An Explicit Martingale Version of the one-dimensional Brenier Theorem*

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Abstract

By investigating model-independent bounds for exotic options in financial mathematics, a martingale version of the Monge-Kantorovich mass transport problem was introduced in [3, 24]. Further, by suitable adaptation of the notion of cyclical monotonicity, [4] obtained an extension of the one-dimensional Brenier's theorem to the present martingale version. In this paper, we complement the previous work by extending the so-called Spence-Mirrlees condition to the case of martingale optimal transport. Under some technical conditions on the starting and the target measures, we provide an explicit characterization of the corresponding optimal martingale transference plans both for the lower and upper bounds. These explicit extremal probability measures coincide with the unique left and right monotone martingale transference plans introduced in [4]. Our approach relies on the (weak) duality result stated in [3], and provides, as a by-product, an explicit expression for the corresponding optimal semi-static hedging strategies. We finally provide an extension to the multiple marginals case.

1 Introduction

Since the seminal paper of Hobson [29], an important literature has developed on the topic of robust or model-free superhedging of some path dependent derivative security with payoff ξ, given the observation of the stochastic process of some underlying financial asset, together with a class of derivatives. See [7, 11, 12, 13, 14, 15, 16, 18, 19, 31, 33, 39] and the survey papers of Obłój [40] and Hobson [30]. In continuous-time models, these papers mainly focus on derivatives whose payoff ξ is stable under time change. Then, the key-observation was

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that, in the idealized context where all $T-$maturity European calls and puts, with all possible strikes, are available for trading, model-free superhedging cost of ξ is closely related to the Skorohod Embedding problem. Indeed, the market prices of all $T-$maturity European calls and puts with all possible strikes allow to recover the marginal distribution of the underlying asset price at time T.

Recently, this problem has been addressed via a new connection to the theory of optimal transportation, see [3, 24, 27, 1, 2, 20, 21]. Our interest in this paper is on the formulation of a Brenier Theorem in the present martingale context. We recall that the Brenier Theorem in the standard optimal transportation theory states that the optimal coupling measure is the gradient of some convex function which identifies in the one-dimensional case to the so-called Fréchet-Hoeffding coupling [6]. A remarkable feature is that this coupling is optimal for the class of coupling cost functions satisfying the so-called Spence-Mirrlees condition.

We first consider the one-period model. Denote by X, Y the prices of some underlying asset at the future maturities 0 and 1, respectively. Then, the possibility of dynamic trading implies that the no-arbitrage condition is equivalent to the non-emptyness of the set $\mathcal{M}_2$ of all joint measures $\mathbb{P}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfying the martingale condition $\mathbb{E}^{\mathbb{P}}[Y|X] = X$. The model-free subhedging and superhedging costs of some derivative security with payoff $c(X, Y)$, given the marginal distributions $X \sim \mu$ and $Y \sim \nu$, is essentially reduced to the martingale transportation problems:

$$\inf_{\mathbb{P}\in\mathbf{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)] \quad \text{and} \quad \sup_{\mathbb{P}\in\mathbf{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)],$$

where $\mathbf{M}_2(\mu, \nu)$ is the collection of all probability measures $\mathbb{P} \in \mathcal{M}_2$ such that $X \sim_{\mathbb{P}} \mu$, $Y \sim_{\mathbb{P}} \nu$. Our main objective is to characterize the optimal coupling measures which solve the above problems. This provides some remarkable extremal points of the convex (and weakly compact) set $\mathbf{M}_2(\mu, \nu)$. In the absence of marginal restrictions, Jacod and Yor [35] (see also Jacod and Shiryaev [34], Dubins and Schwarz [22], for the discrete-time setting) proved that a martingale measure $\mathbb{P} \in \mathcal{M}_2$ is extremal if and only if $\mathbb{P}$-local martingales admit a predictable representation. In the present one-period model, such extremal points of $\mathcal{M}_2$ consist of binomial models. For a specific class of coupling functions c, the extremal points of the corresponding martingale transportation problem turn out to be of the same nature, and our main contribution in this paper is to provide an explicit characterization.

Our starting point is a paper by Hobson and Neuberger [32] who considered the specific case of the coupling function $c(x, y) := |x - y|$, and provided a completely explicit solution of the optimal coupling measure and the corresponding optimal semi-static strategy. In a recent paper, Beiglböck and Juillet [4] address the problem from the viewpoint of optimal transportation. By a convenient extension of the notion of cyclic monotonicity, [4] introduce the notion of left-monotone transference plan. They also introduce the notion of left-curtain as a left-monotone transference plan concentrated on the graph of a binomial map. The remarkable result of [4] is the existence and uniqueness of the left-monotone transference plan which is indeed a left-curtain, together with the optimality of this joint probability measure for some specific class $\mathcal{C}_{\mathrm{BJ}}$ of coupling payoffs $c(x,y)$. Notice that the coupling measure of [32] is not a left-curtain, and $\mathcal{C}_{\mathrm{BJ}}$ does not contain the coupling payoff $|x-y|$.

As a main first contribution, we provide an explicit description of the left-curtain $\mathbb{P}_*$ of [4]. Then, by using the weak duality inequality,

- we provide a larger class $\mathcal{C} \supset \mathcal{C}_{\mathrm{BJ}}$ of payoff functions for which $\mathbb{P}_*$ is optimal,
- we identify explicitly the solution of the dual problem which consists of the optimal semi-static superhedging strategy,
- as a by-product, the strong duality holds true.

Our class $\mathcal{C}$ is the collection of all smooth functions $c : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, with linear growth, such that $c_{xyy} > 0$. We argue that this is essentially the natural class for our martingale version of the Brenier Theorem.

We next explore the multiple marginals extension of our result. In the context of the finite discrete-time model, we provide a direct extension of our result which applies to the context of the discrete monitored variance swap. This answers the open question of optimal model-free upper and lower bounds for this derivative security.

The paper is organized as follows. Section 2 provides a quick review of the Brenier Theorem in the standard one-dimensional optimal transportation problem. The martingale version of the Brenier Theorem is reported in Section 3. We next report our extensions to the multiple marginals case in Section 6. Finally, Section 7 contains the proofs of our main results.

2 The Brenier Theorem in One-dimensional Optimal Transportation

2.1 The two-marginals optimal transportation problem

Let X, Y be two scalar random variables denoting the prices of two financial assets at some future maturity T. The pair (X,Y) takes values in $\mathbb{R}^2$, and its distribution is defined by some $\mathbb{P} \in \mathcal{P}_{\mathbb{R}^2}$, the set of all probability measures on $\mathbb{R}^2$.

We assume that $T-$maturity European call options, on each asset and with all possible strikes, are available for trading at exogenously given market prices. Then, it follows from Breeden and Litzenberger [5] that the marginal distributions of X and Y are completely determined by the second derivative of the corresponding (convex) call price function with respect to the strike. We shall denote by μ and ν the implied marginal distributions of X and Y, respectively, ℓ^μ, r^μ, ℓ^ν, r^ν the left and right endpoints of their supports, and F_μ, F_ν the corresponding cumulative distribution functions.

By definition of the problem, the probability measures μ and ν have finite first moment:

$$\int |x|\mu(dx) + \int |y|\nu(dy) \quad < \quad \infty, \tag{2.1}$$

and although the supports of μ and ν could be restricted to the non-negative real line for the financial application, we shall consider the more general case where μ and ν lie in $\mathcal{P}_{\mathbb{R}}$, the collection of all probability measures on $\mathbb{R}$.

We consider a derivative security defined by the payoff $c(X,Y)$ at maturity T, for some upper semicontinuous function $c:\mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfying the growth condition:

$$c(x,y) \le \varphi(x)+\psi(y) \quad \text{for some} \quad \varphi,\psi:\mathbb{R}\longrightarrow\mathbb{R}, \quad \varphi^+\in\mathbb{L}^1(\mu), \psi^+\in\mathbb{L}^1(\nu). \tag{2.2}$$

The model-independent upper bound for this payoff, consistent with vanilla option prices of maturity T, can then be framed as a Monge-Kantorovich (in short MK) optimal transport problem:

$$P_2^0(\mu,\nu) := \sup_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X,Y)\big] \quad \text{where} \quad \mathcal{P}_2(\mu,\nu) := \big\{\mathbb{P}\in\mathcal{P}_{\mathbb{R}^2} : X\sim_{\mathbb{P}}\mu \text{ and } Y\sim_{\mathbb{P}}\nu\big\}.$$

Here, for the sake of simplicity, we have assumed a zero interest rate. This can easily be relaxed by considering the forwards of X and Y. Notice that $c(X,Y)$ is measurable by the upper semicontinuity condition on c, and $\mathbb{E}^{\mathbb{P}}[c(X,Y)]$ is a well-defined scalar in $\mathbb{R}\cup\{-\infty\}$ by Conditions (2.1) and (2.2).

In the original optimal transportation problem as formulated by Monge, the above maximization problem was restricted to the following subclass of measures.

Definition 2.1. *A probability measure $\mathbb{P}\in\mathcal{P}_2(\mu,\nu)$ is called a transference map if $\mathbb{P}(dx,dy) := \mu(dx)\delta_{\{T(x)\}}(dy)$, for some measurable map $T:\mathbb{R}\longrightarrow\mathbb{R}$.*

The dual problem associated to the MK optimal transportation problem is defined by :

$$D_2^0(\mu,\nu) := \inf_{(\varphi,\psi)\in\mathcal{D}_2^0} \big\{\mu(\varphi)+\nu(\psi)\big\},$$

where $\mu(\varphi) := \int\varphi d\mu$, $\nu(\psi) := \int\psi d\nu$, and denoting $\varphi\oplus\psi(x,y) := \varphi(x)+\psi(y)$:

$$\mathcal{D}_2^0 \;:=\; \big\{(\varphi,\psi) : \varphi^+\in\mathbb{L}^1(\mu), \psi^+\in\mathbb{L}^1(\nu) \text{ and } \varphi\oplus\psi\ge c\big\}.$$

The dual problem $D_2^0(\mu,\nu)$ is the cheapest superhedging strategy of the derivative security $c(X,Y)$ using the market instruments consisting of $T-$maturity European calls and puts with all possible strikes. The weak duality inequality

$$P_2^0(\mu,\nu) \;\le\; D_2^0(\mu,\nu)$$

is immediate. For an upper semicontinuous payoff function c, equality holds and an optimal probability measure $\mathbb{P}^*$ for the MK problem P_2^0 exists, see e.g. Villani [44].

In this paper, our main interest is on the following results of Rachev and Rüschendorf [42], corresponding to the one-dimensional version of the Brenier theorem [6], which provides an interesting characterization of $\mathbb{P}^*$ in terms of the so-called Fréchet-Hoeffding pushing forward μ to ν, defined by the map

$$T_* := F_\nu^{-1}\circ F_\mu, \tag{2.3}$$

where F_ν^{-1} is the right-continuous inverse of F_ν:

$$F_\nu^{-1}(x) := \inf\{y : F_\nu(y) > x\}.$$

In particular, the following result relates the MK optimal transportation problem P_2^0 to the original Monge mass transportation problem for a remarkable class of coupling functions c. We observe that the following result holds in a wider generality, in particular the set of measures $\mathbb{P}_T$ induced by a map T pushing forward μ to ν is dense in $\mathcal{P}_{\mathbb{R}^2}$ whenever μ is atomless and the supports of μ and ν are contained in compact subsets. For the purpose of our financial interpretation, this result characterizes the structure of the worst case financial market that the derivative security hedger may face, and characterizes the optimal hedging strategies by the functions φ_* and ψ_* defined up to an irrelevant constant by

$$\varphi_*(x) := c\big(x, T_*(x)\big) - \psi_* \circ T_*(x), \quad \psi_*'(y) := c_y\big(T_*^{-1}(y), y\big), \quad x, y \in \mathbb{R}. \tag{2.4}$$

Theorem 2.2. (see e.g. [44], Theorem 2.44) *Let c be upper semicontinuous with linear growth. Assume that the partial derivative c_{xy} exists and satisfies the Spence-Mirrlees condition $c_{xy} > 0$. Assume further that μ has no atoms, $\varphi_*^+ \in \mathbb{L}^1(\mu)$ and $\psi_*^+ \in \mathbb{L}^1(\nu)$. Then*
(i) $P_2^0(\mu,\nu) = D_2^0(\mu,\nu) = \int c\big(x, T_*(x)\big)\mu(dx)$,
(ii) $(\varphi_*, \psi_*) \in \mathcal{D}_2^0$, *and is a solution of the dual problem* D_2^0,
(iii) $\mathbb{P}_*(dx, dy) := \mu(dx)\delta_{T_*(x)}(dy)$ *is a solution of the MK optimal transportation problem* P_2^0, *and is the unique optimal transference map.*

Proof. We provide the proof for completeness, as our main result in this paper will be an adaptation of the subsequent argument. First, it is clear that $\mathbb{P}_* \in \mathcal{P}(\mu,\nu)$. Then $\mathbb{E}^{\mathbb{P}_*}[c(X,Y)] \le P_2^0(\mu,\nu)$. We now prove that

$$(\varphi_*, \psi_*) \in \mathcal{D}_2^0 \quad \text{and} \quad \mu(\varphi_*) + \nu(\psi_*) = \mathbb{E}^{\mathbb{P}_*}[c(X,Y)]. \tag{2.5}$$

In view of the weak duality $P_2^0(\mu,\nu) \le D_2^0(\mu,\nu)$, this would imply that $P_2^0(\mu,\nu) = D_2^0(\mu,\nu)$ and that $\mathbb{P}_*$ and (φ_*, ψ_*) are solutions of $P_2^0(\mu,\nu)$ and $D_2^0(\mu,\nu)$, respectively.

Under our assumption that $\varphi_*^+ \in \mathbb{L}^1(\mu)$, $\psi_*^+ \in \mathbb{L}^1(\nu)$, notice that (2.5) is equivalent to:

$$0 = H^0\big(x, T_*(x)\big) = \min_{y\in\mathbb{R}} H^0(x,y), \quad \text{where} \quad H^0 := \varphi_* \oplus \psi_* - c.$$

The first-order condition for the last minimization problem provides the expression of ψ_*' in (2.4), and the expression of φ_* follows from the first equality. Since

$$H_y^0(x,y) \;=\; c_y\big(T_*^{-1}(y), y\big) - c_y(x,y) \;=\; \int_x^{T_*^{-1}(y)} c_{xy}(\xi, y)d\xi,$$

it follows from the Spence-Mirrlees condition that $T_*(x)$ is the unique solution of the first-order condition. Finally, we compute that $H_{yy}^0\big(x, T_*(x)\big)T_*'(x) = c_{xy}\big(x, T_*(x)\big) > 0$ by the Spence-Mirrlees condition, where the derivatives are in the sense of distributions. Hence $T_*(x)$ is the unique global minimizer of $H(x,.)$. □

We observe that we may also formulate sufficient conditions on the coupling function c so as to guarantee that the integrability conditions $\varphi_*^+ \in \mathbb{L}^1(\mu), \psi_*^+ \in \mathbb{L}^1(\nu)$ hold true. See [44], Theorem 2.44.

Remark 2.3 (Symmetry: anti-monotone rearrangement map). (i) *Suppose that the coupling function c satisfies $c_{xy} < 0$. Then, the upper bound $P_2^0(\mu,\nu)$ is attained by the anti-monotone rearrangement map*

$$\overline{\mathbb{P}}_*(dx,dy) := \mu(dx)\delta_{\{\overline{T}_*(x)\}}(dy), \quad \textit{where} \quad \overline{T}_*(x) := F_\nu^{-1} \circ \big(1 - F_\mu(-x)\big).$$

To see this, it suffices to rewrite the optimal transportation problem equivalently with modified inputs:

$$\bar{c}(x,y) := c(-x,y), \quad \overline{\mu}(x) := \mu\big((-x,\infty)\big), \quad \overline{\nu} := \nu,$$

so that $\bar{c}$ satisfies the Spence-Mirrlees condition $\bar{c}_{xy} > 0$.
(ii) *Under the Spence-Mirrlees condition $c_{xy} > 0$, the lower bound problem is explicitly solved by the anti-monotone rearrangement. Indeed, it follows from the first part (i) of the present remark that:*

$$\inf_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X,Y)\big] = -\sup_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[-c(X,Y)\big] = -\mathbb{E}^{\overline{\mathbb{P}}_*}\big[-c(X,Y)\big] = \int c\big(x,\overline{T}_*(x)\big)\mu(dx).$$

Remark 2.4. *The Spence-Mirrlees condition is a natural requirement in the optimal transportation setting in the following sense. The optimization problem is not affected by the modification of the coupling function from c to $\bar{c} := c + a \oplus b$ for any $a \in \mathbb{L}^1(\mu)$ and $b \in \mathbb{L}^1(\nu)$. Since $c_{xy} = \bar{c}_{xy}$, it follows that the Spence-Mirrlees condition is stable for the above transformation of the coupling function.*

Example 2.5 (Basket option). *Let $c(x,y) = (x+y-k)^+$, for some $k \in \mathbb{R}$ (see [17, 38] for multi-asset basket options). The result of Theorem 2.2 applies to this example as well, as it is shown in [44] Chapter 2 that the regularity condition $c \in C^{1,1}$ is not needed. The upper bound is attained by the Fréchet-Hoeffding transference map $T_* := F_\nu^{-1} \circ F_\mu$, and the optimal hedging strategy is:*

$$\psi_*(y) = (y-\bar{y})^+, \quad \varphi_*(x) = \big(T_*(x) + x - k\big)^+ - \big(T_*(x) - \bar{y}\big)^+,$$

where $\bar{y}$ is defined by $T_*(k-\bar{y}) = \bar{y}$.

2.2 The multi-marginals optimal transportation problem

The previous results have been extended to the $n-$marginals optimal transportation problem by Gangbo and Święch [25], Carlier [9], and Pass [41]. Let $X = (X_1,\ldots,X_n)$ be a random variable with values in $\mathbb{R}^n$, representing the prices at some fixed time horizon of n financial assets, and consider some upper semicontinuous payoff function $c : \mathbb{R}^n \longrightarrow \mathbb{R}$ with linear growth.

Let $\mu_1, \ldots, \mu_n \in \mathcal{P}_{\mathbb{R}}$ be the corresponding marginal distributions, and $\mu := (\mu_1, \ldots, \mu_n)$. The upper bound market price on the derivative security with a payoff function c is defined by the optimal transportation problem:

$$P_n^0(\mu) := \sup_{\mathbb{P} \in \mathcal{P}_n(\mu)} \mathbb{E}^{\mathbb{P}}\big[c(X)\big], \quad \text{where} \quad \mathcal{P}_n(\mu) := \big\{\mathbb{P} \in \mathcal{P}_{\mathbb{R}^n} : X_i \sim_{\mathbb{P}} \mu_i, 1 \le i \le n\big\}. \quad (2.6)$$

Then, under convenient conditions on the coupling function c (see Pass [41] for the most general ones), there exists a solution $\mathbb{P}_*$ to the MK optimal transportation problem $P_n^0(\mu)$ which is the unique optimal transference map defined by T_*^i, $i = 2, \ldots, n$:

$$\mathbb{P}^*(dx_1, \ldots, dx_n) = \mu_1(dx_1) \prod_{i=2}^n \delta_{T_*^i(x_1)}(dx_i), \quad \text{where} \quad T_*^i = F_{\mu_i}^{-1} \circ F_{\mu_1}, \quad i = 2, \ldots, n.$$

The optimal upper bound is then given by

$$P_n^0(\mu) = \int c\big(\xi, T_*^2(\xi), \ldots, T_*^n(\xi)\big)\mu_1(d\xi).$$

3 Martingale Transport Problem: Formulation and First Intuitions

The main objective of this paper is to obtain a version of the Brenier theorem for the martingale transportation problem introduced by Beiglböck, Henry-Labordère and Penkner [3] and Galichon, Henry-Labordère and Touzi [24]. A first result in this direction was obtained by Hobson and Neuberger [32] in the context of the coupling function $c(x, y) = |x - y|$. The general case was considered by Beiglböck and Juillet [4] who introduced the martingale version of the cyclic monotonicity condition in standard optimal transport, namely the martingale monotonicity condition, and showed existence and uniqueness of such a monotone martingale measure, and its optimality for a class of coupling functions. Our result complements the last reference by an explicit extension of the Fréchet-Hoeffding optimal coupling. We outline in Sections 5.4 and 5.6 the main differences with [4, 32].

3.1 Probability measures in convex order

In the context of the financial motivation of Subsection 2.1, we interpret the pair of random variables X, Y as the prices of the same financial asset at dates t_1 and t_2, respectively, with $t_1 < t_2$. Then, the no-arbitrage condition states that the price process of the tradable asset is a martingale under the pricing and hedging probability measure. We therefore restrict the set of probability measures to:

$$\mathcal{M}_2(\mu, \nu) \quad := \quad \big\{\mathbb{P} \in \mathcal{P}_2(\mu, \nu) : \mathbb{E}^{\mathbb{P}}[Y|X] = X\big\},$$

where μ, ν have finite first moment as in (2.1). This set of probability measures is clearly convex, and the martingale condition implies that $\ell^\nu \le \ell^\mu \le r^\mu \le r^\nu$. Throughout this paper, we shall denote

$$\delta F := F_\nu - F_\mu.$$

By a classical result of Strassen [43], $\mathcal{M}_2(\mu,\nu)$ is non-empty if and only if $\mu \preceq \nu$ in sense of convex ordering, i.e.

(i) μ, ν have the same mean: $\int \xi d\delta F(\xi) = 0$,

(ii) and $\delta c(k) := \int (\xi - k)^+(\nu - \mu)(d\xi) \ge 0$, for all $k \in \mathbb{R}$.

By direct integration by parts, we see that

$$\delta c(k) \;=\; -\int_{[k,\infty)} \delta F(\xi) d\xi \quad \text{for all} \quad k \in \mathbb{R}. \tag{3.1}$$

Consequently, we may express the last condition (ii) as:

$$\int_{[k,\infty)} \delta F(\xi) d\xi \le 0 \quad \text{or, equivalently,} \quad \int_{[-\infty,k)} \delta F(\xi) d\xi \ge 0, \quad \text{for all} \quad k \in \mathbb{R}, \tag{3.2}$$

where the last equivalence follows from the first property (i). A crucial ingredient for the present paper is the decomposition of the pair (μ,ν) into irreducible components, as introduced by Beiglböck & Juillet [4].

Definition 3.1. *Let $\mu \preceq \nu$. We say that the pair (μ,ν) is irreducible if the set $I := \{\delta c > 0\}$ is connected and $\mu(I) = \mu(\mathbb{R})$. We denote by J the union of I and any endpoints of I that are atoms of ν, and we refer to the pair (I, J) as the domain of (μ,ν).*

The following decomposition result is restated from Beiglböck & Juillet [4], Theorem 8.4.

Proposition 3.2. *Let $\mu \preceq \nu$ and let $(I_k)_{1kN}$ be the (open) components of $\{\delta c > 0\}$, where $N \in \{0, 1, ..., \infty\}$. Set $I_0 := \mathbb{R} \setminus \cup_{k\ge1} I_k$ and $\mu_k = \mu|_{I_k}$ for $k \ge 0$, so that $\mu = \sum_{k\ge0} \mu_k$. Then, there exists a unique decomposition $\nu = \sum_{k\ge0} \nu_k$ such that*
- *$\mu_0 = \nu_0$ and $\mu_k \preceq \nu_k$ for all $k \ge 1$,*
- *and $I_k = \{\delta c_k > 0\}$ for all $k \ge 1$, where $\delta c_k(x) := \int (\xi - x)^+(\nu_k - \mu_k)(d\xi)$.*

Moreover, any $\mathbb{P} \in \mathcal{M}(\mu,\nu)$ admits a unique decomposition $\mathbb{P} = \sum_{k\ge0} \mathbb{P}_k$ such that $\mathbb{P}_k \in \mathcal{M}(\mu_k,\nu_k)$ for all $k \ge 0$.

Observe that the measure P_0 in the last statement is the trivial constant martingale transport from μ_0 to itself. In particular, P_0 does not depend on the choice of $P \in \mathcal{M}(\mu,\nu)$.

3.2 Problem formulation

Let $c : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an upper semicontinuous function satisfying the growth condition (2.2), representing the payoff of a derivative security. In the present context, the model-independent upper bound for the price of the claim can be formulated by the following martingale optimal transportation problem:

$$P_2(\mu,\nu) \;\; := \sup_{\mathbb{P}\in\mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X,Y)\big]. \tag{3.3}$$

Remark 3.3. *When μ and ν have finite second moment, notice that $\mathbb{E}^{\mathbb{P}}[(X-Y)^2] = -\mathbb{E}^{\mathbb{P}}[X^2]+\mathbb{E}^{\mathbb{P}}[Y^2] = \int \xi^2 d\delta F(\xi)$ for all $\mathbb{P}\in\mathcal{M}_2(\mu,\nu)$. Then, the quadratic case, which is the typical example of coupling in the optimal transportation theory, is irrelevant in the present martingale version.*

The Kantorovich dual in the present martingale transport problem is formulated as follows. Because of the possibility of dynamic trading the financial asset between times t_1 and t_2, the set of dual variables is defined by:

$$\mathcal{D}_2 \;\; := \;\; \big\{(\varphi,\psi,h) : \varphi^+\in\mathbb{L}^1(\mu), \psi^+\in\mathbb{L}^1(\nu), h\in\mathbb{L}^0, \text{ and } \varphi\oplus\psi+h^{\otimes}\geq c\big\}, \tag{3.4}$$

where $\varphi\oplus\psi(x,y) := \varphi(x)+\psi(y)$, and $h^{\otimes}(x,y) := h(x)(y-x)$. The dual problem is:

$$D_2(\mu,\nu) \;\; := \inf_{(\varphi,\psi,h)\in\mathcal{D}_2} \big\{\mu(\varphi)+\nu(\psi)\big\}, \tag{3.5}$$

and can be interpreted as the cheapest superhedging strategy of the derivative $c(X,Y)$ by dynamic trading on the underlying asset, and static trading on the European options with maturities t_1 and t_2. Since μ,ν have finite first moment, and c satisfies the growth condition (2.2), the weak duality inequality:

$$P_2(\mu,\nu) \;\; \leq \;\; D_2(\mu,\nu) \tag{3.6}$$

follows immediately from the definition of both problems. The strong duality result (i.e. equality holds), together with the existence of a maximizer $\mathbb{P}_*\in\mathcal{M}_2(\mu,\nu)$ for the martingale transportation problem $P_2(\mu,\nu)$, is proved in [3]. However, existence does not hold in general for the dual problem $D_2(\mu,\nu)$. An example of non-existence is provided in [3]. In the present paper, we shall obtain existence under a martingale version of the Spence-Mirrlees condition.

3.3 Monotone martingale transport plans

Our objective in this paper is to provide explicitly the left-monotone martingale transport plan introduced by Beiglböck and Juillet [4].

Definition 3.4. *We say that $\mathbb{P}\in\mathcal{M}_2(\mu,\nu)$ is left-monotone (resp. right-monotone) if there exists a Borel set $\Gamma\subset\mathbb{R}\times\mathbb{R}$ such that $\mathbb{P}[(X,Y)\in\Gamma]=1$, and for all $(x,y_1),(x,y_2),(x',y')\in\Gamma$ with $x<x'$ (resp. $x>x'$), it must hold that $y'\notin(y_1,y_2)$.*

Similar to [4], we shall consider probability measures μ, ν satisfying the following restriction.

Assumption 3.5. *The probability measures μ and ν have finite first moments, $\mu \preceq \nu$ in convex order, and μ has no atoms.*

Under Assumption 3.5, Theorem 1.5 and Corollary 1.6 of [4] state that there exists a unique left-monotone martingale transport plan $\mathbb{P}_* \in \mathcal{M}_2(\mu,\nu)$, and that the graph of $\mathbb{P}_*$ is concentrated on two maps $T_d, T_u : \mathbb{R} \longrightarrow \mathbb{R}$, $T_d(x) \le x \le T_u(x)$ for all $x \in \mathbb{R}$, i.e.

$$\mathbb{P}_*(dx,dy) = \mu(dx)\big[q(x)\delta_{T_u(x)}+(1-q(x))\delta_{T_d(x)}\big](dy), \text{ with } q(x) = \frac{x-T_d(x)}{(T_u-T_d)(x)}\mathbf{1}_{\{(T_u-T_d)(x)>0\}}. \tag{3.7}$$

Remark 3.6. *By the convex ordering condition* (3.2)*, it follows that δF increases from and to zero at the left and right boundaries of its support, respectively. Moreover, δF is upper-semicontinuous by the continuity of F_μ in Assumption 3.5. Then the local suprema of δF are attained by local maximizers in (ℓ_μ, r_μ).*

Let $\mathbf{M}(\delta F)$ be the collection of all local maximizers of the function δF. Moreover for all local maximizer $m \in \mathbf{M}(\delta F)$, we denote:

$$\begin{array}{rcl} m_- & := & \sup\big\{x < m : \ \delta F(x) < \delta F(m)\big\}, \\ m_+ & := & \inf\big\{x > m : \ \delta F(x) < \delta F(m)\big\}. \end{array} \tag{3.8}$$

The set:

$$\mathbf{M}_0(\delta F) \ := \ \big\{m \in \mathbf{M}(\delta F) : \ m = m_+ \text{ and } \delta F = \delta F(m) \text{ on } [m_-, m]\big\}$$

will play a crucial role in our characterization. Our construction will be performed under the following additional assumption on the pair of measures (μ,ν).

Assumption 3.7. *ν has no atoms, and $\mathbf{M}_0(\delta F)$ is a finite set of points.*

Under this assumption, the unique decomposition $\mathbb{P} = \sum_{k\ge 0}\mathbb{P}_k$ with $\mathbb{P}_k \in \mathcal{M}(\mu_k,\nu_k)$ of Proposition 3.2 corresponds to the irreducible domains (I_k, I_k), i.e. $J_k = I_k$.

Finally, we observe that the construction of the left-monotone martingale transport plan will be elaborated separately on each irreducible component, see Theorem 4.5 (ii) below. Therefore, without loss of generality, it suffices to provide the construction for an irreducible pair (μ,ν), i.e.

$$\delta c(x) := -\int_x^\infty \delta F(\xi)d\xi > 0 \quad \text{for all} \quad x \in I. \tag{3.9}$$

3.4 First intuitions

In this subsection, we provide a construction of the left-monotone transport plan, for an irreducible pair (μ,ν) of measures in convex order, under the simplifying condition

$$\mathbf{M}(\delta F) = \mathbf{M}_0(\delta F) = \{m_1\} \quad \text{for some} \quad \ell_\mu < m_1 < r_\mu, \tag{3.10}$$

so that δF is strictly increasing on $(-\infty, m_1]$.

The definition of the left-monotone transport map suggests that T_u is non-decreasing and T_d non-increasing. This is a first guess which will be verified under our simplifying condition (3.10). However, we emphasize that it will turn out to be wrong in the more general case studied in Section 4, but will serve to guide our intuition.

As a first consequence of the non-increase of T_d and the non-decrease of T_u, we see that they have a countable number of discontinuities. Therefore, since μ has no atoms, we may choose the maps T_d and T_u to be right-continuous. In order to allow for a decreasing map T_d, we guess that there exists some bifurcation point m such that:

$$T_d(x) = T_u(x) \;\text{ for } x \le m, \text{ and}$$
$$T_d : (m,\infty) \longmapsto (-\infty, m), \text{ non-increasing, } T_u : (m,\infty) \longmapsto (m,\infty) \;\text{ non-decreasing.}$$

We denote by T_d^{-1}, T_u^{-1} the right-continuous generalized inverse of T_d and T_u, respectively. Since ν has no atoms, we observe that

$$\big\{x' \ge m : T_u(x') = T_u(x)\big\} = \big\{x' \ge m : T_d(x') = T_d(x)\big\} = \{x\}, \text{ for } \mu-\text{a.e. } x \ge m. \tag{3.11}$$

By the representation (3.7) of the left-monotone transport map, we have $X \sim_{\mathbb{P}} \mu$, and the martingale condition $\mathbb{E}^{\mathbb{P}}[Y|X] = X$ holds true. It remains to impose the mass conservation condition $Y \sim_{\mathbb{P}} \nu$, i.e. $\mathbb{P}[Y \in dy] = \nu(dy)$.

(i) Mass conservation condition. We consider separately the domains on both sides of the bifurcation point m.

• *Upper support.* Let $y > m$ be a point of the support of ν. Then $y := T_u(x)$ for some $x \ge m$, and

$$\mathbb{P}[Y \in dy] \;=\; \mathbb{E}\big[q(X)\mathbf{1}_{\{T_u(X)\in dy\}}\big] \;=\; q(x)dF_\mu(x),$$

by (3.11). Then, the mass conservation condition in this case is:

$$dF_\nu(T_u) \;=\; q dF_\mu. \tag{3.12}$$

• *Lower support.* Let $y < m$ be a point of the support of ν. Then, $y = T_d(x)$ for some $x > m$, and

$$\mathbb{P}[Y \in dy] \;=\; dF_\mu(y) + \mathbb{E}\big[(1-q(X))\mathbf{1}_{\{T_d(X)\in dy\}}\big] \;=\; dF_\mu(y) - (1-q(x))dF_\mu(x),$$

by (3.11), where the last minus sign is due to the decrease of T_d on (m, ∞). The mass conservation condition is then:

$$d\delta F(T_d) \quad = \quad -(1-q)dF_\mu. \tag{3.13}$$

We are then reduced to the system of ODEs (3.12)-(3.13) on $[m, \infty)$, with the boundary condition $T_u(m) = T_d(m) = m$. Recall that we have to solve for the unknowns T_u, T_d, and also for the bifurcation level m.

(ii) Determining the bifurcation point. Subtracting (3.12) and (3.13), we get $dF_\nu(T_u) = dF_\mu + d\delta F(T_d)$. Integrating between m and x, and using the boundary condition $T_u(m) = T_d(m) = m$, we see that:

$$F_\nu(T_u) = F_\mu + \delta F(T_d) \quad \text{on} \quad [m, \infty). \tag{3.14}$$

We expect that T_u and T_d be in one-to-one relation. Since F_ν is non-decreasing, the last equation allows indeed to express T_u in terms of T_d by using the right-continuous inverse F_ν^{-1}. However, expressing T_d in terms of T_u requires that $m \le m_1$ so that T_d takes values in the domain where δF is strictly increasing, and thus has a continuous inverse δF^{-1}. Then, using again (3.14), it follows from the non-decrease of F_ν and the fact that $x \le T_u(x)$ that:

$$\delta F(x) \le F_\nu(T_u(x)) - F_\mu(x) = \delta F(T_d(x)) \le \delta F(m) \quad \text{for all} \quad x \ge m.$$

Consequently, the only possible choice for $m \le m_1$ is

$$m \quad = \quad m_1.$$

(iii) Solving for T_d and T_u. We continue our derivation under the simplifying condition (3.10). First, by (3.14), we express T_u in terms of T_d:

$$T_u(x) = g\big(x, T_d(x)\big), \; x \ge m, \quad \text{with} \quad g(x,y) := F_\nu^{-1}\big(F_\mu(x) + \delta F(y)\big), \tag{3.15}$$

where we extend the definition of F_ν^{-1} by setting $F_\nu^{-1} = \infty$ on $(1, \infty)$ and $F_\nu^{-1} = -\infty$ on $(-\infty, 0)$. Next, by the definition of q together with (3.12)-(3.13) and (3.15), we have

$$\begin{aligned} x dF_\mu \; = \; [qT_u + (1-q)T_d]dF_\mu \; &= \; T_u dF_\nu(T_u) - T_d d\delta F(T_d) \\ &= \; g(x, T_d)\big[dF_\mu + d\delta F(T_d)\big] - T_d d\delta F(T_d). \end{aligned}$$

We are then reduced to the ordinary differential equation:

$$\big[g(x, T_d) - T_d\big] d\delta F(T_d) + \big[g(x, T_d) - x\big] dF_\mu = 0 \quad \text{on} \quad [m, \infty). \tag{3.16}$$

Observe that

$$d_y g(x,y) d\delta F(y) = (dF_\nu^{-1})\big(F_\mu(x) + \delta F(y)\big) dF_\mu(x) d\delta F(y) = d_x g(x,y) dF_\mu(x). \tag{3.17}$$

Then,

$$
\begin{aligned}
d_x \int_m^{T_d} \big[g(x,\xi)-\xi\big] d\delta F(\xi) &= \big[g(x,T_d)-T_d\big] d\delta F(T_d) + \Big[\int_m^{T_d} d_y g(x,y)\Big] dF_\mu(x) \\
&= \big[g(x,T_d)-T_d\big] d\delta F(T_d) + \big[g(x,T_d)-g(x,m)\big] dF_\mu(x),
\end{aligned}
$$

we re-write (3.16) as:

$$
d_x \int_m^{T_d} \big[g(x,\zeta)-\zeta\big] d\delta F(\zeta) + \big[g(x,m)-x\big] dF_\mu(x) \;=\; 0,
$$

which provides by direct integration, and using the boundary condition $T_d(m) = m$,

$$
G^m(T_d, x) = 0, \quad \text{for} \quad x \geq m, \tag{3.18}
$$

where:

$$
G^m(t,x) := -\int_t^m \big[g(x,\zeta)-\zeta\big] d\delta F(\zeta) + \int_m^x \big[g(\xi,m)-\xi\big] dF_\mu(\xi), \quad t \leq m \leq x. \tag{3.19}
$$

We finally verify that equation (3.18) defines uniquely $T_d(x) \in (-\infty, m]$.

- First, the function $t \longmapsto G^m(t,x)$ is continuous and strictly increasing for $x \geq m \geq t$. Indeed, the continuity is inherited from the continuity of δF. Next, for $\zeta \leq m < x$, it follows that $F_\mu(x) > F_\mu(\zeta)$ or, equivalently, $F_\mu(x) + \delta F(\zeta) > F_\nu(\zeta)$. Then, $g(x,\zeta) = F_\nu^{-1}(F_\mu(x) + \delta F(\zeta)) > \zeta$, and the strict increase of G^m in t is inherited from the strict increase of δF on $(-\infty, m_1)$.

- At $t = m$, we compute that $G^m(m,x) = \int_m^x \big[g(\xi,m)-\xi\big] dF_\mu(\xi) > 0$ for $x > m$. The last strict inequality follows from the fact that $g(x,m) > x$ for all $x > m$, under our simplifying condition (3.10), and the strict increase of F_μ at a right neighborhood of m.

- Finally, as $t \searrow -\infty$, we now show that $G^m(-\infty, x) < 0$ for all $x > m$. By (3.17), we observe that

$$
\begin{aligned}
d_x G^m(-\infty, x) &= -\Big[\int_{-\infty}^m d_\zeta g(x,\zeta)\Big] dF_\mu + \big[g(x,m)-x\big] dF_\mu \\
&= \big[g(x,-\infty)-x\big] dF_\mu \;=\; \big[F_\nu^{-1} \circ F_\mu(x) - x\big] dF_\mu.
\end{aligned}
$$

  By direct inegration, this provides,

$$
G^m(-\infty, x) \;=\; G^m(-\infty, m) + \int_m^x \big[F_\nu^{-1} \circ F_\mu(\xi) - \xi\big] dF_\mu(\xi) \;=\; \gamma(x),
$$

  where:

$$
\gamma(x) := \int_{-\infty}^{F_\nu^{-1} \circ F_\mu(x)} \xi dF_\nu(\xi) - \int_{-\infty}^x \xi dF_\mu(\xi), \quad \text{for} \quad x \in \mathbb{R}. \tag{3.20}
$$

Notice that $\gamma(-\infty) = 0$, and, since μ and ν have the same mean, $\gamma(\infty) = 0$. We next analyze the maximum of γ. Since $d\gamma(x) = [F_\nu^{-1} \circ F_\mu(x) - x]dF_\mu(x)$, we may restrict to a point $x^* \in \mathrm{Supp}(\mu)$ of local maximum of γ, so that $F_\nu^{-1}\big(F_\mu(x^*)-\big) \le x^* \le F_\nu^{-1}\big(F_\mu(x^*)\big)$, and therefore $\gamma(x^*) = \int_{-\infty}^{x^*} \xi d\delta F(\xi) = -\int (x^* - \xi)^+ d\delta F(\xi) < 0$ by the irreducibility condition (3.9) of the pair (μ, ν).

4 Explicit Construction of the left-monotone martingale transport plan

4.1 Preliminaries

We recall that our construction will be accomplished separately on each irreducible component, and consequently we may assume without loss of generality that the pair (μ, ν) is irreducible so that (3.9) holds true.

Recall also the function g introduced in (3.15). In order to relax the simplifying condition (3.10), we need to introduce, for a measurable subset $A \in \mathcal{B}_\mathbb{R}$ with δF increasing on A, the analogue of (3.19):

$$G_A^m(t, x) := -\int_t^m \big[g(x,\zeta) - \zeta\big]\mathbf{1}_A(\zeta) d\delta F(\zeta) + \int_m^x \big[g(\xi, m) - \xi\big] dF_\mu(\xi), \quad t \le m \le x. \quad (4.1)$$

Notice that G_A^m is continuous in t, by the continuity of δF. Recall from Assumption 3.7 that $\mathbf{M}_0(\delta F)$ is a finite set:

$$\mathbf{M}_0(\delta F) = \{m_1^0, \ldots, m_n^0\} \quad \text{for some} \quad -\infty < m_1^0 < \ldots, m_n^0 < \infty.$$

We also need to introduce the set

$$B_0 := \{x \in \mathbb{R} : \ \delta F \text{ increasing on a right neighborhood of } x\}, \quad x_0 := \inf B_0,$$

Here, $x \in B_0$ means that, for all $\varepsilon > 0$, we may find $x_\varepsilon \in (x, x+\varepsilon)$ such that $\delta F(x_\varepsilon) > \delta F(x)$. Observe that

$$x_0 < m_1^0 \quad \text{and} \quad \delta F = 0 \quad \text{on} \quad (-\infty, x_0],$$

where the first inequality is a direct consequence of the definition of x_0 and m_1^0, and the second property follows from the characterization (3.2) of the dominance $\mu \preceq \nu$ in the convex order.

Recall the function γ of (3.20). Our construction uses recursively the following ingredients:

($\mathbf{I_1}$) $m_0 \in \{-\infty\} \cup \mathbf{M}_0(\delta F)$, and $A_0 \subset B_0 \cap (-\infty, m_0)$ with $\delta F > 0$ on A_0, satisfying $G_{A_0}^{m_0}(-\infty, .) = \gamma$, and $\int_{-\infty}^{m_0} \mathbf{1}_{A_0} d\phi(\delta F) = \int_{-\infty}^{m_0} d\phi(\delta F)$ for all non-decreasing map ϕ;

($\mathbf{I_2}$) $\bar{x}_0 \in B_0 \cap [m_0, m_n^0)$ and $t_0 \in A_0 \cup \{-\infty\}$ satisfying $\delta F(t_0) = \delta F(\bar{x}_0) \ge 0$ and $G_{A_0}^{m_0}(t_0, \bar{x}_0) = 0$.

Lemma 4.1. *Let* $m_1 := \min\big[\mathbf{M}_0(\delta F)\cap(\bar{x}_0,\infty)\big]$, *and* $A_1 := \big(A_0\setminus[t_0,m_0]\big)\cup(\bar{x}_0,m_1)$. *Then,*

(i) $\delta F > 0$ *on* A_1, $G^{m_1}_{A_1}(-\infty,.) = \gamma$, *and* $\int_{-\infty}^{m_1}\mathbf{1}_{A_1}d\phi(\delta F) = \int_{-\infty}^{m_1}d\phi(\delta F)$, *for all non-decreasing map* ϕ*;*

(ii) *for all* $x \geq m_1$ *with* $\delta F(x) \leq \delta F(m_1)$, *there exists a unique scalar* $t^{m_1}_{A_1}(x) \in A_1$ *such that* $G^{m_1}_{A_1}\big(t^{m_1}_{A_1}(x),x\big) = 0$*;*

(iii) $t^{m_1}_{A_1}$ *is decreasing* $\mu-$*a.e. on* $\big[m_1,x_1\big]$, *where* $x_1 := \inf\{x > m_1 : g\big(x,t^{m_1}_{A_1}(x)\big) \leq x\}$*;*

(iv) *if* $x_1 < \infty$, *then* $x_1 \in B_0\cap[m_1,m^0_n)\setminus\mathbf{M}_0(\delta F)$, *and* $\delta F\big(t^{m_1}_{A_1}(x_1)\big) = \delta F(x_1) \geq 0$.

The proof of this lemma is reported in Subsection 7.1.

4.2 Explicit construction

We start by defining:

$$T_d(x) \;=\; T_u(x) \;=\; x \;\text{ for }\; x \leq x_0, \tag{4.2}$$

and we continue the construction of the maps T_d, T_u along the following steps.

Step 1: Set $m_0 := -\infty$, $A_0 := \emptyset$, $\bar{x}_0 := x_0$, $t_0 = -\infty$, and notice that $(\mathbf{I_1})-(\mathbf{I_2})$ are obviously satisfied by these ingredients. We may then apply Lemma 4.1, and obtain $m_1 := m^0_1$, the smallest point on $\mathbf{M}_0(\delta F)$, and $A_1, x_1, t_1 := t^{m_1}_{A_1}(x_1)$. Define the maps T_d, T_u on (x_0,x_1) by:

$$\begin{array}{c} T_d(x) \;=\; T_u(x) \;=\; x \;\text{ for }\; x_0 < x \leq m_1, \\ T_d(x) := t^{m_1}_{A_1}(x), \quad T_u(x) := g\big(x,T_d(x)\big) \;\text{ for }\; m_1 \leq x < x_1. \end{array} \tag{4.3}$$

If $x_1 = \infty$, this completes the construction, and we set $m_j = x_j = \infty$ for all $j > 1$. See Figure 1 below for such an example. Otherwise, Lemma 4.1 guarantees that the new ingredients (m_1, A_1, x_1, t_1) satisfy Conditions $(\mathbf{I_1})-(\mathbf{I_2})$, and we may continue with the next step.

Step i: Suppose that the pair of maps (T_d, T_u) is defined on $(-\infty, x_{i-1})$ for some quadruple $(m_{i-1}, A_{i-1}, x_{i-1}, t_{i-1})$ satisfying Conditions $(\mathbf{I_1})-(\mathbf{I_2})$. We may then apply Lemma 4.1, and obtain $m_i := \min\big[\mathbf{M}_0(\delta F)\cap(x_{i-1},\infty)\big]$, and $A_i, x_i, t_i := t^{m_i}_{A_i}(x_i)$. Define the maps T_d, T_u on (x_{i-1}, x_i) by:

$$\begin{array}{c} T_d(x) \;=\; T_u(x) \;=\; x \;\text{ for }\; x_{i-1} < x \leq m_i, \\ T_d(x) := t^{m_i}_{A_i}(x), \quad T_u(x) := g\big(x,T_d(x)\big) \;\text{ for }\; m_i \leq x < x_i. \end{array} \tag{4.4}$$

If $x_i = \infty$, this completes the construction, and we set $m_j = x_j = \infty$ for all $j > i$. Otherwise, Lemma 4.1 guarantees that the new ingredients (m_i, A_i, x_i, t_i) satisfy Conditions $(\mathbf{I_1})-(\mathbf{I_2})$, and we may continue with the next step.

Since $\mathbf{M}_0(\delta F)$ is assumed to be finite, the last iteration can only have a finite number of steps. We observe that we may extend to the case where $\mathbf{M}_0(\delta F)$ is countable, the delicate case of an accumulation point of $\mathbf{M}_0(\delta F)$ could be addressed by means of transfinite induction. We deliberately choose to avoid such technicalities in order to focus on the main properties of the above construction.

Remark 4.2 (Some properties of T_d). *From the above construction of T_d, we see that*
(i) *T_d is right-continuous, and decreasing on each interval (m_i, x_i), $\mu-$a.e.*
(ii) *In general, the restriction of T_d to $\cup_{i\geq 0}(m_i, x_i)$ fails to be non-decreasing. However, for $i \neq j$, we have $T_d\big((m_i, x_i)\big) \cap T_d\big((m_j, x_j)\big) = \emptyset$. Consequently, the right-continuous inverse T_d^{-1} of T_d is well defined.*

Remark 4.3 (Some properties of T_u). *From the above construction of T_u, we see that*
(i) *T_u is right continuous, $T_u([m_i, x_i]) \subset [m_i, x_i]$, and $T_u(x) > x$ for $x \in (m_i, x_i)$ for all i.*
(ii) *T_u is nondecreasing, and strictly increasing $\mu-$a.e. The last property will be clear from Theorem 4.5 (ii) below, and implies that the right-continuous inverse T_u^{-1} of T_u is well-defined.*

Remark 4.4. One could extend the above construction to the case where $\mathbf{M}_0(\delta F)$ is countable with no point of right accumulation, thus weakening the conditions of Assumption 3.7. However, the condition that F_ν has no atoms in this assumption is more difficult to by-pass because then the ODE's in Theorem 4.5 (ii) fail, in general, due to the fact that $T_d^{-1} \circ T_d(x)$ and $T_u^{-1} \circ T_u(x)$ may be larger than $\{x\}$.

4.3 The left-monotone martingale transport plan

The last construction provides, under Assumptions 3.5 and 3.7, our martingale version of the Fréchet-Hoeffding coupling for an irreducible pair (μ, ν) with domain (I, I):

$$T_*(x, dy) := \mathbf{1}_D(x)\delta_{\{x\}}(dy) + \mathbf{1}_{I\setminus D}(x)\big[q(x)\delta_{\{T_u(x)\}}(dy) + (1-q)(x)\delta_{\{T_d(x)\}}(dy)\big], \qquad (4.5)$$

with

$$D \ := \ \cup_{i\geq 0}(x_{i-1}, m_i] \quad \text{and} \quad q(x) \ := \ \frac{x - T_d(x)}{T_u(x) - T_d(x)}, \qquad (4.6)$$

We recall that our construction has a finite number of steps, $N \leq n$ say, due to our condition that $\mathbf{M}_0(\delta F)$ is finite, and that the union in the definition of the set D is finite by our convention that $m_{j+1} = x_j = \infty$ for all $j \geq N$. Observe also from our previous construction that $T_d(x) < x < T_u(x)$ on each (x_i, m_i). Therefore, q takes values in $[0, 1]$.

Theorem 4.5. *Let $\mu \preceq \nu$ be two probablity measures on $\mathbb{R}$.*
(i) *Assume that (μ, ν) is irreducible, with domain (I, I), and satisfies Assumptions 3.5 and 3.7. Then, the probability measure $\mathbb{P}_*(dx, dy) := \mu(dx)T_*(x, dy)$ on $I \times I$ is the unique left-monotone transport plan in $\mathcal{M}_2(\mu, \nu)$. Moreover T_u and T_d solve the following ODEs:*

$$d(\delta F \circ T_d) = -(1-q)dF_\mu, \ d(F_\nu \circ T_u) = qdF_\mu, \ \textit{whenever } x \in [m_i, x_i) \textit{ and } T_d(x) \in \mathrm{int}(A_i).$$

(ii) *Let $(\mu_k, \nu_k)_{k\geq 0}$ be the decomposition of (μ, ν) in irreducible components, with corresponding domains $(I_k, J_k)_{k\geq 0}$, as introduced in Proposition 3.2. Consider also the decomposition of $\mathbb{P} = \sum_{k\geq 0} P_k \in \mathcal{M}(\mu, \nu)$ with $\mathbb{P}_k \in \mathcal{M}(\mu_k, \nu_k)$, $k \geq 0$. Then $\mathbb{P}$ is left-monotone if and only if $\mathbb{P}_k$ is left monotone for all $k \geq 1$.*

The proof of part (i) is reported in Section 7.1. Part (ii) is obvious given the decomposition of Proposition 3.2.

We conclude this subsection by the following remarkable property of T_d which uses the notation (3.8).

Proposition 4.6. *Let Let (μ,ν) be an irreducible component satisfying Assumptions 3.5 and 3.7. Let $i \geq 1$ be such that $m_{i-} = m_i$. Then $T_d(m_i) = m_i$. If in addition F_μ, F_ν are twice differentiable near m_i, then T_d is also differentiable on $[m_i, m_i + h)$ for some $h > 0$, with right derivatives at m_i:*

$$T_d'(m_i+) = -1/2 \quad \textit{and} \quad T_d''(m_i+) = +\infty.$$

Proof. We shall denote $f_\mu := F_\mu'$, $f_\nu := F_\nu'$, $\delta f := f_\nu - f_\mu$.

By construction, we have $T_d(m_i) = m_i$ and the differentiation of $G_{A_i}^{m_i}\big(T_d(x), x\big) = 0$ reproduces the mass conservation condition (3.16). This ordinary differential equation shows that T_d inherits the differentiability of F_ν and F_μ on $(m_i, m_i + h)$ for some $h > 0$, with

$$T_d'(x) \;=\; -\frac{g\big(x, T_d(x)\big) - x}{g\big(x, T_d(x)\big) - T_d(x)} \, \frac{f_\mu(x)}{\delta f\big(T_d(x)\big)}, \quad x \in (m_i, m_i + h).$$

Let $\varepsilon := x - T_d(x)$, and recall that $g(x,x) = x$. Then, it follows from direct calculation that

$$\begin{aligned} g(x, T_d) - x \;&=\; -\varepsilon \frac{\delta f}{f_\nu}(x) + \frac{\varepsilon^2}{2}\Big(\frac{\delta f'}{f_\nu} - \Big(\frac{\delta f}{f_\nu}\Big)^2 \frac{f_\nu'}{f_\nu}\Big)(x) + o(\varepsilon^2), \\ \delta f\big(T_d(x)\big) \;&=\; \delta f(x) - \varepsilon \delta f'(x) + o(\varepsilon). \end{aligned}$$

where o is a continuous function with $o(0) = 0$. Then:

$$T_d'(x) \;=\; \frac{-\frac{\delta f}{f_\nu} + \frac{1}{2}\varepsilon\big[\frac{\delta f'}{f_\nu} - \big(\frac{\delta f}{f_\nu}\big)^2 \frac{f_\nu'}{f_\nu}\big] + o(\varepsilon)}{1 - \frac{\delta f}{f_\nu} + \frac{1}{2}\varepsilon\big[\frac{\delta f'}{f_\nu} - \big(\frac{\delta f}{f_\nu}\big)^2 \frac{f_\nu'}{f_\nu}\big] + o(\varepsilon)} \, \frac{f_\mu}{\delta f - \varepsilon \delta f' + o(\varepsilon)}(x), \quad x \in (m_i, m_i + h).$$

Notice that $0 \leq x - m_i \leq \varepsilon$. Then, since $f_\mu(m_i) = f_\nu(m_i)$, we have $\delta f(x) = (x - m_i)\delta f'(x) + \circ(\varepsilon)$, and therefore:

$$\begin{aligned} T_d'(x) \;&=\; \frac{-\delta f(x) + \frac{1}{2}\varepsilon \delta f'(x) + o(\varepsilon)}{\delta f(x) - \varepsilon \delta f(x) + o(\varepsilon)} \;=\; \frac{-(x - m_i) + \frac{1}{2}\varepsilon + o(\varepsilon)}{(x - m_i) - \varepsilon + o(\varepsilon)} \\ &=\; \frac{-\frac{1}{2} + \frac{x - m_i}{\varepsilon} + o(1)}{1 - \frac{x - m_i}{\varepsilon} + o(1)}, \quad x \in (m_i, m_i + h), \end{aligned} \tag{4.7}$$

where we recall that $\varepsilon = x - T_d(x)$. Since T_d is non-increasing, this implies further that $0 \leq x - m_i \leq \frac{1}{2}\varepsilon$. Moreover, by the convergence $T_d \to m$, we see that $m = T_d(x) + (m - x)T_d'(x) + \circ(x - m)$, and thus $\frac{x - T_d(x)}{x - m} = 1 - T_d'(x) + \circ(1)$. Substituting this in (4.7), we get

$$T_d'(x) \;=\; \frac{\frac{1}{2}(1 + T_d'(x)) + \circ(1)}{-T_d'(x) + \circ(1)}, \quad x \in (m_i, m_i + h),$$

from which we conclude that $T_d'(x) \longrightarrow -1/2$ as $x \searrow m_i$.

Finally, we compute $T_d''(m_i)$. By the ODE satisfied by T_d and the smoothness of g, it follows that T_d' is differentiable at any $x > m_i$. We then differentiate the ODE satisfied by T_d, and use Taylor expansions as above. The result follows from direct calculation by sending $x \searrow m_i$. □

5 Martingale one-dimensional Brenier Theorem

5.1 Derivation of the optimal semi-static hedging strategy

Similar to our construction, the optimal semi-static hedging strategy will be obtained separately on each irreducible component. Consequently, we may assume without loss of generality that the pair (μ, ν) is irreducible.

We start by following the same line of argument as in the proof of Theorem 2.2. Our objective is to construct a triple

$$(\varphi_*, \psi_*, h_*) \in \mathcal{D}_2 \quad \text{such that} \quad \mu(\varphi_*) + \nu(\psi_*) = \mathbb{E}^{\mathbb{P}_*}[c(X,Y)]. \tag{5.1}$$

This will provide equality in (3.6) with the optimality of $\mathbb{P}_*$ for the optimal transportation problem P_2 and the optimality of (φ_*, ψ_*, h_*) for the dual problem D_2.

By the definition of the dual set $\mathcal{D}_2$, we observe that the requirement (5.1) is equivalent to

$$\varphi_*(X) + \psi_*(Y) + h_*(X)(Y - X) - c(X,Y) = 0, \quad \mathbb{P}_* - \text{a.s. for some function } h_*, \tag{5.2}$$

and that the function φ_* is determined from (ψ_*, h_*) by:

$$\varphi_*(x) = \max_{y\in\mathbb{R}} H(x,y), \quad \text{where} \quad H(x,y) := c(x,y) - \psi_*(y) - h_*(x)(y-x), \quad x,y \in \mathbb{R}. \tag{5.3}$$

Recall the set D defined in (4.6) on which we have $T_d(x) = T_u(x) = x$, $x \in D$, and the right-continuous inverse functions T_d^{-1}, T_u^{-1} defined in Remark 4.2 (ii) and Remark 4.3 (iii).

From the perfect replication property (5.2), it follows that h_* in determined on D^c in terms of ψ_* by:

$$h_*(x) \;=\; \frac{(c(x,.) - \psi_*) \circ T_u(x) - (c(x,.) - \psi_*) \circ T_d(x)}{(T_u - T_d)(x)} \quad \text{for} \quad x \in D^c. \tag{5.4}$$

Since T_u and T_d are maximizers in (5.3), it follows from the first-order condition that

$$\psi_*' \circ T_u(x) = c_y(x, T_u(x)) - h_*(x), \quad \psi_*' \circ T_d(x) = c_y(x, T_d(x)) - h_*(x), \quad x \in D^c, \tag{5.5}$$

$$\text{and} \quad \psi_*'(x) = c_y(x,x) - h_*(x) \quad \text{for} \quad x \in D. \tag{5.6}$$

Differentiating (5.4), and using (5.5), we see that for $x \in D^c$:

$$h_*' = \frac{d}{dx}\Big\{\frac{c(.,T_u) - c(.,T_d)}{T_u - T_d}\Big\} + \frac{T_u' - T_d'}{T_u - T_d}\,\frac{\psi_*(T_u) - \psi_*(T_d)}{T_u - T_d} + \frac{T_d'\big[c_y(.,T_d) - h_*\big] - T_u'\big[c_y(.,T_u) - h_*\big]}{T_u - T_d}$$

which leads by direct calculation:

$$h'_* \;=\; \frac{c_x(.,T_u)-c_x(.,T_d)}{T_u-T_d} \;\text{ on } D^c. \tag{5.7}$$

This determines h^* on D up to irrelevant constants. By evaluating the second equation in (5.5) at a point $T_d^{-1}(x)\in D$, it follows from (5.6) that:

$$c_y(x,x)-h_*(x) \;=\; c_y(T_d^{-1}(x),x)-h_*\circ T_d^{-1}(x), \quad x\in D. \tag{5.8}$$

Since T_d and T_u take values in D and D^c, respectively, and h_* is determined by (5.8) on D, we see that $h_*|_{D^c}$ is determined by (5.7), and equation (5.5) determines ψ_* on $\mathbb{R}$.

5.2 Main result

The previous formal derivation suggest the following candidate functions for the semi-static hedging strategy. Up to a constant, the dynamic hedging component h_* is defined on each continuity point by:

$$h'_*=\frac{c_x(.,T_u)-c_x(.,T_d)}{T_u-T_d} \text{ on } D^c,\;\; h_*=h_*\circ T_d^{-1}+c_y(.,.)-c_y(T_d^{-1},.) \text{ on } D. \tag{5.9}$$

The payoff function ψ_* is defined up to a constant on each continuity interval by:

$$\psi'_*=c_y(T_u^{-1},.)-h_*\circ T_u^{-1} \;\;\text{ on }\;\; D^c, \;\;\; \psi'_*=c_y(T_d^{-1},.)-h_*\circ T_d^{-1} \;\;\text{ on }\;\; D. \tag{5.10}$$

The corresponding function φ_* is given by:

$$\begin{aligned}\varphi_*(x) \;&=\; \mathbb{E}^{\mathbb{P}_*}\big[c(X,Y)-\psi_*(Y)|X=x\big] \qquad (5.11)\\ &=\; q(x)\big(c(x,.)-\psi_*\big)\circ T_u(x)+\big(1-q(x)\big)\big(c(x,.)-\psi_*\big)\circ T_d(x), \quad x\in\mathbb{R}.\end{aligned}$$

Finally, we define h_* and ψ_* from (5.9)-(5.10) by imposing that

$$\text{the function } c(.,T_u)-\psi_*(T_u)-[c(.,T_d)-\psi_*(T_d)]-(T_u-T_d)h \text{ is continuous.} \tag{5.12}$$

The last requirement is obviously possible as the number of jumps of T_d and T_u is finite, due to our assumption that $\mathbf{M}_0(\delta F)$ is finite. Indeed, (5.12) determines $\psi_*(T_u)$ from $\psi_*(T_d)$ at discontinuity points, from left to right.

Theorem 5.1. *Let (μ,ν) be an irreducible pair (w.l.o.g.) satisfying Assumptions 3.5 and 3.7. Assume further that $\varphi_*^+\in\mathbb{L}^1(\mu)$, $\psi_*^+\in\mathbb{L}^1(\nu)$, and that the partial derivative of the coupling function c_{xyy} exists and $c_{xyy}>0$ on $\mathbb{R}\times\mathbb{R}$. Then:*
(i) $(\varphi_*,\psi_*,h_*)\in\mathcal{D}_2$,
(ii) *the strong duality holds for the martingale transportation problem, $\mathbb{P}_*$ is a solution of $P_2(\mu,\nu)$, and (φ_*,ψ_*,h_*) is a solution of $D_2(\mu,\nu)$:*

$$\int c\big(x,T_*(x,dy)\big)\mu(dx) \;=\; \mathbb{E}^{\mathbb{P}_*}\big[c(X,Y)\big] \;=\; P_2(\mu,\nu) \;=\; D_2(\mu,\nu) \;=\; \mu(\varphi_*)+\nu(\psi_*).$$

Remark 5.2 (Symmetry: the right-monotone martingale transport plan)**.**
(i) *Suppose that $c_{xyy} < 0$. Then, the upper bound $P_2(\mu, \nu)$ is attained by the right-monotone martingale transport map*

$$\bar{\mathbb{P}}_*(dx, dy) := \bar{\mu}(dx)\bar{T}_*(x, dy),$$

where $\bar{T}_*$ is defined as in (4.5) with the pair of probability measures $(\bar{\mu}, \bar{\nu})$:

$$F_{\bar{\mu}}(x) := 1 - F_\mu(-x), \quad \textit{and} \quad F_{\bar{\nu}}(y) := 1 - F_\nu(-y).$$

To see this, we rewrite the optimal transportation problem equivalently with modified inputs:

$$\bar{c}(x, y) := c(-x, -y), \quad \bar{\mu}\big((-\infty, x]\big) := \mu\big([-x, \infty)\big), \quad \bar{\nu}\big((-\infty, y]\big) := \nu\big([-y, \infty)\big),$$

so that $\bar{c}_{xyy} > 0$, as required in Theorem 5.1. Note that the martingale constraint is preserved by the map $(x, y) \to (-x, -y)$.
(ii) *Suppose that $c_{xyy} > 0$. Then, the lower bound problem is explicitly solved by the right-monotone martingale transport plan. Indeed, it follows from the first part (i) of the present remark that:*

$$\inf_{\mathbb{P} \in \mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X, Y)\big] = -\sup_{\mathbb{P} \in \mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[-c(X, Y)\big] = \mathbb{E}^{\bar{\mathbb{P}}_*}\big[c(X, Y)\big] = \int c\big(x, \bar{T}_*(x, dy)\big)\mu(dx).$$

Remark 5.3. *The martingale counterpart of the Spence-Mirrlees condition is $c_{xyy} > 0$. We now argue that this condition is the natural requirement in the present setting. Indeed, the optimization problem is not affected by the modification of the coupling function from c to $\bar{c}(x, y) := c(x, y) + a(x) + b(y) + h(x)(y - x)$ for any $a \in \mathbb{L}^1(\mu)$, $b \in \mathbb{L}^1(\nu)$, and $h \in \mathbb{L}^0$. Since $c_{xyy} = \bar{c}_{xyy}$, it follows that the condition $c_{xyy} > 0$ is stable for the above transformation of the coupling function.*

Remark 5.4 (*Comparison with Beiglböck and Juillet [4]*)**.** The remarkable notion of left-monotone martingale transport was introduced by Beiglböck and Juillet [4], where existence and uniqueness is proved.

1. We first show that their conditions on the coupling function fall in the context of our Theorem 5.1:

   - The first class of coupling functions considered in [4] is of the form $c(x, y) = h(y - x)$ for some differentiable function h whose derivative is strictly concave. Notice that this form of coupling essentially falls under our condition $c_{xyy} > 0$.
   - The second class of coupling functions considered in [4] is of the form $c(x, y) = \psi(x)\phi(y)$ where ψ is a non-negative decreasing function and ϕ a non-negative strict concave function. This class also essentially falls under our condition that $c_{xyy} > 0$.

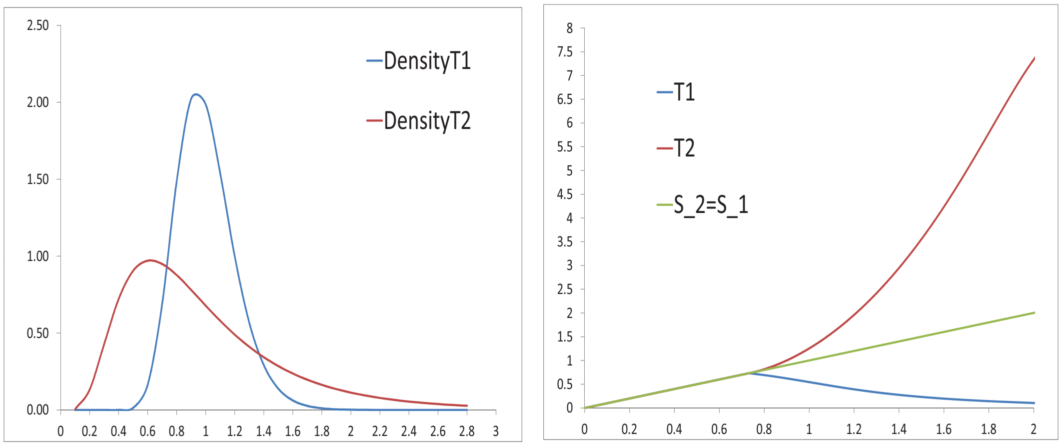

Figure 1: Maps T_d and T_u built from two log-normal densities with variances 0.04 and 0.32. $m_1 = 0.731$.

2. The proof of [4] does not use the dual formulation of the martingale optimal transport problem. They rather extend the concept of cyclical monotonicity to the martingale context, and provide an existence result without explicit characterization of the maps (T_d, T_u). Also, our derivation of the optimal semi-static hedging strategy (φ_*, ψ_*, h_*) is new. We recall however that the result of [4] does not require our Assumption 3.7.

3. Our construction agrees with the example of two Log-normal distributions $\mu_0 = e^{\mathcal{N}(-\sigma_1^2/2,\sigma_1^2)}$ and $\nu_0 = e^{\mathcal{N}(-\sigma_2^2/2,\sigma_2^2)}$, $\sigma_1^2 < \sigma_2^2$, illustrated in Figure 2 of [4]. By using our construction, we reproduce the left-monotone transference map in Figure 1. Indeed, in this case, $x_0 = -\infty$, δF has a unique local (and therefore global) maximizer m_1 of δF, and $x_1 = \infty$. The left-monotone transport plan is explicitly obtained from our construction after Step 1, i.e. no further steps are needed in this case.

Example 5.5. *We provide an example where δF has two local maxima and the construction needs two steps. Let μ and ν be defined by*

$$\mu_1 = \mathcal{N}(1, 0.5) \quad \textit{and} \quad \nu_1(x) = \frac{1}{3}\big[\mathcal{N}(1,2) + \mathcal{N}(0.6, 0.1) + \mathcal{N}(1.4, 0.3)\big].$$

Clearly μ and ν have mean 1, and $\mu \preceq \nu$. We also immediately check that δF has two local maxima $m_1 = -0.15$ and $m_2 = 0.72$. Figure 2 below reports the maps T_u and T_d as obtained from our construction.

Remark 5.6 (*Comparison with Hobson and Neuberger [32]*)**.** Our Theorem 5.1 does not apply to the coupling function $c(x, y) = |x - y|$ considered by Hobson and Neuberger [32]. More importantly, the corresponding maps T_u^{HN} and T_d^{HN} introduced in [32] are both nondecreasing with $T_d^{\mathrm{HN}}(x) < x < T_u^{\mathrm{HN}}(x)$ for all $x \in \mathbb{R}$. So our solution (T_d, T_u) is of a different

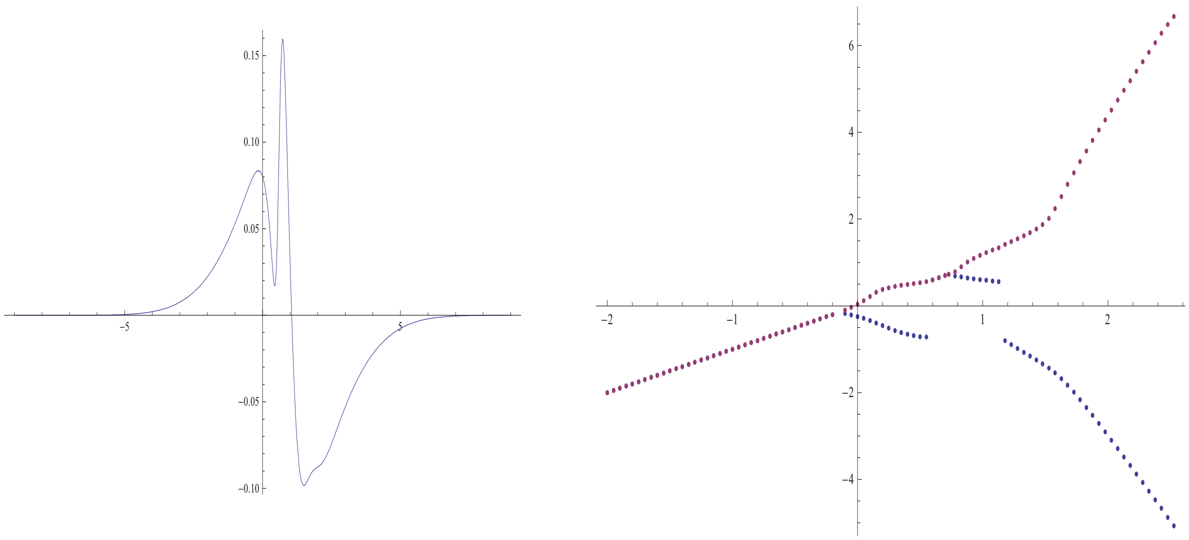

Figure 2: δF has two local maxima (left), and T_d, T_u corresponding to μ_1, ν_1 (right).

nature and in contrast with the above $(T_d^{\mathrm{HN}}, T_u^{\mathrm{HN}})$, our left-monotone martingale transport map T_* does not depend on the nature of the coupling function c as long as $c_{xyy} > 0$.

However, by following the line of argument of the proof of Theorem 5.1, we may recover the solution of Hobson and Neuberger [32]. As a matter of fact, our method of proof is similar to that of [32], as the dual problem D_2 is exactly the Lagrangian obtained by the penalization of the objective function by Lagrange multipliers.

5.3 Some examples

Example 5.7 (Variance swap). *The coupling function in this case is* $c(x,y) = \big(\ln(y/x)\big)^2$ *where* μ *and* ν *have support in* $(0,\infty)$. *In particular, it satisfies the requirement of Theorem 5.1 that* $c_{xyy} > 0$. *Then, the optimal upper bound is given by*

$$P_2(\mu,\nu) \;=\; \int_0^\infty \Big[q(x)\Big(\ln\frac{T_u(x)}{x}\Big)^2 + (1-q)(x)\Big(\ln\frac{T_d(x)}{x}\Big)^2\Big]\mu(dx), \qquad (5.13)$$

where q *is set to an arbitrary value on* D. *In Figure 3, we have plotted* φ_*, ψ_* *and* h_* *with marginal distributions* $\mu_0 = e^{\mathcal{N}(-\sigma_1^2/2,\sigma_1^2)}$ *and* $\nu_0 = e^{\mathcal{N}(-\sigma_2^2/2,\sigma_2^2)}$, $\sigma_1^2 = .04 < \sigma_2^2 = .32$. *We recall that the corresponding maps* T_d, T_u *are plotted in Figure 1. The expression for* ψ_* *is*

$$\psi_*'(x) = \frac{2}{x}\ln\left(\frac{x}{T_u^{-1}(x)}\right) + 2\int_{x_0}^{T_u^{-1}(x)} \frac{\ln\left(\frac{T_u(\xi)}{T_d(\xi)}\right)}{\xi(T_u(\xi)-T_d(\xi))}d\xi.$$

In particular, $\psi_*''(x) = \frac{2}{x^2}$ *for all* $x \le m_1$.

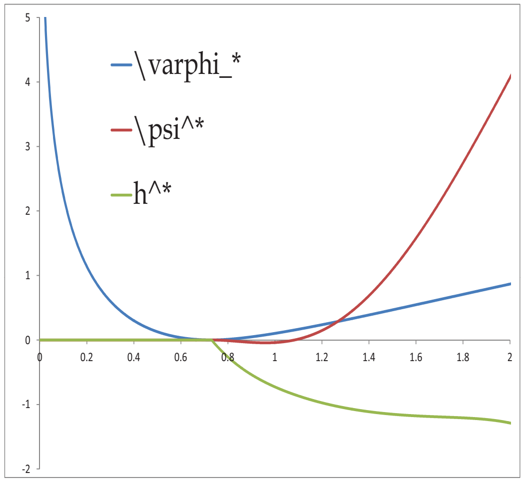

Figure 3: Superreplication strategy for a 2-period variance swap given two log-normal densities with variances 0.04 and 0.32.

Example 5.8. *Consider the coupling function* $c(x,y) = -\left(\frac{y}{x}\right)^p$, $p > 1$, *and let the measures* μ, ν *be supported in* $(0,\infty)$. *This payoff function also satisfies the condition of Theorem 5.1 that* $c_{xyy} > 0$. *The best upper bound is then given by*

$$P_2(\mu,\nu) = -\int_0^\infty \left[q(x)\left(\frac{T_u(x)}{x}\right)^p + (1-q)(x)\left(\frac{T_d(x)}{x}\right)^p\right]\mu(dx).$$

6 The $n-$Marginals Martingale Transport

In this section, we provide a direct extension of our results to the martingale transportation problem under finitely many marginals constraint. Fix an integer $n \geq 2$, and let $X = (X_1,\ldots,X_n)$ be a vector of n random variables denoting the prices of some financial asset at dates $t_1 < \ldots < t_n$. Consider the probability measures $\mu = (\mu_1,\ldots,\mu_n) \in (\mathcal{P}_\mathbb{R})^n$ with $\mu_1 \preceq \ldots \preceq \mu_n$ in the convex order and

$$\int |\xi|\mu_i(d\xi) < \infty \quad \text{and} \quad \int \xi\mu_i(d\xi) = X_0, \quad \text{for all} \quad i = 1,\ldots,n.$$

Similar to the two-marginals case, we introduce the set

$$\mathcal{M}_n(\mu) \;:=\; \left\{\mathbb{P} \in \mathcal{P}_n(\mu) : X \text{ is a } \mathbb{P}-\text{martingale}\right\},$$

where $\mathcal{P}_n(\mu)$ was defined in (2.6). In the present martingale version, we introduce the one-step ahead martingale transport maps defined by means of the n pairs of maps (T^i_d, T^i_u):

$$T^i_*(x_i,.) \;:=\; \mathbf{1}_{D_i}\delta_{\{x_i\}} + \mathbf{1}_{D_i^c}\left(q_i(x_i)\delta_{T^i_u(x_i)} + (1-q_i)(x_i)\delta_{T^i_d(x_i)}\right), \tag{6.1}$$

where $q_i(\xi) := (\xi - T^i_d(\xi))/(T^i_u - T^i_d)(\xi)$ for $\xi \in D^c_i$, and $(D_i, T^i_d, T^i_u)_{i=1,\dots,n-1}$ are defined as in Subsection 4.2 with the pair (μ_i, μ_{i+1}).

The $n-$marginals martingale transport problem is defined by:

$$P_n(\mu) = \sup_{\mathbb{P}\in\mathcal{M}_n(\mu)} \mathbb{E}^{\mathbb{P}}[c(X)],$$

where the map $c : \mathbb{R}^n \longrightarrow \mathbb{R}$ is of the form

$$c(x_1, \dots, x_n) = \textstyle\sum_{i=1}^{n-1} c^i(x_i, x_{i+1})$$

for some upper semicontinuous functions $c^i : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ with linear growth (or Condition (2.2)), $i = 1, \dots, n-1$.

The dual problem is defined by

$$D_n(\mu) \;:=\; \inf_{(u,h)\in\mathcal{D}_n} \sum_{i=1}^{n} \mu_i(u_i),$$

where $u = (u_1, \dots, u_n)$ with components $u^i : \mathbb{R} \longrightarrow \mathbb{R}$, and $h = (h_1, \dots, h_{n-1})$ with components $h_i : \mathbb{R}^i \longrightarrow \mathbb{R}$, taken from the set of dual variables:

$$\mathcal{D}_n := \big\{(u, h) : (u_i)^+ \in \mathbb{L}^1(\mu_i), h_i \in \mathbb{L}^0(\mathbb{R}^i), \text{ and } \oplus_{i=1}^n u_i + \textstyle\sum_{i=1}^{n-1} h_i^{\otimes^i} \geq c\big\}.$$

Here, $\oplus_{i=1}^n u_i(x) = \sum_{i\leq n} u_i(x_i)$ and $h_i^{\otimes^i}(x) = h_i(x_1, \dots, x_i)(x_{i+1} - x_i)$.

Similar to the two-marginals problems, the weak duality inequality $P_n(\mu) \leq D_n(\mu)$ is obvious, and we shall obtain equality in the following result under convenient conditions.

To derive the structure of the optimal hedging strategy, we shall consider the two-marginals (μ_i, μ_{i+1}) problems with coupling functions c^i. By Theorem 5.1, we have for $i = 1, \dots, n-1$:

$$P^i_2(\mu_i, \mu_{i+1}) := \sup_{\mathbb{P}\in\mathcal{M}(\mu_i,\mu_{i+1})} \mathbb{E}^{\mathbb{P}}[c^i(X, Y)] = \inf_{(\varphi,\psi,h)\in\mathcal{D}^i_2} \{\mu_i(\varphi) + \mu_{i+1}(\psi)\} = \mu_i(\varphi^*_i) + \mu_{i+1}(\psi^*_i),$$

where $\mathcal{D}^i_2$ is defined as in (3.4) with c^i substituted to c, and $(\varphi^*_i, \psi^*_i, h^*_i) \in \mathcal{D}^i_2$ are defined as in (5.9)-(5.10)-(5.11) with c^i substituted to c and (T^i_u, T^i_d) substituted to (T_u, T_d). Finally, we define:

$$u^*_i(x_i) := \mathbf{1}_{\{i<n\}}\varphi^*_i(x_i) + \mathbf{1}_{\{i>1\}}\psi^*_{i-1}(x_i), \quad i = 1, \dots, n,$$

and $u^* := \big(u^*_1, \dots, u^*_n\big)$, $h^* := \big(h^*_1, \dots, h^*_{n-1}\big)$.

Theorem 6.1. *Let $(\mu_i)_{1\leq i\leq n}$ be probability measures on $\mathbb{R}$ without atoms, with $\mu_1 \preceq \ldots \preceq \mu_n$ in convex order, (μ_{i-1}, μ_i) irreducible, and $\mathbf{M}_0(F_{\mu_i} - F_{\mu_{i-1}})$ finite for all $1 < i \leq n$. Assume further that*

- *c^i have linear growth, that the cross derivatives c^i_{xyy} exist and satisfy $c^i_{xyy} > 0$,*
- *φ^*_i, ψ^*_i satisfy the integrability conditions $(\varphi^*_i)^+ \in \mathbb{L}^1(\mu_i)$, $(\psi^*_i)^+ \in \mathbb{L}^1(\mu_{i+1})$.*

Then, the strong duality holds, the probability measure $\mathbb{P}^*_n(dx) = \mu_1(dx_1) \prod_{i=1}^{n-1} T^i_*(x_i, dx_{i+1})$ on $\mathbb{R}^n$ is optimal for the martingale transportation problem $P_n(\mu)$, and (u^*, h^*) is optimal for the dual problem $D_n(\mu)$, i.e.

$$\mathbb{P}^*_n \in \mathcal{M}_n(\mu), \quad (u^*, h^*) \in \mathcal{D}_n, \quad \text{and} \quad \mathbb{E}^{\mathbb{P}^*_n}[c(X)] = P_n(\mu) = D_n(\mu) = \textstyle\sum_{i=1}^n \mu_i(u^*_i).$$

Proof. Clearly, we have $\mathbb{P}^*_n \in \mathcal{M}_n(\mu)$, which provides the inequality $\mathbb{E}^{\mathbb{P}^*_n}[c(X)] \le P_n(\mu)$. We next observe that $(u^*, h^*) \in \mathcal{D}_n$ from our construction. Then $D_n(\mu) \le \sum_{i\le n}\mu_i(u^*_i) = \mathbb{E}^{\mathbb{P}^*_n}[c(X)]$. The required result follows from the weak duality inequality $P_n(\mu) \le D_n(\mu)$. $\Box$

Remark 6.2. *The optimal lower bound for a coupling function as in Theorem 6.1 is attained by the mirror solution introduced in Remark 5.2.*

Example 6.3 (Discrete monitoring variance swaps)**.** *This is a continuation of our Example 5.7. Suppose that $(\mu_i)_{1\le i\le n}$ have support in $(0,\infty)$ with mean X_0, satisfy the conditions of Theorem 6.1. Let $c(x_1,\dots,x_n) := \sum_{i=1}^n \big(\ln \frac{x_i}{x_{i-1}}\big)^2$. Then:*

$$P_n(\mu) = \int \Big(\ln \frac{\xi}{X_0}\Big)^2 \mu_1(d\xi) + \sum_{i=1}^{n-1}\int_0^\infty \Big[q_i(\xi)\Big(\ln \frac{T^i_u(\xi)}{\xi}\Big)^2 + (1-q_i)(\xi)\Big(\ln \frac{T^i_d(\xi)}{\xi}\Big)^2\Big]\mu_i(d\xi).$$

This optimal bound depends on all the marginals. The optimal lower bound is attained by our mirror solution, see Remark 6.2.

Remark 6.4. *In a related robust hedging problem, Hobson and Klimmek [31], derived an optimal upper bound for a derivative $c(x_1,\dots,x_n) = \sum_{i=1}^{n-1} c^0(x_i,x_{i+1})$. The difference with our problem above is that they are only given the marginal distribution μ_n for X_n. See also Kahale [36]. We would like to emphasize that [31] assume the variance Kernel c^0 to satisfy the conditions $c^0(x,x) = c^0_y(x,x) = 0$, $(x-y)c^0_{xy} + c^0_x > 0$, together with our Spence-Mirrlees condition $c^0_{xyy} > 0$. In the context of our problem with finitely many given marginals $\mu_1,\dots,\mu_n$, notice that, apart from the Spence-Mirrlees condition, none of these requirements are preserved by the transformation of Remark 5.3.*

7 Proof of the main results

7.1 Construction of the left-monotone map

This section is devoted to the proof of Theorem 4.5.

Proof of Lemma 4.1 (i) That $\delta F > 0$ on A_1 is obvious by construction. Also, for a non-decreasing function ϕ, the equality $\int_{-\infty}^{m_1} \mathbf{1}_{A_1} d\phi(\delta F) = \int_{-\infty}^{m_1} d\phi(\delta F)$ follows immediately from the corresponding property verified by the pair (m_0, A_0), the definition of A_1, and the fact that $\delta F(t_0) = \delta F(\bar{x}_0)$.

We next verify that $G^{m_1}_{A_1}(-\infty, .) = \gamma$, where

$$G^{m_1}_{A_1}(-\infty, x) = -\int_{-\infty}^{m_1}\big[g(x,\xi) - \xi\big]1_{A_1}(\xi) d\delta F + \int_{m_1}^{x}\big[g(\xi, m_1) - \xi\big]dF_\mu(\xi).$$

By direct differentiation, we see that

$$dG^{m_1}_{A_1}(-\infty, x) = \Big(-\int_{-\infty}^{m_1}\mathbf{1}_{A_1}(\zeta)d_\zeta g(x,\zeta) + g(x,m_1) - x\Big)dF_\mu(x) = \big[F_\nu^{-1}\circ F_\mu(x) - x\big]dF_\mu(x),$$

where the last equality follows from the first part of (i). We then re-write

$$G_{A_1}^{m_1}(-\infty, x) \;=\; G_{A_1}^{m_1}(-\infty, \bar{x}_0) + \int_{\bar{x}_0}^{x} \big[F_\nu^{-1} \circ F_\mu(\xi) - \xi\big] dF_\mu(\xi). \tag{7.1}$$

Since $A_1 = \big(A_0 \setminus (t_0, m_0]\big) \cup [\bar{x}_0, m_1]$, and $G_{A_0}^{m_0}(t_0, \bar{x}_0) = 0$, we compute that

$$\begin{aligned}
G_{A_1}^{m_1}(-\infty, \bar{x}_0) &= -\int_{-\infty}^{t_0} \big[g(\bar{x}_0, \zeta) - \zeta\big] \mathbf{1}_{A_0}(\zeta) d\delta F(\zeta) + G_{A_0}^{m_0}(t_0, \bar{x}_0) \\
&\quad - \int_{\bar{x}_0}^{m_1} \big[g(\bar{x}_0, \zeta) - \zeta\big] d\delta F(\zeta) + \int_{m_1}^{\bar{x}_0} \big[g(\xi, m_1) - \xi\big] dF_\mu(\xi) \\
&= G_{A_0}^{m_0}(-\infty, \bar{x}_0) - \int_{\bar{x}_0}^{m_1} \big[g(\bar{x}_0, \zeta) - \zeta\big] d\delta F(\zeta) + \int_{m_1}^{\bar{x}_0} \big[g(\xi, m_1) - \xi\big] dF_\mu(\xi) \\
&= G_{A_0}^{m_0}(-\infty, \bar{x}_0) - \int_{\bar{x}_0}^{m_1} g(\bar{x}_0, \zeta) d\delta F(\zeta) + \int_{m_1}^{\bar{x}_0} g(\xi, m_1) dF_\mu(\xi) + \int_{\bar{x}_0}^{m_1} \zeta dF_\nu(\zeta) \\
&= G_{A_0}^{m_0}(-\infty, \bar{x}_0),
\end{aligned}$$

where the last equality follows from direct change of variables in the second and third terms. Plugging this in (7.1), it follows from direct change of variable in the integral that

$$G_{A_1}^{m_1}(-\infty, x) \;=\; G_{A_0}^{m_0}(-\infty, x) \;=\; \gamma(x)$$

(ii) Since $m_1 \in \mathbf{M}_0(\delta F)$, it follows from the definition of $\mathbf{M}_0(\delta F)$ that $F_\mu(x) > F_\mu(\zeta)$ for all $x > m_1$ and $\zeta \in A_1$. Since ν has no atoms, its right-continuous inverse F_ν^{-1} is strictly increasing, implying that $g(x, \zeta) - \zeta > g(\zeta, \zeta) - \zeta = F_\nu^{-1} \circ F_\nu(\zeta) - \zeta$. Moreover, since δF is strictly increasing on A_1, we see that F_ν is strictly increasing in A_1, and therefore $F_\nu^{-1} \circ F_\nu(\zeta) = \zeta$. Hence, $g(x, \zeta) - \zeta > 0$ on A_1, and it follows that, for $t < m_1 \le x$,

$$t \longmapsto G_{A_1}^{m_1}(t, x) \quad \text{is continuous, strictly increasing on } A_1, \text{ and flat on } (-\infty, m_1] \setminus A_1.$$

We next verify that $G_{A_1}^{m_1}(m_1, x) := \int_{m_1}^{x} \big[g(\xi, m_1) - \xi\big] dF_\mu(\xi) > 0$ as long as $\delta F(m_1) > \delta F(x)$. Indeed, for $\xi \in (m_1, x)$, we have $\delta F(m_1) > \delta F(\xi)$, implying that $g(\xi, m_1) > F_\nu^{-1} \circ F_\nu(\xi)$ by the increase of F_ν^{-1}. Notice that the right-continuous inverse F_ν^{-1} satisfies $F_\nu^{-1} \circ F_\nu(\xi) \ge \xi$. Then $g(\xi, m_1) > \xi$, and we deduce that $G_{A_1}^{m_1}(m_1, x) > 0$ from the fact that F_μ is strictly increasing on a right neighborhood of m_1, by the definition of $\mathbf{M}_0(\delta F)$.

Then, in order to establish the existence and uniqueness of $t_{A_1}^{m_1}(x)$, it remains to verify that $G_{A_1}^{m_1}(-\infty, x) = \gamma(x) < 0$ for all $x \ge m_1$.

Since δF increases from zero at the left extreme of to support, and increases to zero at the right extreme of its support, we see that $\gamma(x) < 0$ near both extremes of its support. Next, let x^* be any possible local maximizer of γ. Then, it follows from the first order condition in the expression (7.1) that γ is flat off $\mathrm{Supp}(\mu)$, and we may assume that x^* is either an interior point of $\mathrm{Supp}(F_\mu)$ or x^* is a left accumulation point of $\mathrm{Supp}(F_\mu)$. In both cases, it follows from the first order condition that

$$F_\nu^{-1}\big(F_\mu(x^*) -\big) \le x^* \le F_\nu^{-1}\big(F_\mu(x^*)\big).$$

If F_ν^{-1} is continuous at the point $F_\mu(x^*)$, then $\delta F(x^*) = 0$, and it follows that

$$\gamma(x^*) = \int_{(-\infty,x^*]} \xi d\delta F(\xi) = -\int_{(-\infty,x^*]} (x^* - \xi) d\delta F(\xi) = -\int (x^* - \xi)^+ d\delta F(\xi).$$

By the fact that the pair (μ, ν) is irreducible, it follows from (3.9) that $\gamma(x^*) < 0$.

In the alternative case that F_ν^{-1} jumps at the point $F_\mu(x^*)$, notice that F_ν is flat at the right of $F_\nu^{-1} \circ F_\mu(x^*)$, and therefore the conclusion $\gamma(x^*) < 0$ holds true in this case as well.
(iii) Direct differentiation reveals that

$$dG_{A_1}^{m_1}\big(t_{A_1}^{m_1}(x), x\big) \;\; = \;\; -\big[g(\big(t_{A_1}^{m_1}(x), x\big) - t_{A_1}^{m_1}(x)\big] d\big\{\delta F \circ t_{A_1}^{m_1}\big\}(x) + \big[g(x, m_1) - x\big] dF_\mu(x).$$

The required result follows immediately from the restriction of $t_{A_1}^{m_1}(x)$ to take values in a set of increase of $\delta\, F$.
(iv) Suppose $x_1 < \infty$. Then, since the possible jumps of F_ν^{-1} are positive, it follows from the definition of x_1 that $g\big(x_1, t_{A_1}^{m_1}(x_1)\big) = x_1$, and $F_\mu(x_1) + \delta F\big(t_{A_1}^{m_1}(x_1)\big)$ is a continuity point of F_ν^{-1}. Consequently, $\delta F\big(t_{A_1}^{m_1}(x_1)\big) = \delta F(x_1)$, and

$$x_1 \;\; = \;\; \inf \big\{x > m_1 : \;\delta F\big(t_{A_1}^{m_1}(x)\big) \leq \delta F(x)\big\}. \tag{7.2}$$

Since $t_1 := t_{A_1}^{m_1}(x_1) \in A_1$, we see that $x_1 \in B_0$ is necessarily a point of (right-)increase of δF, and we have

- either $t_1 \in [\bar{x}_0, m_1]$, implying that $\delta F(x_1) = \delta F(t_1) \geq \delta F(\bar{x}_0) \geq 0$,
- or $t_1 \in A_0 \setminus (t_0, m_0]$, implying again that $\delta F(x_1) \geq 0$.

Finally, since δF increases to zero at the right extreme of its support, it follows from the fact that $x_1 \in B_0$ and $\delta F(x_1) \geq 0$ that $x_1 \leq m_n$, and by (7.2) together with the non-increase of $t_{A_1}^{m_1}$, we see that $x_1 \notin \mathbf{M}_0(\delta F)$. □

Proof of Theorem 4.5 (i) By construction, the probability measure $\mathbb{P}_*$ satisfies the left-monotonicity property of Definition 3.4. In the rest of this proof, we verify that $\mathbb{P}_* \in \mathcal{M}_2(\mu, \nu)$. In particular, by the uniqueness result of Beiglböck and Juillet [4] (Theorem 1.5 and Corollary 1.6), this would imply that $\mathbb{P}_*$ is the unique left monotone transport plan.

First, by the definition of $\mathbb{P}_*$ in (4.5), $X \sim_{\mathbb{P}_*} \mu$, and $\mathbb{E}^{\mathbb{P}_*}[Y|X] = X$. It remains to verify that $Y \sim_{\mathbb{P}_*} \nu$. We argue as in the beginning of Section 7.1 considering separately the following alternatives for any point $y \in \mathbb{R}$:
Case 1: $y = y_d \in D \cap B_0$ corresponds to some point x such that $y_d = T_d(x)$, and we see from the definition of $\mathbb{P}_*$ that:

$$\mathbb{P}_*[Y \in dy] = dF_\mu\big(T_d(x)\big) - (1-q) dF_\mu(x) \quad \text{and} \quad dF_\nu\big(T_u(x)\big) = q dF_\mu.$$

Since $dF_\nu(T_u) = q dF_\mu$, and $T_u(x) = g(x, T_d(x))$, this provides:

$$\mathbb{P}_*[Y \in dy] = d\{F_\mu(T_d) - F_\mu + F_\nu(T_u)\}(x) = dF_\nu(y).$$

Case 2: $y = y_u \in D^c$ corresponds to some x such that $y_u = T_u(x)$. By the definition of $\mathbb{P}_*$, and the fact that $dF_\nu(T_u) = qdF_\mu$, we see that:

$$\mathbb{P}_*[Y \in dy] = qdF_\mu(x) = dF_\nu(y).$$

Case 3: In the remaining alternative $y \in D \setminus B_0$, we observe that the function δF is flat near y, and there is no $x \neq y$ such that $T_d(x) = y$ or $T_u(x) = y$. Then, it follows from the definition of $\mathbb{P}_*$ that:

$$\mathbb{P}_*[Y \in dy] = dF_\mu(y) = dF_\nu(y).$$

(ii) Differentiating the integral equation defined by G^A at a continuity point of T_d, we see that:

$$\begin{aligned} 0 &= -\big[F_\nu^{-1} \circ F_\mu(x) - x\big]dF_\mu(x) + \big[g(x, T_d(x)) - F_\nu^{-1} \circ F_\mu(x)\big]dF_\mu(x) \\ &\quad + \big[g(x, T_d(x)) - T_d(x)\big]d\delta F(T_d(x)) \\ &= \big[g(x, T_d(x)) - x\big]dF_\mu(x) + \big[g(x, T_d(x)) - T_d(x)\big]d\delta F(T_d(x)). \end{aligned}$$

Since $T_u = g(., T_d)$ this is the required ODE. The ODE for T_u is obtained by using the relation $T_u = g(., T_d)$. □

7.2 Optimal semi-static strategy: proof of Theorem 5.1

Following the line of argument of the proof of Theorem 2.2, we see from the weak duality (3.6) that

$$\mathbb{E}^{\mathbb{P}_*}[c(X, Y)] \leq P_2(\mu, \nu) \leq D_2(\mu, \nu).$$

Then, the proof of Theorem 5.1 is completed by the following result.

Lemma 7.1. *Let μ, ν be as in Assumptions 3.5 and 3.7, and suppose that the payoff function c satisfies $c_{xyy} > 0$. Then $\varphi_* \oplus \psi_* + h_*^{\otimes} \geq c$.*

Proof (i) We first verify that the second order condition for a local maximum of $H(x, .)$ is satisfied on D^c. Differentiating (5.5), and using the expression of h'_* in (5.9), we see that

$$H_{yy}(., T_u)dT_u = c_{yy}(., T_u)dT_u - d\psi'_*(T_u) = \frac{c_x(., T_u) - c_x(., T_d)}{T_u - T_d}dx - c_{xy}(., T_u)dx$$

on D^c. Since $c_{xyy} > 0$, this implies that $H_{yy}(., T_u)T'_u = \frac{c_x(.,T_u) - c_x(.,T_d)}{T_u - T_d} - c_{xy}(., T_u) < 0$, and by the non-decrease of T_u, it follows that $H_{yy}(., T_u) < 0$. Similarly,

$$H_{yy}(., T_d)T'_d = \big[c_{yy}(., T_d) - \psi''_* \circ T_d\big]T'_d = \frac{c_x(., T_u) - c_x(., T_d)}{T_u - T_d} - c_{xy}(., T_d) > 0,$$

on D^c, and by the non-increase of T_d, this implies that $H_{yy}(.,T_d)<0$.
(ii) We next show that $y\longmapsto H(.,y)$ is increasing before T_d, and decreasing after T_u. In particular, this implies that:

$$\varphi_*(x)=\max_{y\in[T_d(x),T_u(x)]}H(x,y)\quad\text{for all}\quad x\in\mathbb{R}.$$

Set $y:=T_u(x)$, let m_i be the local maximum from which $(T_d,T_u)(x)$ is constructed, and consider an arbitrary $y'=T_u(x')>y$ for some $x'>x$. We only report the proof for the case $x'\in(m_j,x_j]$ for some $j\geq i$; the remaining case $x'\in(x_j,m_{j+1}]$ for some $j\geq i$ is treated similarly. Recalling that $H_y(x,T_u(x))=0$, we decompose

$$H_y(x,y') \;=\; H_y(x,y')-H_y(x,m_j\vee y)+\sum_{i+1}^{j}(A_k+B_k),$$

where the last sum is set to zero whenever $i=j$, and

$$A_k:=H_y(x,m_k)-H_y(x,x_{k-1}),\quad B_k:=H_y(x,x_{k-1})-H_y(x,m_{k-1}\vee T_u(x)).$$

We next compute from the expression of h_* in (5.9) that:

$$\begin{aligned}
H_y(x,y')-H_y(x,m_j\vee y) &= \int_{m_j\vee y}^{y'}\big[c_{yy}(x,\xi')-\psi_*''(\xi')\big]d\xi'\\
&\leq \int_{m_j\vee y}^{y'}\big[c_{yy}(x,\xi')-c_{yy}(T_u^{-1}(\xi'),\xi')\big]d\xi'\\
&= -\int_{m_j\vee y}^{y'}\int_x^{T_u^{-1}(\xi')}c_{xyy}(\xi,\xi')d\xi d\xi'<0,
\end{aligned}$$

where the second inequality follows from the second order condition verified in (i). Similarly, we compute that

$$\begin{aligned}
A_k &= \int_{(x_{k-1},m_k]}\big[c_{yy}(x,\xi')-\psi_*''(\xi')\big]d\xi'\\
&\leq \int_{x_{k-1}}^{m_k}\big[c_{yy}(x,\xi')-c_{yy}(T_d^{-1}(\xi'),\xi')\big]d\xi'\\
&= -\int_{x_{k-1}}^{m_k}\int_x^{T_d^{-1}(\xi')}c_{xyy}(\xi,\xi')d\xi d\xi'<0,
\end{aligned}$$

where we used again the second order condition verified in (i). Finally,

$$\begin{aligned}
B_k &= \int_{m_{k-1}\vee T_u(x)}^{y}\big[c_{yy}(x,\xi')-\psi_*''(\xi')\big]d\xi'\\
&\leq \int_{m_{k-1}\vee T_u(x)}^{y}\big[c_{yy}(x,\xi')-c_{yy}(T_u^{-1}(\xi'),\xi')\big]d\xi'\\
&= -\int_{m_{k-1}\vee T_u(x)}^{y}\int_x^{T_u^{-1}(y')}c_{xyy}(\xi,\xi')d\xi d\xi'<0.
\end{aligned}$$

A similar argument also shows that $H_y(x,y') < 0$ for $y' < T_d(x)$.
(iii) We next show that $H(.,T_d) = H(.,T_u)$. Denote $\delta H := H(.,T_u) - H(.,T_d)$, and compute:

$$\begin{aligned}\delta H' &:= c_x(.,T_u) - c_x(.,T_d) - (T_u - T_d)h'_* \\ &\quad + \big[c_y(.,T_u) - \psi'_*(T_u) - h_*\big]T'_u - \big[c_y(.,T_d) - \psi'_*(T_d) - h_*\big]T'_d\end{aligned}$$

in the distribution sense. By definition of ψ_* and h_*, it follows that $\delta H' = 0$ at any continuity point. Since δH is continuous by our construction, see (5.12), this shows that $\delta H(x) = \delta H(m_i) = 0$, where m_i is the local maximizer from which $(T_d, T_u)(x)$ is defined.
(iv) We finally show that T_u and T_d are global maximizers of $y \longmapsto H(.,y)$. Let $x \in D^c$, and denote by m the local maximizer from which $T_d(x)$ and $T_u(x)$ are constructed. For fixed $T = T_u(t) \in \big(m, T_u(x)\big)$, it follows from similar calculations as in the previous step that

$$\begin{aligned}\partial_x\big\{H(.,T_u) - H(.,T)\big\} &= c_x(.,T_u) - c_x(.,T) - (T - T_d)h'_* \\ &= (T_u - T)\Big(\frac{c_x(.,T_u) - c_x(.,T)}{T_u - T} - \frac{c_x(.,T_u) - c_x(.,T_d)}{T_u - T_d}\Big) > 0\end{aligned}$$

by the condition $c_{xyy} > 0$. Then $H(.,T_u) - H(.,T) = \int_t^{.} \partial_x\big\{H(.,T_u) - H(.,T)\big\} > 0$.
By a similar calculation, we also show that $H\big(x, T_d(x)\big) - H(x,T) \geq 0$ for all $T \in (T_d(x), m)$. Since $H(x,T_u(x)) = H(x,T_d(x))$ by the previous step, this completes the proof that T_d and T_u are global maximizers of $y \longmapsto H(.,y)$. □

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