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Second Order Backward SDEs, Fully non-linear PDEs, and applications in Finance

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Abstract. The martingale representation theorem in a Brownian filtration represents any square integrable r.v. ξ as a stochastic integral with respect to the Brownian motion. This is the simplest Backward SDE with nul generator and final data ξ, which can be seen as the non-Markov counterpart of the Cauchy problem in second order parabolic PDEs. Similarly, the notion of Second order BSDEs is the non-Markov counterpart of the fully-nonlinear Cauchy problem, and is motivated by applications in finance and probabilistic numerical methods for PDEs.

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1. Introduction

The theory of backward stochastic differential equations (BSDE hereafter) received a considerable attention in the recent literature. The ongoing developments are motivated by financial mathematics, stochastic control, stochastic differential games, and probabilistic numerical methods for partial differential equations (PDEs hereafter). We refer to [12] for a review.

These notes provide an overview on the recent extension to the second order which correspond to second order PDEs. Our objective is to define second order BSDEs in the general non Markov case, which can be viewed as the natural counterpart of PDEs in the non Markovian framework. We put a special emphasis on the examples, mainly from financial mathematics, which acted as a driving line for the progress which was achieved.

Section 2 provides a quick review of the basics of standard BSDEs and their connection to semilinear PDEs. We also provide a non-expert exposition of the main applications in financial mathematics.

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In Section 3, we report our main example of hedging under gamma constraints, which show the main difficulties that one has to solve. The main resukt of this section is the uniqueness result of [8] obtained within a restricted class of integrands.

Section 4 provides a new definition of solutions of 2BSDE motivated by the quasi-sure stochastic analysis developed by Denis and Martini [10] in the context of their analysis of the uncertain volatility model.

Section 5 collects the mains results of these notes, mainly the wellposedness of the quasi-sure formulation of the 2BSDE. We state a representation result which implies uniqueness. With the representation result, comparison becomes trivial. Then, we provide the appropriate a priori estimates. Finally, existence is obtained as follows. First for bounded uniformly continuous final data, the representation suggest a natural candidate for the solution of the 2BSDE, that can be defined by means of the notion of regular conditional probability density. Then, using the a priori estimates, we prove the existence of a solution in an appropriate closure of the space of bounded uniformly continuous random variables. Finally in the Markovian case, under natural condition, the solution of the 2BSDE is a viscosity solution of the corresponding fully nonlinear PDE.

Notations: Scalar products wil be denotes by dots, and transposition of matrices by an exponent $^{\mathrm{T}}$. For a $\sigma-$algebra $\mathcal{F}$, a filtration $\mathbb{F}$, and a probability measure $\mathbb{P}$, we will denote
• $\mathbb{L}^2(\mathcal{F},\mathbb{P})$, the set of $\mathcal{F}-$measurable r.v. with finite second moment under $\mathbb{P}$,
• $\mathbb{H}^2(\mathbb{F},\mathbb{P})$, the set of all $\mathbb{F}-$progressively measurable processes H with $\mathbb{E}\big[\int |H_t|^2 dt\big] < \infty$,
• $\mathbb{S}^2(\mathbb{F},\mathbb{P})$, the subset of $\mathbb{H}^2(\mathbb{F},\mathbb{P})$ with $\mathbb{P}-$a.s. càdlàg sample paths.

2. Review of Standard Backward SDEs

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space supporting a Brownian motion W on $\mathbb{R}^d$, and denote by $\mathbb{F}=\{\mathcal{F}_t, t\geq 0\}$ the corresponding $\mathbb{P}-$augmented canonical filtration.

Consider the two ingredients:
• the generator $F:\mathbb{R}_+\times\Omega\times\mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is such that $(t,\omega)\longmapsto F_t(\omega,y,z)$ is $\mathbb{F}-$progressively measurable for all $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,
• the final data $\xi\in\mathbb{L}^2(\mathbb{P},\mathcal{F}_T)$ for some time horizon $T>0$.

Given a time horizon $T>0$, a (scalar) backward stochastic differential equation (BSDE in short) is defined by:

$$Y_t \quad = \quad \xi + \int_t^T F_s(Y_s,Z_s)ds - \int_t^T Z_s\cdot dW_s, \quad t\leq T. \tag{1}$$

Equations of this type appeared naturally in the work of Bismut [5] on the stochastic maximum Pontryagin principle for stochastic control problems. A systematic study was started by Pardoux and Peng [18], where an existence and uniqueness theory of an $\mathbb{F}-$progressively measurable solution (Y,Z) was introduced. This seminal work generated an extensive literature in stochastic analysis, with natural motivations from financial mathematics.

In this section, we provide a quick review of this theory under the condition

$$F \quad \text{Lipschitz-continuous in} \quad (y,z) \text{ uniformly in } (t,\omega) \tag{2}$$

2.1. The linear case. Consider first the case $F \equiv 0$:

$$Y_t \;=\; \xi - \int_t^T Z_s \cdot dW_s, \quad t \le T. \tag{3}$$

Then, for any $\xi \in \mathbb{L}^2(\mathbb{P}, \mathcal{F}_T)$, there is a unique $\mathbb{F}-$progressively measurable square integrable process Y satisfying (1), given by $Y_t := \mathbb{E}[\xi|\mathcal{F}_t]$, $t \le T$. Moreover, by the martingale representation theorem in the present Brownian filtration, the process Y can be considered in its continuous version, and there exists a unique $\mathbb{F}-$progressively measurable square integrable process Z satisfying (1). By the Doob's maximal inequality, this construction provides a unique solution (Y,Z) of (1) in the space $\mathbb{S}^2(\mathbb{P},\mathbb{F}) \times \mathbb{H}^2(\mathbb{P},\mathbb{F})$, i.e.

$$\mathbb{E}\big[\sup_{t\le T}|Y_t|^2\big] + \mathbb{E}\big[\int_0^T |Z_t|^2 dt\big] \;<\; \infty. \tag{4}$$

We next consider the linear case

$$F_t(y,z) \;=\; -k_t y + \lambda_t \cdot z + \alpha_t, \tag{5}$$

for some $\mathbb{F}-$progressively measurable processes k, λ, α, that we assume to be bounded, for simplicity. Defining

$$\tilde{Y}_t := Y_t e^{-\int_0^t k_s ds}, \; t \in [0,T], \quad \text{and} \quad \tilde{\xi} := \xi e^{-\int_0^t k_s ds} + \int_0^T \alpha_s e^{-\int_0^s k_u du} ds, \tag{6}$$

we can convert the BSDE (1) into a BSDE with nul generator under the equivalent probability measure

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} \;:=\; e^{\int_0^T \lambda_t \cdot dW_t - \frac{1}{2}\int_0^T |\lambda_t|^2 dt}. \tag{7}$$

Example *Hedging contingent claims in frictionless financial markets.* Consider a financial market consisting of d risky assets with price processes:

$$dS_t \;=\; \mathrm{diag}[S_t]\left(b_t dt + \sigma_t dW_t\right), \tag{8}$$

where $\mathrm{diag}[S_t]$ denotes the diagonal matrix with diagonal entries S_t^i, and b, σ, σ^{-1} are $\mathbb{F}-$progressively measurable bounded processes.

- A portfolio strategy is an $\mathbb{F}-$progressively measurable process $\{\theta_t, t \in [0,T]\}$ with values in $\mathbb{R}^d$. Here each component θ_t^i indicates the amount invested in asset

S^i at time t. The self-financing condition defines the dynamics of the liquidation value of the portfolio:

$$dV_t = \sum_{i=1}^{d} \theta_t^i \frac{dS_t^i}{S_t^i} + \left(V_t - \sum_{i=1}^{d} \theta_t^i \right) r_t dt, \tag{9}$$

where the instantaneous interest rate r is $\mathbb{F}-$progressively measurable and bounded. The latter equation is the budget constraint which says that the change in the liquidation value of the portfolio has two components. First, for each asset i the change of value of the holding in asset S^i is given by the change of the corresponding price times the number of shares of this asset held in portfolio at time t. The difference $V_t - \sum_{i=1}^{d} \theta_t^i$ represents the holding in cash on the bank account. Then the second component of the above busget constraint simply says that this investment in the bank has an instantaneous riskless return defined by the instantaneous interest rate.

- A portfolio strategy θ is admissible if $\sigma^{\mathrm{T}}\theta \in \mathbb{H}^2(\mathbb{P}, \mathbb{F})$, so that the process V is well-defined in $\mathbb{H}^2(\mathbb{P}, \mathbb{F})$. We denote by V^θ the corresponding liquidation value process.

- A European contingent claim is a r.v. $\xi \in \mathbb{L}^2(\mathbb{P}, \mathcal{F}_T)$ which indicates the random payoff of a contract between two partes. The seller of such a contract bears the risk of the random payment, and wishes to hedge his position against the bad states of the world. A natural problem is then to

$$\text{Find an admissible portfolio } \theta \quad \text{so that} \quad V_T^\theta = \xi, \ \mathbb{P} - \text{a.s.} \tag{10}$$

This is a BSDE problem with final data ξ, and affine generator $F_t(y, z) = -r_t y - (\sigma^T)^{-1}(b_t - r_t \mathbf{1})$, where $\mathbf{1}$ is the vector of ones in $\mathbb{R}^d$.

2.2. Wellposedness of Backward SDEs.

Next, let F be a generator satisfying (2) and denote $F_t^0 := F_t(0, 0)$. Then assuming $\xi \in \mathbb{L}^2(\mathbb{P}, \mathcal{F}_T)$ and $F^0 \in \mathbb{H}^2(\mathbb{P}, \mathbb{F})$, it follows from a fixed point argument that the BSDE (1) has a unique solution in $\mathbb{S}^2(\mathbb{P}, \mathbb{F}) \times \mathbb{H}^2(\mathbb{P}, \mathbb{F})$.

When the generator is either convex or concave, the solution of the BSDE corresponds to a stochastic control problem in standard form but without diffusion control.

Various extensions of this result have been obtained in the previous literature by weakening the Lipschitz condition (2). The most challenging is probably the case where F has quadratic growth in z, see Kobylanski [16] and Tevzadze [24].

A comparison result is easily obtained, and reads as follows. Suppose that (F, ξ) and (F', ξ') satisfy the above conditions for the existence and uniqueness of solutions (Y, Z) and (Y', Z') of the corresponding BSDEs. Assume that $\xi \leq \xi'$ and $f_t(Y_t, Z_t) \leq f'_t(Y_t, Z_t)$. Then $Y \leq Y'$ on $[0, T]$, $\mathbb{P}-$a.s.

Such a comparison result plays a central role in the theory. For instance, it allows to define the notion of reflected BSDEs (a misleading denomination, to which I prefer the name of *obstacle BSDE*) which are connected to optiomal stopping problems and Dynking games.

Example: *Hedging and different borrowing and lending rates.* Let us turn to the example of the previous subsection. The holding in cash $V_t - \theta_t \cdot \mathbf{1}$ can be either positive, meaning a positive amount on the bank account, or negative, meaning a loan from the bank. In the real life, borrowing and lending rtes are differents are given resectively by $\overline{r}_t \geq \underline{r}_t$. Then, the dynamics of the liquidation value of the portfolio (9) is replaced by:

$$dV_t \;=\; \sum_{i=1}^{d} \theta_t^i \frac{dS_t^i}{S_t^i} + \big((V_t - \theta_t \cdot \mathbf{1})^+ \underline{r}_t - (V_t - \theta_t \cdot \mathbf{1})^- \overline{r}_t\big) dt, \tag{11}$$

which is our simplest example of nonlinear BSDE.

2.3. Markov BSDEs.

The Markov case correspond to the particular specification

$$F_t(\omega, y, z) = f\big(t, X_t(\omega), y, z\big) \quad \text{and} \quad \xi = g\big(X_T(\omega)\big) \tag{12}$$

where X is the solution of some (well-posed) stochastic differential equation

$$X_t \;=\; X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad t \leq T. \tag{13}$$

Moving the time origin to an arbitrary $t \in [t, T]$, we denote by $\{X_s^{t,x}, s \in [t, T]\}$ the solution of the above SDE with initial data $X_t^{t,x} = x$, and by $\{(Y_s^{t,x}, Z_s^{t,x}), s \in [t, T]\}$ the solution of the corresponding BSDE. Then, since the Brownian motion has independent increments and is translation invariant, we easily see that

$$u(t, x) \;:=\; Y_t^{t,x}, \; t \in [0, T], \; x \in \mathbb{R}^d, \tag{14}$$

defines a deterministic function satisfying the semigroup property (or the dynamic programming principle, in the language of stochastic control):

$$u(s, X_s^{t,x}) = Y_s^{t,x} = u(t, x) + \int_t^s f\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big) dr - \int_t^s Z_r^{t,x} dW_r \tag{15}$$

Then, if u is $C^{1,2}$, it follows that $Z_t^{t,x} = \sigma^{\mathrm{T}}(t, x) Du(t, x)$, and u is a classical solution of the semilinear Cauchy problem:

$$-\partial_t u - \frac{1}{2} \mathrm{Tr}\left[\sigma \sigma^{\mathrm{T}} D^2 u\right] - f(t, x, u, Du) = 0, u(T, \cdot) = g. \tag{16}$$

Of course, this equation can be derived in the sense of viscosity solutions when no regularity of u is available.

2.4. Numerical implications.

From the latter connection with the Cauchy problem, one can formuate an extension of the so-called Feynman-Kac representation formula to the semilinear case, which states that whenever the Cauchy problem (16) has a classical solution u, then it has a representation (14) in terms of a corresponding BSDE. Among the various applications of this representation, I would like to highlight its numerical implications.

1. The case of a nul generator $f \equiv 0$ is well known to open the door to probabillistic numerical methods for the approximation of the solution of (16). Indeed, in this case, the BSDE representation reduces to $u(t,x) = Y_t^{t,x} = \mathbb{E}[g(X_T^{t,x})]$ which suggests an approximation based on the law of large numbers. For instance, one can generate independent copies of the r.v. $g(X_T^{t,x})$ (or an appropriate approximation), and define the crude Monte Carlo approximation by simple averaging. A remarkable feature of this approximation is that the rate of convergence, as provided by the central limit theorem, is independent of the dimension d of the state x. This represents a clear advantage of probabilistic schemes.

2. For a general nonlinearity f, let $\pi : t = t_0 < \ldots < t_n = T$ be a partition of the interval $[t,T]$ with time steps $\delta t_k := t_k - t_{k-1}$, and corresponding increments of the Brownian motion $\delta W_{t_k} := W_{t_k} - W_{t_{k-1}}$. Denote by X^π the euler discretization of X along the partition π. The following discretization of (1) was suggested by Bally and Pagès [1] when f does not depend on z, and independently by Bouchard Touzi [6] and Zhang [25] for a general nonlinearity:

$$Y_{t_n}^\pi = g\big(X_{t_n}^\pi\big), \tag{17}$$

and

$$\begin{aligned} Y_{t_{k-1}}^\pi &= \mathbb{E}\big[Y_{t_k}^\pi | X_{t_{k-1}}^\pi\big] + \delta t_k f\big(t_{k-1}, X_{t_{k-1}}^\pi, Y_{t_{k-1}}^\pi, Z_{t_{k-1}}^\pi\big), && (18)\\ Z_{t_{k-1}}^\pi &= \mathbb{E}\big[Y_{t_k}^\pi (\delta t_k \sigma(t_k, X_{t_k}^\pi))^{-1} \delta W_{t_k} | X_{t_{k-1}}^\pi\big]. && (19)\end{aligned}$$

For a feasible scheme, one further needs to introduce an implementable approximation of the regression operator $\mathbb{E}\big[.|X_{t_{k-1}}^\pi = x\big]$. Convergence results of the discrete-time process (Y^π, Z^π) towards the solution (Y,Z) of the Markov BSDE, together with bounds on the rate of convergence are available in the literature, see [6, 14, 9]. Notice however that the asymptotic results in the present nonlinear case depend on the dimension of the state d.

3. Second order BSDEs: difficulties and intuitions

Backward stochastic differential equation are naturally connected to semilinear PDEs of the form (16), i.e. linear dependence of the equation in terms of the hessian matrix. The first objective of the notion of second order BSDEs is to enlarge the notion of BSDEs so as to obtain a connection with fully nonlinear PDEs. This allows to capture more interesting examples. In this section, we provide a simple example which is beyond the scope of standard BSDEs. moreover, this example reveals the difficulty we are facing for our extension.

3.1. Hedging under Gamma constraints. Let us specialize the example of Subsection 2.1 to the one-dimensional case $d = 1$. Denote $\pi_t := \frac{\theta_t}{S_t}$ the number of shares of S held in portfolio at time t, and $V^{\pi} := V^{\theta}$. The practice of the optimal hedging strategy induced by this model leads to a portfolio adjustment at each time t from π_t ti π_{t+dt}, i.e. the investor has to buy or sell (depending on the sign) $\pi_{t+dt} - \pi_t$ shares of the asset S. Although our model assumes that the price process is exogeneous, practioners are fully aware of the nonlinear dependence of the price in terms of the transaction volume, and the impact of their strategies on the price process. This is the so-called illiquidity effect.

To avoid (or at least minimize) such illiquidity costs, we assume that π_t is a continuous semimartingale with

$$d\langle \pi, S\rangle_t \quad = \quad \Gamma_t \langle S\rangle_t, \ \mathbb{P}-\text{a.s.} \tag{20}$$

and we impose some constraints on the process Γ. In fact, the interpretation of Γ, as viewed by practitioners, is the portfolio adjustment consequent to an immediate jump of the underlying price process. Although jumps are not allowed by the model, this is a conservative behavior aiming at building strategies which are robust to such a specification error of the model.

Given a contingent claim $\xi \in \mathbb{L}^2(\mathbb{P}, \mathcal{F}_T)$, our new hedging problem is now:

$$\text{Find an admissible portfolio } \pi \quad \text{so that } \Gamma \in [\underline{\Gamma}, \overline{\Gamma}] \text{ and } \quad V_T^{\pi} = \xi, \ \mathbb{P}-\text{a.s.} \tag{21}$$

where $\underline{\Gamma} < 0 < \overline{\Gamma}$ are given.

We also observe that in the Markov framework, "we expect" that Γ_t should identify the Hessian matrix of the function u defined in (14). Then, this problem is expected to be connected to a fully nonlinear PDE.

However, there is a fundamental difficulty related to the following result due to Bank and Baum [2].

Lemma 3.1. *Let ϕ be a progressively measurable process with $\int_0^T |\phi_t|^2 dt < \infty$, $\mathbb{P}-a.s.$ Then, for every $\varepsilon > 0$, there exists a progressively measurable process ϕ^{ε}, absolutely continuous with respect to the Lebesgue measure, with $\int_0^T |\phi_t^{\varepsilon}|^2 dt < \infty$, and*

$$\sup_{0\le t\le T} \Big\| \int_0^t \phi_t \cdot dW_t - \int_0^t \phi_t^{\varepsilon} \cdot dW_t \Big\|_{\infty} \quad \le \quad \varepsilon. \tag{22}$$

This result shows a high instability of the problem: by accepting to miss the target ξ within a small range of ε, we may approximate the optimal hedging strategy of the frictionless financial market (Subsection 2.1) so that Gamma process of the approximation is zero !

3.2. Non-uniqueness in $\mathbb{L}^2$.

The latter difficulty which appears naturally in the context of the financial application is not exceptional. Let us consider the simplest backward SDE problem involving the Gamma process, similar to the above example:

$$Y_t = c\int_t^T \Gamma_s ds - \int_t^T Z_s dW_s \ \text{ where } \ d\langle Z, W\rangle_t = \Gamma_t dt, \ t\in[0,T], \ \mathbb{P}-\text{a.s.} \tag{23}$$

Obviously, $Y = Z = \Gamma = 0$ is a solution. However, if we admit any square integrable semimartingale Z with square integrable corresponding Γ process, it is shown in Example 6.1 of [23] that, except for the case $c = 0$, the above problem has a non-zero solution.

Consequently, introducing a second order term in the BSDE can not be performed within the classical framework, and one has to face the difficulties due to the instability highlighted in Lemma 3.1. This is the main object of these notes which was dealt with by to approaches

- the first approach, developed in the subsequent subsection 3.3, is to restrict the process Z to an appropriate space, so as to obtain uniqueness. This approach was successful for uniqueness in the Markov framework, but we were not able to have a satisfactory existence theory.
- the second approach is motivated by the example of Subsection 3.4 below, and consists in reinforcing the constraint by requiring the BSDE to be satisfied on a bigger support... This is the content of Section 4 below which contains our main wellposedness results of second order BSDEs.

3.3. A first uniqueness result. In order to involve the process Γ in the problem formulation, we need that the process Z be a semimartingale. Then, we have the following correspondence between the Itô and the Fisk-Stratonovich integrals

$$a\int_0^t Z_t \cdot dW_t \quad = \quad \frac{1}{2}\Gamma_t dt + \int_0^t Z_t \circ dW_t. \tag{24}$$

a We prefer to write the problem using the Fisk-Stratonovich stochastic integral rather than the Itô one. In the present subsection, this is just cosmetic, but it will play a crucial role in Section 4.

Consider the Markov 2BSDE:

$$Y_t \quad = \quad g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s, \Gamma_s)ds - \int_t^T Z_s \circ dX_s, \quad \mathbb{P}-\text{a.s.} \tag{25}$$

where X is defined by the stochastic differential equation

$$dX_t \quad = \quad b(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{26}$$

that we assume wellposed with support in the whole space $\mathbb{R}^d$.

An appropriate class $\mathcal{Z}$ of processes Z is introduced in [8]. Since we will be mainly concentrating on the alternative approach, we do not report the precise description of this class in these notes. To prove the uniqueness result, we introduce the stochastic target problems

$$\mathcal{V}(0, X_0) \quad := \quad \inf\left\{Y_0 : \; Y_T \geq g(X_T), \; \mathbb{P}-\text{a.s. for some } Z \in \mathcal{Z}\right\}, \tag{27}$$

$$\mathcal{U}(0, X_0) \quad := \quad \sup\left\{Y_0 : \; Y_T \leq g(X_T), \; \mathbb{P}-\text{a.s. for some } Z \in \mathcal{Z}\right\}. \tag{28}$$

By moving the time origin to an arbitrary $t \in [0,T]$, we also define the value functions $\mathcal{V}(t,s)$ and $\mathcal{U}(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$. The following result is obtained in [8] by proving that $\mathcal{V}$ and $\mathcal{U}$ are respectively viscosity supersolution and subsolution of the (fully nonlinear) dynamic programming equation:

$$-\partial_t v - h(t,x,v,Dv,D^2v) = 0 \text{ on } [0,T) \times \mathbb{R}^d, \quad \text{and} \quad v(T,.) = g. \tag{29}$$

Theorem 3.2. *Let h be continuous, locally Lipschitz in y, uniformly in all other variables, non-increasing in γ, and has polynomial growth in (x,y,z,γ). Let g be continuous with polynomial growth. Assume further that the nonlinear PDE (29) satisfies a comparison result in the sense of viscosity solutions, within the class of polynomially growing functions. Then there is at most one solution to the backward SDE (25) with $Z \in \mathcal{Z}$.*

3.4. Intuition from uncertain volatility models. The objective of this example is to introduce uncertainty about the volatility process σ in our first example of Subsection 2.1. To do this, we reformulate the problem in the setting of the canonical space $\Omega = \{\omega \in C([0,T]) : \omega(0) = 0\}$ as suggested by Denis and Martini [10]. We denote by B be the coordinate process, $\mathbb{F}$ the corresponding canonical filtration, and $\mathbb{P}_0$ the Wiener measure, so that B is a Brownian motion under $\mathbb{P}_0$.

By obvious discounting, we may reduce to the zero interest rate case. Moreover, after an equivalent change of measure, we may also assume without loss of generality that $b = 0$. The liquidation value process (9) is then given by:

$$V_t \quad := \quad V_0 + \int_0^t \theta_s \cdot dB_s, \tag{30}$$

where the volatility coefficient can be viewed to be absorbed into the canonical process by a time change argument. To model the uncertainty on the volatility, we consider two given constants $0 < \underline{a} \leq \overline{a}$, and we introduce the set $\mathcal{P} = \mathcal{P}_{\underline{a},\overline{a}}$ of all probability measures on Ω such that B is a martingale under $\mathbb{P}$ with quadratic variation absolutely continuous with respect to Lebesgue, and

$$\underline{a} \leq \frac{d\langle B\rangle_t}{dt} \leq \overline{a}, \quad t \in [0,T]. \tag{31}$$

Notice that the family $\mathcal{P}$ has no dominating measure, and all measures contained therein are mutually singular. Since the stochastic integral is defined $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$, it is not clear how to define the liquidation value V in (30) simultaneously under every $\mathbb{P} \in \mathcal{P}$. This achived in [10] by revisiting the stochastic integration theory, replacing the reference probability measure by the capacity

$$ac(A) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[A] \quad \text{for all} \quad A \in \mathcal{F}_T. \tag{32}$$

a An event A is said to be *polar* if $c(A) = 0$, and a property is said to hold *quasi-surely* (q.s. hereafter) if it holds on the complement of a polar set. The first main contribution of [10] is to isolate a set of integrands $\mathcal{H}$, such that the stochastic integral (30) with $\theta \in \mathcal{H}$ is defined quasi-surely, i.e. $\mathbb{P}-$almost surely for all $\mathbb{P} \in \mathcal{P}$.

The superhedging problem can now be formulate rigorously:

$$\mathcal{V}(\xi) \quad := \quad \inf\left\{V_0 : \ V_T \geq \xi, \text{ q.s. for some } \theta \in \mathcal{H}\right\}. \tag{33}$$

This is weaker than the BSDE problem as existence is not required in the formulation (33). The main result of [10] is the following dual formulation of this problem:

$$\mathcal{V}(\xi) \quad = \quad \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi], \tag{34}$$

for random variables ξ in a suitable class.

The interesting feature of this result is that, in the Markov framework $\xi = g(B_T)$, the dynamic programming equation corresponding to the dual problem (34) is fully nonlinear:

$$-\partial_t v - G(D^2 v) = 0, \quad \text{where} \quad G(\gamma) := \sup_{\underline{a}\leq a\leq \overline{a}} \frac{1}{2} a D^2 v = \frac{1}{2}\big(\overline{a}(D^2v)^+ - \underline{a}(D^2v)^-\big). \tag{35}$$

In other words, this observation suggests that the fully nonlinear PDE corresponds to a BSDE defined quasi-surely, similar to the super-hedging problem (33). This is the starting point of our alternative formulation of second order BSDE in the subsequent Section 4, which will turn out to allow for a complete existence and uniqueness theory.

Finally, we observe that the above quasi-sure stochastic analysis is closely related to the $G-$stochastic integral which was recently introduced by Peng [19, 11].

4. A quasi-sure formulation of second order BSDEs

This section introduces the new framework motivated from [10] and [19].

4.1. A nondominated family of singular measures. As in Subsection 3.4, we work on the canonical space Ω. For the purpose of our second order BSDEs, we need to extend the set of non-dominated mutually singular measure $\mathcal{P}$ to the collection of all $\mathbb{P}$ which turn the canonical process B into a local martingale. It follows from Karandikar [15] that there exists an $\mathbb{F}-$progressively measurable process, denoted as $\int_0^t B_s dB_s^{\mathrm{T}}$, which coincides with the Itô's integral, $\mathbb{P}-$a.s. for all local martingale measure $\mathbb{P}$. In particular, this provides a pathwise definition of

$$a\langle B\rangle_t := B_t B_t^{\mathrm{T}} - 2\int_0^t B_s dB_s^{\mathrm{T}} \quad \text{and} \quad \hat{a}_t := \overline{\lim_{\varepsilon\downarrow 0}} \frac{1}{\varepsilon}\Big(\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}\Big), \tag{36}$$

a where the $\overline{\lim}$ is componentwise. Clearly, $\langle B\rangle$ coincides with the $\mathbb{P}-$quadration variation of B, $\mathbb{P}-$a.s. for all local martingale measure $\mathbb{P}$.

For all $\mathbb{F}-$progressively measurable process α taking values in the set $\mathbb{S}_d^{>0}$ of positive definite symmetric matrices and satisfying $\int_0^T |\alpha_t|dt < \infty$, $\mathbb{P}_0-$a.s. we introduce the measure

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, t \in [0,T], \mathbb{P}_0 - \text{a.s.} \tag{37}$$

We denote by $\overline{\mathcal{P}}_S$ the collection of all such measures. It can be shown that

$$\begin{array}{c}\text{every } \mathbb{P} \in \overline{\mathcal{P}}_S \text{ satisfies the Blumenthal zero-one law} \\ \text{and the martingale representation property.}\end{array} \tag{38}$$

4.2. The nonlinear generator. Consider the map $H_t(\omega, y, z, \gamma) : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0. We start with the following natural condition.

Assumption 4.1. For fixed (y, z, γ), H is $\mathbb{F}-$progressively measurable; H is uniformly Lipschitz continuous in (y, z), uniformly continuous in ω under the $\|.\|_\infty-$norm, and lower semi-continuous in γ.

An important role is played by the conjugate of H with respect to γ:

$$F_t(y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2}\text{Tr}[a\gamma] - H_t(y, z, \gamma) \right\}, \quad a \in \mathbb{S}_d^{>0}. \tag{39}$$

and we denote

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t), \quad \hat{F}_t^0 := \hat{F}_t(0, 0). \tag{40}$$

Then F is a $\mathbb{R} \cup \{\infty\}-$valued measurable map. By the above conditions on H, the domain D_{F_t} of F as a function of a is independent of (ω, y, z), and

$$\begin{array}{c}F(\cdot, a) \text{ is uniformly Lischitz continuous in } (y,z) \text{ and uniformly continuous in } \omega, \\ \text{uniformly on } (t,a), \text{ for every } a \in D_{F_t}.\end{array} \tag{41}$$

For every constant $\kappa \in (1, 2]$, we denote by $\mathcal{P}_H^\kappa$ the collection of all those $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that

$$\underline{a}_\mathbb{P} \le \hat{a} \le \overline{a}_\mathbb{P}, \quad dt \times d\mathbb{P} - \text{a.s. for some } \underline{a}_\mathbb{P}, \overline{a}_\mathbb{P} \in \mathbb{S}_d^{>0}, \text{ and } \mathbb{E}^\mathbb{P}\Big[\Big(\int_0^T |\hat{F}_t^0|^\kappa dt\Big)^{2/\kappa}\Big] < \infty. \tag{42}$$

In particular, $\hat{a}_t \in D_{F_t}$, $dt \times d\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H^\kappa$.

By slightly abusing the terminology of Denis and Martini [10], we say a property holds $\mathcal{P}_H^\kappa-$quasi-surely ($\mathcal{P}_H^\kappa-$q.s. for short) if it holds $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H^\kappa$.

Our main results reuire the following conditions on $\hat{F}$.

Assumption 4.2. (i) $\mathcal{P}_H^\kappa$ is not empty.
(ii) The process $\hat{F}^0$ satisfies:

$$\|\hat{F}^0\|^2_{\mathbb{H}_H^{2,\kappa}} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\Big[\operatorname*{ess\,sup}_{0\le t\le 1}^{\mathbb{P}} \Big(\mathbb{E}_t^{H,\mathbb{P}}\Big[\int_0^1 |\hat{F}_s^0|^\kappa ds\Big]\Big)^{2/\kappa}\Big] < \infty. \tag{43}$$

(iii) There exists a constant C such that for all $(y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $\mathbb{P} \in \mathcal{P}_H^\kappa$:

$$\big|\hat{F}_t(y, z_1) - \hat{F}_t(y, z_2)\big| \le C|\hat{a}_t^{1/2}(z_1 - z_2)|, \; dt \times d\mathbb{P} - \text{a.s.} \tag{44}$$

Here we abuse the notation $\mathbb{H}_H^{p,\kappa}$ slightly by noting that, unlike the elements in $\mathbb{H}_H^p$, $\hat{F}^0$ is $1-$dimensional and the norm in (43) does not contain the factor $\hat{a}^{1/2}$.

4.3. The spaces and norms. This subsection collects all norms needed for our results.

• $L_H^{p,\kappa}$: space of all $\mathcal{F}_T-$measurable $\mathbb{R}-$valued random variables ξ with

$$a\|\xi\|^p_{L_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[|\xi|^p] < \infty. \tag{45}$$

a • $\mathbb{H}_H^{2,\kappa}$: space of all $\mathbb{F}^+-$progressively measurable $\mathbb{R}^d-$valued processes Z with

$$a\|Z\|^2_{\mathbb{H}_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\hat{a}_t^{1/2} Z_t|^2 dt\Big] < \infty. \tag{46}$$

a • $\mathbb{D}_H^{2,\kappa}$ the space of all $\mathbb{F}^+-$progressively measurable $\mathbb{R}-$valued processes Y with $\mathcal{P}_H^\kappa-$q.s. càdlàg paths and

$$a\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le T} |Y_t|^2\Big]. \tag{47}$$

a • For $\xi \in L_H^{1,\kappa}$, $\mathbb{P} \in \mathcal{P}_H^\kappa$, and $t \in [0, T]$:

$$a\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})}{}^{\mathbb{P}} \; \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t] \quad \text{where} \quad \mathcal{P}_H^\kappa(t, \mathbb{P}) := \{\mathbb{P}' \in \mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\} \tag{48}$$

a • $\mathbb{L}_H^{2,\kappa}$: subspace of all $\xi \in L_H^2$ such that

$$a\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{0\le t\le 1}{}^{\mathbb{P}} \Big(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\Big)^{2/\kappa}\Big] < \infty. \tag{49}$$

a • $\mathrm{UC}_b(\Omega)$: space of all bounded and uniformly continuous maps $\xi : \Omega \longrightarrow \mathbb{R}$ with respect to the $\|.\|_\infty-$norm.

• $\mathcal{L}_H^{2,\kappa}$: closure of $\mathrm{UC}_b(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}_H^{2,\kappa}}$.

We observe that when $\mathcal{P}_H^\kappa$ is reduced to a singleton:

$$a\mathcal{P}_H^\kappa = \{\mathbb{P}\} \implies \mathcal{L}_H^{2,\kappa} = \mathbb{L}_H^{2,\kappa} = L_H^{2,\kappa} = L^2(\mathbb{P}) \;\; \text{for } 1 \le \kappa < p. \tag{50}$$

a

4.4. Definition. We shall obtain a complete existence and uniqueness theory for the second order BSDE (25) by considering instead the quasi-sure formulation:

$$Y_t = \xi - \int_t^T \hat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s \cdot dB_s + K_1 - K_t, \quad 0 \le t \le T, \quad \mathcal{P}_H^\kappa - \text{q.s.} \tag{51}$$

A solution to the 2BSDE (51) is a pair $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ such that:
- $Y_T = \xi$, $\mathcal{P}_H^\kappa$−q.s.
- For all $\mathbb{P} \in \mathcal{P}_H^\kappa$, the process

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s, \quad 0 \le t \le T, \quad \mathbb{P} - \text{a.s.} \tag{52}$$

has non-decreasing paths, $\mathbb{P}$−a.s.
- The family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ satisfies the minimumality condition:

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})}{\text{ess inf}}{}^{\mathbb{P}} \; \mathbb{E}_t^{\mathbb{P}'}[K_T^{\mathbb{P}'}], \quad \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H^\kappa \text{ and } t \in [0,1]. \tag{53}$$

The above definition is motivated in [22, 23] by the corresponding stochastic target problem. Let us just verify it reduces to the standard notion of BSDE when the generator H is linear in γ:

$$H_t(y, z, \gamma) = \frac{1}{2}\text{Tr}[a_t^0 \gamma] - f_t(y, z), \tag{54}$$

where $a^0 : [0, T] \times \Omega \to \mathbb{S}_d^{>0}$ is $\mathbb{F}$−progressively measurable and has uniform lower and upper bounds. We remark that in this case we do not need to assume that a^0 and f are uniformly continuous in ω. Then, under obvious extension of notations, we have

$$aD_{F_t(\omega)} = \{a_t^0(\omega)\} \quad \text{and} \quad \hat{F}_t(y, z) = f_t(y, z). \tag{55}$$

a Assume further that there exists $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that

$$a\hat{a} = a^0, \quad \mathbb{P} - \text{a.s. and} \quad \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \big(|f_t(0,0)|^2 dt\big)\Big] < \infty, \tag{56}$$

a then $\mathcal{P}_H^\kappa = \mathcal{P}_H^2 = \{\mathbb{P}\}$. In this case, the minimum condition (53) implies

$$a0 = K_0 = \mathbb{E}^{\mathbb{P}}[K_T] \quad \text{and thus} \quad K = 0, \quad \mathbb{P} - \text{a.s.} \tag{57}$$

a Hence, the 2BSDE (51) is equivalent to the following standard BSDE:

$$aY_t = \xi - \int_t^T f_s(Y_s, Z_s)ds - \int_t^1 Z_s dB_s, \quad 0 \le t \le T, \quad \mathbb{P} - \text{a.s.} \tag{58}$$

a Finally, we recall from the previous subsection that in the present case, we have $\mathcal{L}_H^{2,\kappa} = \mathbb{L}_H^{2,\kappa} = L_H^2 = \mathbb{L}^2(\mathbb{P})$ for all $\kappa \in [1, 2)$.

5. Wellposedness of second order BSDEs

This section contains the main results of the papers [20, 21, 22, 23].

For any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}-$stopping time τ, and $\mathcal{F}_\tau-$measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$, we denote by $(\mathcal{Y}^\mathbb{P}, \mathcal{Z}^\mathbb{P}) := (\mathcal{Y}^\mathbb{P}(\tau,\xi), \mathcal{Z}^\mathbb{P}(\tau,\xi))$ the solution to the following standard BSDE:

$$\mathcal{Y}_t^\mathbb{P} = \xi - \int_t^\tau \hat{F}_s(\mathcal{Y}_s^\mathbb{P}, \mathcal{Z}_s^\mathbb{P})ds - \int_t^\tau \mathcal{Z}_s^\mathbb{P} dB_s, \quad 0 \le t \le \tau, \ \mathbb{P}-\text{a.s.} \tag{59}$$

Our first result provides a representation of any solution of the 2BSDE (51).

Theorem 5.1. *Let Assumptions 4.1 and 4.2 hold. Assume that $\xi \in \mathbb{L}_H^{2,\kappa}$ and that $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to 2BSDE (51). Then, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $0 \le t \le T$,*

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})}{\operatorname{ess\,sup}}{}^{\mathbb{P}} \ \mathcal{Y}_t^{\mathbb{P}'}(T,\xi), \quad \mathbb{P}-a.s. \tag{60}$$

Consequently, the 2BSDE (51) has at most one solution in $\mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$.

The above representation, together with the comparison principle for standard BSDEs, implies the following comparison principle for 2BSDEs.

Corollary. *Let Assumptions 4.1 and 4.2 hold. Assume $\xi^i \in \mathbb{L}_H^{2,\kappa}$ and $(Y^i, Z^i) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a corresponding solution of the 2BSDE (51), $i = 1,2$. If $\xi^1 \le \xi^2$, $\mathcal{P}_H^\kappa-q.s.$ then $Y^1 \le Y^2$, $\mathcal{P}_H^\kappa-q.s.$*

We next state the a priori estimates which will be used in the subsequent existence result.

Theorem 5.2. *Let Assumptions 4.1 and 4.2 hold.*
(i) *Assume that $\xi \in \mathbb{L}_H^{2,\kappa}$ and that $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to 2BSDE (51). Then there exist a constant C_κ such that*

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}[|K_1^\mathbb{P}|^2] \quad \le \quad C_\kappa\big(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \|\hat{F}^0\|^2_{\mathbb{H}^{2,\kappa}_{\mathcal{P}_H^\kappa}}\big). \tag{61}$$

(ii) *Assume that $\xi^i \in \mathbb{L}_H^{2,\kappa}$ and that $(Y^i, Z^i) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a corresponding solution to 2BSDE (51), $i = 1,2$. Denote $\delta\xi := \xi^1 - \xi^2$, $\delta Y := Y^1 - Y^2$, $\delta Z := Z^1 - Z^2$, and $\delta K^\mathbb{P} := K^{1,\mathbb{P}} - K^{2,\mathbb{P}}$. Then there exists a constant C_κ such that*

$$\begin{gathered}\|\delta Y\|_{\mathbb{D}_H^{2,\kappa}} \le C_\kappa \|\delta\xi\|_{\mathbb{L}_H^{2,\kappa}}, \\ \|\delta Z\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\Big[\sup_{0\le t\le 1} |\delta K_t^\mathbb{P}|^2\Big] \le C_\kappa \|\delta\xi\|_{\mathbb{L}_H^{2,\kappa}} \Big(\|\xi^1\|_{\mathbb{L}_H^{2,\kappa}} + \|\xi^2\|_{\mathbb{L}_H^{2,\kappa}} + \|\hat{F}^0\|_{\mathbb{H}^{2,\kappa}_{\mathcal{P}_H^\kappa}}\Big).\end{gathered} \tag{62}$$

The main result of this paper is:

Theorem 5.3. *Let Assumptions 4.1 and 4.2 hold. Then for any $\xi \in \mathcal{L}_H^{2,\kappa}$, the 2BSDE (51) has a unique solution $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$.*

Our final result concern the connection between the 2BSDE (51) and the corresponding fully nonlinear PDE in the Markov case:

$$aH_t(\omega, y, z, \gamma) = h(t, B_t(\omega), y, z, \gamma) \quad \text{and} \quad \xi = g(\omega). \tag{63}$$

a Observe that h may not be nondecreasing in γ, but the following $\hat{h}$ is:

$$\hat{h}(t, x, y, z, \gamma) = \sup_{a \in \mathbb{S}_d^{>0}} \left\{ \frac{1}{2}\mathrm{Tr}[a\gamma] - f(t, x, y, z, a) \right\}, \quad \gamma \in \mathbb{R}^{d\times d}. \tag{64}$$

To obtain the connection with the corresponding fully nonlinear PDE, we need more assumptions which are detailed in [23]. Let us just mention that under those assumptions, we have

$$Y_t = u(t, B_t), \ t \in [0, T], \tag{65}$$

where
(i) u is a viscosity subsolution of

$$-\partial_t u^* - \hat{h}^*(\cdot, u^*, Du^*, D^2 u^*) \leq 0 \quad \text{on} \quad [0, 1) \times \mathbb{R}^d. \tag{66}$$

(ii) u is a viscosity supersolution of

$$-\partial_t u_* - \hat{h}_*(\cdot, u_*, Du_*, D^2 u_*) \geq 0 \quad \text{on} \quad [0, 1) \times \mathbb{R}^d. \tag{67}$$

Here, we used the classical notation in the theory of viscosity solutions:

$$au_*(\theta) := \underline{\lim}_{\theta' \to \theta} u(\theta) \quad \text{and} \quad u^*(\theta) := \overline{\lim}_{\theta' \to \theta} u(\theta'), \quad \text{for} \quad \theta = (t, x), \tag{68}$$

$$\hat{h}_*(\theta) := \underline{\lim}_{\theta' \to \theta} \hat{h}(\theta') \quad \text{and} \quad \hat{h}^*(\theta) := \overline{\lim}_{\theta' \to \theta} \hat{h}(\theta'), \quad \text{for} \quad \theta = (t, x, y, z, \gamma). \tag{69}$$

a
Example *Hedging under Gamma constraints.* Consider the quasi-sure reformulation of the problem of Subsection 3.1. The generator is given by

$$ah(t, x, y, z, \gamma) := \frac{1}{2}\gamma \text{ if } \gamma \in [\underline{\Gamma}, \overline{\Gamma}], \quad \text{and} \quad \infty \text{ otherwise}, \tag{70}$$

a where $\underline{\Gamma} < 0 < \overline{\Gamma}$ are given constants. By direct calculation, we see that

$$af(a) = \frac{1}{2}\big(\overline{\Gamma}(a-1)^+ - \underline{\Gamma}(a-1)^-\big), \qquad a \geq 0, \tag{71}$$

a and

$$a\hat{h}(\gamma) = \frac{1}{2}(\gamma \vee \underline{\Gamma}) \text{ if } \gamma \leq \overline{\Gamma}, \quad \text{and} \quad \infty \text{ otherwise}. \tag{72}$$

a Then,

$$a\hat{h}_* = \hat{h} \quad \text{and} \quad \hat{h}^*(\gamma) = \frac{1}{2}(\gamma \vee \underline{\Gamma})\mathbf{1}_{\{\gamma < \overline{\Gamma}\}} + \infty \mathbf{1}_{\{\gamma \geq \overline{\Gamma}\}}. \tag{73}$$

a In view of this, the above viscosity properties (66)-(67) are equivalent to

$$a \min\big\{ -\partial_t u^* - \frac{1}{2}(D^2u^* \vee \underline{\Gamma}),\ \bar{\Gamma} - D^2u^* \big\} \le 0, \tag{74}$$

$$\min\big\{ -\partial_t u_* - \frac{1}{2}(D^2u_* \vee \underline{\Gamma}),\ \bar{\Gamma} - D^2u_* \big\} \ge 0. \tag{75}$$

a

6. A probabilistic scheme for fully nonlinear PDEs

Consider the fully nonlinear Cauchy problem:

$$-\mathcal{L}^X v - h_0\big(\cdot, v, Dv, D^2v\big) = 0, \quad \text{on } [0,T) \times \mathbb{R}^d, \tag{76}$$

$$v(T,\cdot) = g, \quad \text{on } \in \mathbb{R}^d. \tag{77}$$

where

$$a\mathcal{L}^X\varphi \quad := \quad \partial_t\varphi + \mu \cdot D\varphi + \frac{1}{2}\mathrm{Tr}[aD^2\varphi] \tag{78}$$

a is the Dynkin operator of some Markov diffusion process. Similar to Subsection 2.4, a probabilistic numerical scheme for fully nonlinear PDEs was suggested in [8] and analyzed later in [13].

To simplify the notation, we consider the case $\sigma = I_d$. let $\pi : t = t_0 < \ldots < t_n = T$ be a partition of the interval $[t,T]$ with time steps $\delta t_k := t_k - t_{k-1}$, and corresponding increments of the Brownian motion $\delta W_{t_k} := W_{t_k} - W_{t_{k-1}}$. Denote by X^π the euler discretization of X along the partition π. Then the probabilistic numerical scheme for the fully nonlinear PDE is defined by:

$$Y^\pi_{t_n} = g\big(X^\pi_{t_n}\big), \tag{79}$$

and

$$Y^\pi_{t_{k-1}} \quad = \quad \mathbb{E}\big[Y^\pi_{t_k} | X^\pi_{t_{k-1}}\big] + \delta t_k h_0\big(t_{k-1}, X^\pi_{t_{k-1}}, Y^\pi_{t_{k-1}}, Z^\pi_{t_{k-1}}, \Gamma^\pi_{t_{k-1}}\big), \tag{80}$$

$$Z^\pi_{t_{k-1}} \quad = \quad \mathbb{E}\Big[Y^\pi_{t_k}\frac{\delta W_{t_k}}{\delta t_k} | X^\pi_{t_{k-1}}\Big], \tag{81}$$

$$\Gamma^\pi_{t_{k-1}} \quad = \quad \mathbb{E}\Big[Y^\pi_{t_k}\frac{\delta W_{t_k}\delta W_{t_k}^{\mathrm{T}} - \delta t_k}{(\delta t_k)^2} | X^\pi_{t_{k-1}}\Big]. \tag{82}$$

The convergence of this probabilistic numerical scheme is analyzed in [13] by the method of monotonic schemes introduced by Barles and Souganidis [4] and further developed by Krylov [17], Barles and Jakobsen [3].

Moreover, a numerical implementation is reported in [13] for the 3-dimensional mean curvature flow, and a five dimensional stochastic control problem.

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