming that λ is smooth, we obtain by direct calculation that the free boundary ψ must verify the ordinary differential equation (ODE)

$$(3.20) \qquad \lambda''(\psi(m))\psi'(m) = \frac{g'(m)}{m - \psi(m)} \qquad \text{for all } m \in \mathbb{R}.$$

For technical reasons, we need to consider this ODE in the relaxed sense. This contrasts our analysis with that of Peskir [32] and Oblój [30]. Since λ is convex, its second derivative λ'' is well defined as measure on $\mathbb{R}_+$. We then introduce the weak formulation of the ODE (3.20),

$$(3.21) \qquad \int_{\psi(B)} \lambda''(dy) = \int_B \frac{g'(m)}{m-\psi(m)}\, dm \qquad \text{for all } B \in \mathcal{B}(\mathbb{R})$$

and we introduce the collection of all relaxed solutions of (3.20),

$$
\begin{aligned}
\Psi^\lambda := \big\{ &\psi \text{ right-continuous: (3.21) holds and} \\
&\psi(m) < m \text{ for all } m \in \mathbb{R} \big\}.
\end{aligned}
$$
(3.22)

REMARK 3.2. For later use, we observe that (3.21) implies that all functions $\psi \in \Psi^\lambda$ are nondecreasing. Indeed, for $y_1 \le y_2$, it follows from (3.21), together with the nondecrease of g in (3.2) and the convexity of λ, that

$$
\begin{aligned}
\psi(y_2) &= (\lambda')^{-1}\bigg(\lambda'(\psi(y_1)+) + \int_{y_1}^{y_2} \frac{g'(m)}{m-\psi(m)}\, dm\bigg) \\
&\ge (\lambda')^{-1}\big(\lambda'(\psi(y_1)+)\big) \ge \psi(y_1),
\end{aligned}
$$

where $(\lambda')^{-1}$ is the right-continuous inverse of the nondecreasing function λ'. Then, by direct integration that

$$\text{the function } x \longmapsto \lambda(x) - \int_{X_0}^x \int_{X_0}^{\psi^{-1}(y)} \frac{g'(\xi)}{\xi - \psi(\xi)}\, d\xi\, dy \text{ is affine,}$$

where ψ^{-1} is the right-continuous inverse of ψ. This follows from direct differentiation of the above function in the sense of generalized derivatives.

A remarkable feature of the present problem is that there is no natural boundary condition for the ODE (3.20) or its relaxation (3.21). The following result extends the easy part of the elegant maximality principle proved in Peskir [32] by allowing for possibly nonsmooth functions λ. We emphasize the fact that our approach does not need the full strength of Peskir's maximality principle.

LEMMA 3.1. *Let $\lambda \in \hat{\Lambda}_0^\mu$ and $\psi \in \Psi^\lambda$ be arbitrary. Then $u^\lambda \leq v^\psi$.*

PROOF. We organize the proof in three steps:

(1) We first prove that v^ψ is differentiable in m on the diagonal with

$$(3.23) \qquad v_m^\psi(m, m) = 0 \qquad \text{for all } m \in \mathbb{R}.$$

Indeed, since $\psi \in \Psi_\lambda$, it follows from Remark 3.2 that

$$\lambda(x) = c_0 + c_1 x + \int_{X_0}^{x} \int_{X_0}^{\psi^{-1}(y)} \frac{g'(\xi)}{\xi - \psi(\xi)} \, d\xi \, dy$$

for some scalar constants c_0, c_1. Plugging this expression into (3.18), we see that

$$v^\psi(x, m) = g(m) - \left(c_0 + c_1 \psi(m) + \int_{X_0}^{\psi(m)} \int_{X_0}^{\psi^{-1}(y)} \frac{g'(\xi)}{\xi - \psi(\xi)} \, d\xi \, dy \right)$$

$$- \left(c_1 + \int_{X_0}^{m} \frac{g'(\xi)}{\xi - \psi(\xi)} \, d\xi \right)(x - \psi(m))$$

$$= g(m) - c_0 - c_1 x + \int_{X_0}^{m} \frac{g'(\xi)}{\xi - \psi(\xi)} (\psi(\xi) - x) \, d\xi,$$

where the last equality follows from the Fubini theorem together with the fact that g is nondecreasing and $\psi(\xi) < \xi$. Since g is differentiable, (3.23) follows by direct differentiation with respect to m.

(2) For an arbitrary stopping time $\tau \in \mathcal{T}_\infty$, we introduce the stopping times $\tau_n := \tau \wedge \inf\{t > 0 : |X_t - x| > n\}$. Since v^ψ is concave in x, as a consequence of the convexity of λ, it follows from the Itô–Tanaka formula that

$$v^\psi(x, m) \geq v^\psi(X_{\tau_n}, M_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t, M_t) \, dB_t - \int_0^{\tau_n} v_m^\psi(X_t, M_t) \, dM_t$$

$$\geq g(M_{\tau_n}) - \lambda(X_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t, M_t) \, dB_t - \int_0^{\tau_n} v_m^\psi(X_t, M_t) \, dM_t$$

by the fact that $v^\psi \geq g - \lambda$. Notice that $(M_t - X_t) \, dM_t = 0$. Then by the Neumann condition (3.23), we have $v_m^\psi(X_t, M_t) \, dM_t = v_m^\psi(M_t, M_t) \, dM_t = 0$. Taking expectations in the last inequality, we see that

$$(3.24) \qquad v^\psi(x, m) \geq \mathbb{E}_{x,m}\big[g(M_{\tau_n}) - \lambda(X_{\tau_n})\big].$$

(3) We finally take the limit as $n \to \infty$ in the last inequality. First, recall that $(X_{t \wedge \tau})_{t \geq 0}$ is a uniformly integrable martingale. Then, by the Jensen inequality, $\lambda(X_{\tau_n}) \leq \mathbb{E}[\lambda(X_\tau) \mid \mathcal{F}_{\tau_n}]$. Since $\lambda(X_\tau)^- \in \mathbb{L}^1(\mathbb{P}^0)$, this implies that $\mathbb{E}[\lambda(X_{\tau_n})] \leq \mathbb{E}[\lambda(X_\tau)]$ where we also used the tower property of conditional expectations. We then deduce from (3.24) that

$$v^\psi(x, m) \geq \lim_{n \to \infty} \mathbb{E}_{x,m}\big[g(M_{\tau_n}) - \lambda(X_\tau)\big] = \mathbb{E}_{x,m}\big[g(M_\tau) - \lambda(X_\tau)\big]$$

by the nondecrease of the process M and the function g together with the monotone convergence theorem. By the arbitrariness of $\tau \in \mathcal{T}_\infty$, the last inequality shows that $v^\psi \geq u^\lambda$. $\square$

Our next result involves the function

$$\varphi(x, m) := \frac{c(x) - c_0(x)\mathbf{1}_{m < X_0}}{m - x}$$

(3.25)
$$\text{with } c_0(x) := (X_0 - x)^+, (x, m) \in \mathbf{\Delta}$$

and we recall that $c(x) := \int (\xi - x)^+ \mu(d\xi)$ is the (given) European call price with strike x.

LEMMA 3.2.   *For $\lambda \in \hat{\Lambda}_0^\mu$ and $\psi \in \Psi^\lambda$, we have*

$$\mu(\lambda) + u^\lambda(X_0, X_0) \leq g(X_0) + \int \varphi\big(\psi(m), m\big)g'(m)\,dm.$$

PROOF.   (1) Let $\alpha \in \mathbb{R}_+$ be an arbitrary point of differentiability of λ. Then

$$\lambda(x) = \lambda(\alpha) + \lambda'(\alpha)(x - \alpha) + \int_\alpha^x (x - y)\lambda''(dy).$$

Integrating with respect to $\mu - \delta_{X_0}$ and taking $\alpha < X_0$, this provides

$$\mu(\lambda) - \lambda(X_0)$$
$$= \lambda'(\alpha)\left(\int x\mu(dx) - X_0\right) + \int \left(\int_\alpha^x (x - y)\lambda''(dy)\right)(\mu - \delta_{X_0})(dx)$$
$$= -\int_\alpha^{X_0} (X_0 - y)\lambda''(dy) + \int \mathbf{1}_{\{x \geq \alpha\}} \int_\alpha^x (x - y)^+ \lambda''(dy)\mu(dx)$$
$$+ \int \mathbf{1}_{\{x < \alpha\}} \int_x^\alpha (y - x)\lambda''(dy)\mu(dx).$$

Then sending α to ℓ^μ, it follows from the convexity of λ together with the monotone convergence theorem that

$$\mu(\lambda) - \lambda(X_0) = \int (c - c_0)(y)\lambda''(dy).$$

(2) By the inequality in Lemma 3.1, together with (3.19), we now compute that

$$\mu(\lambda) + u^\lambda(X_0, X_0)$$

$$\leq g(X_0) + \int \big(c(y) - c_0(y)(1_{\{y<X_0\}} - 1_{\{\psi(X_0)<y<X_0\}})\big)\lambda''(dy)$$

$$= g(X_0) + \int \big(c(y) - c_0(y)1_{\{y<\psi(X_0)\}}\big)\lambda''(dy).$$

We next use the ODE (3.20) satisfied by ψ in the distribution sense. This provides

$$\mu(\lambda) + u^\lambda(X_0, X_0) \leq g(X_0) + \int \frac{c(\psi(m)) - c_0(\psi(m))\mathbf{1}_{\{m<X_0\}}}{m - \psi(m)} g'(m)\, dm.$$

Here, we observe that the endpoints in the last integral can be taken to 0 and ∞ by the nonnegativity of the integrand. $\quad\square$

We now have all ingredients to express the upper bound (3.15) explicitly in terms of the barycenter function b of (3.9).

LEMMA 3.3. *For a nondecreasing C^1 payoff function g, we have*

$$\inf_{\lambda \in \Lambda_0^\mu} \big\{\mu(\lambda) + u^\lambda(X_0, X_0)\big\} \leq \mu^{\mathrm{HL}}(g).$$

PROOF. Since $\hat{\Lambda}_0^\mu \subset \Lambda_0^\mu$, we compute from Lemma 3.2 that

$$\inf_{\lambda \in \Lambda_0^\mu} \big\{\mu(\lambda) + u^\lambda(X_0, X_0)\big\}$$

(3.26)
$$\leq \inf_{\lambda \in \hat{\Lambda}_0^\mu} \big\{\mu(\lambda) + u^\lambda(X_0, X_0)\big\}$$

$$\leq g(X_0) + \inf_{\lambda \in \hat{\Lambda}_0^\mu} \inf_{\psi \in \Psi^\lambda} \int \varphi\big(\psi(m), m\big) g'(m)\, dm.$$

In the next two steps, we prove that the last minimization problem on the right-hand side of (3.26) can be solved by pointwise minimization inside the integral. Then, in step (3), we compute the induced upper bound.

(1) For all $\lambda \in \hat{\Lambda}_0^\mu$ and $\psi \in \Psi^\lambda$,

$$\int \varphi\big(\psi(m), m\big) g'(m)\, dm \geq \int \inf_{\xi < m} \varphi(\xi, m) g'(m)\, dm.$$

Observe that $c(x) \geq c_0(x)$ for all $x \geq 0$, and $\lim_{x\to 0} c(x) - c_0(x) = 0$. Then

(3.27)
$$\inf_{\xi < m} \varphi(\xi, m) = \varphi(0, m) = 0 \qquad \text{for } m < X_0.$$

On the other hand, it follows from Remark 3.1 that

$$(3.28) \qquad \inf_{\xi < m} \varphi(\xi, m) = \inf_{\xi < m} \frac{c(\xi)}{m - \xi} = \frac{c(\beta(m))}{m - \beta(m)} \qquad \text{for } m \geq X_0.$$

By (3.27) and (3.28), we obtain the lower bound

$$\int \varphi(\psi(m), m) g'(m) \, dm \geq \int \varphi(\beta(m), m) g'(m) \, dm.$$

(2) We now observe that the function β, obtained by pointwise minimization in the previous step, solves the ODE (3.21). Therefore, in order to complete the proof, it remains to verify that $\lambda^* \in \hat{\Lambda}_0^\mu$. The convexity of λ^* is obvious. Also, since $\lambda^* \geq 0$, we only need to prove that $\lambda^* \in \mathbb{L}^1(\mu)$. By step (1) of the proof of Lemma 3.2, we are reduced to verifying that $\int c(x)(\lambda^*)''(dx) < \infty$. Since, by definition, λ^* satisfies the ODE (3.21) with $\psi = b^{-1}$, we directly compute that

$$\int c(x)(\lambda^*)''(dx) = \int \frac{c(b^{-1}(m))}{m - b^{-1}(m)} g'(m) \, dm$$

$$= \int g'(m) \mu^{\mathrm{HL}}([m, \infty)) \, dm < \infty$$

by our assumption that $\mu^{\mathrm{HL}}(g) < \infty$.

(3) From (3.26) and the previous two steps, we have

$$\inf_{\lambda \in \Lambda_0^\mu} \{\mu(\lambda) + u^\lambda(X_0, X_0)\} \leq g(X_0) + \int \frac{c(\beta(x))}{x - \beta(x)} g'(x) \, dx$$

$$= g(X_0) + \int \mu^{\mathrm{HL}}([y, \infty)) g'(x) \, dx = \mu^{\mathrm{HL}}(g)$$

by a direct integration by parts. $\square$

3.4. *Completing the proof of Theorem* 3.1.   To complete the proof of the theorem, it remains to prove that

$$\inf_{\lambda \in \Lambda_0^\mu} \{\mu(\lambda) + u^\lambda(X_0, X_0)\} \geq \mu^{\mathrm{HL}}(g).$$

To see this, we use the fact that the stopping time τ^* defined in (3.13) is a solution of the Skorohod embedding problem, that is, $X_{\tau^*} \sim \mu$ and $(X_{t \wedge \tau^*})_{t \geq 0}$ is a uniformly integrable martingale; see Azéma and Yor [1, 2]. Moreover $X_{\tau^*}^* \sim \mu^{\mathrm{HL}}$. Then, for all $\lambda \in \Lambda_0^\mu$, it follows from the definition of u^λ that $u^\lambda(X_0, X_0) \geq J(\lambda, \tau^*)$, and therefore

$$\mu(\lambda) + u^\lambda(X_0, X_0) \geq \mu(\lambda) + \mathbb{E}_{X_0, X_0}[g(X_{\tau^*}^*) - \lambda(X_{\tau^*})]$$

$$= \mathbb{E}_{X_0, X_0}[g(X_{\tau^*}^*)] = \mu^{\mathrm{HL}}(g).$$

4. Forward start lookback options. In this section, we provide a second application to the case where the derivative security is defined by the payoff

$$\xi = g(B^*_{t_1,t_2}) \qquad \text{where } B^*_{t_1,t_2} := \max_{t_1 \le t \le t_2} B_t$$

and g satisfies the same conditions as in the previous section. We assume that the prices of call options $c_1(k)$ and $c_2(k)$ for the maturities t_1 and t_2 are given for all strikes,

$$c_1(k) = \int (x - k)^+ \mu_1(dx) \quad \text{and} \quad c_2(k) = \int (x - k)^+ \mu_2(dx), \qquad k \ge 0.$$

We also assume that $\mu_1 \preceq \mu_2$ are in convex order:

$$c_1(0) = c_2(0) \quad \text{and} \quad c_1(k) \le c_2(k) \qquad \text{for all } k \ge 0.$$

The model-free superhedging cost is defined as the minimal initial capital which allows to superhedge the payoff ξ, quasi-surely, by means of some dynamic trading strategy in the underlying stock, and a static strategy in the calls $(c_1(k))_{k \ge 0}$ and $(c_2(k))_{k \ge 0}$.

This problem was solved in Hobson [20] in the case $g(x) = x$. Our objective here is to recover his results by means of our stochastic control approach.

A direct adaptation of Proposition 2.1 provides the dual formulation of this problem as

$$U^{\mu_1,\mu_2}(\xi) = \sup_{(\lambda_1,\lambda_2) \in \Lambda^{\mu_1} \times \Lambda^{\mu_2}} \mu_1(\lambda_1) + \mu_2(\lambda_2) + u^{\lambda_1,\lambda_2}(X_0, X_0),$$

where

$$u^{\lambda_1,\lambda_2}(x, m) := \sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}_{x,m}[g(B^*_{t_1,t_2}) - \lambda_1(B_{t_1}) - \lambda_2(B_{t_2})].$$

We next observe that the dynamic value function corresponding to the stochastic control problem u^{λ_1,λ_2} reduces to our previously studied problem u^{λ_2} at time t_1. Then, it follows from the dynamic programming principle that

$$U^{\mu_1,\mu_2}(\xi) = \inf_{(\lambda_1,\lambda_2) \in \Lambda^{\mu_1} \times \Lambda^{\mu_2}} \mu_1(\lambda_1) + \mu_2(\lambda_2)$$
$$+ \sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[u^{\lambda_2}(B_{t_1}, B_{t_1}) - \lambda_1(B_{t_1})].$$

Since the expression to be maximized only involves the distribution of B_{t_1}, it follows from Remark 2.3 together with the Dambis–Dubins–Schwarz time change formula that

$$U^{\mu_1,\mu_2}(\xi) = \inf_{\lambda_2 \in \Lambda^{\mu_2}_0} \mu_2(\lambda_2) + \int u^{\lambda_2}(x, x)\mu_1(dx).$$

We next obtain an upper bound by restricting attention to the subset $\hat{\Lambda}_0^{\mu_2}$ of convex multipliers of $\Lambda_0^{\mu_2}$. For such multipliers, we use the inequality $u^{\lambda_2} \leq v^{\psi_2}$ for all $\psi_2 \in \Psi^{\lambda_2}$ as derived in Lemma 3.1. This provides

$$U^{\mu_1,\mu_2}(\xi)$$

$$\leq \inf_{\lambda_2 \in \hat{\Lambda}_0^{\mu_2}} \mu_2(\lambda_2) + \int v^{\psi_2}(x,x)\mu_1(dx)$$

$$= \mu_1(g) + \inf_{\lambda_2 \in \hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2 \in \Psi^\lambda} \mu_2(\lambda_2) - \mu_1(\lambda_2) + \int \int_{\psi_2(x)}^x (x-y)\lambda_2''(dy)\mu_1(dx)$$

$$= \mu_1(g) + \inf_{\lambda_2 \in \hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2 \in \Psi^\lambda} \int \left(c_2(y) - c_1(y) \right.$$

$$\left. + \int (x-y)\mathbf{1}_{\{\psi_2(x) < y < x\}}\mu_1(dx) \right)\lambda_2''(dy)$$

$$= \mu_1(g) + \inf_{\lambda_2 \in \hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2 \in \Psi^\lambda} \int \left(c_2(y) - \int (x-y)\mathbf{1}_{\{y \leq \psi_2(x)\}}\mu_1(dx) \right)\lambda_2''(dy)$$

$$= \mu_1(g) + \inf_{\lambda_2 \in \hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2 \in \Psi^\lambda} \int \left(c_2(\psi_2(m)) - \int (x-\psi_2(m))\mathbf{1}_{\{m \leq x\}}\mu_1(dx) \right)$$

$$\times \frac{g'(m)\,dm}{m - \psi_2(m)}$$

$$= \mu_1(g) + \inf_{\lambda_2 \in \hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2 \in \Psi^\lambda} \int \left(\frac{c_2(\psi_2(m)) - c_1(m)}{m - \psi_2(m)} - \mu_1([m,\infty)) \right)g'(m)\,dm,$$

where the last equalities follow from similar manipulations as in Lemma 3.2, and in particular make use of the ODE (3.21). Since $g' \geq 0$, we may prove, as in the case of lookback options, that the above minimization problem reduces to the pointwise minimization of the integrand, so that the optimal obstacle is given by

$$\psi_2^*(x) = \max\left\{ \operatorname*{arg\,min}_{\xi < x} h(\xi) \right\} \qquad \text{where } h(\xi) := \frac{c_2(\xi) - c_1(m)}{m - \xi}, \xi < m.$$

Notice that h has left and right derivative at every $\xi < m$, with

$$h'(\xi) = \frac{c_2(\xi) + (x-\xi)c_2'(\xi) - c_1(x)}{(x-\xi)^2}, \qquad \text{a.e.,}$$

where the numerator is a nondecreasing function of ξ, takes the positive value $c_2(x) - c_1(x)$ at $\xi = x$, and takes the negative value $X_0 - x - c_1(x)$ at $\xi = 0$. Then $\psi_2^*(x)$ is the largest root of the equation

$$(4.1) \qquad c_2(\psi_2^*(x)) + (x - \psi_2^*(x))c_2'(\psi_2^*(x)) = c_1(x), \qquad \text{a.e.}$$

so that h is nonincreasing to the left of $\psi_2^*(m)$ and nondecreasing to its right.

At this point, we recognize exactly the solution derived by Hobson [20]. In particular, ψ_2^* induces a solution τ_2^* to the Skorohod embedding problem, and we may use the expression of u^{λ_2} as the value function of an optimal stopping problem. Then, we may conclude the proof that the upper bound derived above is the optimal upper bound by arguing as in Section 3.4 that

$$u^{\lambda_2}(x, x) \geq \mathbb{E}_{x,x}\big[g\big(X_{\tau_2^*}^*\big) - \lambda_2(X_{\tau_2^*})\big].$$

We get that the upper bound is given by

(4.2)
$$\begin{aligned}
U^{\mu_1,\mu_2}(\xi) &= \mu_1(g) - \int g'(m)\mu_1\big([m,\infty)\big)\,dm \\
&\quad + \int \left(\frac{c_2(\psi_2^*(m)) - c_1(m)}{m - \psi_2^*(m)}\right)g'(x)\,dx \\
&= \mu_1(g) - \int \big(c_2'(\psi_2^*(m)) - c_1'(m)\big)g'(m)\,dm \\
&= g\big(\ell^\mu\big) - \int c_2'(\psi_2^*(m))g'(m)\,dm
\end{aligned}$$

by (4.1).

5. Proof of the duality result. Let $\xi : \Omega \longrightarrow \mathbb{R}$ be a measurable map with $\sup_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi^+] < \infty$. If $\mathcal{P}_\infty(\xi) = \varnothing$, the result is trivial. We then continue assuming that $\mathcal{P}_\infty(\xi) \neq \varnothing$ and therefore $U_0(\xi) > -\infty$. Let $X_0 \in \mathbb{R}$ be such that

(5.1) $$X_T^H \geq \xi \qquad \text{for some } H \in \mathcal{H}.$$

By definition of the admissibility set $\mathcal{H}(\xi)$, it follows that the process X^H is a $\mathbb{P}$-local martingale and a $\mathbb{P}$-supermartingale for any $\mathbb{P} \in \mathcal{P}_\infty(\xi)$. Then, it follows from (5.1) that $X_0 \geq \mathbb{E}^{\mathbb{P}}[\xi]$ for all $\mathbb{P} \in \mathcal{P}_\infty(\xi)$. From the arbitrariness of X_0 and $\mathbb{P}$, this shows that

(5.2) $$U^0(\xi) \geq \sup_{\mathbb{P}\in\mathcal{P}_\infty(\xi)} \mathbb{E}^{\mathbb{P}}[\xi] = \sup_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi].$$

In the subsequent sections, we prove that the converse inequality holds under the additional requirement that $\xi \in \mathrm{UC}(\Omega_{X_0})$. Following [35], this result is obtained by introducing a *dynamic version* of the problem which is then proved to have a decomposition leading to the required result. Due to the fact that family of probability measures $\mathcal{P}_\infty$ is nondominated, we need to define conditional distributions on all of the probability space without excepting any zero measure set.

5.1. *Regular conditional probability distribution.*   Let $\mathbb{P}$ be an arbitrary probability measure on Ω, and τ be an $\mathbb{F}$-stopping time. The regular conditional probability distribution (r.c.p.d.) $\mathbb{P}_\tau^\omega$ is defined by:

– for all $\omega \in \Omega$, $\mathbb{P}_\tau^\omega$ is a probability measure on $\mathcal{F}_T$;
– for all $E \in \mathcal{F}_T$, the mapping $\omega \longmapsto \mathbb{P}_\tau^\omega(E)$ is $\mathcal{F}_\tau$-measurable;
– for every bounded $\mathcal{F}_T$-measurable random variable ξ, we have $\mathbb{E}^{\mathbb{P}}[\xi \mid \mathcal{F}_\tau](\omega) = \mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi]$, $\mathbb{P}$-a.s.;
– for all $\omega \in \Omega$, $\mathbb{P}_\tau^\omega[\omega' \in \Omega : \omega'(s) = \omega(s), 0 \le s \le \tau(\omega)] = 1$.

The existence of the r.c.p.d. is justified in Stroock and Varadhan [36]. For a better understanding of this notion, we introduce the shifted canonical space

$$\Omega^t := \{\omega \in C([t, T], \mathbb{R}^d) : \omega(t) = 0\} \qquad \text{for all } t \in [0, T],$$

we denote by B^t the shifted canonical process on Ω^t, $\mathbb{P}_0^t$ the shifted Wiener measure and $\mathbb{F}^t$ the shifted filtration generated by B^t. For $0 \le s \le t \le T$ and $\omega \in \Omega^s$:

– the shifted path $\omega^t \in \Omega^t$ is defined by

$$\omega_r^t := \omega_r - \omega_t \qquad \text{for all } r \in [t, T];$$

– the concatenation path $\omega \otimes_t \tilde{\omega} \in \Omega^s$, for some $\tilde{\omega} \in \Omega^t$, is defined by

$$(\omega \otimes_t \tilde{\omega})(r) := \omega_r \mathbf{1}_{[s,t)}(r) + (\omega_t + \tilde{\omega}_r)\mathbf{1}_{[t,T]}(r) \qquad \text{for all } r \in [s, T];$$

– the shifted $\mathcal{F}_T^t$-measurable r.v. $\xi^{t,\omega}$ of some $\mathcal{F}_T^s$-measurable r.v. ξ on Ω^s is defined by

$$\xi^{t,\omega}(\tilde{\omega}) := \xi(\omega \otimes_t \tilde{\omega}) \qquad \text{for all } \tilde{\omega} \in \Omega^t.$$

Similarly, for an $\mathbb{F}^s$-progressively measurable process X on $[s, T]$, the shifted process $\{X_r^{t,\omega}, r \in [t, T]\}$ is $\mathbb{F}^t$-progressively measurable.

For notational simplicity, we set

$$\omega \otimes_\tau \tilde{\omega} := \omega \otimes_{\tau(\omega)} \tilde{\omega}, \qquad \xi^{\tau,\omega} := \xi^{\tau(\omega),\omega}, \qquad X^{\tau,\omega} := X^{\tau(\omega),\omega}.$$

The r.c.p.d. $\mathbb{P}_\tau^\omega$ induces a probability measure $\mathbb{P}^{\tau,\omega}$ on $\mathcal{F}_T^{\tau(\omega)}$ such that the $\mathbb{P}^{\tau,\omega}$-distribution of $B^{\tau(\omega)}$ is equal to the $\mathbb{P}_\tau^\omega$-distribution of $\{B_t - B_{\tau(\omega)}, t \in [\tau(\omega), T]\}$. Then, the r.c.p.d. can be understood by the identity

$$\mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}] \qquad \text{for all } \mathcal{F}_T\text{-measurable r.v. } \xi.$$

We shall also call $\mathbb{P}^{\tau,\omega}$ the r.c.p.d. of $\mathbb{P}$.

For $0 \le t \le T$, we follow the same construction as in Section 2.1 to define the martingale measures $\mathbb{P}^{t,\alpha}$ for each $\mathbb{F}^t$-progressively measurable $\mathcal{S}_d^{>0}$-valued process α such that $\int_t^T |\alpha_r| \, dr < \infty$, $\mathbb{P}_0^t$-a.s. The collection of all such measures is denoted $\overline{\mathcal{P}}_S^t$. The subset $\mathcal{P}_\infty^t$ and the density process $\hat{a}^t$ of the quadratic variation process $\langle B^t \rangle$ are also defined similarly.

5.2. *The duality result for uniformly continuous payoffs.* Since $\xi \in \mathrm{UC}(\Omega_{X_0})$, there exists a modulus of continuity function ρ such that for all $t \in [0, T]$ and $\omega, \omega' \in \Omega$, $\tilde{\omega} \in \Omega^t$,

$$\left| \xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega}) \right| \leq \rho \left(\|\omega - \omega'\|_t \right),$$

where $\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega_s|$, $0 \leq t \leq T$. The main object in the present proof is the following dynamic value process:

$$(5.3) \qquad V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}^t_\infty} \mathbb{E}^{\mathbb{P}^\omega_t}[\xi] \qquad \text{for all } (t, \omega) \in [0, T] \times \Omega.$$

It follows from the uniform continuity property of ξ that

$$(5.4) \qquad \{V_t, t \in [0, T]\} \text{ is a right-continuous } \mathbb{F}\text{-adapted process.}$$

Moreover, since $\sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi^+] < \infty$, it follows that for all $\mathbb{P} \in \mathcal{P}_\infty(\xi)$ that $V_t \in \mathbb{L}^1(\mathbb{P})$, and by following exactly the proof of Proposition 4.7 in [35], we see that $\{V_t, t \in [0, T]\}$ is a $\mathbb{P}$-supermartingale. We may then apply the Doob–Meyer decomposition, and deduce the existence of a pair of processes $(H^{\mathbb{P}}, K^{\mathbb{P}})$, with $H^{\mathbb{P}} \in \mathbb{H}^2_{\mathrm{loc}}(\mathbb{P})$ and $K^{\mathbb{P}}$ $\mathbb{P}$-integrable nondecreasing, such that

$$V_t = V_0 + \int_0^t H^{\mathbb{P}}_s \, dB_s - K^{\mathbb{P}}_t, \qquad t \in [0, T], \mathbb{P}\text{-a.s.}$$

Since V is a right-continuous semimartingale under each $\mathbb{P} \in \mathcal{P}_\infty(\xi)$, it follows from Karandikar [22] that the family of processes $\{H^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_\infty(\xi)\}$ (defined $\mathbb{P}$-a.s.) can be aggregated into a process $\hat{H}$ defined on $[0, T] \times \Omega$ by $d\langle V, B \rangle_t = \hat{H}_t \, d\langle B \rangle_t$, in the sense that $\hat{H} = H^{\mathbb{P}}$, $dt \times d\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_\infty(\xi)$. Thus we have

$$V_t = V_0 + \int_0^t \hat{H}_s \, dB_s - K^{\mathbb{P}}_t, \qquad t \in [0, T], \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_\infty(\xi).$$

With $X_0 := V_0$, we see that:

– the process $X^{\hat{H}} := X_0 + \int_0^\cdot \hat{H}_s \, dB_s$ is bounded from below by V which is in turn bounded from below by $M^{\mathbb{P}}_t := \mathbb{E}^{\mathbb{P}}_t[\xi]$, $t \in [0, T]$; since $\xi \in \mathbb{L}^1(\mathbb{P})$, the latter is a $\mathbb{P}$-martingale; consequently, $X^{\hat{H}}$ is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}_\infty(\xi)$,

– and $X^{\hat{H}}_T = V_T + K^{\mathbb{P}}_T = \xi + K^{\mathbb{P}}_T \geq \xi$, $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathcal{P}_\infty(\xi)$.

Then $V_0 \geq U^0(\xi)$ by the definition of U^0.

Notice that, as a consequence of the supermartingale property of $X^{\hat{H}}$ under every $\mathbb{P} \in \mathcal{P}_\infty(\xi)$, we have

$$V_0 + \sup_{\mathbb{P} \in \mathcal{P}_\infty(\xi)} \mathbb{E}^{\mathbb{P}}\left[-K^{\mathbb{P}}_T\right] \geq \sup_{\mathbb{P} \in \mathcal{P}_\infty(\xi)} \mathbb{E}^{\mathbb{P}}\left[X^{\hat{H}}_T - K^{\mathbb{P}}_T\right] = \sup_{\mathbb{P} \in \mathcal{P}_\infty(\xi)} \mathbb{E}^{\mathbb{P}}[\xi] = V_0.$$

Since $K^{\mathbb{P}}_0 = 0$ and $K^{\mathbb{P}}$ is nondecreasing, this implies that

$$X^{\hat{H}} \text{ is a } \mathbb{P}\text{-martingale for all } \mathbb{P} \in \mathcal{P}_\infty(\xi)$$

and the nondecreasing process $K^{\mathbb{P}}$ satisfies the minimality condition

$$\inf_{\mathbb{P} \in \mathcal{P}_{\infty}(\xi)} \mathbb{E}^{\mathbb{P}}\big[K^{\mathbb{P}}_T\big] = 0.$$

REMARK 5.1. A possible extension of Theorem 2.1 can be obtained for a larger class of payoff functions ξ. Indeed, notice that the uniform continuity assumption on ξ is essentially used to obtain the measurability (5.4) of dynamic value process V defined in (5.3). In the context of the application of Section 3, such an extension is needed in order to avoid restricting our framework to those measures μ which induce a uniformly continuous optimal static hedging λ^*. A convenient extension of Theorem 2.1, relaxing the uniform integrability condition, is obtained in Possamaï, Royer and Touzi [33], building on the recent results of Nutz and van Handel [29] and Neufeld and Nutz [27].

REFERENCES

[1] AZÉMA, J. and YOR, M. (1979). Une solution simple au problème de Skorokhod. In *Séminaire de Probabilités*, *XIII*. *Lecture Notes in Math.* **721** 90–115. Springer, Berlin. MR0544782

[2] AZÉMA, J. and YOR, M. (1979). Le problème de Skorokhod: Compléments à "Une solution simple au problème de Skorokhod". In *Séminaire de Probabilités*, *XIII*. *Lecture Notes in Math.* **721** 625–633. Springer, Berlin. MR0544832

[3] BEIGLBÖCK, M., HENRY-LABORDÈRE, P. and PENKNER, F. (2011). Model independent bounds for option prices: A mass transport approach. Preprint. Available at arXiv:1106.5929.

[4] BENAMOU, J.-D. and BRENIER, Y. (2000). A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem. *Numer. Math.* **84** 375–393. MR1738163

[5] BONNANS, F. and TAN, X. (2011). A model-free no-arbitrage price bound for variance options. Preprint.

[6] BREEDEN, D. T. and LITZENBERGER, R. H. (1978). Prices of state-contingent claims implicit in options prices. *J. Business* **51** 621–651.

[7] BROWN, H., HOBSON, D. and ROGERS, L. C. G. (2001). The maximum maximum of a martingale constrained by an intermediate law. *Probab. Theory Related Fields* **119** 558–578. MR1826407

[8] CARR, P. and CHOU, A. (1997). Breaking barriers. *Risk* **10** 139–145.

[9] CARRARO, L., EL KAROUI, N. and OBŁÓJ, J. (2012). On Azéma–Yor processes, their optimal properties and the Bachelier-drawdown equation. *Ann. Probab.* **40** 372–400. MR2917776

[10] COX, A. M. G., HOBSON, D. and OBŁÓJ, J. (2008). Pathwise inequalities for local time: Applications to Skorokhod embeddings and optimal stopping. *Ann. Appl. Probab.* **18** 1870–1896. MR2462552

[11] COX, A. M. G. and OBŁÓJ, J. (2011). Robust pricing and hedging of double no-touch options. *Finance Stoch.* **15** 573–605. MR2833100

[12] CVITANIĆ, J., PHAM, H. and TOUZI, N. (1999). Super-replication in stochastic volatility models under portfolio constraints. *J. Appl. Probab.* **36** 523–545. MR1724796

[13] DAVIS, M., OBLÓJ, J. and RAVAL, V. (2013). Arbitrage bounds for weighted variance swap prices. *Math. Finance*. To appear. Available at arXiv:1001.2678v2.

[14] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520. MR1304434

[15] DENIS, L. and MARTINI, C. (2006). A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *Ann. Appl. Probab.* **16** 827–852. MR2244434

[16] FLEMING, W. H. and SONER, H. M. (1993). *Controlled Markov Processes and Viscosity Solutions. Applications of Mathematics* (*New York*) **25**. Springer, New York. MR1199811

[17] FREY, R. (2000). Superreplication in stochastic volatility models and optimal stopping. *Finance Stoch.* **4** 161–187. MR1780325

[18] HARRISON, J. M. and PLISKA, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11** 215–260. MR0622165

[19] HENRY-LABORDÈRE, P., OBLÓJ, J., SPOIDA, P. and TOUZI, N. (2012). Maximum maximum of martingales given finitely-many intermediate marginals. Preprint.

[20] HOBSON, D. (1998). Robust hedging of the lookback option. *Finance Stoch.* **2** 329–347.

[21] HOBSON, D. (2011). The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris-Princeton Lectures on Mathematical Finance* 2010. *Lecture Notes in Math.* **2003** 267–318. Springer, Berlin. MR2762363

[22] KARANDIKAR, R. L. (1995). On pathwise stochastic integration. *Stochastic Process. Appl.* **57** 11–18. MR1327950

[23] KARATZAS, I. and SHREVE, S. E. (1988). *Brownian Motion and Stochastic Calculus. Graduate Texts in Mathematics* **113**. Springer, New York. MR0917065

[24] KREPS, D. M. (1981). Arbitrage and equilibrium in economies with infinitely many commodities. *J. Math. Econom.* **8** 15–35. MR0611252

[25] MADAN, D. B. and YOR, M. (2002). Making Markov martingales meet marginals: With explicit constructions. *Bernoulli* **8** 509–536. MR1914701

[26] MIKAMI, T. and THIEULLEN, M. (2008). Optimal transportation problem by stochastic optimal control. *SIAM J. Control Optim.* **47** 1127–1139. MR2407010

[27] NEUFELD, A. and NUTZ, M. (2012). Superreplication under volatility uncertainty for measurable claims. Preprint. Available at arXiv:1208.6486.

[28] NUTZ, M. (2012). Pathwise construction of stochastic integrals. *Electron. Commun. Probab.* **17** 7. MR2950190

[29] NUTZ, M. and VON HANDEL, R. (2012). Constructing sublinear expectations on path space. Preprint available at arXiv:1205.2415v2.

[30] OBŁÓJ, J. (2005). The maximality principle revisited: On certain optimal stopping problems. Available at arXiv:0405252v2.

[31] OBLÓJ, J. and SPOIDA, P. (2012). An iterated Azéma–Yor type embedding for finitely many marginals. Preprint.

[32] PESKIR, G. (1998). Optimal stopping of the maximum process: The maximality principle. *Ann. Probab.* **26** 1614–1640. MR1675047

[33] POSSAMAÏ, D., ROYER, G. and TOUZI, N. (2013). On the robust superhedging of measurable claims. Preprint.

[34] SONER, H. M., TOUZI, N. and ZHANG, J. (2011). Quasi-sure stochastic analysis through aggregation. *Electron. J. Probab.* **16** 1844–1879. MR2842089

[35] SONER, H. M., TOUZI, N. and ZHANG, J. (2013). Dual formulation of second order target problems. *Ann. Appl. Probab.* **23** 308–347. MR3059237

[36] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR0532498

[37] SZPIRGLAS, J. and MAZZIOTTO, G. (1979). Théorème de séparation dans le problème d'ârret optimal. In *Séminaire de Probabilités*, *XIII*. *Lecture Notes in Math*. **721** 378–384. Springer, Berlin. MR0544808

[38] TAN, X. and TOUZI, N. (2013). Optimal transportation under controlled stochastic dynamics. *Ann*. *Probab*. **41** 3201–3240.

[39] VILLANI, C. (2003). *Topics in Optimal Transportation*. *Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483

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