# STOCHASTIC CONTROL,

# AND APPLICATION TO FINANCE

Nizar Touzi
nizar.touzi@polytechnique.edu

Ecole Polytechnique Paris
Département de Mathématiques Appliquées

This version: 10 November 2018

# Contents


**1** CONDITIONAL EXPECTATION AND LINEAR PARABOLIC PDEs 5
- 1.1 Stochastic differential equations with random coefficients . . . . . 5
- 1.2 Markov solutions of SDEs . . . . . . . . . . . . . . . . . . . . . . 10
- 1.3 Connection with linear partial differential equations . . . . . . . 10
  - 1.3.1 Generator . . . . . . . . . . . . . . . . . . . . . . . . . . . 10
  - 1.3.2 Cauchy problem and the Feynman-Kac representation . . 11
  - 1.3.3 Representation of the Dirichlet problem . . . . . . . . . . 13
- 1.4 The stochastic control approach to the Black-Scholes model . . . 14
  - 1.4.1 The continuous-time financial market . . . . . . . . . . . 14
  - 1.4.2 Portfolio and wealth process . . . . . . . . . . . . . . . . . 15
  - 1.4.3 Admissible portfolios and no-arbitrage . . . . . . . . . . . 16
  - 1.4.4 Super-hedging and no-arbitrage bounds . . . . . . . . . . 17
  - 1.4.5 The no-arbitrage valuation formula . . . . . . . . . . . . . 18
  - 1.4.6 PDE characterization of the Black-Scholes price . . . . . . 18

**2** STOCHASTIC CONTROL AND DYNAMIC PROGRAMMING 21
- 2.1 Stochastic control problems in standard form . . . . . . . . . . . 21
- 2.2 The dynamic programming principle . . . . . . . . . . . . . . . . 24
  - 2.2.1 A weak dynamic programming principle . . . . . . . . . . 24
  - 2.2.2 Dynamic programming without measurable selection . . . 26
- 2.3 The dynamic programming equation . . . . . . . . . . . . . . . . 29
- 2.4 On the regularity of the value function . . . . . . . . . . . . . . . 32
  - 2.4.1 Continuity of the value function for bounded controls . . 32
  - 2.4.2 A deterministic control problem with non-smooth value function . . . . . . . . . . . . . . . . . . . . . . . . . . . . 34
  - 2.4.3 A stochastic control problem with non-smooth value function . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 35

**3** OPTIMAL STOPPING AND DYNAMIC PROGRAMMING 37
- 3.1 Optimal stopping problems . . . . . . . . . . . . . . . . . . . . . 37
- 3.2 The dynamic programming principle . . . . . . . . . . . . . . . . 39
- 3.3 The dynamic programming equation . . . . . . . . . . . . . . . . 40
- 3.4 Regularity of the value function . . . . . . . . . . . . . . . . . . . 42
  - 3.4.1 Finite horizon optimal stopping . . . . . . . . . . . . . . . . 42

3.4.2 Infinite horizon optimal stopping . . . . . . . . . . . . . . 43
3.4.3 An optimal stopping problem with nonsmooth value . . . 46

**4** Solving Control Problems by Verification 49
4.1 The verification argument for stochastic control problems . . . . 49
4.2 Examples of control problems with explicit solutions . . . . . . . 52
4.2.1 Optimal portfolio allocation . . . . . . . . . . . . . . . . . 52
4.2.2 Law of iterated logarithm for double stochastic integrals . 54
4.3 The verification argument for optimal stopping problems . . . . . 57
4.4 Examples of optimal stopping problems with explicit solutions . 59
4.4.1 Perpetual American options . . . . . . . . . . . . . . . . . 59
4.4.2 Finite horizon American options . . . . . . . . . . . . . . 60

**5** Introduction to Viscosity Solutions 63
5.1 Intuition behind viscosity solutions . . . . . . . . . . . . . . . . . 63
5.2 Definition of viscosity solutions . . . . . . . . . . . . . . . . . . . 64
5.3 First properties . . . . . . . . . . . . . . . . . . . . . . . . . . . 65
5.3.1 Change of variable / function . . . . . . . . . . . . . . . . 65
5.3.2 Stability of viscosity solutions . . . . . . . . . . . . . . . . 66
5.3.3 Parameter variables . . . . . . . . . . . . . . . . . . . . . 68
5.4 Comparison result and uniqueness . . . . . . . . . . . . . . . . . 69
5.4.1 Comparison of classical solutions in a bounded domain . . 70
5.4.2 Comparison of viscosity solutions of first order equations . 70
5.4.3 Semijets definition of viscosity solutions . . . . . . . . . . 72
5.4.4 The Crandall-Ishii's lemma . . . . . . . . . . . . . . . . . 73
5.4.5 Comparison of viscosity solutions in a bounded domain . 77
5.5 Comparison in unbounded domains . . . . . . . . . . . . . . . . . 79
5.6 Useful applications . . . . . . . . . . . . . . . . . . . . . . . . . . 82
5.7 Proof of the Crandall-Ishii's lemma . . . . . . . . . . . . . . . . . 84

**6** Dynamic Programming Equation in Viscosity Sense 89
6.1 DPE for stochastic control problems . . . . . . . . . . . . . . . . 89
6.2 DPE for optimal stopping problems . . . . . . . . . . . . . . . . 95
6.3 A comparison result for obstacle problems . . . . . . . . . . . . . 97

**7** Stochastic Target Problems 99
7.1 Stochastic target problems . . . . . . . . . . . . . . . . . . . . . . 99
7.1.1 Formulation . . . . . . . . . . . . . . . . . . . . . . . . . . 99
7.1.2 Geometric dynamic programming principle . . . . . . . . 100
7.1.3 The dynamic programming equation . . . . . . . . . . . . 102
7.1.4 Application: hedging under portfolio constraints . . . . . 106
7.2 Stochastic target problem with controlled probability of success . 109
7.2.1 Reduction to a stochastic target problem . . . . . . . . . 109
7.2.2 The dynamic programming equation . . . . . . . . . . . . 110
7.2.3 Application: quantile hedging in the Black-Scholes model 111

**8** Backward SDEs and Stochastic Control 117
8.1 Motivation and examples . . . . . . . . . . . . . . . . . . . . . . 117
8.1.1 The stochastic Pontryagin maximum principle . . . . . . 118
8.1.2 BSDEs and stochastic target problems . . . . . . . . . . . 120
8.1.3 BSDEs and finance . . . . . . . . . . . . . . . . . . . . . 120
8.2 Wellposedness of BSDEs . . . . . . . . . . . . . . . . . . . . . . . 121
8.2.1 Martingale representation for zero generator . . . . . . . . 122
8.2.2 BSDEs with affine generator . . . . . . . . . . . . . . . . 123
8.2.3 The main existence and uniqueness result . . . . . . . . . 124
8.3 Comparison and stability . . . . . . . . . . . . . . . . . . . . . . 126
8.4 BSDEs and stochastic control . . . . . . . . . . . . . . . . . . . . 128
8.5 BSDEs and semilinear PDEs . . . . . . . . . . . . . . . . . . . . 130
8.6 Appendix: essential supremum . . . . . . . . . . . . . . . . . . . 131

# Chapter 1

# Conditional Expectation and Linear Parabolic PDEs

Throughout this chapter, $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ satisfying the usual conditions. Let $W = \{W_t,\ t \geq 0\}$ be a Brownian motion valued in $\mathbb{R}^d$, defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Throughout this chapter, a maturity $T > 0$ will be fixed. By $\mathbb{H}^2$, we denote the collection of all progressively measurble processes $\phi$ with appropriate (finite) dimension such that $\mathbb{E}\left[\int_0^T |\phi_t|^2 dt\right] < \infty$.

## 1.1 Stochastic differential equations with random coefficients

In this section, we recall the basic tools from stochastic differential equations

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t,\ t \in [0,T], \tag{1.1}$$

where $T > 0$ is a given maturity date. Here, $b$ and $\sigma$ are $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}^n)$-progressively measurable functions from $[0,T] \times \Omega \times \mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathcal{M}_{\mathbb{R}}(n,d)$, respectively. In particular, for every fixed $x \in \mathbb{R}^n$, the processes $\{b_t(x), \sigma_t(x), t \in [0,T]\}$ are $\mathbb{F}-$progressively measurable.

**Definition 1.1.** *A strong solution of* (1.1) *is an $\mathbb{F}-$progressively measurable process $X$ such that $\int_0^T (|b_t(X_t)| + |\sigma_t(X_t)|^2)dt < \infty$, a.s. and*

$$X_t = X_0 + \int_0^t b_s(X_s)ds + \int_0^t \sigma_s(X_s)dW_s, \quad t \in [0,T].$$

Let us mention that there is a notion of weak solutions which relaxes some conditions from the above definition in order to allow for more general stochastic differential equations. Weak solutions, as opposed to strong solutions, are defined on some probabilistic structure (which becomes part of the solution), and not necessarily on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$. Thus, for a weak solution we search for a probability structure $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W})$ and a process $\tilde{X}$ such that the requirement of the above definition holds true. Obviously, any strong solution is a weak solution, but the opposite claim is false.

The main existence and uniqueness result is the following.

**Theorem 1.2.** *Let $X_0 \in \mathbb{L}^2$ be a r.v. independent of $W$. Assume that the processes $b_.(0)$ and $\sigma_.(0)$ are in $\mathbb{H}^2$, and that for some $K > 0$:*

$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \leq K|x - y| \quad \textit{for all} \quad t \in [0, T], \ x, y \in \mathbb{R}^n.$$

*Then, for all $T > 0$, there exists a unique strong solution of (1.1) in $\mathbb{H}^2$. Moreover,*

$$\mathbb{E}\left[\sup_{t \leq T} |X_t|^2\right] \quad \leq \quad C\left(1 + \mathbb{E}|X_0|^2\right) e^{CT}, \tag{1.2}$$

*for some constant $C = C(T, K)$ depending on $T$ and $K$.*

*Proof.* We first establish the existence and uniqueness result, then we prove the estimate (1.2).

<u>*Step 1*</u> For a constant $c > 0$, to be fixed later, we introduce the norm

$$\|\phi\|_{\mathbb{H}^2_c} := \mathbb{E}\left[\int_0^T e^{-ct}|\phi_t|^2 dt\right]^{1/2} \quad \text{for every} \quad \phi \in \mathbb{H}^2.$$

Clearly , the norms $\|.\|_{\mathbb{H}^2}$ and $\|.\|_{\mathbb{H}^2_c}$ on the Hilbert space $\mathbb{H}^2$ are equivalent. Consider the map $U$ on $\mathbb{H}^2$ by:

$$U(X)_t \ := \ X_0 + \int_0^t b_s(X_s)ds + \int_0^t \sigma_s(X_s)dW_s, \qquad 0 \leq t \leq T.$$

By the Lipschitz property of $b$ and $\sigma$ in the $x-$variable and the fact that $b_.(0), \sigma_.(0) \in \mathbb{H}^2$, it follows that this map is well defined on $\mathbb{H}^2$. In order to prove existence and uniqueness of a solution for (1.1), we shall prove that $U(X) \in \mathbb{H}^2$ for all $X \in \mathbb{H}^2$ and that $U$ is a contracting mapping with respect to the norm $\|.\|_{\mathbb{H}^2_c}$ for a convenient choice of the constant $c > 0$.

1- We first prove that $U(X) \in \mathbb{H}^2$ for all $X \in \mathbb{H}^2$. To see this, we decompose:

$$\begin{aligned} \|U(X)\|^2_{\mathbb{H}^2} \quad &\leq \quad 3T\|X_0\|^2_{\mathbb{L}^2} + 3T\mathbb{E}\left[\int_0^T \left|\int_0^t b_s(X_s)ds\right|^2 dt\right] \\ &\qquad +3\mathbb{E}\left[\int_0^T \left|\int_0^t \sigma_s(X_s)dW_s\right|^2 dt\right] \end{aligned}$$

By the Lipschitz-continuity of $b$ and $\sigma$ in $x$, uniformly in $t$, we have $|b_t(x)|^2 \leq K(1+|b_t(0)|^2+|x|^2)$ for some constant $K$. We then estimate the second term by:

$$\mathbb{E}\left[\int_0^T \left|\int_0^t b_s(X_s)ds\right|^2 dt\right] \quad \leq \quad KT\mathbb{E}\left[\int_0^T (1+|b_t(0)|^2+|X_s|^2)ds\right] < \infty,$$

since $X \in \mathbb{H}^2$, and $b(.,0) \in \mathbb{L}^2([0,T])$.

As, for the third term, we use the Doob maximal inequality together with the fact that $|\sigma_t(x)|^2 \leq K(1+|\sigma_t(0)|^2+|x|^2)$, a consequence of the Lipschitz property on $\sigma$:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left|\int_0^t \sigma_s(X_s)dW_s\right|^2 dt\right] &\leq T\mathbb{E}\left[\max_{t\leq T}\left|\int_0^t \sigma_s(X_s)dW_s\right|^2 dt\right] \\
&\leq 4T\mathbb{E}\left[\int_0^T |\sigma_s(X_s)|^2 ds\right] \\
&\leq 4TK\mathbb{E}\left[\int_0^T (1+|\sigma_s(0)|^2+|X_s|^2)ds\right] < \infty.
\end{aligned}$$

2- To see that $U$ is a contracting mapping for the norm $\|.\|_{\mathbb{H}^2_c}$, for some convenient choice of $c>0$, we consider two process $X, Y \in \mathbb{H}^2$ with $X_0 = Y_0$, and we estimate that:

$$\begin{aligned}
& \mathbb{E}\left|U(X)_t - U(Y)_t\right|^2 \\
\leq\ & 2\mathbb{E}\left|\int_0^t (b_s(X_s)-b_s(Y_s))\,ds\right|^2 + 2\mathbb{E}\left|\int_0^t (\sigma_s(X_s)-\sigma_s(Y_s))\,dW_s\right|^2 \\
=\ & 2\mathbb{E}\left|\int_0^t (b_s(X_s)-b_s(Y_s))\,ds\right|^2 + 2\mathbb{E}\int_0^t \left|\sigma_s(X_s)-\sigma_s(Y_s)\right|^2 ds \\
=\ & 2t\mathbb{E}\int_0^t \left|b_s(X_s)-b_s(Y_s)\right|^2 ds + 2\mathbb{E}\int_0^t \left|\sigma_s(X_s)-\sigma_s(Y_s)\right|^2 ds \\
\leq\ & 2(T+1)K \int_0^t \mathbb{E}\left|X_s - Y_s\right|^2 ds.
\end{aligned}$$

Hence, $\|U(X)-U(Y)\|_c \leq \dfrac{2K(T+1)}{c}\|X-Y\|_c$, and therefore $U$ is a contracting mapping for sufficiently large $c$.

*Step 2* We next prove the estimate (1.2). We shall alleviate the notation writing $b_s := b_s(X_s)$ and $\sigma_s := \sigma_s(X_s)$. We directly estimate:

$$\begin{aligned}
\mathbb{E}\left[\sup_{u\le t}|X_u|^2\right] &= \mathbb{E}\left[\sup_{u\le t}\left|X_0+\int_0^u b_s ds+\int_0^u \sigma_s dW_s\right|^2\right]\\
&\le 3\left(\mathbb{E}|X_0|^2+t\mathbb{E}\left[\int_0^t |b_s|^2 ds\right]+\mathbb{E}\left[\sup_{u\le t}\left|\int_0^u \sigma_s dW_s\right|^2\right]\right)\\
&\le 3\left(\mathbb{E}|X_0|^2+t\mathbb{E}\left[\int_0^t |b_s|^2 ds\right]+4\mathbb{E}\left[\int_0^t |\sigma_s|^2 ds\right]\right)
\end{aligned}$$

where we used the Doob's maximal inequality. Since $b$ and $\sigma$ are Lipschitz-continuous in $x$, uniformly in $t$ and $\omega$, this provides:

$$\mathbb{E}\left[\sup_{u\le t}|X_u|^2\right] \le C(K,T)\left(1+\mathbb{E}|X_0|^2+\int_0^t \mathbb{E}\left[\sup_{u\le s}|X_u|^2\right]ds\right)$$

and we conclude by using the Gronwall lemma. $\diamondsuit$

The following exercise shows that the Lipschitz-continuity condition on the coefficients $b$ and $\sigma$ can be relaxed. We observe that further relaxation of this assumption is possible in the one-dimensional case, see e.g. Karatzas and Shreve [8].

**Exercise 1.3.** *In the context of this section, assume that the coefficients $\mu$ and $\sigma$ are locally Lipschitz and linearly growing in $x$, uniformly in $(t,\omega)$. By a localization argument, prove that strong existence and uniqueness holds for the stochastic differential equation* (1.1).

In addition to the estimate (1.2) of Theorem 1.2, we have the following flow continuity results of the solution of the SDE.

**Theorem 1.4.** *Let the conditions of Theorem 1.2 hold true, and consider some $(t,x)\in[0,T)\times\mathbb{R}^n$ with $t\le t'\le T$.*
(i) *There is a constant $C$ such that:*

$$\mathbb{E}\left[\sup_{t\le s\le t'}\left|X_s^{t,x}-X_s^{t,x'}|^2\right|\right] \le Ce^{Ct'}|x-x'|^2. \tag{1.3}$$

(ii) *Assume further that $B:=\sup_{t<t'\le T}(t'-t)^{-1}\mathbb{E}\int_t^{t'}\left(|b_r(0)|^2+|\sigma_r(0)|^2\right)dr<\infty$. Then for all $t'\in[t,T]$:*

$$\mathbb{E}\left[\sup_{t'\le s\le T}\left|X_s^{t,x}-X_s^{t',x}|^2\right|\right] \le Ce^{CT}(B+|x|^2)|t'-t|. \tag{1.4}$$

*Proof.* (i) To simplify the notations, we set $X_s := X_s^{t,x}$ and $X'_s := X_s^{t,x'}$ for all $s\in[t,T]$. We also denote $\delta x := x-x'$, $\delta X := X-X'$, $\delta b := b(X)-b(X')$ and

$\delta\sigma := \sigma(X) - \sigma(X')$. We first decompose:

$$\begin{aligned}
|\delta X_s|^2 &\leq 3\left(|\delta x|^2 + \Big|\int_t^s \delta b_u du\Big|^2 + \Big|\int_t^s \delta\sigma_u dW_u\Big|^2\right) \\
&\leq 3\left(|\delta x|^2 + (s-t)\int_t^s \big|\delta b_u\big|^2 du + \int_t^s \delta\sigma_u dW_u\Big|^2\right).
\end{aligned}$$

Then, it follows from the Doob maximal inequality and the Lipschitz property of the coefficients $b$ and $\sigma$ that:

$$\begin{aligned}
h(t') := \mathbb{E}\left[\sup_{t\leq s\leq t'}|\delta X_s|^2\right] &\leq 3\left(|\delta x|^2 + (s-t)\int_t^s \mathbb{E}\big|\delta b_u\big|^2 du + 4\int_t^s \mathbb{E}\big|\delta\sigma_u\big|^2 d_u\right) \\
&\leq 3\left(|\delta x|^2 + K^2(t'+4)\int_t^s \mathbb{E}|\delta X_u|^2 du\right) \\
&\leq 3\left(|\delta x|^2 + K^2(t'+4)\int_t^s h(u)du\right).
\end{aligned}$$

Then the required estimate follows from the Gronwall inequality.

**2.** We next prove (1.4). We again simplify the notation by setting $X_s := X_s^{t,x}$, $s \in [t,T]$, and $X'_s := X_s^{t',x}$, $s \in [t',T]$. We also denote $\delta t := t'-t$, $\delta X := X - X'$, $\delta b := b(X) - b(X')$ and $\delta\sigma := \sigma(X) - \sigma(X')$. Then following the same arguments as in the previous step, we obtain for all $u \in [t',T]$:

$$\begin{aligned}
h(u) := \mathbb{E}\left[\sup_{t'\leq s\leq u}|\delta X_s|^2\right] &\leq 3\left(\mathbb{E}|X_{t'}-x|^2 + K^2(T+4)\int_{t'}^u \mathbb{E}|\delta X_r|^2 dr\right) \\
&\leq 3\left(\mathbb{E}|X_{t'}-x|^2 + K^2(T+4)\int_{t'}^u h(r)dr\right) \quad (1.5)
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathbb{E}|X_{t'}-x|^2 &\leq 2\left(\mathbb{E}\Big|\int_t^{t'} b_r(X_r)dr\Big|^2 + \mathbb{E}\Big|\int_t^{t'} \sigma_r(X_r)dr\Big|^2\right) \\
&\leq 2\left(T\int_t^{t'} \mathbb{E}|b_r(X_r)|^2 dr + \int_t^{t'} \mathbb{E}|\sigma_r(X_r)|^2 dr\right) \\
&\leq 6(T+1)\int_t^{t'} \left(K^2\mathbb{E}|X_r - x|^2 + |x|^2 + \mathbb{E}|b_r(0)|^2\right)dr \\
&\leq 6(T+1)\Big((t'-t)(|x|^2+B) + K^2\int_t^{t'} \mathbb{E}|X_r-x|^2 dr\Big).
\end{aligned}$$

By the Gronwall inequality, this shows that

$$\mathbb{E}|X_{t'}-x|^2 \quad \leq \quad C(|x|^2+B)|t'-t|e^{C(t'-t)}.$$

Plugging this estimate in (1.5), we see that:

$$h(u) \quad \leq \quad 3\left(C(|x|^2+B)|t'-t|e^C(t'-t) + K^2(T+4)\int_{t'}^u h(r)dr\right), \quad (1.6)$$

and the required estimate follows from the Gronwall inequality. $\diamondsuit$

## 1.2 Markov solutions of SDEs

In this section, we restrict the coefficients $b$ and $\sigma$ to be deterministic functions of $(t,x)$. In this context, we write

$$b_t(x) = b(t,x),\ \ \sigma_t(x) = \sigma(t,x) \quad \text{for} \quad t \in [0,T],\ x \in \mathbb{R}^n,$$

where $b$ and $\sigma$ are continuous functions, Lipschitz in $x$ uniformly in $t$. Let $X^{t,x}_{.}$ denote the solution of the stochastic differential equation

$$X^{t,x}_s = x + \int_t^s b\big(u, X^{t,x}_u\big)du + \int_t^s \sigma\big(u, X^{t,x}_u\big)dW_u \ \ s \geq t$$

The two following properties are obvious:

- Clearly, $X^{t,x}_s = F\left(t, x, s, (W_. - W_t)_{t\leq u\leq s}\right)$ for some deterministic function $F$.
- For $t \leq u \leq s$: $X^{t,x}_s = X^{u, X^{t,x}_u}_s$. This follows from the pathwise uniqueness, and holds also when $u$ is a stopping time.

With these observations, we have the following Markov property for the solutions of stochastic differential equations.

**Proposition 1.5.** *(Markov property) For all $0 \leq t \leq s$:*

$$\mathbb{E}\left[\Phi\left(X_u, t \leq u \leq s\right)|\mathcal{F}_t\right] \quad = \quad \mathbb{E}\left[\Phi\left(X_u, t \leq u \leq s\right)|X_t\right]$$

*for all bounded function $\Phi : \ C([t,s]) \longrightarrow \mathbb{R}$.*

## 1.3 Connection with linear partial differential equations

### 1.3.1 Generator

Let $\{X^{t,x}_s, s \geq t\}$ be the unique strong solution of

$$X^{t,x}_s \quad = \quad x + \int_t^s \mu(u, X^{t,x}_u)du + \int_t^s \sigma(u, X^{t,x}_u)dW_u,\ \ s \geq t,$$

where $\mu$ and $\sigma$ satisfy the required condition for existence and uniqueness of a strong solution.

For a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we define the function $\mathcal{A}f$ by

$$\mathcal{A}f(t,x) = \lim_{h\to 0} \frac{\mathbb{E}[f(X^{t,x}_{t+h})] - f(x)}{h} \quad \text{if the limit exists.}$$

Clearly, $\mathcal{A}f$ is well-defined for all bounded $C^2-$ function with bounded derivatives and

$$\mathcal{A}f(t,x) \quad = \quad \mu(t,x)\cdot f(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}(t,x)\frac{\partial^2 f}{\partial x \partial x^{\mathrm{T}}}\right], \tag{1.7}$$

(Exercise !). The linear differential operator $\mathcal{A}$ is called the *generator* of $X$. It turns out that the process $X$ can be completely characterized by its generator or, more precisely, by the generator and the corresponding domain of definition...

As the following result shows, the generator provides an intimate connection between conditional expectations and linear partial differential equations.

**Proposition 1.6.** *Assume that the function $(t,x) \longmapsto v(t,x) := \mathbb{E}\left[g(X_T^{t,x}\right]$ is $C^{1,2}\left([0,T)\times\mathbb{R}^n\right)$. Then $v$ solves the partial differential equation:*

$$\frac{\partial v}{\partial t} + \mathcal{A}v = 0 \quad \text{and} \quad v(T,.) = g.$$

*Proof.* Given $(t,x)$, let $\tau_1 := T \wedge \inf\{s > t : |X_s^{t,x} - x| \geq 1\}$. By the law of iterated expectation together with the Markov property of the process $X$, it follows that

$$v(t,x) \quad = \quad \mathbb{E}\left[v\left(s\wedge\tau_1, X_{s\wedge\tau_1}^{t,x}\right)\right].$$

Since $v \in C^{1,2}([0,T),\mathbb{R}^n)$, we may apply Itô's formula, and we obtain by taking expectations:

$$\begin{aligned} 0 \quad &= \quad \mathbb{E}\left[\int_t^{s\wedge\tau_1}\left(\frac{\partial v}{\partial t} + \mathcal{A}v\right)(u, X_u^{t,x})du\right] \\ &\quad +\mathbb{E}\left[\int_t^{s\wedge\tau_1}\frac{\partial v}{\partial x}(u, X_s^{t,x})\cdot\sigma(u, X_u^{t,x})dW_u\right] \\ &= \quad \mathbb{E}\left[\int_t^{s\wedge\tau_1}\left(\frac{\partial v}{\partial t} + \mathcal{A}v\right)(u, X_u^{t,x})du\right], \end{aligned}$$

where the last equality follows from the boundedness of $(u, X_u^{t,x})$ on $[t, s\wedge\tau_1]$. We now send $s \searrow t$, and the required result follows from the dominated convergence theorem. $\diamondsuit$

### 1.3.2 Cauchy problem and the Feynman-Kac representation

In this section, we consider the following linear partial differential equation

$$\begin{array}{l} \frac{\partial v}{\partial t} + \mathcal{A}v - k(t,x)v + f(t,x) = 0, \quad (t,x)\in[0,T)\times\mathbb{R}^d \\ v(T,.) = g \end{array} \tag{1.8}$$

where $\mathcal{A}$ is the generator (1.7), $g$ is a given function from $\mathbb{R}^d$ to $\mathbb{R}$, $k$ and $f$ are functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}$, $b$ and $\sigma$ are functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}^d$ and and $\mathcal{M}_\mathbb{R}(d,d)$, respectively. This is the so-called Cauchy problem.

For example, when $k = f \equiv 0$, $b \equiv 0$, and $\sigma$ is the identity matrix, the above partial differential equation reduces to the heat equation.

Our objective is to provide a representation of this purely deterministic problem by means of stochastic differential equations. We then assume that $\mu$ and $\sigma$ satisfy the conditions of Theorem 1.2, namely that

$$\mu,\sigma \quad \text{Lipschitz in } x \text{ uniformly in } t, \qquad \int_0^T\left(|\mu(t,0)|^2 + |\sigma(t,0)|^2\right)dt < \infty \tag{1.9}$$

**Theorem 1.7.** *Let the coefficients $\mu, \sigma$ be continuous and satisfy* (1.9)*. Assume further that the function $k$ is uniformly bounded from below, and $f$ has quadratic growth in $x$ uniformly in $t$. Let $v$ be a $C^{1,2}\left([0,T), \mathbb{R}^d\right) \cap C^0\left([0,T) \times \mathbb{R}^d\right)$ solution of* (1.8) *with quadratic growth in $x$ uniformly in $t$. Then*

$$v(t,x) \ = \ \mathbb{E}\left[\int_t^T \beta_s^{t,x} f(s, X_s^{t,x})ds + \beta_T^{t,x} g\left(X_T^{t,x}\right)\right], \qquad t \le T, \ x \in \mathbb{R}^d,$$

*where $X_s^{t,x} := x + \int_t^s \mu(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dW_u$ and $\beta_s^{t,x} := e^{-\int_t^s k(u, X_u^{t,x})du}$ for $t \le s \le T$.*

*Proof.* We first introduce the sequence of stopping times

$$\tau_n \ := \ \left(T - \frac{1}{n}\right) \wedge \wedge \inf\left\{s > t \ : \ \left|X_s^{t,x} - x\right| \ge n\right\},$$

and we oberve that $\tau_n \longrightarrow T$ $\mathbb{P}-$a.s. Since $v$ is smooth, it follows from Itô's formula that for $t \le s < T$:

$$\begin{aligned} d\left(\beta_s^{t,x} v\left(s, X_s^{t,x}\right)\right) \ &= \ \beta_s^{t,x}\left(-kv + \frac{\partial v}{\partial t} + \mathcal{A}v\right)\left(s, X_s^{t,x}\right) ds \\ &\quad + \beta_s^{t,x} \frac{\partial v}{\partial x}\left(s, X_s^{t,x}\right) \cdot \sigma\left(s, X_s^{t,x}\right) dW_s \\ &= \ \beta_s^{t,x}\left(-f(s, X_s^{t,x})ds + \frac{\partial v}{\partial x}\left(s, X_s^{t,x}\right) \cdot \sigma\left(s, X_s^{t,x}\right) dW_s\right), \end{aligned}$$

by the PDE satisfied by $v$ in (1.8). Then:

$$\begin{aligned} & \mathbb{E}\left[\beta_{\tau_n}^{t,x} v\left(\tau_n, X_{\tau_n}^{t,x}\right)\right] - v(t,x) \\ = \ & \mathbb{E}\left[\int_t^{\tau_n} \beta_s^{t,x}\left(-f(s, X_s)ds + \frac{\partial v}{\partial x}\left(s, X_s^{t,x}\right) \cdot \sigma\left(s, X_s^{t,x}\right) dW_s\right)\right]. \end{aligned}$$

Now observe that the integrands in the stochastic integral is bounded by definition of the stopping time $\tau_n$, the smoothness of $v$, and the continuity of $\sigma$. Then the stochastic integral has zero mean, and we deduce that

$$v(t,x) \ = \ \mathbb{E}\left[\int_t^{\tau_n} \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_{\tau_n}^{t,x} v\left(\tau_n, X_{\tau_n}^{t,x}\right)\right]. \tag{1.10}$$

Since $\tau_n \longrightarrow T$ and the Brownian motion has continuous sample paths $\mathbb{P}-$a.s. it follows from the continuity of $v$ that, $\mathbb{P}-$a.s.

$$\begin{aligned} & \int_t^{\tau_n} \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_{\tau_n}^{t,x} v\left(\tau_n, X_{\tau_n}^{t,x}\right) \\ & \qquad \stackrel{n \to \infty}{\longrightarrow} \int_t^T \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_T^{t,x} v\left(T, X_T^{t,x}\right) \\ & \qquad = \int_t^T \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_T^{t,x} g\left(X_T^{t,x}\right) \end{aligned} \tag{1.11}$$

by the terminal condition satisfied by $v$ in (1.8). Moreover, since $k$ is bounded from below and the functions $f$ and $v$ have quadratic growth in $x$ uniformly in $t$, we have

$$\left|\int_t^{\tau_n} \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_{\tau_n}^{t,x} v\left(\tau_n, X_{\tau_n}^{t,x}\right)\right| \quad \leq \quad C\left(1 + \max_{t \leq T} |X_t|^2\right) .$$

By the estimate stated in the existence and uniqueness theorem 1.2, the latter bound is integrable, and we deduce from the dominated convergence theorem that the convergence in (1.11) holds in $\mathbb{L}^1(\mathbb{P})$, proving the required result by taking limits in (1.10). $\diamondsuit$

The above Feynman-Kac representation formula has an important numerical implication. Indeed it opens the door to the use of Monte Carlo methods in order to obtain a numerical approximation of the solution of the partial differential equation (1.8). For sake of simplicity, we provide the main idea in the case $f = k = 0$. Let $\left(X^{(1)}, \ldots, X^{(k)}\right)$ be an iid sample drawn in the distribution of $X_T^{t,x}$, and compute the mean:

$$\hat{v}_k(t,x) \quad := \quad \frac{1}{k}\sum_{i=1}^{k} g\left(X^{(i)}\right) .$$

By the Law of Large Numbers, it follows that $\hat{v}_k(t,x) \longrightarrow v(t,x)$ $\mathbb{P}-$a.s. Moreover the error estimate is provided by the Central Limit Theorem:

$$\sqrt{k}\left(\hat{v}_k(t,x) - v(t,x)\right) \overset{k\to\infty}{\longrightarrow} \mathcal{N}\left(0, \mathbb{V}ar\left[g\left(X_T^{t,x}\right)\right]\right) \qquad \text{in distribution,}$$

and is remarkably independent of the dimension $d$ of the variable $X$ !

### 1.3.3 Representation of the Dirichlet problem

Let $D$ be an open bounded subset of $\mathbb{R}^d$. The *Dirichlet problem* is to find a function $u$ solving:

$$\mathcal{A}u - ku + f = 0 \text{ on } D \quad \text{and} \quad u = g \text{ on } \partial D, \tag{1.12}$$

where $\partial D$ denotes the boundary of $D$, $f$ and $k$ are continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$, and $\mathcal{A}$ is the generator of the process $X^{0,X_0}$ defined as the unique strong solution of the homogeneous (time independent coefficients) stochastic differential equation

$$X_t^{0,X_0} = X_0 + \int_0^t \mu(X_s^{0,X_0})ds + \int_0^t \sigma(X_s^{0,X_0})dW_s,\ t \geq 0.$$

Similarly to the the representation result of the Cauchy problem obtained in Theorem 1.7, we have the following representation result for the Dirichlet problem.

**Theorem 1.8.** *Let $u$ be a $C^2-$solution of the Dirichlet problem* (1.12). *Assume that $k$ is nonnegative, and*

$$\mathbb{E}[\tau_D^x] < \infty,\ x \in \mathbb{R}^d, \quad \text{where} \quad \tau_D^x := \inf\left\{t \geq 0 :\ X_t^{0,x} \notin D\right\}.$$

*Then, we have the representation:*

$$u(x) \quad = \quad \mathbb{E}\left[g\left(X_{\tau_D^x}^{0,x}\right)e^{-\int_0^{\tau_D^x} k(X_s)ds} + \int_0^{\tau_D^x} f\left(X_t^{0,x}\right)e^{-\int_0^t k(X_s)ds}dt\right].$$

**Exercise 1.9.** *Provide a proof of Theorem 1.8 by imitating the arguments in the proof of Theorem 1.7.*

## 1.4 The stochastic control approach to the Black-Scholes model

### 1.4.1 The continuous-time financial market

Let $T$ be a finite horizon, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting a Brownian motion $W = \{(W_t^1, \ldots, W_t^d),\ 0 \leq t \leq T\}$ with values in $\mathbb{R}^d$. We denote by $\mathbb{F} = \mathbb{F}^W = \{\mathcal{F}_t,\ 0 \leq t \leq T\}$ the canonical augmented filtration of $W$, i.e. the canonical filtration augmented by zero measure sets of $\mathcal{F}_T$.

We consider a financial market consisting of $d+1$ assets :

(i) The first asset $S^0$ is non-risky, and is defined by

$$S_t^0 \ = \ \exp\left(\int_0^t r_u du\right), \qquad 0 \leq t \leq T,$$

where $\{r_t, t \in [0,T]\}$ is a non-negative adapted processes with $\int_0^T r_t dt < \infty$ a.s., and represents the instantaneous interest rate.

(ii) The $d$ remaining assets $S^i$, $i = 1, \ldots, d$, are risky assets with price processes defined by the dynamics

$$\frac{dS_t^i}{S_t^i} \quad = \quad \mu_t^i dt + \sum_{j=1}^d \sigma_t^{i,j} dW_t^j,\ t \in [0,T],$$

for $1 \leq i \leq d$, where $\mu, \sigma$ are $\mathbb{F}-$adapted processes with $\int_0^T |\mu_t^i|dt + \int_0^T |\sigma_t^{i,j}|^2 dt < \infty$ for all $i, j = 1, \ldots, d$. It is convenient to use the matrix notations to represent the dynamics of the price vector $S = (S^1, \ldots, S^d)$:

$$dS_t \quad = \quad S_t \star (\mu_t dt + \sigma_t dW_t),\ t \in [0,T],$$

where, for two vectors $x, y \in \mathbb{R}^d$, we denote $x \star y$ the vector of $\mathbb{R}^d$ with components $(x \star y)_i = x_i y_i$, $i = 1, \ldots, d$, and $\mu, \sigma$ are the $\mathbb{R}^d-$vector with components $\mu^i$'s, and the $\mathcal{M}_{\mathbb{R}}(d,d)-$matrix with entries $\sigma^{i,j}$.

We assume that the $\mathcal{M}_{\mathbb{R}}(d,d)-$matrix $\sigma_t$ is invertible for every $t \in [0,T]$ a.s., and we introduce the process

$$\lambda_t := \sigma_t^{-1}\left(\mu_t - r_t\mathbf{1}\right), \qquad 0 \le t \le T,$$

called the *risk premium process*. Here $\mathbf{1}$ is the vector of ones in $\mathbb{R}^d$. We shall frequently make use of the discounted processes

$$\tilde{S}_t \;:=\; \frac{S_t}{S_t^0} \;=\; S_t \exp\left(-\int_0^t r_u du\right),$$

Using the above matrix notations, the dynamics of the process $\tilde{S}$ are given by

$$d\tilde{S}_t \;=\; \tilde{S}_t \star \big((\mu_t - r_t\mathbf{1})dt + \sigma_t dW_t\big) \;=\; \tilde{S}_t \star \sigma_t \left(\lambda_t dt + dW_t\right).$$

### 1.4.2 Portfolio and wealth process

A portfolio strategy is an $\mathbb{F}-$adapted process $\pi = \{\pi_t, 0 \le t \le T\}$ with values in $\mathbb{R}^d$. For $1 \le i \le n$ and $0 \le t \le T$, $\pi_t^i$ is the amount (in Euros) invested in the risky asset $S^i$.

We next recall the self-financing condition in the present framework. Let $X_t^\pi$ denote the portfolio value, or wealth, process at time $t$ induced by the portfolio strategy $\pi$. Then, the amount invested in the non-risky asset is $X_t^\pi - \sum_{i=1}^n \pi_t^i$ $= X_t^\pi - \pi_t \cdot \mathbf{1}$.

Under the self-financing condition, the dynamics of the wealth process is given by

$$dX_t^\pi \;=\; \sum_{i=1}^n \frac{\pi_t^i}{S_t^i}\, dS_t^i \;+\; \frac{X_t^\pi - \pi_t \cdot \mathbf{1}}{S_t^0}\, dS_t^0.$$

Let $\tilde{X}^\pi$ be the discounted wealth process

$$\tilde{X}_t^\pi \;:=\; X_t^\pi \exp\left(-\int_0^t r(u)du\right), \qquad 0 \le t \le T.$$

Then, by an immediate application of Itô's formula, we see that

$$d\tilde{X}_t \;=\; \tilde{\pi}_t \cdot \sigma_t \left(\lambda_t dt + dW_t\right),\ 0 \le t \le T, \tag{1.13}$$

where $\tilde{\pi}_t := e^{-rt}\pi_t$. We still need to place further technical conditions on $\pi$, at least in order for the above wealth process to be well-defined as a stochastic integral.

Before this, let us observe that, assuming that the risk premium process satisfies the Novikov condition:

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T |\lambda_t|^2 dt}\right] \;<\; \infty,$$

it follows from the Girsanov theorem that the process

$$B_t \ := \ W_t + \int_0^t \lambda_u du\,, \qquad 0 \le t \le T\,, \tag{1.14}$$

is a Brownian motion under the equivalent probability measure

$$\mathbb{Q} := Z_T \cdot \mathbb{P} \text{ on } \mathcal{F}_T \quad \text{where} \quad Z_T := \exp\left(-\int_0^T \lambda_u \cdot dW_u - \frac{1}{2}\int_0^T |\lambda_u|^2 du\right).$$

In terms of the $\mathbb{Q}$ Brownian motion $B$, the discounted price process satisfies

$$d\tilde{S}_t \quad = \quad \tilde{S}_t \star \sigma_t dB_t, \quad t \in [0,T],$$

and the discounted wealth process induced by an initial capital $X_0$ and a portfolio strategy $\pi$ can be written in

$$\tilde{X}_t^\pi \ = \ \tilde{X}_0 + \int_0^t \tilde{\pi}_u \cdot \sigma_u dB_u, \quad \text{for} \quad 0 \le t \le T. \tag{1.15}$$

**Definition 1.10.** *An admissible portfolio process $\pi = \{\theta_t, t \in [0,T]\}$ is an $\mathbb{F}-$progressively measurable process such that $\int_0^T |\sigma_t^{\mathrm{T}} \pi_t|^2 dt < \infty$, a.s. and the corresponding discounted wealth process is bounded from below by a $\mathbb{Q}-$martingale*

$$\tilde{X}_t^\pi \ge M_t^\pi, \ 0 \le t \le T, \quad \textit{for some } \mathbb{Q}-\textit{martingale} \quad M^\pi.$$

*The collection of all admissible portfolio processes will be denoted by $\mathcal{A}$.*

The lower bound $M^\pi$, which may depend on the portfolio $\pi$, has the interpretation of a finite credit line imposed on the investor. This natural generalization of the more usual constant credit line corresponds to the situation where the total credit available to an investor is indexed by some financial holding, such as the physical assets of the company or the personal home of the investor, used as collateral. From the mathematical viewpoint, this condition is needed in order to exclude any arbitrage opportunity, and will be justified in the subsequent subsection.

### 1.4.3 Admissible portfolios and no-arbitrage

We first define precisely the notion of no-arbitrage.

**Definition 1.11.** *We say that the financial market contains no arbitrage opportunities if for any admissible portfolio process $\theta \in \mathcal{A}$,*

$$X_0 = 0 \textit{ and } X_T^\theta \ge 0 \ \mathbb{P}-a.s. \quad \textit{implies} \quad X_T^\theta \ = \ 0 \ \mathbb{P}-a.s.$$

The purpose of this section is to show that the financial market described above contains no arbitrage opportunities. Our first observation is that, by the very definition of the probability measure $\mathbb{Q}$, the discounted price process $\tilde{S}$ satisfies:

$$\text{the process} \quad \left\{\tilde{S}_t,\ 0 \le t \le T\right\} \quad \text{is a } \mathbb{Q}-\text{local martingale.} \tag{1.16}$$

For this reason, $\mathbb{Q}$ is called a *risk neutral measure*, or an *equivalent local martingale measure*, for the price process $S$.

We also observe that the discounted wealth process satisfies:

$$\tilde{X}^\pi \quad \text{is a } \mathbb{Q}-\text{local martingale for every} \quad \pi \in \mathcal{A}, \tag{1.17}$$

as a stochastic integral with respect to the $\mathbb{Q}-$Brownian motion $B$.

**Theorem 1.12.** *The continuous-time financial market described above contains no arbitrage opportunities, i.e. for every $\pi \in \mathcal{A}$:*

$$X_0 = 0 \text{ and } X_T^\pi \ge 0\ \mathbb{P} - a.s. \quad \Longrightarrow \quad X_T^\pi = 0\ \mathbb{P} - a.s.$$

*Proof.* For $\pi \in \mathcal{A}$, the discounted wealth process $\tilde{X}^\pi$ is a $\mathbb{Q}-$local martingale bounded from below by a $\mathbb{Q}-$martingale. Then $\tilde{X}^\pi$ is a $\mathbb{Q}-$super-martingale. In particular, $\mathbb{E}^{\mathbb{Q}}\left[\tilde{X}_T^\pi\right] \le \tilde{X}_0 = X_0$. Recall that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $S^0$ is strictly positive. Then, this inequality shows that, whenever $X_0^\pi = 0$ and $X_T^\pi \ge 0$ $\mathbb{P}-$a.s. (or equivalently $\mathbb{Q}-$a.s.), we have $\tilde{X}_T^\pi = 0$ $\mathbb{Q}-$a.s. and therefore $X_T^\pi = 0$ $\mathbb{P}-$a.s. ♢

### 1.4.4 Super-hedging and no-arbitrage bounds

Let $G$ be an $\mathcal{F}_T-$measurable random variable representing the payoff of a derivative security with given maturity $T > 0$. The *super-hedging* problem consists in finding the minimal initial cost so as to be able to face the payment $G$ without risk at the maturity of the contract $T$:

$$V(G) \quad := \quad \inf\left\{X_0 \in \mathbb{R}\ :\ X_T^\pi \ge G\ \mathbb{P} - \text{a.s. for some } \pi \in \mathcal{A}\right\}.$$

**Remark 1.13.** Notice that $V(G)$ depends on the reference measure $\mathbb{P}$ only by means of the corresponding null sets. Therefore, the super-hedging problem is not changed if $\mathbb{P}$ is replaced by any equivalent probability measure.

We now show that, under the no-arbitrage condition, the super-hedging problem provides *no-arbitrage bounds* on the market price of the derivative security.

Assume that the buyer of the contingent claim $G$ has the same access to the financial market than the seller. Then $V(G)$ *is the maximal amount that the buyer of the contingent claim contract is willing to pay.* Indeed, if the seller requires a premium of $V(G) + 2\varepsilon$, for some $\varepsilon > 0$, then the buyer would not accept to pay this amount as he can obtain at least $G$ by trading on the financial market with initial capital $V(G) + \varepsilon$.

Now, since selling of the contingent claim $G$ is the same as buying the contingent claim $-G$, we deduce from the previous argument that

$$-V(-G) \quad \le \quad \text{market price of } G \quad \le \quad V(G). \tag{1.18}$$

### 1.4.5 The no-arbitrage valuation formula

We denote by $p(G)$ the market price of a derivative security $G$.

**Theorem 1.14.** *Let $G$ be an $\mathcal{F}_T-$measurabel random variable representing the payoff of a derivative security at the maturity $T>0$, and recall the notation $\tilde{G} := G\exp\left(-\int_0^T r_t dt\right)$. Assume that $\mathbb{E}^{\mathbb{Q}}[|\tilde{G}|]<\infty$. Then*

$$p(G) \;=\; V(G) \;\;=\;\; \mathbb{E}^{\mathbb{Q}}[\tilde{G}].$$

*Moreover, there exists a portfolio $\pi^* \in \mathcal{A}$ such that $X_0^{\pi^*} = p(G)$ and $X_T^{\pi^*} = G$, a.s., that is $\pi^*$ is a perfect replication strategy.*

*Proof.* 1- We first prove that $V(G) \geq \mathbb{E}^{\mathbb{Q}}[\tilde{G}]$. Let $X_0$ and $\pi \in \mathcal{A}$ be such that $X_T^\pi \geq G$, a.s. or, equivalently, $\tilde{X}_T^\pi \geq \tilde{G}$ a.s. Notice that $\tilde{X}^\pi$ is a $\mathbb{Q}-$super-martingale, as a $\mathbb{Q}-$local martingale bounded from below by a $\mathbb{Q}-$martingale. Then $X_0 = \tilde{X}_0 \geq \mathbb{E}^{\mathbb{Q}}[\tilde{X}_T^\pi] \geq \mathbb{E}^{\mathbb{Q}}[\tilde{G}]$.
2- We next prove that $V(G) \leq \mathbb{E}^{\mathbb{Q}}[\tilde{G}]$. Define the $\mathbb{Q}-$martingale $Y_t := \mathbb{E}^{\mathbb{Q}}[\tilde{G}|\mathcal{F}_t]$ and observe that $\mathbb{F}^W = \mathbb{F}^B$. Then, it follows from the martingale representation theorem that $Y_t = Y_0 + \int_0^T \phi_t \cdot dB_t$ for some $\mathbb{F}-$adapted process $\phi$ with $\int_0^T |\phi_t|^2 dt < \infty$ a.s. Setting $\tilde{\pi}^* := (\sigma^{\mathrm{T}})^{-1}\phi$, we see that

$$\pi^* \in \mathcal{A} \quad \text{and} \quad Y_0 + \int_0^T \tilde{\pi}^* \cdot \sigma_t dB_t = \tilde{G} \;\mathbb{P}-\text{a.s.}$$

which implies that $Y_0 \geq V(G)$ and $\pi^*$ is a perfect hedging stratgey for $G$, starting from the initial capital $Y_0$.
3- From the previous steps, we have $V(G) = \mathbb{E}^{\mathbb{Q}}[\tilde{G}]$. Applying this result to $-G$, we see that $V(-G) = -V(G)$, so that the no-arbitrage bounds (1.18) imply that the no-arbitrage market price of $G$ is given by $V(G)$. ♢

### 1.4.6 PDE characterization of the Black-Scholes price

In this subsection, we specialize further the model to the case where the risky securities price processes are Markov diffusions defined by the stochastic differential equations:

$$dS_t \;=\; S_t \star \big(r(t,S_t)dt + \sigma(t,S_t)dB_t\big).$$

Here $(t,s) \longmapsto s \star r(t,s)$ and $(t,s) \longmapsto s \star \sigma(t,s)$ are Lipschitz-continuous functions from $\mathbb{R}_+ \times [0,\infty)^d$ to $\mathbb{R}^d$ and $\mathcal{S}_d$, successively. We also consider a *Vanilla* derivative security defined by the payoff

$$G \;=\; g(S_T),$$

where $g : [0,\infty)^d \to \mathbb{R}$ is a measurable function bounded from below. From the previous subsection, the no-arbitrage price at time $t$ of this derivative security is given by

$$V(t,S_t) \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u,S_u)du} g(S_T)|\mathcal{F}_t\right] \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u,S_u)du} g(S_T)|S_t\right],$$

where the last equality follows from the Markov property of the process $S$. Assuming further that $g$ has linear growth, it follows that $V$ has linear growth in $s$ uniformly in $t$. Since $V$ is defined by a conditional expectation, it is expected to satisfy the linear PDE:

$$-\partial_t V - rs \star DV - \frac{1}{2}\mathrm{Tr}\left[(s \star \sigma)^2 D^2 V\right] - rV \quad = \quad 0. \tag{1.19}$$

More precisely, if $V \in C^{1,2}(\mathbb{R}_+, \mathbb{R}^d)$, the $V$ is a classical solution of (1.19) and satisfies the final condition $V(T,.) = g$. Coversely, if the PDE (1.19) combined with the final condition $v(T,.) = g$ has a classical solution $v$ with linear growth, then $v$ coincides with the derivative security price $V$.

# Chapter 2

# Stochastic Control and Dynamic Programming

In this chapter, we assume that the filtration $\mathbb{F}$ is the $\mathbb{P}-$augmentation of the canonical filtration of the Brownian motion $W$. This restriction is only needed in order to simplify the presentation of the proof of the dynamic programming principle. We will also denote by

$$\mathbf{S} := [0,T) \times \mathbb{R}^n \quad \text{where} \quad T \in [0,\infty].$$

The set $\mathbf{S}$ is called the *parabolic* interior of the state space. We will denote by $\bar{\mathbf{S}} := \mathrm{cl}(\mathbf{S})$ its closure, i.e. $\bar{\mathbf{S}} = [0,T] \times \mathbb{R}^n$ for finite $T$, and $\bar{\mathbf{S}} = \mathbf{S}$ for $T = \infty$.

## 2.1 Stochastic control problems in standard form

Control processes. Given a subset $U$ of $\mathbb{R}^k$, we denote by $\mathcal{U}$ the set of all progressively measurable processes $\nu = \{\nu_t,\, t < T\}$ valued in $U$. The elements of $\mathcal{U}$ are called control processes.

Controlled Process. Let

$$b \;:\; (t,x,u) \in \mathbf{S} \times U \quad \longrightarrow \quad b(t,x,u) \in \mathbb{R}^n$$

and

$$\sigma \;:\; (t,x,u) \in \mathbf{S} \times U \quad \longrightarrow \quad \sigma(t,x,u) \in \mathcal{M}_{\mathbb{R}}(n,d)$$

be two continuous functions satisfying the conditions

$$\begin{aligned} |b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| &\leq K\,|x-y|, && (2.1)\\ |b(t,x,u)| + |\sigma(t,x,u)| &\leq K\,(1+|x|+|u|). && (2.2) \end{aligned}$$

for some constant $K$ independent of $(t, x, y, u)$. For each control process $\nu \in \mathcal{U}$, we consider the controlled stochastic differential equation :

$$dX_t \;=\; b(t, X_t, \nu_t)dt + \sigma(t, X_t, \nu_t)dW_t. \tag{2.3}$$

If the above equation has a unique solution $X$, for a given initial data, then the process $X$ is called the controlled process, as its dynamics is driven by the action of the control process $\nu$.

We shall be working with the following subclass of control processes :

$$\mathcal{U}_0 \;:=\; \mathcal{U} \cap \mathbb{H}^2, \tag{2.4}$$

where $\mathbb{H}^2$ is the collection of all progressively measurable processes with finite $\mathbb{L}^2(\Omega \times [0, T))-$norm. Then, for every finite maturity $T' \le T$, it follows from the above uniform Lipschitz condition on the coefficients $b$ and $\sigma$ that

$$\mathbb{E}\left[\int_0^{T'} \left(|b| + |\sigma|^2\right)(s, x, \nu_s)ds\right] < \infty \quad \text{for all} \quad \nu \in \mathcal{U}_0,\ x \in \mathbb{R}^n,$$

which guarantees the existence of a controlled process on the time interval $[0, T']$ for each given initial condition and control. The following result is an immediate consequence of Theorem 1.2.

**Theorem 2.1.** *Let $\nu \in \mathcal{U}_0$ be a control process, and $\xi \in \mathbb{L}^2(\mathbb{P})$ be an $\mathcal{F}_0-$measurable random variable. Then, there exists a unique $\mathbb{F}-$adapted process $X^\nu$ satisfying (6.3) together with the initial condition $X_0^\nu = \xi$. Moreover for every $T > 0$, there is a constant $C > 0$ such that*

$$\mathbb{E}\left[\sup_{0 \le s \le t} |X_s^\nu|^2\right] \;<\; C(1 + \mathbb{E}[|\xi|^2])e^{Ct} \quad \textit{for all} \quad t \in \mathrm{cl}([0, T)). \tag{2.5}$$

<u>*Cost functional.*</u> Let

$$f, k \;:\; [0, T) \times \mathbb{R}^n \times U \longrightarrow \mathbb{R} \quad \text{and} \quad g \;:\; \mathbb{R}^n \longrightarrow \mathbb{R}$$

be given functions. We assume that $f, k$ are continuous and $\|k^-\|_\infty < \infty$ (i.e. $\max(-k, 0)$ is uniformly bounded). Moreover, we assume that $f$ and $g$ satisfy the quadratic growth condition :

$$|f(t, x, u)| + |g(x)| \;\le\; K(1 + |u| + |x|^2),$$

for some constant $K$ independent of $(t, x, u)$. We define the cost function $J$ on $[0, T] \times \mathbb{R}^n \times \mathcal{U}$ by :

$$J(t, x, \nu) \;:=\; \mathbb{E}\left[\int_t^T \beta^\nu(t, s) f(s, X_s^{t,x,\nu}, \nu_s)ds + \beta^\nu(t, T) g(X_T^{t,x,\nu}) \mathbf{1}_{T<\infty}\right],$$

when this expression is meaningful, where

$$\beta^\nu(t, s) \;:=\; e^{-\int_t^s k(r, X_r^{t,x,\nu}, \nu_r)dr},$$

and $\{X_s^{t,x,\nu}, s \geq t\}$ is the solution of (6.3) with control process $\nu$ and initial condition $X_t^{t,x,\nu} = x$.

*Admissible control processes.* In the finite horizon case $T < \infty$, the quadratic growth condition on $f$ and $g$ together with the bound on $k^-$ ensure that $J(t, x, \nu)$ is well-defined for all control process $\nu \in \mathcal{U}_0$. We then define the set of admissible controls in this case by $\mathcal{U}_0$.

More attention is needed for the infinite horizon case. In particular, the discount term $k$ needs to play a role to ensure the finiteness of the integral. In this setting the largest set of admissible control processes is given by

$$\mathcal{U}_0 := \left\{\nu \in \mathcal{U} : \mathbb{E}\big[\int_0^\infty \beta^\nu(t,s)\big(1+|X_s^{t,x,\nu}|^2+|\nu_s)|\big)ds\big] < \infty \text{ for all } x\right\} \text{ when } T = \infty.$$

*The stochastic control problem.* The purpose of this section is to study the minimization problem

$$V(t,x) \;:=\; \sup_{\nu \in \mathcal{U}_0} J(t,x,\nu) \quad \text{for} \quad (t,x) \in \mathbf{S}.$$

Our main concern is to describe the local behavior of the value function $V$ by means of the so-called *dynamic programming equation*, or *Hamilton-Jacobi-Bellman* equation. We continue with some remarks.

**Remark 2.2.** (i) If $V(t,x) = J(t,x,\hat{\nu}_{t,x})$, we call $\hat{\nu}_{t,x}$ an *optimal control* for the problem $V(t,x)$.

(ii) The following are some interesting subsets of controls :

- a process $\nu \in \mathcal{U}_0$ which is adapted to the natural filtration $\mathbb{F}^X$ of the associated state process is called *feedback control*,

- a process $\nu \in \mathcal{U}_0$ which can be written in the form $\nu_s = \tilde{u}(s, X_s)$ for some measurable map $\tilde{u}$ from $[0,T] \times \mathbb{R}^n$ into $U$, is called *Markovian control*; notice that any Markovian control is a feedback control,

- the deterministic processes of $\mathcal{U}_0$ are called *open loop controls*.

(iii) Suppose that $T < \infty$, and let $(Y, Z)$ be the controlled processes defined by

$$dY_s \;=\; Z_s f(s, X_s, \nu_s)ds \quad \text{and} \quad dZ_s \;=\; -Z_s k(s, X_s, \nu_s)ds\;,$$

and define the augmented state process $\bar{X} := (X, Y, Z)$. Then, the above value function $V$ can be written in the form :

$$V(t,x) \;=\; \bar{V}(t,x,0,1)\;,$$

where $\bar{x} = (x, y, z)$ is some initial data for the augmented state process $\bar{X}$,

$$\bar{V}(t,\bar{x}) \;:=\; \mathbb{E}_{t,\bar{x}}\left[\bar{g}(\bar{X}_T)\right] \quad \text{and} \quad \bar{g}(x,y,z) \;:=\; y + g(x)z\;.$$

Hence the stochastic control problem $V$ can be reduced without loss of generality to the case where $f = k \equiv 0$. We shall appeal to this reduced form whenever convenient for the exposition.

(iv) For notational simplicity we consider the case $T < \infty$ and $f = k = 0$. The previous remark shows how to immediately adapt the following argument so that the present remark holds true without the restriction $f = k = 0$. The extension to the infinite horizon case is also immediate.

Consider the value function

$$\tilde{V}(t,x) \quad := \quad \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[g(X_T^{t,x,\nu})\right], \tag{2.6}$$

differing from $V$ by the restriction of the control processes to

$$\mathcal{U}_t \quad := \quad \{\nu \in \mathcal{U}_0 : \nu \text{ independent of } \mathcal{F}_t\}. \tag{2.7}$$

Since $\mathcal{U}_t \subset \mathcal{U}_0$, it is obvious that $\tilde{V} \leq V$. We claim that

$$\tilde{V} \quad = \quad V, \tag{2.8}$$

so that both problems are indeed equivalent. To see this, fix $(t,x) \in \mathbf{S}$ and $\nu \in \mathcal{U}_0$. Then, $\nu$ can be written as a measurable function of the canonical process $\nu((\omega_s)_{0 \leq s \leq t}, (\omega_s - \omega_t)_{t \leq s \leq T})$, where, for fixed $(\omega_s)_{0 \leq s \leq t}$, the map $\nu_{(\omega_s)_{0 \leq s \leq t}} : (\omega_s - \omega_t)_{t \leq s \leq T} \mapsto \nu((\omega_s)_{0 \leq s \leq t}, (\omega_s - \omega_t)_{t \leq s \leq T})$ can be viewed as a control independent on $\mathcal{F}_t$. Using the independence of the increments of the Brownian motion, together with Fubini's Lemma, it thus follows that

$$\begin{aligned} J(t,x;\nu) \quad &= \quad \int \mathbb{E}\left[g(X_T^{t,x,\nu_{(\omega_s)_{0 \leq s \leq t}}})\right] d\mathbb{P}((\omega_s)_{0 \leq s \leq t}) \\ &\leq \quad \int \tilde{V}(t,x) d\mathbb{P}((\omega_s)_{0 \leq s \leq t}) \;=\; \tilde{V}(t,x). \end{aligned}$$

By arbitrariness of $\nu \in \mathcal{U}_0$, this implies that $\tilde{V}(t,x) \geq V(t,x)$.

## 2.2 The dynamic programming principle

### 2.2.1 A weak dynamic programming principle

The dynamic programming principle is the main tool in the theory of stochastic control. In these notes, we shall prove rigorously a weak version of the dynamic programming which will be sufficient for the derivation of the dynamic programming equation. We denote:

$$V_*(t,x) := \liminf_{(t',x') \to (t,x)} V(t',x') \quad \text{and} \quad V^*(t,x) := \limsup_{(t',x') \to (t,x)} V(t',x'),$$

for all $(t,x) \in \bar{\mathbf{S}}$. We also recall the subset of controls $\mathcal{U}_t$ introduced in (2.7) above.

**Theorem 2.3.** *Assume that $V$ is locally bounded and fix $(t,x) \in \mathbf{S}$. Let $\{\theta^\nu,\ \nu \in \mathcal{U}_t\}$ be a family of finite stopping times independent of $\mathcal{F}_t$ with values in $[t,T]$. Then:*

$$V(t,x) \quad \geq \quad \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[\int_t^{\theta^\nu} \beta^\nu(t,s) f(s, X_s^{t,x,\nu}, \nu_s) ds + \beta^\nu(t,\theta^\nu) V_*(\theta^\nu, X_{\theta^\nu}^{t,x,\nu})\right].$$

*Assume further that $g$ is lower-semicontinuous and $X_{t,x}^\nu \mathbf{1}_{[t,\theta^\nu]}$ is $\mathbb{L}^\infty-$bounded for all $\nu \in \mathcal{U}_t$. Then*

$$V(t,x) \quad \leq \quad \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[\int_t^{\theta^\nu} \beta^\nu(t,s) f(s, X_s^{t,x,\nu}, \nu_s) ds + \beta^\nu(t,\theta^\nu) V^*(\theta^\nu, X_{\theta^\nu}^{t,x,\nu})\right].$$

We shall provide an intuitive justification of this result after the following comments. A rigorous proof is reported in Section 2.2.2 below.

(i) If $V$ is continuous, then $V = V_* = V^*$, and the above weak dynamic programming principle reduces to the classical dynamic programming principle:

$$V(t,x) \quad = \quad \sup_{\nu \in \mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^\theta \beta(t,s) f(s, X_s, \nu_s) ds + \beta(t,\theta) V(\theta, X_\theta)\right] \tag{2.9}$$

(ii) In the discrete-time framework, the dynamic programming principle (2.9) can be stated as follows :

$$V(t,x) \quad = \quad \sup_{u \in U} \mathbb{E}_{t,x}\left[f(t, X_t, u) + e^{-k(t+1, X_{t+1}, u)} V(t+1, X_{t+1})\right].$$

Observe that the supremum is now taken over the subset $U$ of the finite dimensional space $R^k$. Hence, the dynamic programming principle allows to reduce the initial maximization problem, over the subset $\mathcal{U}$ of the infinite dimensional set of $\mathbb{R}^k-$valued processes, into a finite dimensional maximization problem. However, we are still facing an infinite dimensional problem since the dynamic programming principle relates the value function at time $t$ to the value function at time $t+1$.

(iii) In the context of the above discrete-time framework with finite horizon $T < \infty$, notice that the dynamic programming principle suggests the following backward algorithm to compute $V$ as well as the associated optimal strategy (when it exists). Since $V(T,\cdot) = g$ is known, the above dynamic programming principle can be applied recursively in order to deduce the value function $V(t,x)$ for every $t$.

(iv) In the continuous time setting, there is no obvious counterpart to the above backward algorithm. But, as the stopping time $\theta$ approaches $t$, the above dynamic programming principle implies a special local behavior for the value function $V$. When $V$ is known to be smooth, this will be obtained by means of Itô's formula.

(v) It is usually very difficult to determine *a priori* the regularity of $V$. The situation is even worse since there are many counter-examples showing that the value function $V$ can not be expected to be smooth in general; see Section 2.4. This problem is solved by appealing to the notion of viscosity solutions, which provides a weak local characterization of the value function $V$.

(vi) Once the local behavior of the value function is characterized, we are faced to the important uniqueness issue, which implies that $V$ is completely characterized by its local behavior together with some convenient boundary condition.

**Intuitive justification of** (2.9). Let us assume that $V$ is continuous. In particular, $V$ is measurable and $V = V_* = V^*$. Let $\tilde{V}(t,x)$ denote the right hand-side of (2.9).

By the tower Property of the conditional expectation operator, it is easily checked that

$$J(t,x,\nu) \quad = \quad \mathbb{E}_{t,x}\left[\int_t^\theta \beta(t,s)f(s,X_s,\nu_s)ds + \beta(t,\theta)J(\theta,X_\theta,\nu)\right].$$

Since $J(\theta,X_\theta,\nu) \leq V(\theta,X_\theta)$, this proves that $V \leq \tilde{V}$. To prove the reverse inequality, let $\mu \in \mathcal{U}$ and $\varepsilon > 0$ be fixed, and consider an $\varepsilon-$optimal control $\nu^\varepsilon$ for the problem $V(\theta,X_\theta)$, i.e.

$$J(\theta,X_\theta,\nu^\varepsilon) \quad \geq \quad V(\theta,X_\theta) - \varepsilon.$$

Clearly, one can choose $\nu^\varepsilon = \mu$ on the stochastic interval $[t,\theta]$. Then

$$\begin{aligned} V(t,x) \quad &\geq \quad J(t,x,\nu^\varepsilon) \;=\; \mathbb{E}_{t,x}\left[\int_t^\theta \beta(t,s)f(s,X_s,\mu_s)ds + \beta(t,\theta)J(\theta,X_\theta,\nu^\varepsilon)\right] \\ &\geq \quad \mathbb{E}_{t,x}\left[\int_t^\theta \beta(t,s)f(s,X_s,\mu_s)ds + \beta(t,\theta)V(\theta,X_\theta)\right] - \varepsilon\,\mathbb{E}_{t,x}[\beta(t,\theta)]\,. \end{aligned}$$

This provides the required inequality by the arbitrariness of $\mu \in \mathcal{U}$ and $\varepsilon > 0$. $\diamondsuit$

**Exercise**. Where is the gap in the above sketch of the proof ?

### 2.2.2 Dynamic programming without measurable selection

In this section, we provide a rigorous proof of Theorem 2.3. Notice that, we have no information on whether $V$ is measurable or not. Because of this, the right-hand side of the classical dynamic programming principle (2.9) is not even known to be well-defined.

The formulation of Theorem 2.3 avoids this measurability problem since $V_*$ and $V^*$ are lower- and upper-semicontinuous, respectively, and therefore measurable. In addition, it allows to avoid the typically heavy technicalities related to measurable selection arguments needed for the proof of the classical (2.9) after a convenient relaxation of the control problem, see e.g. El Karoui and Jeanblanc [5].

**Proof of Theorem 2.3** For simplicity, we consider the finite horizon case $T < \infty$, so that, without loss of generality, we assume $f = k = 0$, See Remark 2.2 (iii). The extension to the infinite horizon framework is immediate.

**1.** Let $\nu \in \mathcal{U}_t$ be arbitrary and set $\theta := \theta^\nu$. Then:

$$\mathbb{E}\left[g\left(X_T^{t,x,\nu}\right)|\mathcal{F}_\theta\right](\omega) = J(\theta(\omega), X_\theta^{t,x,\nu}(\omega); \tilde{\nu}_\omega),$$

where $\tilde{\nu}_\omega$ is obtained from $\nu$ by freezing its trajectory up to the stopping time $\theta$. Since, by definition, $J(\theta(\omega), X_\theta^{t,x,\nu}(\omega); \tilde{\nu}_\omega) \le V^*(\theta(\omega), X_\theta^{t,x,\nu}(\omega))$, it follows from the tower property of conditional expectations that

$$\mathbb{E}\left[g\left(X_T^{t,x,\nu}\right)\right] \;=\; \mathbb{E}\left[\mathbb{E}\left[g\left(X_T^{t,x,\nu}\right)|\mathcal{F}_\theta\right]\right] \;\le\; \mathbb{E}\left[V^*\left(\theta, X_\theta^{t,x,\nu}\right)\right],$$

which provides the second inequality of Theorem 2.3 by the arbitrariness of $\nu \in \mathcal{U}_t$.

**2.** Let $\varepsilon > 0$ be given, and consider an arbitrary function

$$\varphi : \mathbf{S} \longrightarrow \mathbb{R} \quad \text{such that} \quad \varphi \text{ upper-semicontinuous and } V \ge \varphi.$$

2.a. There is a family $(\nu^{(s,y),\varepsilon})_{(s,y)\in\mathbf{S}} \subset \mathcal{U}_0$ such that:

$$\nu^{(s,y),\varepsilon} \in \mathcal{U}_s \text{ and } J(s,y;\nu^{(s,y),\varepsilon}) \ge V(s,y) - \varepsilon, \quad \text{for every} \quad (s,y) \in \mathbf{S} \tag{2.10}$$

Since $g$ is lower-semicontinuous and has quadratic growth, it follows from Theorem 2.1 that the function $(t',x') \mapsto J(t',x';\nu^{(s,y),\varepsilon})$ is lower-semicontinuous, for fixed $(s,y) \in \mathbf{S}$. Together with the upper-semicontinuity of $\varphi$, this implies that we may find a family $(r_{(s,y)})_{(s,y)\in\mathbf{S}}$ of positive scalars so that, for any $(s,y) \in \mathbf{S}$,

$$\begin{gathered}\varphi(s,y) - \varphi(t',x') \ge -\varepsilon \text{ and } J(s,y;\nu^{(s,y),\varepsilon}) - J(t',x';\nu^{(s,y),\varepsilon}) \le \varepsilon \\ \text{for } (t',x') \in B(s,y;r_{(s,y)}),\end{gathered} \tag{2.11}$$

where, for $r > 0$ and $(s,y) \in \mathbf{S}$,

$$B(s,y;r) := \{(t',x') \in \mathbf{S} \;:\; t' \in (s-r,s),\; |x'-y| < r\}\;.$$

Clearly, $\left\{B(s,y;r) : (s,y) \in \mathbf{S},\; 0 < r \le r_{(s,y)}\right\}$ forms an open covering of $[0,T) \times \mathbb{R}^d$. It then follows from the Lindelöf covering Theorem, see e.g. [4] Theorem 6.3 Chap. VIII, that we can find a countable sequence $(t_i, x_i, r_i)_{i\ge1}$ of elements of $\mathbf{S} \times \mathbb{R}$, with $0 < r_i \le r_{(t_i,x_i)}$ for all $i \ge 1$, such that $\mathbf{S} \subset$

$\{T\} \times \mathbb{R}^d \cup (\cup_{i\geq 1} B(t_i, x_i; r_i))$. Set $A_0 := \{T\} \times \mathbb{R}^d$, $C_{-1} := \emptyset$, and define the sequence

$$A_{i+1} := B(t_{i+1}, x_{i+1}; r_{i+1}) \setminus C_i \quad \text{where} \quad C_i := C_{i-1} \cup A_i,\ i \geq 0.$$

With this construction, it follows from (2.10), (2.11), together with the fact that $V \geq \varphi$, that the countable family $(A_i)_{i\geq 0}$ satisfies

$$\begin{array}{c}(\theta, X_\theta^{t,x,\nu}) \in \cup_{i\geq 0} A_i\ \mathbb{P}-\text{a.s.},\ A_i \cap A_j = \emptyset \text{ for } i \neq j \in \mathbb{N}, \\ \text{and } J(\cdot; \nu^{i,\varepsilon}) \geq \varphi - 3\varepsilon \text{ on } A_i \text{ for } i \geq 1,\end{array} \tag{2.12}$$

where $\nu^{i,\varepsilon} := \nu^{(t_i,x_i),\varepsilon}$ for $i \geq 1$.
2.b. We now prove the first inequality in Theorem 2.3. We fix $\nu \in \mathcal{U}_t$ and $\theta \in \mathcal{T}_{[t,T]}^t$. Set $A^n := \cup_{0\leq i\leq n} A_i$, $n \geq 1$. Given $\nu \in \mathcal{U}_t$, we define for $s \in [t, T]$:

$$\nu_s^{\varepsilon,n} \quad := \quad \mathbf{1}_{[t,\theta]}(s)\nu_s + \mathbf{1}_{(\theta,T]}(s)\Big(\nu_s \mathbf{1}_{(A^n)^c}(\theta, X_\theta^{t,x,\nu}) + \sum_{i=1}^n \mathbf{1}_{A_i}(\theta, X_\theta^{t,x,\nu})\nu_s^{i,\varepsilon}\Big).$$

Notice that $\{(\theta, X_\theta^{t,x,\nu}) \in A_i\} \in \mathcal{F}_\theta^t$. Then, it follows that $\nu^{\varepsilon,n} \in \mathcal{U}_t$. Then, it follows from (2.12) that:

$$\begin{aligned}
\mathbb{E}\left[g\left(X_T^{t,x,\nu^{\varepsilon,n}}\right)|\mathcal{F}_\theta\right]\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right) \quad &= \quad V\left(T, X_T^{t,x,\nu^{\varepsilon,n}}\right)\mathbf{1}_{A_0}\left(\theta, X_\theta^{t,x,\nu}\right) \\
&+ \quad \sum_{i=1}^n J(\theta, X_\theta^{t,x,\nu}, \nu^{i,\varepsilon})\mathbf{1}_{A_i}\left(\theta, X_\theta^{t,x,\nu}\right) \\
&\geq \quad \sum_{i=0}^n \left(\varphi(\theta, X_\theta^{t,x,\nu} - 3\varepsilon\right)\mathbf{1}_{A_i}\left(\theta, X_\theta^{t,x,\nu}\right) \\
&= \quad \left(\varphi(\theta, X_\theta^{t,x,\nu}) - 3\varepsilon\right)\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right),
\end{aligned}$$

which, by definition of $V$ and the tower property of conditional expectations, implies

$$\begin{aligned}
V(t,x) \quad &\geq \quad J(t,x,\nu^{\varepsilon,n}) \\
&= \quad \mathbb{E}\left[\mathbb{E}\Big[g\left(X_T^{t,x,\nu^{\varepsilon,n}}\right)|\mathcal{F}_\theta\Big]\right] \\
&\geq \quad \mathbb{E}\left[\left(\varphi\left(\theta, X_\theta^{t,x,\nu}\right) - 3\varepsilon\right)\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right)\right] \\
&\quad\ + \mathbb{E}\left[g\left(X_T^{t,x,\nu}\right)\mathbf{1}_{(A^n)^c}\left(\theta, X_\theta^{t,x,\nu}\right)\right].
\end{aligned}$$

Since $g\left(X_T^{t,x,\nu}\right) \in \mathbb{L}^1$, it follows from the dominated convergence theorem that:

$$\begin{aligned}
V(t,x) \quad &\geq \quad -3\varepsilon + \liminf_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right)\right] \\
&= \quad -3\varepsilon + \lim_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})^+\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right)\right] \\
&\qquad - \lim_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})^-\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right)\right] \\
&= \quad -3\varepsilon + \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})\right],
\end{aligned}$$

where the last equality follows from the left-hand side of (2.12) and from the monotone convergence theorem, due to the fact that either $\mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})^+\right] < \infty$ or $\mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})^-\right] < \infty$. By the arbitrariness of $\nu \in \mathcal{U}_t$ and $\varepsilon > 0$, this shows that:

$$V(t,x) \quad \geq \quad \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})\right]. \tag{2.13}$$

**3.** It remains to deduce the first inequality of Theorem 2.3 from (2.13). Fix $r > 0$. It follows from standard arguments, see e.g. Lemma 3.5 in [12], that we can find a sequence of continuous functions $(\varphi_n)_n$ such that $\varphi_n \leq V_* \leq V$ for all $n \geq 1$ and such that $\varphi_n$ converges pointwise to $V_*$ on $[0,T] \times B_r(0)$. Set $\phi_N := \min_{n\geq N} \varphi_n$ for $N \geq 1$ and observe that the sequence $(\phi_N)_N$ is non-decreasing and converges pointwise to $V_*$ on $[0,T] \times B_r(0)$. By (2.13) and the monotone convergence Theorem, we then obtain:

$$V(t,x) \quad \geq \quad \lim_{N\to\infty} \mathbb{E}\left[\phi_N(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right] = \mathbb{E}\left[V_*(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right].$$

$\diamondsuit$

## 2.3 The dynamic programming equation

The dynamic programming equation is the infinitesimal counterpart of the dynamic programming principle. It is also widely called the *Hamilton-Jacobi-Bellman* equation. In this section, we shall derive it under strong smoothness assumptions on the value function. Let $\mathcal{S}^d$ be the set of all $d \times d$ symmetric matrices with real coefficients, and define the map $H : \mathbf{S} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^d$ by :

$$\begin{aligned} &H(t,x,r,p,\gamma) \\ := \ &\sup_{u\in U} \left\{ -k(t,x,u)r + b(t,x,u)\cdot p + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}}(t,x,u)\gamma] + f(t,x,u) \right\}. \end{aligned}$$

We also need to introduce the linear second order operator $\mathcal{L}^u$ associated to the controlled process $\{\beta(0,t)X_t^u,\, t \geq 0\}$ controlled by the constant control process $u$ :

$$\begin{aligned} \mathcal{L}^u\varphi(t,x) \quad := \quad & -k(t,x,u)\varphi(t,x) + b(t,x,u)\cdot D\varphi(t,x) \\ & + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}(t,x,u)D^2\varphi(t,x)\right], \end{aligned}$$

where $D$ and $D^2$ denote the gradient and the Hessian operators with respect to the $x$ variable. With this notation, we have by Itô's formula:

$$\begin{aligned} \beta^\nu(0,s)\varphi(s,X_s^\nu) - \beta^\nu(0,t)\varphi(t,X_t^\nu) \quad = \quad & \int_t^s \beta^\nu(0,r)\left(\partial_t + \mathcal{L}^{\nu_r}\right)\varphi(r,X_r^\nu)dr \\ & + \int_t^s \beta^\nu(0,r)D\varphi(r,X_r^\nu)\cdot\sigma(r,X_r^\nu,\nu_r)dW_r \end{aligned}$$

for every $s \geq t$ and smooth function $\varphi \in C^{1,2}([t,s],\mathbb{R}^n)$ and each admissible control process $\nu \in \mathcal{U}_0$.

**Proposition 2.4.** *Assume the value function $V \in C^{1,2}([0,T),\mathbb{R}^n)$, and let the coefficients $k(\cdot,\cdot,u)$ and $f(\cdot,\cdot,u)$ be continuous in $(t,x)$ for all fixed $u \in U$. Then, for all $(t,x) \in \mathbf{S}$:*

$$-\partial_t V(t,x) - H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \quad \geq \quad 0. \tag{2.14}$$

*Proof.* Let $(t,x) \in \mathbf{S}$ and $u \in U$ be fixed and consider the constant control process $\nu = u$, together with the associated state process $X$ with initial data $X_t = x$. For all $h > 0$, Define the stopping time :

$$\theta_h \quad := \quad \inf\{s > t \;:\; (s-t, X_s - x) \notin [0,h) \times \alpha B\} \;,$$

where $\alpha > 0$ is some given constant, and $B$ denotes the unit ball of $\mathbb{R}^n$. Notice that $\theta_h \longrightarrow t$, $\mathbb{P}-$a.s. when $h \searrow 0$, and $\theta_h = h$ for $h \leq \bar{h}(\omega)$ sufficiently small.
**1.** From the first inequality of the dynamic programming principle, it follows that :

$$\begin{aligned}
0 \quad &\leq \quad \mathbb{E}_{t,x}\left[\beta(0,t)V(t,x) - \beta(0,\theta_h)V(\theta_h, X_{\theta_h}) - \int_t^{\theta_h} \beta(0,r)f(r,X_r,u)dr\right] \\
&= \quad -\mathbb{E}_{t,x}\left[\int_t^{\theta_h} \beta(0,r)(\partial_t V + \mathcal{L}^{\cdot} V + f)(r,X_r,u)dr\right] \\
&\qquad -\mathbb{E}_{t,x}\left[\int_t^{\theta_h} \beta(0,r)DV(r,X_r)\cdot\sigma(r,X_r,u)dW_r\right],
\end{aligned}$$

the last equality follows from Itô's formula and uses the crucial smoothness assumption on $V$.
**2.** Observe that $\beta(0,r)DV(r,X_r)\cdot\sigma(r,X_r,u)$ is bounded on the stochastic interval $[t,\theta_h]$. Therefore, the second expectation on the right hand-side of the last inequality vanishes, and we obtain :

$$-\mathbb{E}_{t,x}\left[\frac{1}{h}\int_t^{\theta_h} \beta(0,r)(\partial_t V + \mathcal{L}^{\cdot} V + f)(r,X_r,u)dr\right] \quad \geq \quad 0$$

We now send $h$ to zero. The a.s. convergence of the random value inside the expectation is easily obtained by the mean value Theorem; recall that $\theta_h = h$ for sufficiently small $h > 0$. Since the random variable $h^{-1}\int_t^{\theta_h} \beta(0,r)(\mathcal{L}^{\cdot} V + f)(r,X_r,u)dr$ is essentially bounded, uniformly in $h$, on the stochastic interval $[t,\theta_h]$, it follows from the dominated convergence theorem that :

$$-\partial_t V(t,x) - \mathcal{L}^u V(t,x) - f(t,x,u) \quad \geq \quad 0.$$

By the arbitrariness of $u \in U$, this provides the required claim. $\diamondsuit$

We next wish to show that $V$ satisfies the nonlinear partial differential equation (2.15) with equality. This is a more technical result which can be proved by different methods. We shall report a proof, based on a contradiction argument, which provides more intuition on this result, although it might be slightly longer than the usual proof reported in standard textbooks.

**Proposition 2.5.** *Assume the value function* $V \in C^{1,2}([0,T), \mathbb{R}^n)$*, and let the function* $H$ *be upper semicontinuous, and* $\|k^+\|_\infty < \infty$*. Then, for all* $(t,x) \in$ $\mathbf{S}$*:*

$$-\partial_t V(t,x) - H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \quad \leq \quad 0. \tag{2.15}$$

*Proof.* Let $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$ be fixed, assume to the contrary that

$$\partial_t V(t_0,x_0) + H\left(t_0,x_0,V(t_0,x_0),DV(t_0,x_0),D^2V(t_0,x_0)\right) \quad < \quad 0, \tag{2.16}$$

and let us work towards a contradiction.
**1.** For a given parameter $\varepsilon > 0$, define the smooth function $\varphi \geq V$ by

$$\varphi(t,x) \quad := \quad V(t,x) + \varepsilon\left(|t-t_0|^2 + |x-x_0|^4\right).$$

Then

$$(V-\varphi)(t_0,x_0) = 0, \quad (DV - D\varphi)(t_0,x_0) = 0, \quad (\partial_t V - \partial_t\varphi)(t_0,x_0) = 0,$$
$$\text{and} \quad (D^2V - D^2\varphi)(t_0,x_0) = 0,$$

and (2.16) says that:

$$h(t_0,x_0) \;:=\; \partial_t\varphi(t_0,x_0) + H\left(t_0,x_0,\varphi(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0)\right) \quad < \quad 0.$$

**2.** By upper semicontinuity of $H$, we have:

$$h(t,x) < 0 \;\text{ on }\; \mathcal{N}_\eta := (-\eta,\eta) \times \eta B \quad \text{for } \eta > 0 \text{ sufficiently small,}$$

where $B$ denotes the unit ball centered at $x_0$. We next observe that the parameter $\gamma$ defined by the following is positive:

$$-\gamma e^{\eta\|k^+\|_\infty} \quad := \quad \max_{\partial\mathcal{N}_\eta}\,(V-\varphi) \;<\; 0. \tag{2.17}$$

**3.** Let $\nu$ be an arbitrary control process, and denote by $X$ and $\beta$ the controlled process and the discount factor defined by $\nu$ and the initial data $X_{t_0} = x_0$. Consider the stopping time

$$\theta \quad := \quad \inf\left\{s > t \;:\; (s,X_s) \not\in \mathcal{N}_\eta\right\},$$

and observe that, by continuity of the state process, $(\theta,X_\theta) \in \partial\mathcal{N}_\eta$, so that :

$$(V-\varphi)(\theta,X_\theta) \quad \leq \quad -\gamma e^{\eta\|k^+\|_\infty}$$

by (2.17). Recalling that $\beta(t_0,t_0) = 1$, we now compute that:

$$\begin{aligned}\beta(t_0,\theta)V(\theta,X_\theta) - V(t_0,x_0) \quad &\leq \quad \int_{t_0}^{\theta} d[\beta(t_0,r)\varphi(r,X_r)] - \gamma e^{\eta\|k^+\|_\infty}\beta(t_0,\theta)\\ &\leq \quad \int_{t_0}^{\theta} d[\beta(t_0,r)\varphi(r,X_r)] - \gamma.\end{aligned}$$

By Itô's formula, this provides :

$$V(t_0, x_0) \quad \geq \quad \gamma + \mathbb{E}_{t_0,x_0}\Big[\beta(t_0,\theta)V(\theta, X_\theta) - \int_{t_0}^{\theta} \beta(t_0, r)(\partial_t\varphi + \mathcal{L}^{\nu_r}\varphi)(r, X_r)dr\Big],$$

where the "$dW$" integral term has zero mean, as its integrand is bounded on the stochastic interval $[t_0, \theta]$. Observe also that $(\partial_t\varphi + \mathcal{L}^{\nu_r}\varphi)(r, X_r) + f(r, X_r, \nu_r) \leq h(r, X_r) \leq 0$ on the stochastic interval $[t_0, \theta]$. We therefore deduce that :

$$V(t_0, x_0) \quad \geq \quad \gamma + \mathbb{E}_{t_0,x_0}\Big[\int_{t_0}^{\theta} \beta(t_0, r)f(r, X_r, \nu_r)dr + \beta(t_0, \theta)V(\theta, X_\theta)\Big].$$

As $\gamma$ is a positive constant independent of $\nu$, this implies that:

$$V(t_0, x_0) \quad \geq \quad \gamma + \sup_{\nu\in\mathcal{U}_t} \mathbb{E}_{t_0,x_0}\Big[\int_{t_0}^{\theta} \beta(t_0, r)f(r, X_r, \nu_r)dr + \beta(t_0, \theta)V(\theta, X_\theta)\Big],$$

which is the required contradiction of the second part of the dynamic programming principle, and thus completes the proof. ◊

As a consequence of Propositions 2.4 and 2.5, we have the main result of this section :

**Theorem 2.6.** *Let the conditions of Propositions 2.5 and 2.4 hold. Then, the value function $V$ solves the Hamilton-Jacobi-Bellman equation*

$$-\partial_t V - H\left(., V, DV, D^2V\right) \; = \; 0 \quad on \quad \mathbf{S}. \tag{2.18}$$

## 2.4 On the regularity of the value function

The purpose of this paragraph is to show that the value function should not be expected to be smooth in general. We start by proving the continuity of the value function under strong conditions; in particular, we require the set $U$ in which the controls take values to be bounded. We then give a simple example in the deterministic framework where the value function is not smooth. Since it is well known that stochastic problems are “more regular” than deterministic ones, we also give an example of stochastic control problem whose value function is not smooth.

### 2.4.1 Continuity of the value function for bounded controls

For notational simplicity, we reduce the stochastic control problem to the case $f = k \equiv 0$, see Remark 2.2 (iii). Our main concern, in this section, is to show the standard argument for proving the continuity of the value function. Therefore, the following results assume strong conditions on the coefficients of the model in order to simplify the proofs. We first start by examining the value function $V(t, \cdot)$ for fixed $t \in [0, T]$.

**Proposition 2.7.** *Let $f = k \equiv 0$, $T < \infty$, and assume that $g$ is Lipschitz continuous. Then:*
(i) *$V$ is Lipschitz in $x$, uniformly in $t$.*
(ii) *Assume further that $U$ is bounded. Then $V$ is $\frac{1}{2}-$Hölder-continuous in $t$, and there is a constant $C > 0$ such that:*

$$\left|V(t,x) - V(t',x)\right| \;\leq\; C(1+|x|)\sqrt{|t-t'|};\quad t,t' \in [0,T],\ x \in \mathbb{R}^n.$$

*Proof.* (i) For $x, x' \in \mathbb{R}^n$ and $t \in [0,T)$, we first estimate that:

$$\begin{aligned}
|V(t,x) - V(t,x')| &\leq \sup_{\nu\in\mathcal{U}_0} \mathbb{E}\left|g\left(X_T^{t,x,\nu}\right) - g\left(X_T^{t,x',\nu}\right)\right| \\
&\leq Const \sup_{\nu\in\mathcal{U}_0} \mathbb{E}\left|X_T^{t,x,\nu} - X_T^{t,x',\nu}\right| \\
&\leq Const\,|x-x'|,
\end{aligned}$$

where we used the Lipschitz-continuity of $g$ together with the flow estimates of Theorem 1.4, and the fact that the coefficients $b$ and $\sigma$ are Lipschitz in $x$ uniformly in $(t,u)$. This compltes the proof of the Lipschitz property of the value function $V$.
(ii) To prove the Hölder continuity in $t$, we shall use the dynamic programming principle.

(ii-1) We first make the following important observation. A careful review of the proof of Theorem 2.3 reveals that, whenever the stopping times $\theta^\nu$ are constant (i.e. deterministic), the dynamic programming principle holds true with the semicontinuous envelopes taken only with respect to the $x-$variable. Since $V$ was shown to be continuous in the first part of this proof, we deduce that:

$$V(t,x) \;=\; \sup_{\nu\in\mathcal{U}_0} \mathbb{E}\left[V\left(t', X_{t'}^{t,x,\nu}\right)\right] \tag{2.19}$$

for all $x \in \mathbb{R}^n$, $t < t' \in [0,T]$.

(ii-2) Fix $x \in \mathbb{R}^n$, $t < t' \in [0,T]$. By the dynamic programming principle (2.19), we have :

$$\begin{aligned}
|V(t,x) - V(t',x)| &= \left|\sup_{\nu\in\mathcal{U}_0} \mathbb{E}\left[V\left(t', X_{t'}^{t,x,\nu}\right)\right] - V(t',x)\right| \\
&\leq \sup_{\nu\in\mathcal{U}_0} \mathbb{E}\left|V\left(t', X_{t'}^{t,x,\nu}\right) - V(t',x)\right|.
\end{aligned}$$

By the Lipschitz-continuity of $V(s,\cdot)$ established in the first part of this proof, we see that :

$$|V(t,x) - V(t',x)| \;\leq\; Const \sup_{\nu\in\mathcal{U}_0} \mathbb{E}\left|X_{t'}^{t,x,\nu} - x\right|. \tag{2.20}$$

We shall now prove that

$$\sup_{\nu\in\mathcal{U}} \mathbb{E}\left|X_{t'}^{t,x,\nu} - x\right| \;\leq\; Const\,(1+|x|)|t-t'|^{1/2}, \tag{2.21}$$

which provides the required $(1/2)-$Hölder continuity in view of (2.20). By definition of the process $X$, and assuming $t < t'$, we have

$$\begin{aligned} \mathbb{E}\left|X_{t'}^{t,x,\nu} - x\right|^2 &= \mathbb{E}\left|\int_t^{t'} b(r, X_r, \nu_r)dr + \int_t^{t'} \sigma(r, X_r, \nu_r)dW_r\right|^2 \\ &\leq Const\, \mathbb{E}\left[\int_t^{t'} |h(r, X_r, \nu_r)|^2\, dr\right] \end{aligned}$$

where $h := [b^2 + \sigma^2]^{1/2}$. Since $h$ is Lipschitz-continuous in $(t, x, u)$ and has quadratic growth in $x$ and $u$, this provides:

$$\mathbb{E}\left|X_{t'}^{t,x,\nu} - x\right|^2 \leq Const\left(\int_t^{t'} (1 + |x|^2 + |\nu_r|^2)dr + \int_t^{t'} \mathbb{E}\left|X_r^{t,x,\nu} - x\right|^2 dr\right).$$

Since the control process $\nu$ is uniformly bounded, we obtain by the Gronwall lemma the estimate:

$$\mathbb{E}\left|X_{t'}^{t,x,\nu} - x\right|^2 \leq Const\,(1 + |x|^2)|t' - t|, \tag{2.22}$$

where the constant does not depend on the control $\nu$. ♢

**Remark 2.8.** When $f$ and/or $k$ are non-zero, the conditions required on $f$ and $k$ in order to obtain the $(1/2)-$Hölder continuity of the value function can be deduced from the reduction of Remark 2.2 (iii).

**Remark 2.9.** Further regularity results can be proved for the value function under convenient conditions. Typically, one can prove that $\mathcal{L}^u V$ exists in the generalized sense, for all $u \in U$. This implies immediately that the result of Proposition 2.5 holds in the generalized sense. More technicalities are needed in order to derive the result of Proposition 2.4 in the generalized sense. We refer to [6], §IV.10, for a discussion of this issue.

### 2.4.2 A deterministic control problem with non-smooth value function

Let $\sigma \equiv 0$, $b(x, u) = u$, $U = [-1, 1]$, and $n = 1$. The controlled state is then the one-dimensional deterministic process defined by :

$$X_s = X_t + \int_t^s \nu_t dt \quad \text{for} \quad 0 \leq t \leq s \leq T\,.$$

Consider the deterministic control problem

$$V(t, x) := \sup_{\nu \in \mathcal{U}} (X_T)^2.$$

The value function of this problem is easily seen to be given by :

$$V(t, x) = \begin{cases} (x + T - t)^2 & \text{for } x \geq 0 \quad \text{with optimal control } \hat{u} = 1\,, \\ (x - T + t)^2 & \text{for } x \leq 0 \quad \text{with optimal control } \hat{u} = -1\,. \end{cases}$$

This function is continuous. However, a direct computation shows that it is not differentiable at $x = 0$.

### 2.4.3 A stochastic control problem with non-smooth value function

Let $U = \mathbb{R}$, and the controlled process $X$ be the scalar process defined by the dynamics:

$$dX_t \;=\; \nu_t dW_t,$$

where $W$ is a scalar Brownian motion. Let $g$ be a lower semicontinuous mapping on $\mathbb{R}$, with $-\alpha' - \beta'|x| \le g(x) \le \alpha + \beta x$, $x \in \mathbb{R}$, for some constants $\alpha, \alpha'\beta, \beta' \in \mathbb{R}$. We consider the stochastic control problem

$$V(t,x) \;:=\; \sup_{\nu \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[g(X_T^\nu)\right].$$

Let us assume that $V$ is smooth, and work towards a contradiction.

**1.** If $V$ is $C^{1,2}([0,T),\mathbb{R})$, then it follows from Proposition 2.4 that $V$ satisfies

$$-\partial_t V - \frac{1}{2}u^2 D^2 V \ge 0 \quad \text{for all} \quad u \in \mathbb{R},$$

and all $(t,x) \in [0,T) \times \mathbb{R}$. By sending $u$ to infinity, it follows that

$$V(t,\cdot) \quad \text{is concave for all} \quad t \in [0,T). \tag{2.23}$$

**2.** Notice that $V(t,x) \ge \mathbb{E}_{t,x}\left[g(X_T^0)\right] = g(x)$. Then, it follows from (2.23) that:

$$V(t,x) \;\ge\; g^{\mathrm{conc}}(x) \quad \text{for all} \quad (t,x) \in [0,T) \times \mathbb{R}, \tag{2.24}$$

where $g^{\mathrm{conc}}$ is the concave envelope of $g$, i.e. the smallest concave majorant of $g$. Notice that $g^{\mathrm{conc}} < \infty$ as $g$ is bounded from above by a line.

**3.** Since $g \le g^{\mathrm{conc}}$, we see that

$$V(t,x) \;:=\; \sup_{\nu \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[g(X_T^\nu)\right] \;\le\; \sup_{\nu \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[g^{\mathrm{conc}}(X_T^\nu)\right] \;=\; g^{\mathrm{conc}}(x),$$

by the martingale property of $X^\nu$. In view of (2.24), we have then proved that

$$\begin{aligned} V \;&\in\; C^{1,2}([0,T),\mathbb{R}) \\ \Longrightarrow \quad V(t,x) \;&=\; g^{\mathrm{conc}}(x) \text{ for all } (t,x) \in [0,T) \times \mathbb{R}. \end{aligned}$$

Now recall that this implication holds for any arbitrary non-negative lower semi-continuous function $g$. We then obtain a contradiction whenever the function $g^{\mathrm{conc}}$ is not $C^2(\mathbb{R})$. Hence

$$g^{\mathrm{conc}} \;\notin\; C^2(\mathbb{R}) \quad \Longrightarrow \quad V \;\notin\; C^{1,2}([0,T),\mathbb{R}^2).$$

# Chapter 3

# Optimal Stopping and Dynamic Programming

As in the previous chapter, we assume here that the filtration $\mathbb{F}$ is defined as the $\mathbb{P}-$augmentation of the canonical filtration of the Brownian motion $W$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Our objective is to derive similar results, as those obtained in the previous chapter for standard stochastic control problems, in the context of optimal stopping problems. We will then first start by the formulation of optimal stopping problems, then the corresponding dynamic programming principle, and dynamic programming equation.

## 3.1 Optimal stopping problems

For $0 \le t \le T \le \infty$, we denote by $\mathcal{T}_{[t,T]}$ the collection of all $\mathbb{F}-$stopping times with values in $[t, T]$. We also recall the notation $\mathbf{S} := [0, T) \times \mathbb{R}^n$ for the parabolic state space of the underlying state process $X$ defined by the stochastic differential equation:

$$dX_t \quad = \quad \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{3.1}$$

where $\mu$ and $\sigma$ are defined on $\bar{\mathbf{S}}$ and take values in $\mathbb{R}^n$ and $\mathcal{S}_n$, respectively. We assume that $\mu$ and $\sigma$ satisfies the usual Lipschitz and linear growth conditions so that the above SDE has a unique strong solution satisfying the integrability proved in Theorem 1.2.

The infinitesimal generator of the Markov diffusion process $X$ is denoted by

$$\mathcal{A}\varphi := \mu \cdot D\varphi + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}} D^2\varphi\right].$$

Let $g$ be a measurable function from $\mathbb{R}^n$ to $\mathbb{R}$, and assume that:

$$\mathbb{E}\left[\sup_{0\le t<T} |g(X_t)|\right] \quad < \quad \infty. \tag{3.2}$$

For instance, if $g$ has polynomial growth, the latter integrability condition is automatically satisfied. Under this condition, the following criterion:

$$J(t,x,\tau) := \mathbb{E}\left[g\left(X_\tau^{t,x}\right)\mathbf{1}_{\tau<\infty}\right] \tag{3.3}$$

is well-defined for all $(t,x) \in \mathbf{S}$ and $\tau \in \mathcal{T}_{[t,T]}$. Here, $X^{t,x}$ denotes the unique strong solution of (3.1) with initial condition $X_t^{t,x} = x$.

The optimal stopping problem is now defined by:

$$V(t,x) := \sup_{\tau\in\mathcal{T}_{[t,T]}} J(t,x,\tau) \quad \text{for all} \quad (t,x) \in \mathbf{S}. \tag{3.4}$$

A stopping time $\hat{\tau} \in \mathcal{T}_{[t,T]}$ is called an optimal stopping rule if $V(t,x) = J(t,x,\hat{\tau})$.

The set

$$\mathcal{S} \quad := \quad \{(t,x) : \ V(t,x) = g(x)\} \tag{3.5}$$

is called the *stopping region* and is of particular interest: whenever the state is in this region, it is optimal to stop immediately. Its complement $\mathcal{S}^c$ is called the *continuation region*.

**Remark 3.1.** As in the previous chapter, we could have considered an appearently more general criterion

$$V(t,x) \quad := \quad \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}\left[\int_t^\tau \beta(t,s)f(s,X_s)ds + \beta(t,\tau)g\left(X_\tau^{t,x}\right)\mathbf{1}_{\tau<\infty}\right],$$

with

$$\beta(t,s) := e^{-\int_t^s k(s,X_s)ds} \quad \text{for} \quad 0 \le t \le s < T.$$

However by introducing the additional state

$$\begin{aligned} Y_t &:= Y_0 + \int_0^t \beta_s f(s,X_s)ds, \\ Z_t &:= Z_0 + \int_0^t Z_s k(s,X_s)ds, \end{aligned}$$

we see immediately that we may reduce this problem to the context of (3.4).

**Remark 3.2.** Consider the subset of stopping rules:

$$\mathcal{T}_{[t,T]}^t \quad := \quad \left\{\tau \in \mathcal{T}_{[t,T]} : \ \tau \text{ independent of } \mathcal{F}_t\right\}. \tag{3.6}$$

By a similar argument as in Remark 2.2 (iv), we can see that the maximization in the optimal stopping problem (3.4) can be restricted to this subset, i.e.

$$V(t,x) := \sup_{\tau\in\mathcal{T}_{[t,T]}^t} J(t,x,\tau) \quad \text{for all} \quad (t,x) \in \mathbf{S}. \tag{3.7}$$

## 3.2 The dynamic programming principle

In the context of optimal stopping problems, the proof of the dynamic programming principle is easier than in the context of stochastic control problems of the previous chapter. The reader may consult the excellent exposition in the book of Karatzas and Shreve [9], Appendix D, where the following dynamic programming principle is proved:

$$V(t,x) \quad = \quad \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<\theta\}} g(X^{t,x}_\tau) + \mathbf{1}_{\{\tau\geq\theta\}} V(\theta, X^{t,x}_\theta)\right], \qquad (3.8)$$

for all $(t,x) \in \mathbf{S}$ and $\tau \in \mathcal{T}_{[t,T]}$. In particular, the proof in the latter reference does not require any heavy measurable selection, and is essentially based on the supermartingale nature of the so-called Snell envelope process. Moreover, we observe that it does not require any Markov property of the underlying state process.

We report here a different proof in the sprit of the weak dynamic programming principle for stochastic control problems proved in the previous chapter. The subsequent argument is specific to our Markovian framework and, in this sense, is weaker than the classical dynamic programming principle. However, the combination of the arguments of this chapter with those of the previous chapter allow to derive a dynamic programming principle for mixed stochastic control and stopping problem.

The following claim will be making using of the subset $\mathcal{T}^t_{[t,T]}$, introduced in (3.6), of all stopping times in $\mathcal{T}_{[t,T]}$ which are independent of $\mathcal{F}_t$, and the notations:

$$V_*(t,x) := \liminf_{(t',x')\to(t,x)} V(t',x') \quad \text{and} \quad V^*(t,x) := \limsup_{(t',x')\to(t,x)} V(t',x')$$

for all $(t,x) \in \bar{\mathbf{S}}$. We recall that $V_*$ and $V^*$ are the lower and upper semicontinuous envelopes of $V$, and that $V_* = V^* = V$ whenever $V$ is continuous.

**Theorem 3.3.** *Assume that $V$ is locally bounded. For $(t,x) \in \mathbf{S}$, let $\theta \in \bar{\mathcal{T}}^t_{[t,T]}$ be a stopping time such that $X^{t,x}_\theta$ is bounded. Then:*

$$V(t,x) \quad \leq \quad \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<\theta\}} g(X^{t,x}_\tau) + \mathbf{1}_{\{\tau\geq\theta\}} V^*(\theta, X^{t,x}_\theta)\right], \qquad (3.9)$$

$$V(t,x) \quad \geq \quad \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<\theta\}} g(X^{t,x}_\tau) + \mathbf{1}_{\{\tau\geq\theta\}} V_*(\theta, X^{t,x}_\theta))\right]. \qquad (3.10)$$

*Proof.* Inequality (3.9) follows immediately from the tower property and the fact that $J \leq V^*$.

We next prove inequality (3.10) with $V_*$ replaced by an arbitrary function

$$\varphi : \mathbf{S} \longrightarrow \mathbb{R} \quad \text{such} \quad \varphi \text{ is upper-semicontinuous and } V \geq \varphi,$$

which implies (3.10) by the same argument as in Step 3 of the proof of Theorem 2.3.

Arguying as in Step 2 of the proof of Theorem 2.3, we first observe that, for every $\varepsilon > 0$, we can find a countable family $\bar{A}_i \subset (t_i - r_i, t_i] \times A_i \subset \mathbf{S}$, together with a sequence of stopping times $\tau^{i,\varepsilon}$ in $\mathcal{T}^{t_i}_{[t_i,T]}$, $i \geq 1$, satisfying $\bar{A}_0 = \{T\} \times \mathbb{R}^d$ and

$$\cup_{i\geq 0}\bar{A}_i = \mathbf{S}, \;\; \bar{A}_i \cap \bar{A}_j = \emptyset \text{ for } i \neq j \in \mathbb{N}, \;\; \bar{J}(\cdot;\tau^{i,\varepsilon}) \geq \varphi - 3\varepsilon \text{ on } \bar{A}_i \text{ for } i \geq 1. \tag{3.11}$$

Set $\bar{A}^n := \cup_{i\leq n}\bar{A}_i$, $n \geq 1$. Given two stopping times $\theta, \tau \in \mathcal{T}^t_{[t,T]}$, it is clear that

$$\tau^{n,\varepsilon} := \tau \mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\geq\theta\}} \left( T\mathbf{1}_{(\bar{A}^n)^c}\left(\theta, X^{t,x}_\theta\right) + \sum_{i=1}^n \tau^{i,\varepsilon}\mathbf{1}_{\bar{A}_i}\left(\theta, X^{t,x}_\theta\right)\right)$$

defines a stopping time in $\mathcal{T}^t_{[t,T]}$. We then deduce from the tower property and (3.11) that

$$\begin{aligned}
\bar{V}(t,x) &\geq \bar{J}(t,x;\tau^{n,\varepsilon}) \\
&\geq \mathbb{E}\left[g\left(X^{t,x}_\tau\right)\mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\geq\theta\}}\left(\varphi(\theta,X^{t,x}_\theta) - 3\varepsilon\right)\mathbf{1}_{\bar{A}^n}(\theta,X^{t,x}_\theta)\right] \\
&\quad + \mathbb{E}\left[\mathbf{1}_{\{\tau\geq\theta\}}g(X^{t,x}_T)\mathbf{1}_{(\bar{A}^n)^c}(\theta,X^{t,x}_\theta)\right].
\end{aligned}$$

By sending $n \to \infty$ and arguing as in the end of Step 2 of the proof of Theorem 2.3, we deduce that

$$\bar{V}(t,x) \;\;\geq\;\; \mathbb{E}\left[g\left(X^{t,x}_\tau\right)\mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\geq\theta\}}\varphi(\theta,X^{t,x}_\theta)\right] - 3\varepsilon,$$

and the result follows from the arbitrariness of $\varepsilon > 0$ and $\tau \in \mathcal{T}^t_{[t,T]}$. ♢

## 3.3 The dynamic programming equation

In this section, we explore the infinitesimal counterpart of the dynamic programming principle of Theorem 3.3, when the value function $V$ is a priori known to be smooth. The smoothness that will be required in this chapter must be so that we can apply Itô's formula to $V$. In particular, $V$ is continuous, and the dynamic programming principle of Theorem 3.3 reduces to the classical dynamic programming principle (3.8).

Loosely speaking, the following dynamic programming equation says the following:

- In the stopping region $\mathcal{S}$ defined in (3.5), continuation is sub-optimal, and therefore the linear PDE must hold with inequality in such a way that the value function is a submartingale.

- In the continuation region $\mathcal{S}^c$, it is optimal to delay the stopping decision after some small moment, and therefore the value function must solve a linear PDE as in Chapter 1.

**Theorem 3.4.** *Assume that $V \in C^{1,2}([0,T),\mathbb{R}^n)$, and let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous. Then $V$ solves the obstacle problem:*

$$\min\{-(\partial_t + \mathcal{A})V \ , \ V - g\} \ = \ 0 \quad on \quad \mathbf{S}. \tag{3.12}$$

*Proof.* We organize the proof into two steps.
**1.** We first show that:

$$\min\{-(\partial_t + \mathcal{A})V \ , \ V - g\} \ \geq \ 0 \quad \text{on} \quad \mathbf{S}. \tag{3.13}$$

The inequality $V - g \geq 0$ is obvious as the constant stopping rule $\tau = t \in \mathcal{T}_{[t,T]}$ is admissible. Next, for $(t_0, x_0) \in \mathbf{S}$, consider the stopping times

$$\theta_h := \inf\left\{t > t_0 : \ (t, X_t^{t_0,x_0}) \notin [t_0, t_0 + h] \times B\right\}, h > 0,$$

where $B$ is the unit ball of $\mathbb{R}^n$ centered at $x_0$. Then $\theta_h \in \mathcal{T}_{[t,T]}^t$ for sufficiently small $h$, and it follows from (3.10)that:

$$V(t_0, x_0) \quad \geq \quad \mathbb{E}\left[V\left(\theta_h, X_{\theta_h}\right)\right].$$

We next apply Itô's formula, and observe that the expected value of the diffusion term vanishes because $(t, X_t)$ lies in the compact subset $[t_0, t_0 + h] \times B$ for $t \in [t_0, \theta_h]$. Then:

$$\mathbb{E}\left[\frac{-1}{h}\int_{t_0}^{\theta_h}(\partial_t + \mathcal{A})V(t, X_t^{t_0,x_0})dt\right] \quad \geq \quad 0.$$

Clearly, there exists $\hat{h}_\omega > 0$, depending on $\omega$, $\theta_h = h$ for $h \leq \hat{h}_\omega$. Then, it follows from the mean value theorem that the expression inside the expectation converges $\mathbb{P}-$a.s. to $-(\partial_t + \mathcal{A})V(t_0, x_0)$, and we conclude by dominated convergence that $-(\partial_t + \mathcal{A})V(t_0, x_0) \geq 0$.
**2.** In order to complete the proof, we use a contradiction argument, assuming that

$$V(t_0, x_0) > g(x_0) \text{ and } -(\partial_t + \mathcal{A})V(t_0, x_0) > 0 \text{ at some } (t_0, x_0) \in \mathbf{S}, \tag{3.14}$$

and we work towards a contradiction of (3.9). Introduce the function

$$\varphi(t, x) := V(t, x) + \frac{\varepsilon}{2}|x - x_0|^2 \quad \text{for} \quad (t, x) \in \mathbf{S}.$$

Then, it follows from (3.14) that for a sufficiently small $\varepsilon > 0$, we may find $h > 0$ and $\delta > 0$ such that

$$V \geq g + \delta \text{ and } -(\partial_t + \mathcal{A})\varphi \geq 0 \quad \text{on} \quad \mathcal{N}_h := [t_0, t_0 + h] \times hB. \tag{3.15}$$

Moreover:

$$-\gamma \quad := \quad \max_{\partial\mathcal{N}_h}(V - \varphi) \ < \ 0. \tag{3.16}$$

Next, let

$$\theta \quad := \quad \inf\left\{t > t_0 : \quad \left(t, X_t^{t_0,x_0}\right) \not\in \mathcal{N}_h\right\}.$$

For an arbitrary stopping rule $\tau \in \mathcal{T}_{[t,T]}^t$, we compute by Itô's formula that:

$$\begin{aligned}
\mathbb{E}\left[V\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right) - V(t_0,x_0)\right] &= \mathbb{E}\left[(V-\varphi)\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right)\right] \\
&\quad +\mathbb{E}\left[\varphi\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right) - \varphi(t_0,x_0)\right] \\
&= \mathbb{E}\left[(V-\varphi)\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right)\right] \\
&\quad +\mathbb{E}\left[\int_{t_0}^{\tau\wedge\theta} (\partial_t + \mathcal{A})\varphi(t, X_t^{t_0,x_0})dt\right],
\end{aligned}$$

where the diffusion term has zero expectation because the process $(t, X_t^{t_0,x_0})$ is confined to the compact subset $\mathcal{N}_h$ on the stochastic interval $[t_0, \tau\wedge\theta]$. Since $-\mathcal{L}\varphi \geq 0$ on $\mathcal{N}_h$ by (3.15), this provides:

$$\begin{aligned}
\mathbb{E}\left[V\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right) - V(t_0,x_0)\right] &\leq \mathbb{E}\left[(V-\varphi)\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right)\right] \\
&\leq -\gamma\mathbb{P}[\tau \geq \theta],
\end{aligned}$$

by (3.16). Then, since $V \geq g+\delta$ on $\mathcal{N}_h$ by (3.15):

$$\begin{aligned}
V(t_0,x_0) &\geq \gamma\mathbb{P}[\tau\geq\theta] + \mathbb{E}\left[\left(g(X_\tau^{t_0,x_0}) + \delta\right)\mathbf{1}_{\{\tau<\theta\}} + V\left(\theta, X_\theta^{t_0,x_0}\right)\mathbf{1}_{\{\tau\geq\theta\}}\right] \\
&\geq (\gamma\wedge\delta) + \mathbb{E}\left[g(X_\tau^{t_0,x_0})\mathbf{1}_{\{\tau<\theta\}} + V\left(\theta, X_\theta^{t_0,x_0}\right)\mathbf{1}_{\{\tau\geq\theta\}}\right].
\end{aligned}$$

By the arbitrariness of $\tau \in \mathcal{T}_{[t,T]}^t$, this provides the desired contradiction of (3.9). ♢

## 3.4 Regularity of the value function

### 3.4.1 Finite horizon optimal stopping

In this subsection, we consider the case $T < \infty$. Similar to the continuity result of Proposition 2.7 for the stochastic control framework, the following continuity result is obtained as a consequence of the flow continuiy of Theorem 1.4 together with the dynamic programming principle.

**Proposition 3.5.** *Assume $g$ is Lipschitz-continuous, and let $T < \infty$. Then, there is a constant $C$ such that:*

$$\left|V(t,x) - V(t',x')\right| \leq C\left(|x-x'| + \sqrt{|t-t'|}\right) \quad \textit{for all} \quad (t,x),(t',x') \in \mathbf{S}.$$

*Proof.* (i) For $t \in [0,T]$ and $x, x' \in \mathbb{R}^n$, it follows from the Lipschitz property of $g$ that:

$$\begin{aligned}
|V(t,x) - V(t,x')| &\leq Const \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}\left|X_\tau^{t,x} - X_\tau^{t,x'}\right| \\
&\leq Const\, \mathbb{E}\sup_{t\leq s\leq T}\left|X_\tau^{t,x} - X_\tau^{t,x'}\right| \\
&\leq Const\,|x-x'|
\end{aligned}$$

by the flow continuity result of Theorem 1.4.
ii) To rpove the H"older continuity result in $t$, we argue as in the proof of Proposition 2.7 using the dynamic programming principle of Theorem 3.3.

(ii-1) We first observe that, whenever the stopping time $\theta = t' > t$ is constant (i.e. deterministic), the dynamic programming principle (3.9)-(3.10) holds true if the semicontinuous envelopes are taken with respect to the variable $x$, with fixed time variable. Since $V$ is continuous in $x$ by the first part of this proof, we deduce that

$$V(t,x) = \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<t'\}} g\left(X^{t,x}_\tau\right) + \mathbf{1}_{\{\tau \ge t'\}} V\left(t', X^{t,x}_{t'}\right)\right] \tag{3.17}$$

(ii) We then estimate that

$$\begin{aligned} 0 \le V(t,x) - \mathbb{E}\left[V\left(t', X^{t,x}_{t'}\right)\right] &\le \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<t'\}}\left(g\left(X^{t,x}_\tau\right) - V\left(t', X^{t,x}_{t'}\right)\right)\right] \\ &\le \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<t'\}}\left(g\left(X^{t,x}_\tau\right) - g\left(X^{t,x}_{t'}\right)\right)\right], \end{aligned}$$

where the last inequality follows from the fact that $V \ge g$. Using the Lipschitz property of $g$, this provides:

$$\begin{aligned} 0 \le V(t,x) - \mathbb{E}\left[V\left(t', X^{t,x}_{t'}\right)\right] &\le \mathit{Const}\, \mathbb{E}\left[\sup_{t \le s \le t'} \left|X^{t,x}_s - X^{t,x}_{t'}\right|\right] \\ &\le \mathit{Const}\,(1+|x|)\sqrt{t'-t} \end{aligned}$$

by the flow continuity result of Theorem 1.4. Using this estimate together with the Lipschitz property proved in (i) above, this provides:

$$\begin{aligned} |V(t,x) - V(t',x)| &\le \left|V(t,x) - \mathbb{E}\left[V\left(t', X^{t,x}_{t'}\right)\right]\right| + \left|\mathbb{E}\left[V\left(t', X^{t,x}_{t'}\right)\right] - V(t',x)\right| \\ &\le \mathit{Const}\left((1+|x|)\sqrt{t'-t} + \mathbb{E}\left|X^{t,x}_{t'} - x\right|\right) \\ &\le \mathit{Const}\,(1+|x|)\sqrt{t'-t}, \end{aligned}$$

by using again Theorem 1.4. ♢

### 3.4.2 Infinite horizon optimal stopping

In this section, the state process $X$ is defined by a homogeneous scalar diffusion:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \tag{3.18}$$

We introduce the hitting times:

$$H^x_b := \inf\left\{t > 0 : \; X^{0,x} = b\right\},$$

and we assume that the process $X$ is regular, i.e.

$$\mathbb{P}\left[H^x_b < \infty\right] > 0 \quad \text{for all} \quad x, b \in \mathbb{R}, \tag{3.19}$$

which means that there is no subinterval of $\mathbb{R}$ from which the process $X$ can not exit.

We consider the infinite horizon optimal stopping problem:

$$V(x) \quad := \quad \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\beta\tau} g\left(X_\tau^{0,x}\right) \mathbf{1}_{\{\tau<\infty\}}\right], \tag{3.20}$$

where $\mathcal{T} := \mathcal{T}_{[0,\infty]}$, and $\beta > 0$ is the discount rate parameter.

According to Theorem 3.3, the dynamic programming equation corresponding to this optimal stopping problem is the obstacle problem:

$$\min\left\{\beta v - \mathcal{A}v, v - g\right\} \quad = \quad 0,$$

where the differential operator in the present homogeneous context is given by the generator of the diffusion:

$$\mathcal{A}v \quad := \quad \mu v' + \frac{1}{2}\sigma^2 v''. \tag{3.21}$$

The ordinary differential equation

$$\mathcal{A}v - \beta v \quad = \quad 0 \tag{3.22}$$

has two positive linearly independent solutions

$$\psi, \phi \geq 0 \quad \text{such that} \quad \psi \text{ strictly increasing, } \phi \text{ strictly decreasing.} \tag{3.23}$$

Clearly $\psi$ and $\phi$ are uniquely determined up to a positive constant, and all other solution of (3.22) can be expressed as a linear combination of $\psi$ and $\phi$.

The following result follows from an immediate application of Itô's formula.

**Lemma 3.6.** *For any $b_1 < b_2$, we have:*

$$\begin{aligned}
\mathbb{E}\left[e^{-\beta H_{b_1}^x} \mathbf{1}_{\{H_{b_1}^x \leq H_{b_2}^x\}}\right] &= \frac{\psi(x)\phi(b_2) - \psi(b_2)\phi(x)}{\psi(b_1)\phi(b_2) - \psi(b_2)\phi(b_1)}, \\
\mathbb{E}\left[e^{-\beta H_{b_2}^x} \mathbf{1}_{\{H_{b_1}^x \geq H_{b_2}^x\}}\right] &= \frac{\psi(b_1)\phi(x) - \psi(x)\phi(b_1)}{\psi(b_1)\phi(b_2) - \psi(b_2)\phi(b_1)}.
\end{aligned}$$

We now show that the value function $V$ is concave up to some change of variable, and provides conditions under which $V$ is $C^1$ across the exercise boundary, i.e. the boundary between the exercise and the continuation regions. For the next result, we observe that the fnction $(\psi/\phi)$ is continuous and strictly increasing by (3.23), and therefore invertible.

**Theorem 3.7.** (i) *The function $(V/\phi) \circ (\psi/\phi)^{-1}$ is concave. In particular, $V$ is continuous on $\mathbb{R}$.*
(ii) *Let $x_0$ be such that $V(x_0) = g(x_0)$, and assume that $g$, $\psi$ and $\phi$ are differentiable at $x_0$. Then $V$ is differentiable at $x_0$, and $V'(x_0) = g'(x_0)$.*

*Proof.* For (i), it is sufficient to prove that:

$$\frac{\frac{V}{\phi}(x)-\frac{V}{\phi}(b_1)}{\frac{\psi}{\phi}(x)-\frac{\psi}{\phi}(b_1)} \;\geq\; \frac{\frac{V}{\phi}(b_2)-\frac{V}{\phi}(x)}{\frac{\psi}{\phi}(b_2)-\frac{\psi}{\phi}(x)} \quad \text{for all} \quad b_1<x<b_2. \tag{3.24}$$

For $\varepsilon>0$, consider the $\varepsilon-$optimal stopping rules $\tau_1,\tau_2\in\mathcal{T}$ for the problems $V(b_1)$ and $V(b_2)$:

$$\mathbb{E}\left[e^{-\beta\tau_i}g\left(X^{0,b_i}_{\tau_i}\right)\right]\geq V(b_i)-\varepsilon \quad\text{for}\quad i=1,2.$$

We next define the stopping time

$$\tau^\varepsilon \;:=\; \left(H^x_{b_1}+\tau_1\circ\theta_{H^x_{b_1}}\right)\mathbf{1}_{\{H^x_{b_1}<H^x_{b_2}\}}+\left(H^x_{b_2}+\tau_2\circ\theta_{H^x_{b_2}}\right)\mathbf{1}_{\{H^x_{b_2}<H^x_{b_1}\}},$$

where $\theta$ denotes the shift operator on the canonical space, i.e. $\theta_t(\omega)(s)=\omega(t+s)$. In words, the stopping rule $\tau^\varepsilon$ uses the $\varepsilon-$optimal stopping rule $\tau_1$ if the level $b_1$ is reached before the level $b_2$, and the $\varepsilon-$optimal stopping rule $\tau_2$ otherwise. Then, it follows from the strong Markov property that

$$\begin{aligned}
V(x) \;&\geq\; \mathbb{E}\left[e^{-\beta\tau^\varepsilon}g\left(X^{0,x}_{\tau^\varepsilon}\right)\right]\\
&=\; \mathbb{E}\left[e^{-\beta H^x_{b_1}}\mathbb{E}\left[e^{-\beta\tau_1}g\left(X^{0,b_1}_{\tau_1}\right)\right]\mathbf{1}_{\{H^x_{b_1}<H^x_{b_2}\}}\right]\\
&\quad+\mathbb{E}\left[e^{-\beta H^x_{b_2}1}\mathbb{E}\left[e^{-\beta\tau_2}g\left(X^{0,b_2}_{\tau_2}\right)\right]\mathbf{1}_{\{H^x_{b_2}<H^x_{b_1}\}}\right]\\
&\geq\; \left(V(b_1)-\varepsilon\right)\mathbb{E}\left[e^{-\beta H^x_{b_1}}\mathbf{1}_{\{H^x_{b_1}<H^x_{b_2}\}}\right]\\
&\quad+\left(V(b_2)-\varepsilon\right)\mathbb{E}\left[e^{-\beta H^x_{b_2}}\mathbf{1}_{\{H^x_{b_2}<H^x_{b_1}\}}\right].
\end{aligned}$$

Sending $\varepsilon\searrow 0$, this provides

$$V(x) \;\geq\; V(b_1)\mathbb{E}\left[e^{-\beta H^x_{b_1}}\mathbf{1}_{\{H^x_{b_1}<H^x_{b_2}\}}\right]+V(b_2)\mathbb{E}\left[e^{-\beta H^x_{b_2}}\mathbf{1}_{\{H^x_{b_2}<H^x_{b_1}\}}\right].$$

By using the explicit expressions of Lemma 3.6 above, this provides:

$$\frac{V(x)}{\phi(x)} \;\geq\; \frac{V(b_1)}{\phi(b_1)}\,\frac{\frac{\psi}{\phi}(b_2)-\frac{\psi}{\phi}(x)}{\frac{\psi}{\phi}(b_2)-\frac{\psi}{\phi}(b_1)}\;+\;\frac{V(b_2)}{\phi(b_2)}\,\frac{\frac{\psi}{\phi}(x)-\frac{\psi}{\phi}(b_1)}{\frac{\psi}{\phi}(b_2)-\frac{\psi}{\phi}(b_1)},$$

which implies (3.24).

(ii) We next prove the smoothfit result. Let $x_0$ be such that $V(x_0)=g(x_0)$. Then, since $V\geq g$, $\psi$ is strictly increasing, $\phi\geq 0$ is strictly decreasing, it follows from (3.24) that:

$$\begin{aligned}
\frac{\frac{g}{\phi}(x_0+\varepsilon)-\frac{g}{\phi}(x_0)}{\frac{\psi}{\phi}(x_0+\varepsilon)-\frac{\psi}{\phi}(x_0)} \;&\leq\; \frac{\frac{V}{\phi}(x_0+\varepsilon)-\frac{V}{\phi}(x_0)}{\frac{\psi}{\phi}(x_0+\varepsilon)-\frac{\psi}{\phi}(x_0)} \qquad (3.25)\\
&\leq\; \frac{\frac{V}{\phi}(x_0-\delta)-\frac{V}{\phi}(x_0)}{\frac{\psi}{\phi}(x_0-\delta)-\frac{\psi}{\phi}(x_0)}\;\leq\;\frac{\frac{g}{\phi}(x_0-\delta)-\frac{g}{\phi}(x_0)}{\frac{\psi}{\phi}(x_0-\delta)-\frac{\psi}{\phi}(x_0)}
\end{aligned}$$

for all $\varepsilon > 0$, $\delta > 0$. Multiplying by $((\psi/\phi)(x_0+\varepsilon) - (\psi/\phi)(x_0))/\varepsilon$, this implies that:

$$\frac{\frac{g}{\phi}(x_0+\varepsilon) - \frac{g}{\phi}(x_0)}{\varepsilon} \le \frac{\frac{V}{\phi}(x_0+\varepsilon) - \frac{V}{\phi}(x_0)}{\varepsilon} \le \frac{\Delta^+(\varepsilon)}{\Delta^-(\delta)} \frac{\frac{g}{\phi}(x_0-\delta) - \frac{g}{\phi}(x_0)}{\delta}, \tag{3.26}$$

where

$$\Delta^+(\varepsilon) := \frac{\frac{\psi}{\phi}(x_0+\varepsilon) - \frac{\psi}{\phi}(x_0)}{\varepsilon} \quad \text{and} \quad \Delta^-(\delta) := \frac{\frac{\psi}{\phi}(x_0-\delta) - \frac{\psi}{\phi}(x_0)}{\delta}.$$

We next consider two cases:

- If $(\psi/\phi)'(x_0) \neq 0$, then we may take $\varepsilon = \delta$ and send $\varepsilon \searrow 0$ in (3.26) to obtain:

$$\frac{d^+(\frac{V}{\phi})}{dx}(x_0) = \left(\frac{g}{\phi}\right)'(x_0). \tag{3.27}$$

- If $(\psi/\phi)'(x_0) = 0$, then, we use the fact that for every sequence $\varepsilon_n \searrow 0$, there is a subsequence $\varepsilon_{n_k} \searrow 0$ and $\delta_k \searrow 0$ such that $\Delta^+(\varepsilon_{n_k}) = \Delta^-(\delta_k)$. Then (3.26) reduces to:

$$\frac{\frac{g}{\phi}(x_0+\varepsilon_{n_k}) - \frac{g}{\phi}(x_0)}{\varepsilon_{n_k}} \le \frac{\frac{V}{\phi}(x_0+\varepsilon_{n_k}) - \frac{V}{\phi}(x_0)}{\varepsilon_{n_k}} \le \frac{\frac{g}{\phi}(x_0-\delta_k) - \frac{g}{\phi}(x_0)}{\delta_k},$$

  and therefore

$$\frac{\frac{V}{\phi}(x_0+\varepsilon_{n_k}) - \frac{V}{\phi}(x_0)}{\varepsilon_{n_k}} \longrightarrow \left(\frac{g}{\phi}\right)'(x_0).$$

  By the arbitrariness of the sequence $(\varepsilon_n)_n$, this provides (3.27).

Similarly, multiplying (3.25) by $((\psi/\phi)(x_0) - (\psi/\phi)(x_0-\delta))/\delta$, and arguing as above, we obtain:

$$\frac{d^-(\frac{V}{\phi})}{dx}(x_0) = \left(\frac{g}{\phi}\right)'(x_0),$$

thus completing the proof. ♢

### 3.4.3 An optimal stopping problem with nonsmooth value

We consider the example

$$X_s^{t,x} := x + (W_t - W_s) \quad \text{for} \quad s \ge t.$$

Let $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a measurable nonnegative function with $\liminf_{x\to\infty} g(x) = 0$, and consider the infinite horizon optimal stopping problem:

$$\begin{aligned} V(t,x) &:= \sup_{\tau\in\mathcal{T}_{[t,\infty]}} \mathbb{E}\left[g\left(X_\tau^{t,x}\right)\mathbf{1}_{\{\tau<\infty\}}\right] \\ &= \sup_{\tau\in\mathcal{T}_{[t,\infty)}} \mathbb{E}\left[g\left(X_\tau^{t,x}\right)\right]. \end{aligned}$$

Let us assume that $V \in C^{1,2}(\mathbf{S})$, and work towards a contradiction. We first observe by the homogeneity of the problem that $V(t,x) = V(x)$ is independent of $t$. Moreover, it follows from Theorem 3.4 that $V$ is concave in $x$ and $V \geq g$. Then

$$V \geq g^{\text{conc}}, \tag{3.28}$$

where $g^{\text{conc}}$ is the concave envelope of $g$. If $g^{\text{conc}} = \infty$, then $V = \infty$. We then continue in the more inetersting case where $g^{\text{conc}} < \infty$.

By the Jensen inequality and the non-negativity of $g$, the process $\{g\left(X_s^{t,x}\right), s \geq t\}$ is a supermartingale, and:

$$V(t,x) \;\leq\; \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}\left[g^{\text{conc}}\left(X_\tau^{t,x}\right)\right] \;\leq\; g^{\text{conc}}(x).$$

Hence, $V = g^{\text{conc}}$, and we obtain the required contradiction whenever $g^{\text{conc}}$ is not differentiable at some point of $\mathbb{R}$.

# Chapter 4

# Solving Control Problems by Verification

In this chapter, we present a general argument, based on Itô's formula, which allows to show that some "guess" of the value function is indeed equal to the unknown value function. Namely, given a smooth solution $v$ of the dynamic programming equation, we give sufficient conditions which allow to conclude that $v$ coincides with the value function $V$. This is the so-called *verification argument*. The statement of this result is heavy, but its proof is simple and relies essentially on Itô's formula. However, depending on the problem in hand, the verification of the conditions which must be satisfied by the candidate solution can be difficult.

The verification argument will be provided in the contexts of stochastic control and optimal stopping problems. We conclude the chapter with some examples.

## 4.1 The verification argument for stochastic control problems

We recall the stochastic control problem formulation of Section 2.1. The set of admissible control processes $\mathcal{U}_0 \subset \mathcal{U}$ is the collection of all progressively measurable processes with values in the subset $U \subset \mathbb{R}^k$. For every admissible control process $\nu \in \mathcal{U}_0$, the controlled process is defined by the stochastic differential equation:

$$dX_t^\nu \quad = \quad b(t, X_t^\nu, \nu_t)dt + \sigma(t, X_t^\nu, \nu_t)dW_t.$$

The gain criterion is given by

$$J(t,x,\nu) := \mathbb{E}\left[\int_t^T \beta^\nu(t,s) f(s, X_s^{t,x,\nu}, \nu_s)ds + \beta^\nu(t,T) g(X_T^{t,x,\nu})\right],$$

with

$$\beta^\nu(t,s) \quad := \quad e^{-\int_t^s k(r,X_r^{t,x,\nu},\nu_r)dr}.$$

The stochastic control problem is defined by the value function:

$$V(t,x) := \sup_{\nu\in\mathcal{U}_0} J(t,x,\nu), \quad \text{for} \quad (t,x)\in\mathbf{S}. \tag{4.1}$$

We follow the notations of Section 2.3. We recall the Hamiltonian $H : \mathbf{S}\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}_d$ defined by :

$$H(t,x,r,p,\gamma)$$
$$:= \sup_{u\in U}\left\{-k(t,x,u)r + b(t,x,u)\cdot p + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}}(t,x,u)\gamma] + f(t,x,u)\right\},$$

where $b$ and $\sigma$ satisfy the conditions (2.1)-(2.2), and the coefficients $f$ and $k$ are measurable. From the results of the previous section, the dynamic programming equation corresponding to the stochastic control problem (4.1) is:

$$-\partial_t v - H(.,v,Dv,D^2v) = 0 \quad \text{and} \quad v(T,.) = g. \tag{4.2}$$

A function $v$ will be called a *supersolution* (resp. *subsolution*) of the equation (4.2) if

$$-\partial_t v - H(.,v,Dv,D^2v) \geq \ (\text{resp.} \leq)\ 0 \quad \text{and} \quad v(T,.) \geq \ (\text{resp.} \leq)\ g.$$

The proof of the subsequent result will make use of the following linear second order operator

$$\begin{aligned}\mathcal{L}^u\varphi(t,x) \quad := \quad & -k(t,x,u)\varphi(t,x) + b(t,x,u)\cdot D\varphi(t,x)\\ & +\frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}(t,x,u)D^2\varphi(t,x)\right],\end{aligned}$$

which corresponds to the controlled process $\{\beta^u(0,t)X_t^u,\ t\geq 0\}$ controlled by the constant control process $u$, in the sense that

$$\begin{aligned}\beta^\nu(0,s)\varphi(s,X_s^\nu) - \beta^\nu(0,t)\varphi(t,X_t^\nu) \quad = \quad & \int_t^s \beta^\nu(0,r)\left(\partial_t + \mathcal{L}^{\nu_r}\right)\varphi(r,X_r^\nu)dr\\ & +\int_t^s \beta^\nu(0,r)D\varphi(r,X_r^\nu)\cdot\sigma(r,X_r^\nu,\nu_r)dW_r\end{aligned}$$

for every $t\leq s$ and smooth function $\varphi\in C^{1,2}([t,s],\mathbb{R}^d)$ and each admissible control process $\nu\in\mathcal{U}_0$. The last expression is an immediate application of Itô's formula.

**Theorem 4.1.** *Let $T<\infty$, and $v\in C^{1,2}([0,T),\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)$. Assume that $\|k^-\|_\infty<\infty$ and $v$ and $f$ have quadratic growth, i.e. there is a constant $C$ such that*

$$|f(t,x,u)| + |v(t,x)| \ \leq\ C(1+|x|^2+|u|), \quad (t,x,u)\in[0,T)\times\mathbb{R}^d\times U.$$

(i) *Suppose that $v$ is a supersolution of* (4.2)*. Then $v \geq V$ on $[0,T] \times \mathbb{R}^d$.*
(ii) *Let $v$ be a solution of* (4.2)*, and assume that there exists a minimizer $\hat{u}(t,x)$ of $u \longmapsto \mathcal{L}^u v(t,x) + f(t,x,u)$ such that*

- $0 = \partial_t v(t,x) + \mathcal{L}^{\hat{u}(t,x)} v(t,x) + f\big(t,x,\hat{u}(t,x)\big)$,
- *the stochastic differential equation*

$$dX_s \;=\; b\left(s, X_s, \hat{u}(s,X_s)\right) ds + \sigma\left(s, X_s, \hat{u}(s,X_s)\right) dW_s$$

  *defines a unique solution $X$ for each given initial data $X_t = x$,*
- *the process $\hat{\nu}_s := \hat{u}(s, X_s)$ is a well-defined control process in $\mathcal{U}_0$.*

*Then $v = V$, and $\hat{\nu}$ is an optimal Markov control process.*

*Proof.* Let $\nu \in \mathcal{U}_0$ be an arbitrary control process, $X$ the associated state process with initial date $X_t = x$, and define the stopping time

$$\theta_n \;:=\; (T - n^{-1}) \wedge \inf\{s > t \;:\; |X_s - x| \geq n\}.$$

By Itô's formula, we have

$$v(t,x) \;=\; \beta(t,\theta_n) v\left(\theta_n, X_{\theta_n}\right) - \int_t^{\theta_n} \beta(t,r)(\partial_t + \mathcal{L}^{\nu_r}) v(r, X_r) dr - \int_t^{\theta_n} \beta(t,r) Dv(r,X_r) \cdot \sigma(r, X_r, \nu_r) dW_r$$

Observe that $(\partial_t + \mathcal{L}^{\nu_r})v + f(\cdot,\cdot,u) \leq \partial_t v + H(\cdot,\cdot,v,Dv,D^2v) \leq 0$, and that the integrand in the stochastic integral is bounded on $[t,\theta_n]$, a consequence of the continuity of $Dv$, $\sigma$ and the condition $\|k^-\|_\infty < \infty$. Then :

$$v(t,x) \;\geq\; \mathbb{E}\left[\beta(t,\theta_n) v\left(\theta_n, X_{\theta_n}\right) + \int_t^{\theta_n} \beta(t,r) f(r, X_r, \nu_r) dr\right]. \tag{4.3}$$

We now take the limit as $n$ increases to infinity. Since $\theta_n \longrightarrow T$ a.s. and

$$\begin{aligned} &\left|\beta(t,\theta_n) v\left(\theta_n, X_{\theta_n}\right) + \int_t^{\theta_n} \beta(t,r) f(r,X_r,\nu_r) dr\right| \\ &\quad\leq\; C e^{T\|k^-\|_\infty}\big(1 + |X_{\theta_n}|^2 + T + \textstyle\int_t^T |X_s|^2 ds\big) \\ &\quad\leq\; C e^{T\|k^-\|_\infty}(1+T)\big(1 + \sup_{t\leq s\leq T} |X_s|^2 + \textstyle\int_t^T |\nu_s|^2 ds\big) \;\in\; \mathbb{L}^1, \end{aligned}$$

by the estimate (2.5) of Theorem 2.1, it follows from the dominated convergence that

$$\begin{aligned} v(t,x) \;&\geq\; \mathbb{E}\left[\beta(t,T) v(T,X_T) + \int_t^T \beta(t,r) f(r,X_r,\nu_r) dr\right] \\ &\geq\; \mathbb{E}\left[\beta(t,T) g(X_T) + \int_t^T \beta(t,r) f(r,X_r,\nu_r) dr\right], \end{aligned}$$

where the last inequality uses the condition $v(T,\cdot) \geq g$. Since the control $\nu \in \mathcal{U}_0$ is arbitrary, this completes the proof of (i).

Statement (ii) is proved by repeating the above argument and observing that the control $\hat{\nu}$ achieves equality at the crucial step (4.3). $\diamondsuit$

**Remark 4.2.** When $U$ is reduced to a singleton, the optimization problem $V$ is degenerate. In this case, the DPE is linear, and the verification theorem reduces to the so-called *Feynman-Kac formula.*

Notice that the verification theorem assumes the existence of such a solution, and is by no means an existence result. However, it provides uniqueness in the class of functions with quadratic growth.

We now state without proof an existence result for the DPE together with the terminal condition $V(T,\cdot) = g$ (see [8] and the references therein). The main assumption is the so-called *uniform parabolicity* condition :

$$\begin{array}{c}\text{there is a constant } c>0 \text{ such that}\\ \xi'\ \sigma\sigma'(t,x,u)\ \xi\ \geq\ c|\xi|^2 \text{ for all } (t,x,u)\in[0,T]\times\mathbb{R}^n\times U\ .\end{array} \tag{4.4}$$

In the following statement, we denote by $C_b^k(\mathbb{R}^n)$ the space of bounded functions whose partial derivatives of orders $\leq k$ exist and are bounded continuous. We similarly denote by $C_b^{p,k}([0,T],\mathbb{R}^n)$ the space of bounded functions whose partial derivatives with respect to $t$, of orders $\leq p$, and with respect to $x$, of order $\leq k$, exist and are bounded continuous.

**Theorem 4.3.** *Let Condition 4.4 hold, and assume further that :*
- *$U$ is compact;*
- *$b$, $\sigma$ and $f$ are in $C_b^{1,2}([0,T],\mathbb{R}^n)$;*
- *$g \in C_b^3(\mathbb{R}^n)$.*

*Then the DPE* (2.18) *with the terminal data $V(T,\cdot) = g$ has a unique solution $V \in C_b^{1,2}([0,T]\times\mathbb{R}^n)$.*

## 4.2 Examples of control problems with explicit solutions

### 4.2.1 Optimal portfolio allocation

We now apply the verification theorem to a classical example in finance, which was introduced by Merton [10, 11], and generated a huge literature since then.

Consider a financial market consisting of a non-risky asset $S^0$ and a risky one $S$. The dynamics of the price processes are given by

$$dS_t^0\ =\ S_t^0 r dt \quad \text{and} \quad dS_t\ =\ S_t[\mu dt + \sigma dW_t]\ .$$

Here, $r$, $\mu$ and $\sigma$ are some given positive constants, and $W$ is a one-dimensional Brownian motion.

The investment policy is defined by an $\mathbb{F}-$adapted process $\pi = \{\pi_t,\ t \in [0,T]\}$, where $\pi_t$ represents the amount invested in the risky asset at time $t$;

The remaining wealth $(X_t - \pi_t)$ is invested in the risky asset. Therefore, the liquidation value of a self-financing strategy satisfies

$$\begin{aligned} dX_t^\pi &= \pi_t \frac{dS_t}{S_t} + (X_t^\pi - \pi_t)\frac{dS_t^0}{S_t^0} \\ &= (rX_t + (\mu - r)\pi_t)\, dt + \sigma \pi_t dW_t. \end{aligned} \tag{4.5}$$

Such a process $\pi$ is said to be admissible if it lies in $\mathcal{U}_0 = \mathbb{H}^2$ which will be refered to as the set of all admissible portfolios. Observe that, in view of the particular form of our controlled process $X$, this definition agrees with (2.4).

Let $\gamma$ be an arbitrary parameter in $(0,1)$ and define the *power utility function* :

$$U(x) \;:=\; x^\gamma \quad \text{for} \quad x \geq 0\,.$$

The parameter $\gamma$ is called the relative risk aversion coefficient.

The objective of the investor is to choose an allocation of his wealth so as to maximize the expected utility of his terminal wealth, i.e.

$$V(t,x) \;:=\; \sup_{\pi \in \mathcal{U}_0} \mathbb{E}\left[U(X_T^{t,x,\pi})\right],$$

where $X^{t,x,\pi}$ is the solution of (4.5) with initial condition $X_t^{t,x,\pi} = x$.

The dynamic programming equation corresponding to this problem is :

$$\frac{\partial w}{\partial t}(t,x) + \sup_{u \in \mathbb{R}} \mathcal{A}^u w(t,x) \;=\; 0, \tag{4.6}$$

where $\mathcal{A}^u$ is the second order linear operator :

$$\mathcal{A}^u w(t,x) \;:=\; (rx + (\mu - r)u)\, \frac{\partial w}{\partial x}(t,x) + \frac{1}{2}\sigma^2 u^2\, \frac{\partial^2 w}{\partial x^2}(t,x).$$

We next search for a solution of the dynamic programming equation of the form $v(t,x) = x^\gamma h(t)$. Plugging this form of solution into the PDE (4.6), we get the following ordinary differential equation on $h$ :

$$0 \;=\; h' + \gamma h \sup_{u \in \mathbb{R}} \left\{ r + (\mu - r)\frac{u}{x} + \frac{1}{2}(\gamma - 1)\sigma^2 \frac{u^2}{x^2} \right\} \tag{4.7}$$

$$\;=\; h' + \gamma h \sup_{\delta \in \mathbb{R}} \left\{ r + (\mu - r)\delta + \frac{1}{2}(\gamma - 1)\sigma^2 \delta^2 \right\} \tag{4.8}$$

$$\;=\; h' + \gamma h \left[ r + \frac{1}{2}\frac{(\mu - r)^2}{(1-\gamma)\sigma^2} \right], \tag{4.9}$$

where the maximizer is :

$$\hat{u} \;:=\; \frac{\mu - r}{(1-\gamma)\sigma^2}\, x.$$

Since $v(T,\cdot) = U(x)$, we seek for a function $h$ satisfying the above ordinary differential equation together with the boundary condition $h(T) = 1$. This induces the unique candidate:

$$h(t) \;:=\; e^{a(T-t)} \quad \text{with} \quad a \;:=\; \gamma\left[r + \frac{1}{2}\frac{(\mu-r)^2}{(1-\gamma)\sigma^2}\right].$$

Hence, the function $(t,x) \longmapsto x^\gamma h(t)$ is a classical solution of the HJB equation (4.6). It is easily checked that the conditions of Theorem 4.1 are all satisfied in this context. Then $V(t,x) = x^\gamma h(t)$, and the optimal portfolio allocation policy is given by the linear control process:

$$\hat{\pi}_t \;=\; \frac{\mu - r}{(1-\gamma)\sigma^2}\; X_t^{\hat{\pi}}.$$

### 4.2.2 Law of iterated logarithm for double stochastic integrals

The main object of this paragraph is Theorem 4.5 below, reported from [2], which describes the local behavior of double stochastic integrals near the starting point zero. This result will be needed in the problem of hedging under gamma constraints which will be discussed later in these notes. An interesting feature of the proof of Theorem 4.5 is that it relies on a verification argument. However, the problem does not fit exactly in the setting of Theorem 4.1. Therefore, this is an interesting exercise on the verification concept.

Given a bounded predictable process $b$, we define the processes

$$Y_t^b \;:=\; Y_0 + \int_0^t b_r dW_r \quad \text{and} \quad Z_t^b \;:=\; Z_0 + \int_0^t Y_r^b dW_r\,, \quad t \geq 0\,,$$

where $Y_0$ and $Z_0$ are some given initial data in $\mathbb{R}$.

**Lemma 4.4.** *Let $\lambda$ and $T$ be two positive parameters with $2\lambda T < 1$. Then :*

$$E\left[e^{2\lambda Z_T^b}\right] \;\leq\; E\left[e^{2\lambda Z_T^1}\right] \quad \textit{for each predictable process } b \textit{ with } \quad \|b\|_\infty \;\leq\; 1\,.$$

*Proof.* We split the argument into three steps.
**1.** We first directly compute that

$$E\left[e^{2\lambda Z_T^1}\middle|\,\mathcal{F}_t\right] \;=\; v(t, Y_t^1, Z_t^1)\,,$$

where, for $t \in [0,T]$, and $y, z \in \mathbb{R}$, the function $v$ is given by :

$$\begin{aligned}
v(t,y,z) \;&:=\; E\left[\exp\left(2\lambda\left\{z + \int_t^T (y + W_u - W_t)\, dW_u\right\}\right)\right] \\
&=\; e^{2\lambda z} E\left[\exp\left(\lambda\{2yW_{T-t} + W_{T-t}^2 - (T-t)\}\right)\right] \\
&=\; \mu \exp\left[2\lambda z - \lambda(T-t) + 2\mu^2\lambda^2(T-t)y^2\right]\,,
\end{aligned}$$

where $\mu := [1-2\lambda(T-t)]^{-1/2}$. Observe that

$$\text{the function } v \text{ is strictly convex in } y, \tag{4.10}$$

and

$$yD^2_{yz}v(t,y,z) \;=\; 8\mu^2\lambda^3(T-t)\,v(t,y,z)\,y^2 \;\geq\; 0\,. \tag{4.11}$$

**2.** For an arbitrary real parameter $\beta$, we denote by $\mathcal{A}^\beta$ the generator the process $\left(Y^b, Z^b\right)$ :

$$\mathcal{A}^\beta \;:=\; \frac{1}{2}\beta^2 D^2_{yy} + \frac{1}{2}y^2 D^2_{zz} + \beta y D^2_{yz}\,.$$

In this step, we intend to prove that for all $t \in [0,T]$ and $y,z \in \mathbb{R}$ :

$$\max_{|\beta|\leq 1} \mathcal{A}^\beta v(t,y,z) \;=\; \mathcal{A}^1 v(t,y,z) \;=\; 0\,. \tag{4.12}$$

The second equality follows from the fact that $\{v(t,Y^1_t,Z^1_t),\ t\leq T\}$ is a martingale . As for the first equality, we see from (4.10) and (4.11) that 1 is a maximizer of both functions $\beta \longmapsto \beta^2 D^2_{yy}v(t,y,z)$ and $\beta \longmapsto \beta y D^2_{yz}v(t,y,z)$ on $[-1,1]$.

**3.** Let $b$ be some given predictable process valued in $[-1,1]$, and define the sequence of stopping times

$$\tau_k \;:=\; T \wedge \inf\left\{t\geq 0 \;:\; (|Y^b_t| + |Z^b_t| \geq k\right\}\;,\quad k\in\mathbb{N}\,.$$

By Itô's lemma and (4.12), it follows that :

$$\begin{aligned}
v(0,Y_0,Z_0) \;&=\; v\left(\tau_k, Y^b_{\tau_k}, Z^b_{\tau_k}\right) - \int_0^{\tau_k}\left[bD_y v + yD_z v\right]\left(t,Y^b_t,Z^b_t\right)dW_t \\
&\qquad - \int_0^{\tau_k}(\partial_t + \mathcal{A}^{b_t})v\left(t,Y^b_t,Z^b_t\right)dt \\
&\geq\; v\left(\tau_k, Y^b_{\tau_k}, Z^b_{\tau_k}\right) - \int_0^{\tau_k}\left[bD_y v + yD_z v\right]\left(t,Y^b_t,Z^b_t\right)dW_t\,.
\end{aligned}$$

Taking expected values and sending $k$ to infinity, we get by Fatou's lemma :

$$\begin{aligned}
v(0,Y_0,Z_0) \;&\geq\; \liminf_{k\to\infty}\; E\left[v\left(\tau_k, Y^b_{\tau_k}, Z^b_{\tau_k}\right)\right] \\
&\geq\; E\left[v\left(T, Y^b_T, Z^b_T\right)\right] \;=\; E\left[e^{2\lambda Z^b_T}\right]\,,
\end{aligned}$$

which proves the lemma. ◇

We are now able to prove the law of the iterated logarithm for double stochastic integrals by a direct adaptation of the case of the Brownian motion. Set

$$h(t) \;:=\; 2t\log\log\frac{1}{t} \quad \text{for} \quad t>0\,.$$

**Theorem 4.5.** *Let $b$ be a predictable process valued in a bounded interval $[\beta_0, \beta_1]$ for some real parameters $0 \leq \beta_0 < \beta_1$, and $X_t^b := \int_0^t \int_0^u b_v dW_v dW_u$. Then :*

$$\beta_0 \;\leq\; \limsup_{t \searrow 0} \frac{2X_t^b}{h(t)} \;\leq\; \beta_1 \qquad a.s.$$

*Proof.* We first show that the first inequality is an easy consequence of the second one. Set $\bar\beta := (\beta_0 + \beta_1)/2 \geq 0$, and set $\delta := (\beta_1 - \beta_0)/2$. By the law of the iterated logarithm for the Brownian motion, we have

$$\bar\beta \;=\; \limsup_{t \searrow 0} \frac{2X_t^{\bar\beta}}{h(t)} \quad\leq\quad \delta \limsup_{t \searrow 0} \frac{2X_t^{\tilde b}}{h(t)} + \limsup_{t \searrow 0} \frac{2X_t^{b}}{h(t)}\;,$$

where $\tilde b := \delta^{-1}(\bar\beta - b)$ is valued in $[-1, 1]$. It then follows from the second inequality that :

$$\limsup_{t \searrow 0} \frac{2X_t^b}{h(t)} \quad\geq\quad \bar\beta - \delta \;=\; \beta_0\;.$$

We now prove the second inequality. Clearly, we can assume with no loss of generality that $\|b\|_\infty \leq 1$. Let $T > 0$ and $\lambda > 0$ be such that $2\lambda T < 1$. It follows from Doob's maximal inequality for submartingales that for all $\alpha \geq 0$,

$$\begin{aligned} P\left[\max_{0\leq t\leq T} 2X_t^b \;\geq\; \alpha\right] \quad&=\quad P\left[\max_{0\leq t\leq T} \exp(2\lambda X_t^b) \;\geq\; \exp(\lambda\alpha)\right] \\ &\leq\quad e^{-\lambda\alpha} E\left[e^{2\lambda X_T^b}\right]\;. \end{aligned}$$

In view of Lemma 4.4, this provides :

$$\begin{aligned} P\left[\max_{0\leq t\leq T} 2X_t^b \;\geq\; \alpha\right] \quad&\leq\quad e^{-\lambda\alpha} E\left[e^{2\lambda X_T^1}\right] \\ &=\quad e^{-\lambda(\alpha+T)}(1 - 2\lambda T)^{-\frac{1}{2}}\;. \end{aligned} \tag{4.13}$$

We have then reduced the problem to the case of the Brownian motion, and the rest of this proof is identical to the first half of the proof of the law of the iterated logarithm for the Brownian motion. Take $\theta$, $\eta \in (0, 1)$, and set for all $k \in \mathbb{N}$,

$$\alpha_k \;:=\; (1+\eta)^2 h(\theta^k) \quad\text{ and }\quad \lambda_k \;:=\; [2\theta^k(1+\eta)]^{-1}\;.$$

Applying (4.13), we see that for all $k \in \mathbb{N}$,

$$P\left[\max_{0\leq t\leq \theta^k} 2X_t^b \;\geq\; (1+\eta)^2 h(\theta^k)\right] \;\leq\; e^{-1/2(1+\eta)}\left(1+\eta^{-1}\right)^{\frac{1}{2}}(-k\log\theta)^{-(1+\eta)}\,.$$

Since $\sum_{k\geq 0} k^{-(1+\eta)} < \infty$, it follows from the Borel-Cantelli lemma that, for almost all $\omega \in \Omega$, there exists a natural number $K^{\theta,\eta}(\omega)$ such that for all $k \geq K^{\theta,\eta}(\omega)$,

$$\max_{0\leq t\leq \theta^k} 2X_t^b(\omega) \quad<\quad (1+\eta)^2 h(\theta^k)\,.$$

In particular, for all $t \in (\theta^{k+1}, \theta^k]$,

$$2X_t^b(\omega) \quad < \quad (1+\eta)^2 h(\theta^k) \;\leq\; (1+\eta)^2 \frac{h(t)}{\theta}\,.$$

Hence,

$$\limsup_{t \searrow 0} \frac{2X_t^b}{h(t)} \quad < \quad \frac{(1+\eta)^2}{\theta} \quad \text{a.s.}$$

and the required result follows by letting $\theta$ tend to 1 and $\eta$ to 0 along the rationals. ♢

## 4.3 The verification argument for optimal stopping problems

In this section, we develop the verification argument for finite horizon optimal stopping problems. Let $T > 0$ be a finite time horizon, and $X^{t,x}$ denote the solution of the stochastic differential equation:

$$X_s^{t,x} = x + \int_t^s b(s, X_s^{t,x})ds + \int_t^s \sigma(s, X_s^{t,x})dW_s, \tag{4.14}$$

where $b$ and $\sigma$ satisfy the usual Lipschitz and linear growth conditions. Given the functions $k, f : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^d \longrightarrow \mathbb{R}$, we consider the optimal stopping problem

$$V(t,x) \quad := \quad \sup_{\tau \in \mathcal{T}_{[t,T]}^t} \mathbb{E}\left[\int_t^\tau \beta(t,s) f(s, X_s^{t,x})ds + \beta(t,\tau) g(X_\tau^{t,x})\right], \tag{4.15}$$

whenever this expected value is well-defined, where

$$\beta(t,s) := e^{-\int_t^s k(r, X_r^{t,x})dr}, \quad 0 \leq t \leq s \leq T.$$

By the results of the previous chapter, the corresponding dynamic programmin equation is:

$$\min\left\{-\partial_t v - \mathcal{L}v - f, v - g\right\} = 0 \quad \text{on } [0,T) \times \mathbb{R}^d, \quad v(T,.) = g, \tag{4.16}$$

where $\mathcal{L}$ is the second order differential operator

$$\mathcal{L}v \quad := \quad b \cdot Dv + \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^{\mathrm{T}} D^2 v\big] - kv.$$

Similar to Section 4.1, a function $v$ will be called a *supersolution* (resp. *subsolution*) of (4.16) if

$$\min\left\{-\partial_t v - \mathcal{L}v - f, v - g\right\} \geq \text{ (resp. } \leq \text{) } 0 \quad \text{and} \quad v(T,.) \geq \text{ (resp. } \leq \text{) } g.$$

Before stating the main result of this section, we observe that for many interesting examples, it is known that the value function $V$ does not satisfy the $C^{1,2}$ regularity which we have been using so far for the application of Itô's formula. Therefore, in order to state a result which can be applied to a wider class of problems, we shall enlarge in the following remark the set of function for which Itô's formula still holds true.

**Remark 4.6.** Let $v$ be a function in the Sobolev space $W^{1,2}(\mathbf{S})$. By definition, for such a function $v$, there is a sequence of functions $(v^n)_{n\geq 1} \subset C^{1,2}(\mathbf{S})$ such that $v^n \longrightarrow v$ uniformly on compact subsets of $\mathbf{S}$, and

$$\|\partial_t v^n - \partial_t v^m\|_{\mathbb{L}^2(\mathbf{S})} + \|Dv^n - Dv^m\|_{\mathbb{L}^2(\mathbf{S})} + \|D^2 v^n - D^2 v^m\|_{\mathbb{L}^2(\mathbf{S})} \longrightarrow 0.$$

Then, Itô's formula holds true for $v^n$ for all $n \geq 1$, and is inherited by $v$ by sending $n \to \infty$.

**Theorem 4.7.** *Let $T < \infty$ and $v \in W^{1,2}([0,T), \mathbb{R}^d)$. Assume further that $v$ and $f$ have quadratic growth. Then:*
(i) *If $v$ is a supersolution of* (4.16)*, then $v \geq V$.*
(ii) *If $v$ is a solution of* (4.16)*, then $v = V$ and*

$$\tau_t^* \quad := \quad \inf\{s > t : \; v(s, X_s) = g(X_s)\}$$

*is an optimal stopping time.*

*Proof.* Let $(t,x) \in [0,T) \times \mathbb{R}^d$ be fixed and denote $\beta_s := \beta(t,s)$.
(i) For an arbitrary stopping time $\tau \in \mathcal{T}^t_{[t,T)}$ , we denote

$$\tau_n \quad := \quad \tau \wedge \inf\left\{s > t : \; |X_s^{t,x} - x| > n\right\}.$$

By our regularity conditions on $v$, notice that Itô's formula can be applied to it piecewise. Then:

$$\begin{aligned} v(t,x) \;&=\; \beta_{\tau_n} v(\tau_n, X_{\tau_n}^{t,x}) - \int_t^{\tau_n} \beta_s (\partial_t + \mathcal{L}) v(s, X_s^{t,x}) ds - \int_t^{\tau_n} \beta_s (\sigma^{\mathrm{T}} D v)(s, X_s^{t,x}) dW_s \\ &\geq\; \beta_{\tau_n} v(\tau_n, X_{\tau_n}^{t,x}) + \int_t^{\tau_n} \beta_s f(s, X_s^{t,x}) ds - \int_t^{\tau_n} \beta_s (\sigma^{\mathrm{T}} D v)(s, X_s^{t,x}) dW_s \end{aligned}$$

by the supersolution property of $v$. Since $(s, X_s^{t,x})$ is bounded on the stochastic interval $[t, \tau_n]$, this provides:

$$v(t,x) \quad \geq \quad \mathbb{E}\Big[\beta_{\tau_n} v(\tau_n, X_{\tau_n}^{t,x}) + \int_t^{\tau_n} \beta_s f(s, X_s^{t,x}) ds\Big].$$

Notice that $\tau_n \longrightarrow \tau$ a.s. Then, since $f$ and $v$ have quadratic growth, we may pass to the limit $n \to \infty$ invoking the dominated convergence theorem, and we get:

$$v(t,x) \quad \geq \quad \mathbb{E}\Big[\beta_T v(T, X_T^{t,x}) + \int_t^T \beta_s f(s, X_s^{t,x}) ds\Big].$$

Since $v(T,.) \geq g$ by the supersolution property, this concludes the proof of (i).
(ii) Let $\tau_t^*$ be the stopping time introduced in the theorem. Then, since $v(T,.) = g$, it follows that $\tau_t^* \in \mathcal{T}_{[t,T]}^t$. Set

$$\tau_t^n := \tau_t^* \wedge \big\{\inf\{s > t : \ |X_s^{t,x} - x| > n\big\}.$$

Observe that $v > g$ on $[t, \tau_t^n) \subset [t, \tau_t^*)$ and therefore $-\partial_t v - \mathcal{L}v - f = 0$ on $[t, \tau_t^n)$. Then, proceeding as in the previous step, it follows from Itô's formula that:

$$v(t,x) \quad = \quad \mathbb{E}\Big[\beta_{\tau_t^n} v(\tau_t^n, X_{\tau_t^n}^{t,x}) + \int_t^{\tau_t^n} \beta_s f(s, X_s^{t,x}) ds\Big].$$

Since $\tau_t^n \longrightarrow \tau_t^*$ a.s. and $f, v$ have quadratic growth, we may pass to the limit $n \to \infty$ invoking the dominated convergence theorem. This leads to:

$$v(t,x) \quad = \quad \mathbb{E}\Big[\beta_T v(T, X_T^{t,x}) + \int_t^T \beta_s f(s, X_s^{t,x}) ds\Big],$$

and the required result follows from the fact that $v(T,.) = g$. $\diamondsuit$

# 4.4 Examples of optimal stopping problems with explicit solutions

## 4.4.1 Perpetual American options

The pricing problem of perpetual American put options reduces to the infinite horizon optimal stopping problem:

$$P(t,s) \quad := \quad \sup_{\tau \in \mathcal{T}_{[t,\infty)}^t} \mathbb{E}\big[e^{-r(\tau - t)}(K - S_\tau^{t,s})^+\big],$$

where $K > 0$ is a given exercise price, $S^{t,s}$ is defined by the Black-Scholes constant coefficients model:

$$S_u^{t,s} \quad := \quad s \exp\big(r - \frac{\sigma^2}{2}\big)(u-t) + \sigma(W_u - W_t), \quad u \geq t,$$

and $r \geq 0$, $\sigma > 0$ are two given constants. By the time-homogeneity of the problem, we see that

$$P(t,s) = P(s) := \sup_{\tau \in \mathcal{T}_{[0,\infty)}} \mathbb{E}\big[e^{-r\tau}(K - S_\tau^{0,s})^+\big]. \tag{4.17}$$

In view this time independence, it follows that the dynamic programming corresponding to this problem is:

$$\min\{v - (K-s)^+, rv - rsDv - \frac{1}{2}\sigma^2 D^2 v\} \quad = \quad 0. \tag{4.18}$$

In order to proceed to a verification argument, we now guess a solution to the previous obstacle problem. From the nature of the problem, we search for a solution of this obstacle problem defined by a parameter $s_0 \in (0, K)$ such that:

$$p(s) = K - s \text{ for } s \in [0, s_0] \quad \text{and} \quad rp - rsp' - \frac{1}{2}\sigma^2 s^2 p'' = 0 \text{ on } [s_0, \infty).$$

We are then reduced to solving a linear second order ODE on $[s_0, \infty)$, thus determining $v$ by

$$p(s) = As + Bs^{-2r/\sigma^2} \quad \text{for} \quad s \in [s_0, \infty),$$

up to the two constants $A$ and $B$. Notice that $0 \le p \le K$. Then the constant $A = 0$ in our candidate solution, because otherwise $v \longrightarrow \infty$ at infinity. We finally determine the constants $B$ and $s_0$ by requiring our candidate solution to be continuous and differentiable at $s^*$. This provides two equations:

$$Bs_0^{-2r/\sigma^2} = K - s_0 \quad \text{and} \quad \frac{-2r/\sigma^2}{B} s_0^{-2r/\sigma^2 - 1} = -1,$$

which provide our final candidate

$$s_0 = \frac{2rK}{2r + \sigma^2}, \quad p(s) = (K - s)\mathbf{1}_{[0,s_0]}(s) + \mathbf{1}_{[s_0,\infty)}(s)\frac{\sigma^2 s_0}{2r}\left(\frac{s}{s_0}\right)^{\frac{-2r}{\sigma^2}}. \qquad (4.19)$$

Notice that our candidate $p$ is not twice differentiable at $s_0$ as $p''(s_0-) = 0 \neq p''(s_0+)$. However, by Remark 4.6, Itô's formula still applies to $p$, and $p$ satisfies the dynamic programming equation (4.18). We now show that

$$p = P \text{ with optimal stopping time } \tau^* := \inf\left\{t > 0 : \; p(S_t^{0,s}) = (K - S_t^{0,s})^+\right\}. \qquad (4.20)$$

Indeed, for an arbitrary stopping time $\tau \in \mathcal{T}_{[0,\infty)}$, it follows from Itô's formula that:

$$\begin{aligned} p(s) &= e^{-r\tau} p(S_\tau^{0,s}) - \int_0^\tau e^{-rt}(-rp + rsp' + \frac{1}{2}\sigma^2 s^2 p'')(S_t)dt - \int_0^\tau p'(S_t)\sigma S_t dW_t \\ &\ge e^{-r\tau}(K - S_\tau^{t,s})^+ - \int_0^\tau p'(S_t)\sigma S_t dW_t \end{aligned}$$

by the fact that $p$ is a supersolution of the dynamic programming equation. Since $p'$ is bounded, there is no need to any localization to get rid of the stochastic integral, and we directly obtain by taking expected values that $p(s) \ge \mathbb{E}[e^{-r\tau}(K - S_\tau^{t,s})^+]$. By the arbitrariness of $\tau \in \mathcal{T}_{[0,\infty)}$, this shows that $p \ge P$.

We next repeat the same argument with the stopping time $\tau^*$, and we see that $p(s) = \mathbb{E}[e^{-r\tau^*}(K - S_{\tau^*}^{0,s})^+]$, completing the proof of (4.20).

### 4.4.2 Finite horizon American options

Finite horizon optimal stopping problems rarely have an explicit solution. So the following example can be seen as a sanity check. In the context of the financial market of the previous subsection, we assume the instanteneous interest rate $r = 0$, and we consider an American option with payoff function $g$ and maturity $T > 0$. Then the price of the corresponding American option is given by the optimal stopping problem:

$$P(t,s) \quad := \quad \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\big[g(S^{t,s}_\tau)\big]. \tag{4.21}$$

The corresponding dynamic programming equation is:

$$\min\big\{v-g, -\partial_t v - \frac{1}{2}D^2 v\big\} = 0 \ \text{ on } \ [0,T)\times\mathbb{R}_+ \quad \text{and} \quad v(T,.) = g. \tag{4.22}$$

Assuming further that $g \in W^{1,2}$ and concave, we see that $g$ is a solution of the dynamic programming equation. Then, provided that $g$ satisfies suitable growth condition, we see by a verification argument that $P = p$.

Notice that the previous result can be obtained directly by the Jensen inequality together with the fact that $S$ is a martingale.

# Chapter 5

# Introduction to Viscosity Solutions

Throughout this chapter, we provide the main tools from the theory of viscosity solutions for the purpose of our applications to stochastic control problems. For a deeper presentation, we refer to the excellent overview paper by Crandall, Ischii and Lions [3].

## 5.1 Intuition behind viscosity solutions

We consider a non-linear second order partial differential equation

$$\text{(E)} \quad F\left(x, u(x), Du(x), D^2u(x)\right) \;=\; 0 \;\text{ for }\; x \in \mathcal{O}$$

where $\mathcal{O}$ is an open subset of $\mathbb{R}^d$ and $F$ is a continuous map from $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \longrightarrow \mathbb{R}$. A crucial condition on $F$ is the so-called *ellipticity* condition :

**Standing Assumption** For all $(x, r, p) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d$ and $A, B \in \mathcal{S}_d$:

$$F(x, r, p, A) \leq F(x, r, p, B) \quad \text{whenever} \quad A \geq B.$$

The full importance of this condition will be made clear in Proposition 5.2 below.

The first step towards the definition of a notion of weak solution to (E) is the introduction of sub and supersolutions.

**Definition 5.1.** *A function $u : \mathcal{O} \longrightarrow \mathbb{R}$ is a classical supersolution (resp. subsolution) of (E) if $u \in C^2(\mathcal{O})$ and*

$$F\left(x, u(x), Du(x), D^2u(x)\right) \;\geq\; \textit{(resp. } \leq\textit{)} \;\; 0 \;\; \textit{for} \;\; x \in \mathcal{O}\, .$$

The theory of viscosity solutions is motivated by the following result, whose simple proof is left to the reader.

**Proposition 5.2.** *Let $u$ be a $C^2(\mathcal{O})$ function. Then the following claims are equivalent.*
(i) *$u$ is a classical supersolution (resp. subsolution) of (E)*
(ii) *for all pairs $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that $x_0$ is a minimizer (resp. maximizer) of the difference $u - \varphi$) on $\mathcal{O}$, we have*

$$F\left(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \quad \geq \textit{(resp. } \leq\textit{)} \quad 0 \; .$$

## 5.2 Definition of viscosity solutions

For the convenience of the reader, we recall the definition of the semicontinuous envelopes. For a locally bounded function $u : \mathcal{O} \longrightarrow \mathbb{R}$, we denote by $u_*$ and $u^*$ the lower and upper semicontinuous envelopes of $u$. We recall that $u_*$ is the largest lower semicontinuous minorant of $u$, $u^*$ is the smallest upper semicontinuous majorant of $u$, and

$$u_*(x) \;=\; \liminf_{x' \to x} u(x') \;, \qquad u^*(x) \;=\; \limsup_{x' \to x} u(x') \;.$$

We are now ready for the definition of viscosity solutions. Observe that Claim (ii) in the above proposition does not involve the regularity of $u$. It therefore suggests the following weak notion of solution to (E).

**Definition 5.3.** *Let $u : \mathcal{O} \longrightarrow \mathbb{R}$ be a locally bounded function.*
(i) *We say that $u$ is a (discontinuous) viscosity supersolution of (E) if*

$$F\left(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \quad \geq \quad 0$$

*for all pairs $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that $x_0$ is a minimizer of the difference $(u_* - \varphi)$ on $\mathcal{O}$.*
(ii) *We say that $u$ is a (discontinuous) viscosity subsolution of (E) if*

$$F\left(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \quad \leq \quad 0$$

*for all pairs $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that $x_0$ is a maximizer of the difference $(u^* - \varphi)$ on $\mathcal{O}$.*
(iii) *We say that $u$ is a (discontinuous) viscosity solution of (E) if it is both a viscosity supersolution and subsolution of (E).*

**Notation** We will say that $F\left(x, u_*(x), Du_*(x), D^2u_*(x)\right) \geq 0$ in the viscosity sense whenever $u_*$ is a viscosity supersolution of (E). A similar notation will be used for subsolution.

**Remark 5.4.** An immediate consequence of Proposition 5.2 is that any classical solution of (E) is also a viscosity solution of (E).

**Remark 5.5.** Clearly, the above definition is not changed if the minimum or maximum are local and/or strict. Also, by a density argument, the test function can be chosen in $C^\infty(\mathcal{O})$.

**Remark 5.6.** Consider the equation (E$^+$): $|u'(x)| - 1 = 0$ on $\mathbb{R}$. Then

- The function $f(x) := |x|$ is not a viscosity supersolution of (E$^+$). Indeed the test function $\varphi \equiv 0$ satisfies $(f-\varphi)(0) = 0 \leq (f-\varphi)(x)$ for all $x \in \mathbb{R}$. But $|\varphi'(0)| = 0 \not\geq 1$.

- The function $g(x) := -|x|$ is a viscosity solution of (E$^+$). To see this, we concentrate on the origin which is the only critical point. The supersolution property is obviously satisfied as there is no smooth function which satisfies the minimum condition. As for the subsolution property, we observe that whenever $\varphi \in C^1(\mathbb{R})$ satisfies $(g-\varphi)(0) = \max(g-\varphi)$, then $|\varphi'(0)| \geq 1$, which is exactly the viscosity subsolution property of $g$.

- Similarly, the function $f$ is a viscosity solution of the equation (E$^-$): $-|u'(x)| + 1 = 0$ on $\mathbb{R}$.

In Section 6.1, we will show that the value function $V$ is a viscosity solution of the DPE (2.18) under the conditions of Theorem 2.6 (except the smoothness assumption on $V$). We also want to emphasize that proving that the value function is a viscosity solution is almost as easy as proving that it is a classical solution when $V$ is known to be smooth.

The main difficulty in the theory of viscosity solutions is the interpretation of the equation in the viscosity sense. First, by weakening the notion of solution to the second order nonlinear PDE (E), we are enlarging the set of solutions, and one has to guarantee that uniqueness still holds (in some convenient class of functions). This issue will be discussed in the subsequent Section 5.4.

## 5.3 First properties

### 5.3.1 Change of variable / function

We start with two useful properties of viscosity solutions which allow to apply standard change of variable techniques for classical solutions in the context of viscosity solutions.

**Proposition 5.7.** *Let $u$ be a locally bounded (discontinuous) viscosity supersolution of (E). If $f$ is a $C^1(\mathbb{R})$ function with $Df \neq 0$ on $\mathbb{R}$, then the function $v := f^{-1} \circ u$ is a (discontinuous)*
*- viscosity supersolution, when $Df > 0$,*
*- viscosity subsolution, when $Df < 0$,*
*of the equation*

$$K(x, v(x), Dv(x), D^2v(x)) \;=\; 0 \quad \textit{for} \quad x \in \mathcal{O}\,,$$

*where*

$$K(x, r, p, A) \quad := \quad F\left(x, f(r), Df(r)p, D^2f(r)pp' + Df(r)A\right).$$

We leave the easy proof of this proposition to the reader. The next result shows how limit operations with viscosity solutions can be performed very easily.

A particular change of function in the previous proposition is the multiplication by a scalar. Another useful tool is the addition, or convex combination, of viscosity sub or supersolutions. Of course, this is possible only if the equation is suitable for such operations. For this reason, the following result specializes the discussion to the case where the nonlinearity $F(x, r, p, A)$ is convex in $(r, p, A)$. By standard convex analysis, this means that $F$ can be expressed as:

$$F(x, r, p, A) = \sup_{\gamma \in G} \Big\{ -f_\gamma(x) + k_\gamma(x) r - b_\gamma(x) \cdot p - \frac{1}{2} \mathrm{Tr}\big[\sigma_\gamma(x)^2 A\big] \Big\} \tag{5.1}$$

for some family of functions $(f_\gamma, k_\gamma, b_\gamma, \sigma_\gamma)_\gamma$.

**Proposition 5.8.** *Assume $F$ is convex in $(r, p, A)$, with functions $(f_\gamma, k_\gamma, b_\gamma)$ in the representation (5.1) continuous in $x$, and $\sigma_\gamma$ Lipschitz in $x$, for all $\gamma \in G$.*

*Let $u_1$ and $u_2$ be two upper semicontinuous viscosity subsolutions of (E). Then, $\lambda u_1 + (1 - \lambda) u_2$ is a viscosity subsolution of (E).*

We only provide the proof in the case where one either one of the subsolutions is classical. The general case requires more technical tools which will be developed later, and the corresponding proof is reported in Section 5.4.4.

*Proof. (Assuming $u_2 \in C^2$)* Denote $\lambda_1 := \lambda$ and $\lambda_2 := 1 - \lambda$, and $u := \lambda_1 u_1 + \lambda_2 u_2$. Let $x_0$ in $\mathcal{O}$ and $\varphi \in C^2(\mathcal{O})$ be such that $(u - \varphi)(x_0) = \max_{\mathcal{O}}(u - \varphi)$. Denote $\psi := \lambda_1^{-1}(\varphi - \lambda_2 u_2)$, or equivalently $\varphi = \lambda_1 \psi + \lambda_2 u_2$. Then $(u_1 - \psi)(x_0) = \max_{\mathcal{O}}(u_1 - \psi)$, and it follows from the viscosity subsolution property of $u_1$ that

$$F\big(x_0, u_1(x_0), D\psi(x_0), D^2\psi(x_0)\big) \leq 0.$$

Since $u_2$ is a classical subsolution of the equation (E), and $F$ is convex in $(r, p, A)$, we now compute directly that

$$\begin{aligned} F\big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) &\leq \lambda_1 F\big(x_0, u_1(x_0), D\psi(x_0), D^2\psi(x_0)\big) \\ &\quad + \lambda_2 F\big(x_0, u_2(x_0), Du_2(x_0), D^2 u_2(x_0)\big) \leq 0. \end{aligned}$$

$\diamondsuit$

Notice that, in the last simplified proof, the technical condition on the coefficients $(f_\gamma, k_\gamma, b_\gamma, \sigma_\gamma)_\gamma$ of the representation (5.1) of $F$ has not been used.

### 5.3.2 Stability of viscosity solutions

The following result is crucial in the theory of viscosity solutions, and is in fact the reason for the "viscosity" denomination. Loosely speaking, we now show that limiting operations with viscosity solutions are always possible under minimal assumptions, namely that all limiting quantities are well-defined !

**Theorem 5.9.** *Let $u_\varepsilon$ be a lower semicontinuous viscosity supersolution of the equation*

$$F_\varepsilon\left(x, u_\varepsilon(x), Du_\varepsilon(x), D^2u_\varepsilon(x)\right) = 0 \quad \text{for} \quad x \in \mathcal{O},$$

*where $(F_\varepsilon)_{\varepsilon>0}$ is a sequence of continuous functions satisfying the ellipticity condition. Suppose that $(\varepsilon, x) \longmapsto u_\varepsilon(x)$ and $(\varepsilon, z) \longmapsto F_\varepsilon(z)$ are locally bounded, and define*

$$\underline{u}(x) := \liminf_{(\varepsilon,x')\to(0,x)} u_\varepsilon(x') \quad \text{and} \quad \overline{F}(z) := \limsup_{(\varepsilon,z')\to(0,z)} F_\varepsilon(z').$$

*Then, $\underline{u}$ is a lower semicontinuous viscosity supersolution of the equation*

$$\overline{F}\left(x, \underline{u}(x), D\underline{u}(x), D^2\underline{u}(x)\right) = 0 \quad \text{for} \quad x \in \mathcal{O}.$$

*A similar statement holds for subsolutions.*

*Proof.* The fact that $\underline{u}$ is a lower semicontinuous function is left as an exercise for the reader. Let $\varphi \in C^2(\mathcal{O})$ and $\bar{x}$, be a strict minimizer of the difference $\underline{u} - \varphi$. By definition of $\underline{u}$, there is a sequence $(\varepsilon_n, x_n) \in (0,1] \times \mathcal{O}$ such that

$$(\varepsilon_n, x_n) \longrightarrow (0, \bar{x}) \quad \text{and} \quad u_{\varepsilon_n}(x_n) \longrightarrow \underline{u}(\bar{x}).$$

Consider some $r > 0$ together with the closed ball $\bar{B}$ with radius $r$, centered at $\bar{x}$. Of course, we may choose $|x_n - \bar{x}| < r$ for all $n \geq 0$. Let $\bar{x}_n$ be a minimizer of $u_{\varepsilon_n} - \varphi$ on $\bar{B}$. We claim that

$$\bar{x}_n \longrightarrow \bar{x} \text{ and } u_{\varepsilon_n}(\bar{x}_n) \longrightarrow \underline{u}(\bar{x}) \quad \text{as} \quad n \to \infty. \tag{5.2}$$

Before verifying this, let us complete the proof. We first deduce that $\bar{x}_n$ is an interior point of $\bar{B}$ for large $n$, so that $\bar{x}_n$ is a local minimizer of the difference $u_{\varepsilon_n} - \varphi$. Then :

$$F_{\varepsilon_n}\left(\bar{x}_n, u_{\varepsilon_n}(\bar{x}_n), D\varphi(\bar{x}_n), D^2\varphi(\bar{x}_n)\right) \geq 0,$$

and the required result follows by taking limits and using the definition of $\overline{F}$.

It remains to prove Claim (5.2). Recall that $(x_n)_n$ is valued in the compact set $\bar{B}$. Then, there is a subsequence, still named $(x_n)_n$, which converges to some $\tilde{x} \in \bar{B}$. We now prove that $\tilde{x} = \bar{x}$ and obtain the second claim in (5.2) as a by-product. Using the fact that $\bar{x}_n$ is a minimizer of $u_{\varepsilon_n} - \varphi$ on $\bar{B}$, together with the definition of $\underline{u}$, we see that

$$\begin{aligned}
0 = (\underline{u} - \varphi)(\bar{x}) &= \lim_{n\to\infty} \left(u_{\varepsilon_n} - \varphi\right)(x_n) \\
&\geq \limsup_{n\to\infty} \left(u_{\varepsilon_n} - \varphi\right)(\bar{x}_n) \\
&\geq \liminf_{n\to\infty} \left(u_{\varepsilon_n} - \varphi\right)(\bar{x}_n) \\
&\geq (\underline{u} - \varphi)(\tilde{x}).
\end{aligned}$$

We now obtain (5.2) from the fact that $\bar{x}$ is a strict minimizer of the difference $(\underline{u} - \varphi)$. $\diamondsuit$

Observe that the passage to the limit in partial differential equations written in the classical or the generalized sense usually requires much more technicalities, as one has to ensure convergence of all the partial derivatives involved in the equation. The above stability result provides a general method to pass to the limit when the equation is written in the viscosity sense, and its proof turns out to be remarkably simple.

A possible application of the stability result is to establish the convergence of numerical schemes. In view of the simplicity of the above statement, the notion of viscosity solutions provides a nice framework for such questions.

### 5.3.3 Parameter variables

The following result is trivial in the classical case, but needs some technicalities when stated in the viscosity sense.

**Proposition 5.10.** *Let $A \subset \mathbb{R}^{d_1}$ and $B \subset \mathbb{R}^{d_2}$ be two open subsets, and let $u : A \times B \longrightarrow \mathbb{R}$ be a lower semicontinuous viscosity supersolution of the equation :*

$$F\left(x, y, u(x,y), D_y u(x,y), D_y^2 u(x,y)\right) \ \geq \ 0 \quad \text{on} \quad A \times B,$$

*where $F$ is a continuous elliptic operator. Then, for all fixed $x_0 \in A$, the function $v(y) := u(x_0, y)$ is a viscosity supersolution of the equation :*

$$F\left(x_0, y, v(y), Dv(y), D^2 v(y)\right) \ \geq \ 0 \quad \text{on} \quad B.$$

*A similar statement holds for the subsolution property.*

*Proof.* Fix $x_0 \in A$, set $v(y) := u(x_0, y)$, and let $y_0 \in B$ and $f \in C^2(B)$ be such that

$$(v - f)(y_0) \ < \ (v - f)(y) \quad \text{for all} \quad y \in J \setminus \{y_0\} \, , \tag{5.3}$$

where $J$ is an arbitrary compact subset of $B$ containing $y_0$ in its interior. For each integer $n$, define

$$\varphi_n(x, y) \ := \ f(y) - n|x - x_0|^2 \quad \text{for} \quad (x, y) \in A \times B \, ,$$

and let $(x_n, y_n)$ be defined by

$$(u - \varphi_n)(x_n, y_n) \ = \ \min_{I \times J} (u - \varphi_n) \, ,$$

where $I$ is a compact subset of $A$ containing $x_0$ in its interior. We claim that

$$(x_n, y_n) \longrightarrow (x_0, y_0) \quad \text{and} \quad u(x_n, y_n) \longrightarrow u(x_0, y_0) \quad \text{as} \quad n \longrightarrow \infty. \tag{5.4}$$

Before proving this, let us complete the proof. Since $(x_0, y_0)$ is an interior point of $A \times B$, it follows from the viscosity property of $u$ that

$$\begin{aligned} 0 &\leq F\left(x_n, y_n, u(x_n, y_n), D_y\varphi_n(x_n, y_n), D_y^2\varphi_n(x_n, y_n)\right) \\ &= F\left(x_n, y_n, u(x_n, y_n), Df(y_n), D^2 f(y_n)\right), \end{aligned}$$

and the required result follows by sending $n$ to infinity.

We now turn to the proof of (5.4). Since the sequence $(x_n, y_n)_n$ is valued in the compact subset $A \times B$, we have $(x_n, y_n) \longrightarrow (\bar{x}, \bar{y}) \in A \times B$, after passing to a subsequence. Observe that

$$\begin{aligned} u(x_n, y_n) - f(y_n) &\leq u(x_n, y_n) - f(y_n) + n|x_n - x_0|^2 \\ &= (u - \varphi_n)(x_n, y_n) \\ &\leq (u - \varphi_n)(x_0, y_0) \;=\; u(x_0, y_0) - f(y_0)\,. \end{aligned}$$

Taking the limits, this provides:it follows from the lower semicontinuity of $u$ that

$$\begin{aligned} u(\bar{x}, \bar{y}) - f(\bar{y}) &\leq \liminf_{n\to\infty} u(x_n, y_n) - f(y_n) + n|x_n - x_0|^2 \\ &\leq \limsup_{n\to\infty} u(x_n, y_n) - f(y_n) + n|x_n - x_0|^2 \qquad (5.5) \\ &\leq u(x_0, y_0) - f(y_0). \end{aligned}$$

Since $u$ is lower semicontinu, this implies that $u(\bar{x}, \bar{y}) - f(\bar{y}) + \liminf_{n\to\infty} n|x_n - x_0|^2 \leq u(x_0, y_0) - f(y_0)$. Then, we must have $\bar{x} = x_0$, and

$$(v - f)(\bar{y}) \;=\; u(x_0, \bar{y}) - f(\bar{y}) \quad\leq\quad (v - f)(y_0),$$

which implies that $\bar{y} = y_0$ in view of (5.3), and $n|x_n - x_0|^2 \longrightarrow 0$. We also deduce from inequalities (5.5) that $u(x_n, y_n) \longrightarrow u(x_0, y_0)$, concluding the proof of (5.4). $\diamondsuit$

## 5.4 Comparison result and uniqueness

In this section, we show that the notion of viscosity solutions is consistent with the maximum principle for a wide class of equations. Once we will have such a result, the reader must be convinced that the notion of viscosity solutions is a good weakening of the notion of classical solution.

We recall that the maximum principle is a stronger statement than uniqueness, i.e. any equation satisfying a comparison result has no more than one solution.

In the viscosity solutions literature, the maximum principle is rather called *comparison principle*.

### 5.4.1 Comparison of classical solutions in a bounded domain

Let us first review the maxium principle in the simplest classical sense.

**Proposition 5.11.** *Assume that $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^d$, and the nonlinearity $F(x,r,p,A)$ is elliptic and strictly increasing in $r$. Let $u,v \in C^2\big(\mathrm{cl}(\mathcal{O})\big)$ be classical subsolution and supersolution of* (E)*, respectively, with $u \le v$ on $\partial\mathcal{O}$. Then $u \le v$ on* $\mathrm{cl}(\mathcal{O})$.

*Proof.* Our objective is to prove that

$$M \;:=\; \sup_{\mathrm{cl}(\mathcal{O})} (u-v) \;\le\; 0. \tag{5.6}$$

Assume to the contrary that $M>0$. Then since $\mathrm{cl}(\mathcal{O})$ is a compact subset of $\mathbb{R}^d$, and $u-v \le 0$ on $\partial\mathcal{O}$, we have:

$$M=(u-v)(x_0) \text{ for some } x_0\in\mathcal{O} \text{ with } D(u-v)(x_0)=0,\; D^2(u-v)(x_0)\le 0. \tag{5.7}$$

Then, it follows from the viscosity properties of $u$ and $v$ that:

$$\begin{aligned} F\big(x_0,u(x_0),Du(x_0),D^2u(x_0)\big) \;\le\; 0 \quad &\le \quad F\big(x_0,v(x_0),Dv(x_0),D^2v(x_0)\big) \\ &\le \quad F\big(x_0,u(x_0)-M,Du(x_0),D^2u(x_0)\big), \end{aligned}$$

where the last inequality follows crucially from the ellipticity of $F$. This provides the desired contradiction, under our condition that $F$ is strictly increasing in $r$. $\diamondsuit$

The objective of this section is to mimic the previous proof in the sense of viscosity solutions. We first start by the case of first order equations where the beautiful trick of doubling variables allows for an immediate adaptation of the previous argument. However, more work is needed for the second order case.

### 5.4.2 Comparison of viscosity solutions of first order equations

A crucial idea in the theory of viscosity solutions is to replace the maximization problem in (5.6) by:

$$M_n := \sup_{x,y\in\mathrm{cl}(\mathcal{O})} \Big\{ u(x)-v(y)-\frac{n}{2}|x-y|^2 \Big\}, \quad n\ge 1, \tag{5.8}$$

where the concept of doubling variables separates the dependence of the functions $u$ and $v$ on two different variables, and the penalization term involves the parameter $n$ which is intended to mimick the maximization problem (5.6) for large $n$.

**Proposition 5.12.** *Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^d$. Assume that the nonlineary $F(x, r, p)$ is independent of the $A-$variable, Lipshitz in the $x-$variable uniformly in $(r, p)$, and that there is a constant $c > 0$ such that*

$$F(x, r', p) - F(x, r, p) \geq c(r' - r) \text{ for all } r' \geq r, \ x \in \mathrm{cl}(\mathcal{O}), \ p \in \mathbb{R}^d. \tag{5.9}$$

*Let $u, v \in C^0\big(\mathrm{cl}(\mathcal{O})\big)$ be viscosity subsolution and supersolution of* (E)*, respectively, with $u \leq v$ on $\partial\mathcal{O}$. Then $u \leq v$ on* $\mathrm{cl}(\mathcal{O})$.

*Proof.* Suppose to the contrary that $\eta := (u - v)(x_0) > 0$ for some $x_0 \in \mathcal{O}$, so that the maximum value in (5.8) $M_n \geq \eta$. Since $u$ and $v$ are continuous, we may find for each $n \geq 1$ a maximizer $(x_n, y_n) \in \mathrm{cl}(\mathcal{O})^2$ of the problem (5.8):

$$M_n = u(x_n) - v(y_n) - \frac{n}{2}|x_n - y_n|^2.$$

We shall prove later that

$$2\varepsilon_n := n|x_n - y_n|^2 \longrightarrow 0, \text{ as } n \to \infty, \text{ and } x_n, y_n \in \mathcal{O} \text{ for large } n. \tag{5.10}$$

Observe that (5.8) implies that $x_n$ is a maximizer of the difference $(u - \varphi)$, and $y_n$ is a minimizer of the difference $(v - \psi)$, where

$$\varphi(x) := v(y_n) + \frac{n}{2}|x - y_n|^2, \ \ x \in \mathrm{cl}(\mathcal{O}), \ \ \psi(y) := u(x_n) - \frac{n}{2}|x_n - y|^2, \ \ x \in \mathrm{cl}(\mathcal{O}).$$

Then, it follows from the viscosity properties of $u$ and $v$ that

$$F\big(x_n, u(x_n), D\varphi(x_n)\big) \leq 0 \leq F\big(y_n, v(y_n), D\psi(y_n)\big), \quad \text{for large} \quad n.$$

Using the Lipshitz property of $F$ in $x$ and the increaseness property (5.9), this implies that

$$\begin{aligned} 0 &\geq F\big(x_n, v(y_n) + M_n + \varepsilon_n, n(x_n - y_n)\big) - F\big(y_n, v(y_n), n(x_n - y_n)\big) \\ &\geq -|F_x|_{\mathbb{L}^\infty}|x_n - y_n| + c(M_n + \varepsilon_n) \end{aligned}$$

In view of (5.10) and the fact that $M_n \geq \eta > 0$, this leads to the required contradiction.

It remains to justify (5.10). Let $x^* \in \mathrm{cl}(\mathcal{O})$ ba a maximizer of $(u - v)$ on $\mathrm{cl}(\mathcal{O})$, and denote by $m_n(x, y)$ the objective function in the maximization problem (5.8). Let $(\bar{x}, \bar{y})$ be any accumulation point of the bounded sequence $(x_n, y_n)_n$, i.e. $(x_{n_k}, y_{n_k}) \longrightarrow (\bar{x}, \bar{y})$ for some subsequence $(n_k)_k$. Then, it follows from the obvious inequality $M_n = m_n(x_n, y_n) \geq m_n(x^*, x^*)$ that

$$\begin{aligned} \limsup_{k\to\infty} \frac{n_k}{2}|x_{n_k} - y_{n_k}|^2 &\leq u(x_{n_k}) - v(y_{n_k}) - (u - v)(x^*) \\ &\leq u(\bar{x}) - v(\bar{y}) - (u - v)(x^*), \end{aligned}$$

by the upper-semicontinuity of $u$ and the slower-semicontinuity of $v$. This shows that $\bar{x} = \bar{y}$, and it follows from the definition of $x^*$ as a maximizer of $(u - v)$ that

$$\limsup_{k\to\infty} \frac{n_k}{2}|x_{n_k} - y_{n_k}|^2 \leq (u - v)(\bar{x}) - (u - v)(x^*) \leq 0,$$

and hence $n_k|x_{n_k}-y_{n_k}|^2 \longrightarrow 0$ as $k\to\infty$. By the arbitrariness of the converging subsequence $(x_{n_k},y_{n_k})_k$, we obtain that $n|x_n-y_n|^2 \longrightarrow 0$ as $n\to\infty$.

Finally, suppose that there is a subsequence $(x_{n_k})_{k\ge 1}\subset \partial\mathcal{O}$. Then, since $u\le v$ on $\partial\mathcal{O}$:

$$\begin{aligned}\eta \ \le\ M_{n_k} \ \le\ u(x_{n_k})-v(y_{n_k}) \quad &\le \quad v(x_{n_k})-v(y_{n_k})+\sup_{\partial\mathcal{O}}(u-v)\\ &\le \quad \rho\big(|x_{x_k}-y_{n_k}|\big) \ \longrightarrow\ 0, \text{ as } k\to\infty,\end{aligned}$$

contradicting the positivity of $\eta$. A similar argument shows that $y_n\in\mathcal{O}$ for large $n$. ♢

In the previous proof, one can easily see that the condition that $F$ is uniformly Lipshitz in $x$ can be weakened by exploiting the stronger convergence of $|x_n-y_n|$ to 0 in (5.10). For instance, the previous comparison result holds true under the monotonicity condition (5.9) together with the locally Lipschitz condition:

$$|F(x',r,p)-F(x,r,p)| \le C(1+|p|)|x-x'|.$$

### 5.4.3 Semijets definition of viscosity solutions

In order to extend the comparison result to second order equations, we first need to develop a convenient alternative definition of viscosity solutions. For $p\in\mathbb{R}^d$, and $A\in\mathcal{S}_d$, we introduce the quadratic function:

$$q^{p,A}(y) \quad := \quad p\cdot y+\frac{1}{2}Ay\cdot y, \quad y\in\mathbb{R}^d.$$

For $v\in\mathrm{LSC}(\mathcal{O})$, let $(x_0,\varphi)\in\mathcal{O}\times C^2(\mathcal{O})$ be such that $x_0$ is a local minimizer of the difference $(v-\varphi)$ in $\mathcal{O}$. Then, defining $p:=D\varphi(x_0)$ and $A:=D^2\varphi(x_0)$, it follows from a second order Taylor expansion that:

$$v(x)\ge v(x_0)+q^{p,A}(x-x_0)+\circ\big(|x-x_0|^2\big).$$

Motivated by this observation, we introduce the *subjet* $J^-_{\mathcal{O}}v(x_0)$ by

$$J^-_{\mathcal{O}}v(x_0) := \Big\{(p,A)\in\mathbb{R}^d\times\mathcal{S}_d : v(x)\ge v(x_0)+q^{p,A}(x-x_0)+\circ\big(|x-x_0|^2\big)\Big\}. \tag{5.11}$$

Similarly, we define the *superjet* $J^+_{\mathcal{O}}u(x_0)$ of a function $u\in\mathrm{USC}(\mathcal{O})$ at the point $x_0\in\mathcal{O}$ by

$$J^+_{\mathcal{O}}u(x_0) := \Big\{(p,A)\in\mathbb{R}^d\times\mathcal{S}_d : u(x)\le u(x_0)+q^{p,A}(x-x_0)+\circ\big(|x-x_0|^2\big)\Big\} \tag{5.12}$$

Then, it can be proved that a function $v\in\mathrm{LSC}(\mathcal{O})$ is a viscosity supersolution of the equation (E) if and only if

$$F(x,v(x),p,A)\ge 0 \quad \text{for all} \quad (p,A)\in J^-_{\mathcal{O}}v(x).$$

The nontrivial implication of the previous statement requires to construct, for every $(p, A) \in J_{\mathcal{O}}^- v(x_0)$, a smooth test function $\varphi$ such that the difference $(v-\varphi)$ has a local minimum at $x_0$. We refer to Fleming and Soner [6], Lemma V.4.1 p211.

A symmetric statement holds for viscosity subsolutions. By continuity considerations, we can even enlarge the semijets $J_{\mathcal{O}}^{\pm} w(x_0)$ to the folowing closure

$$\begin{aligned}\bar{J}_{\mathcal{O}}^{\pm} w(x) \quad := \quad & \Big\{(p, A) \in \mathbb{R}^d \times \mathcal{S}_d : (x_n, w(x_n), p_n, A_n) \longrightarrow (x, w(x), p, A) \\ & \qquad \text{for some sequence } (x_n, p_n, A_n)_n \subset \mathrm{Graph}(J_{\mathcal{O}}^{\pm} w)\Big\},\end{aligned}$$

where $(x_n, p_n, A_n) \in \mathrm{Graph}(J_{\mathcal{O}}^{\pm} w)$ means that $(p_n, A_n) \in J_{\mathcal{O}}^{\pm} w(x_n)$. The following result is obvious, and provides an equivalent definition of viscosity solutions.

**Proposition 5.13.** *Consider an elliptic nonlinearity $F$, and let $u \in \mathrm{USC}(\mathcal{O})$, $v \in \mathrm{LSC}(\mathcal{O})$.*
(i) *Assume that $F$ is lower-semicontinuous. Then, $u$ is a viscosity subsolution of* (E) *if and only if:*

$$F(x, u(x), p, A) \leq 0 \quad \text{for all} \quad x \in \mathcal{O} \text{ and } (p, A) \in \bar{J}_{\mathcal{O}}^+ u(x).$$

(ii) *Assume that $F$ is upper-semicontinuous. Then, $v$ is a viscosity supersolution of* (E) *if and only if:*

$$F(x, v(x), p, A) \geq 0 \quad \text{for all} \quad x \in \mathcal{O} \text{ and } (p, A) \in \bar{J}_{\mathcal{O}}^- v(x).$$

### 5.4.4 The Crandall-Ishii's lemma

The major difficulty in mimicking the proof of Proposition 5.11 is to derive an analogous statement to (5.7) without involving the smoothness of $u$ and $v$, as these functions are only known to be upper- and lower-semicontinuous in the context of viscosity solutions.

This is provided by the following result due to M. Crandall and I. Ishii. For a symmetric matrix, we denote by $|A| := \sup\{(A\xi) \cdot \xi : |\xi| \leq 1\}$.

**Lemma 5.14.** *Let $\mathcal{O}$ be an open locally compact subset of $\mathbb{R}^d$. Given $u \in USC(\mathcal{O})$ and $v \in LSC(\mathcal{O})$, set $m(x, y) := u(x) - v(y)$, $x, y \in \mathcal{O}$, and assume that:*

$$(m-\phi)(x_0, y_0) = \max_{\mathcal{O}^2}(m-\phi) \text{ for some } (x_0, y_0) \in \mathcal{O}^2,\ \phi \in C^2\left(cl(\mathcal{O})^2\right). \tag{5.13}$$

*Then, for each $\varepsilon > 0$, there exist $A, B \in \mathcal{S}_d$ such that*

$$(D_x\phi(x_0, y_0), A) \in \bar{J}_{\mathcal{O}}^+ u(x_0), \qquad (-D_y\phi(x_0, y_0), B) \in \bar{J}_{\mathcal{O}}^- v(y_0),$$

*and the following inequality holds in the sense of symmetric matrices in $\mathcal{S}_{2d}$:*

$$-\left(\varepsilon^{-1} + \left|D^2\phi(x_0, y_0)\right|\right) I_{2d} \leq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq D^2\phi(x_0, y_0) + \varepsilon D^2\phi(x_0, y_0)^2.$$

*Proof.* See Section 5.7. $\diamondsuit$

We will be applying Lemma 5.14 in the particular case

$$\phi(x,y) := \frac{\alpha}{2}|x-y|^2 \quad \text{for} \quad x,y \in \mathcal{O}. \tag{5.14}$$

Intuitively, sending $\alpha$ to $\infty$, we expect that the maximization of $(u(x)-v(y)-\phi(x,y)$ on $\mathcal{O}^2$ reduces to the maximization of $(u-v)$ on $\mathcal{O}$ as in (5.7). Then, taking $\varepsilon^{-1}=\alpha$, we directly compute that the conclusions of Lemma 5.14 reduce to

$$(\alpha(x_0-y_0),A) \in \bar{J}^+_{\mathcal{O}}u(x_0), \qquad (\alpha(x_0-y_0),B) \in \bar{J}^-_{\mathcal{O}}v(y_0), \tag{5.15}$$

and

$$-3\alpha\begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} \le \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \le 3\alpha\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}. \tag{5.16}$$

**Remark 5.15.** If $u$ and $v$ were $C^2$ functions in Lemma 5.14, the first and second order condition for the maximization problem (5.13) with the test function (5.14) is $Du(x_0)=\alpha(x_0-y_0)$, $Dv(x_0)=\alpha(x_0-y_0)$, and

$$\begin{pmatrix} D^2u(x_0) & 0 \\ 0 & -D^2v(y_0) \end{pmatrix} \le \alpha\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}.$$

Hence, the right-hand side inequality in (5.16) is worsening the previous second order condition by replacing the coefficient $\alpha$ by $3\alpha$. $\diamondsuit$

**Remark 5.16.** The right-hand side inequality of (5.16) implies that

$$A \le B. \tag{5.17}$$

To see this, take an arbitrary $\xi \in \mathbb{R}^d$, and denote by $\xi^{\mathrm{T}}$ its transpose. From right-hand side inequality of (5.16), it follows that

$$0 \ge (\xi^{\mathrm{T}},\xi^{\mathrm{T}})\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix} = (A\xi)\cdot\xi - (B\xi)\cdot\xi.$$

$\diamondsuit$

Before turning to the comparison result for second order equations, let us go back to the

**Proof of Proposition 5.8** Introduce the linear maps in $(r,p,A)$:

$$F_\gamma(x,r,p,A) = -f_\gamma(x) + k_\gamma(x)r - b_\gamma(x)\cdot p - \frac{1}{2}\mathrm{Tr}\big[\sigma_\gamma(x)^2A\big] \quad \text{for all} \quad \gamma \in G.$$

In view of the representation (5.1), the functions $u_1$ and $u_2$ are viscosity subsolutions of the linear equations

$$F_\gamma\big(x,u_i(x),Du_i(x),D^2u_i(x)\big) \le 0, \quad \text{for all} \quad \gamma \in G; i=1,2.$$

**1.** Denote again $\lambda_1 := \lambda$, $\lambda_2 := 1-\lambda$, and set $u = \lambda_1 u_1 + \lambda_2 u_2$. Let $r > 0$, $x^0 \in B_0 := B_r(x^0) \subset \mathcal{O}$, with closure $\bar{B}_0$, and $\varphi \in C^2(\mathcal{O})$ be such that

$$(u-\varphi)(x^0) \quad = \quad \text{strict}\max_{\bar{B}_0}(u-\varphi).$$

We shall use the doubling variable technique, and thus introduce:

$$\psi(x,y) := \lambda_1(u_1-\varphi)(x) + \lambda_2(u_2-\varphi)(y) - \phi(x,y), \quad \text{with} \quad \phi(x,y) := \frac{\alpha}{2}|x-y|^2.$$

By the compactness of the closed ball $\bar{B}_0$, we may find a maximizer $(x_\alpha, y_\alpha)$ of $\phi$ on $\bar{B}_0 \times \bar{B}_0$:

$$M_\alpha := \sup_{\bar{B}_0 \times \bar{B}_0} \phi = \phi(x_\alpha, y_\alpha) \quad \text{for all} \quad \alpha \geq 0.$$

We continue using the following claim, which will be proved later,

$$\begin{array}{c} \alpha|x_\alpha - y_\alpha|^2 \longrightarrow 0, \quad x_\alpha \longrightarrow x^0, \quad M_\alpha \longrightarrow (u-\varphi)(x^0), \\ \text{and} \quad u_1(x_\alpha) \longrightarrow u_1(x^0), \quad u_2(y_\alpha) \longrightarrow u_2(x^0). \end{array} \tag{5.18}$$

In particular, this implies that $x_\alpha, y_\alpha \in B_0$ for sufficiently large $\alpha$.
**2.** We are now in a position to apply the Crandall-Ishii Lemma 5.14. Then, we may find matrices $A_1, A_2 \in \mathcal{S}_d$ such that

$$\begin{array}{l} \big(D_x\phi(x_\alpha,y_\alpha) + \lambda_1 D\varphi(x_\alpha), A_1\big) \in \bar{J}^+_{\mathcal{O}}(\lambda_1 u_1)(x_\alpha) = \lambda_1 \bar{J}^+_{\mathcal{O}}(u_1)(x_\alpha), \\ \big(D_y\phi(x_\alpha,y_\alpha) + \lambda_2 D\varphi(y_\alpha), A_2\big) \in \bar{J}^-_{\mathcal{O}}(\lambda_2 u_2)(y_\alpha) = \lambda_2 \bar{J}^-_{\mathcal{O}}(u_2)(y_\alpha), \end{array}$$

with the matrix inequalities

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \leq \quad k\alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + \begin{pmatrix} \lambda_1 D^2\varphi(x_\alpha) & 0 \\ 0 & \lambda_2 D^2\varphi(y_\alpha) \end{pmatrix}.$$

As $u_1$ and $u_2$ are both viscosity subsolutions of the equation defined by the nonlinearity $F_\gamma$, this implies that

$$\begin{array}{rcl} 0 & \geq & \lambda_1 F_\gamma\big(x_\alpha, u_1(x_\alpha), \lambda_1^{-1} D_x\phi(x_\alpha,y_\alpha) + D\varphi(x_\alpha), \lambda_1^{-1}A_1\big) \\ & & +\lambda_2 F_\gamma\big(y_\alpha, u_2(y_\alpha), \lambda_2^{-1} D_y\phi(x_\alpha,y_\alpha) + D\varphi(y_\alpha), \lambda_2^{-1}A_2\big) \\ & = & -\lambda_1 f_\gamma(x_\alpha) - \lambda_2 f_\gamma(y_\alpha) + \lambda_1 k_\gamma(x_\alpha)u_1(x_\alpha) + \lambda_2 k_\gamma(y_\alpha)u_2(y_\alpha) \\ & & -\lambda_1 b_\gamma(x_\alpha)\cdot D\varphi(x_\alpha) - \lambda_2 b_\gamma(y_\alpha)\cdot D\varphi(y_\alpha) \\ & & -\dfrac{1}{2}\mathrm{Tr}\big[\sigma_\gamma^2(x_\alpha)A_1 + \sigma_\gamma^2(y_\alpha)A_2\big], \end{array}$$

where we used the fact that $D_x\phi = -D_y\phi = \alpha(x-y)$. Then, $f_\gamma, k_\gamma, b_\gamma$ are continuous, (5.18) yields

$$\begin{array}{rcl} 0 & \geq & -f_\gamma(x^0) + k_\gamma(x^0)u(x^0) - b_\gamma(x^0)\cdot D\varphi(x^0) \\ & & -\dfrac{1}{2}\liminf\limits_{\alpha\to\infty} \mathrm{Tr}\big[\sigma_\gamma^2(x_\alpha)A_1 + \sigma_\gamma^2(y_\alpha)A_2\big]. \end{array}$$

Denoting $A_1' := A_1 - D^2\varphi(x_\alpha)$, and $A_2' := A_2 - D^2\varphi(y_\alpha)$, it follows from the calculation in Example 5.21 below that

$$\begin{aligned}
\mathrm{Tr}\big[\sigma_\gamma^2(x_\alpha)A_1 + \sigma_\gamma^2(y_\alpha)A_2\big] &= \mathrm{Tr}\big[\sigma_\gamma^2(x_\alpha)D^2\varphi(x_\alpha) + \sigma_\gamma^2(y_\alpha)D^2\varphi(y_\alpha)\big] \\
&\quad + \mathrm{Tr}\big[\sigma_\gamma^2(x_\alpha)A_1' + \sigma_\gamma^2(y_\alpha)A_2'\big] \\
&\le \mathrm{Tr}\big[\sigma_\gamma^2(x_\alpha)D^2\varphi(x_\alpha) + \sigma_\gamma^2(y_\alpha)D^2\varphi(y_\alpha)\big] \\
&\quad + 3\alpha\mathrm{Tr}\big[(\sigma_\gamma(x_\alpha) - \sigma_\gamma(y_\alpha))^2\big] \\
&\le \mathrm{Tr}\big[\sigma_\gamma^2(x_\alpha)D^2\varphi(x_\alpha) + \sigma_\gamma^2(y_\alpha)D^2\varphi(y_\alpha)\big] \\
&\quad + 3L_\gamma\alpha|x_\alpha - y_\alpha|^2,
\end{aligned}$$

where $L_\gamma$ is the Lipschitz constant of the coefficient $\sigma_\gamma$. Using again (5.18) , this provides

$$0 \;\ge\; -f_\gamma(x^0) + k_\gamma(x^0)u(x^0) - b_\gamma(x^0)\cdot D\varphi(x^0) - \frac{1}{2}\mathrm{Tr}[\sigma_\gamma^2(x^0)D^2\varphi(x^0)].$$

The required viscosity subsolution property follows from the arbitrariness of $\gamma \in G$.

**3.** It remains to justify (5.18). As $(x_\alpha, y_\alpha)_\alpha$ is bounded, we may find a converging sequence $(x_{\alpha_n}, y_{\alpha_n})_n$ to some imiting point $(\hat{x}, \hat{y})$. By the definition of $(x_\alpha, y_\alpha)$ as the maximizer of $\phi$, we have

$$\begin{aligned}
(u-\varphi)(x^0, y^0) &\le \limsup_{n\to\infty} \phi(x_{\alpha_n}, y_{\alpha_n}) \\
&\le \lambda_1(u_1-\varphi)(\hat{x}) + \lambda_2(u_2-\varphi)(\hat{y}) - \liminf_{n\to\infty} \alpha_n\big|x_{\alpha_n} - y_{\alpha_n}\big|^2.
\end{aligned}$$

This implies that $\hat{x} = \hat{y}$. Then,

$$\begin{aligned}
(u-\varphi)(x^0) &\le \liminf_{n\to\infty} \phi(x_{\alpha_n}, y_{\alpha_n}) \\
&\le \limsup_{n\to\infty} \phi(x_{\alpha_n}, y_{\alpha_n}) \;\le\; (u-\varphi)(\hat{x}).
\end{aligned}$$

As $x^0$ is a strict maximizer of the difference $u - \varphi$, we conclude that $\hat{x} = x^0$. In particular, since any subsequence of $(x_\alpha, y_\alpha)$ converges to $(x^0, y^0)$, we deduce that $(x_\alpha, y_\alpha) \longrightarrow (x^0, y^0)$, and we may repeat the previous sequence of inequalities with $\alpha \to \infty$, and obtain that all inequalities are in fact equalities. Consequently, we also have $\alpha|x_\alpha - y_\alpha|^2 \longrightarrow 0$, and $\limsup_{\alpha\to\infty} \lambda_1 u_1(x_\alpha) + \lambda_2 u_2(y_\alpha) = u(x^0)$. The last convergence result, together with the upper semicontinuity of $u_2$, implies that

$$\lambda_1 \liminf_{\alpha\to\infty} u_1(x_\alpha) = \lim_{\alpha\to\infty} \lambda_1 u_(x_\alpha) + \lambda_2 u_2(y_\alpha) - \lambda_2 \limsup_{\alpha\to\infty} u_2(y_\alpha) \le \lambda_1 u_1(x^0).$$

As $u_1$ is upper semicontinuous, this implies that $u_1(x_\alpha) \longrightarrow u_1(x^0)$. By the same argument, we obtain that $u_2(y_\alpha) \longrightarrow u_2(x^0)$. $\diamondsuit$

### 5.4.5 Comparison of viscosity solutions in a bounded domain

We now prove a comparison result for viscosity sub- and supersolutions by using Lemma 5.14 to mimic the proof of Proposition 5.11. The statement will be proved under the following conditions on the nonlinearity $F$ which will be used at the final Step 3 of the subsequent proof.

**Assumption 5.17.** (i) *There exists $\gamma > 0$ such that*

$$F(x,r,p,A) - F(x,r',p,A) \geq \gamma(r-r') \textit{ for all } r \geq r',\ (x,p,A) \in \mathcal{O} \times \mathbb{R}^d \times \mathcal{S}_d.$$

(ii) *There is a function $\varpi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\varpi(0+) = 0$, such that*

$$F(y,r,\alpha(x-y),B) - F(x,r,\alpha(x-y),A) \leq \varpi\left(\alpha|x-y|^2 + |x-y|\right)$$
$$\textit{for all } x,y \in \mathcal{O},\ r \in \mathbb{R} \textit{ and } A,B \textit{ satisfying } (5.16).$$

**Remark 5.18.** Assumption 5.17 (ii) implies that the nonlinearity $F$ is elliptic. To see this, notice that for $A \leq B$, $\xi, \eta \in \mathbb{R}^d$, and $\varepsilon > 0$, we have

$$\begin{aligned} A\xi\cdot\xi - (B+\varepsilon I_d)\eta\cdot\eta &\leq B\xi\cdot\xi - (B+\varepsilon I_d)\eta\cdot\eta \\ &= 2\eta\cdot B(\xi-\eta) + B(\xi-\eta)\cdot(\xi-\eta) - \varepsilon|\eta|^2 \\ &\leq \varepsilon^{-1}|B(\xi-\eta)|^2 + B(\xi-\eta)\cdot(\xi-\eta) \\ &\leq |B|\left(1+\varepsilon^{-1}|B|\right)|\xi-\eta|^2. \end{aligned}$$

For $3\alpha \geq (1+\varepsilon^{-1}|B|)|B|$, the latter inequality implies the right-hand side of (5.16) holds true with $(A, B+\varepsilon I_d)$. For $\varepsilon$ sufficiently small, the left-hand side of (5.16) is also true with $(A, B+\varepsilon I_d)$ if in addition $\alpha > |A| \vee |B|$. Then

$$F(x-\alpha^{-1}p, r, p, B+\varepsilon I) - F(x,r,p,A) \quad \leq \quad \varpi\left(\alpha^{-1}(|p|^2+|p|)\right),$$

which provides the ellipticity of $F$ by sending $\alpha \to \infty$ and $\varepsilon \to 0$. ♢

**Theorem 5.19.** *Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^d$ and let $F$ be an elliptic operator satisfying Assumption 5.17. Let $u \in USC(\mathcal{O})$ and $v \in LSC(\mathcal{O})$ be viscosity subsolution and supersolution of the equation (E), respectively. Then*

$$u \leq v \textit{ on } \partial\mathcal{O} \quad \Longrightarrow \quad u \leq v \textit{ on } \bar{\mathcal{O}} := cl(\mathcal{O}).$$

*Proof.* As in the proof of Proposition 5.11, we assume to the contrary that

$$\delta := (u-v)(z) > 0 \quad \text{for some} \quad z \in \mathcal{O}. \tag{5.19}$$

*Step 1:* For every $\alpha > 0$, it follows from the upper-semicontinuity of the difference $(u-v)$ and the compactness of $\bar{\mathcal{O}}$ that

$$\begin{aligned} M_\alpha &:= \sup_{\mathcal{O}\times\mathcal{O}} \left\{u(x) - v(y) - \frac{\alpha}{2}|x-y|^2\right\} \\ &= u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \end{aligned} \tag{5.20}$$

for some $(x_\alpha, y_\alpha) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}}$. Since $\bar{\mathcal{O}}$ is compact, there is a subsequence $(x_n, y_n) := (x_{\alpha_n}, y_{\alpha_n})$, $n \geq 1$, which converges to some $(\hat{x}, \hat{y}) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}}$. We shall prove in Step 4 below that

$$\hat{x} = \hat{y}, \ \alpha_n |x_n - y_n|^2 \longrightarrow 0, \text{ and } M_{\alpha_n} \longrightarrow (u-v)(\hat{x}) = \sup_{\mathcal{O}} (u-v). \tag{5.21}$$

Then, since $u \leq v$ on $\partial\mathcal{O}$ and

$$\delta \leq M_{\alpha_n} = u(x_n) - v(y_n) - \frac{\alpha_n}{2}|x_n - y_n|^2 \tag{5.22}$$

by (5.19), it follows from the first claim in (5.21) that $(x_n, y_n) \in \mathcal{O} \times \mathcal{O}$.
*Step 2:* Since the maximizer $(x_n, y_n)$ of $M_{\alpha_n}$ defined in (5.20) is an interior point to $\mathcal{O} \times \mathcal{O}$, it follows from Lemma 5.14 that there exist two symmetric matrices $A_n, B_n \in \mathcal{S}_n$ satisfying (5.16) such that $(x_n, \alpha_n(x_n - y_n), A_n) \in \bar{J}^+_{\mathcal{O}} u(x_n)$ and $(y_n, \alpha_n(x_n - y_n), B_n) \in \bar{J}^-_{\mathcal{O}} v(y_n)$. Then, since $u$ and $v$ are viscosity subsolution and supersolution, respectively, it follows from the alternative definition of viscosity solutions in Proposition 5.13 that:

$$F\left(x_n, u(x_n), \alpha_n(x_n - y_n), A_n\right) \leq 0 \leq F\left(y_n, v(y_n), \alpha_n(x_n - y_n), B_n\right). \tag{5.23}$$

*Step 3:* We first use the strict monotonicity Assumption 5.17 (i) to obtain:

$$\begin{aligned} \gamma\delta \ \leq \ \gamma\big(u(x_n) - v(y_n)\big) \quad &\leq \quad F\left(x_n, u(x_n), \alpha_n(x_n - y_n), A_n\right) \\ &\quad\; -F\left(x_n, v(y_n), \alpha_n(x_n - y_n), A_n\right). \end{aligned}$$

By (5.23), this provides:

$$\gamma\delta \quad \leq \quad F\left(y_n, v(y_n), \alpha_n(x_n - y_n), B_n\right) - F\left(x_n, v(y_n), \alpha_n(x_n - y_n), A_n\right).$$

Finally, in view of Assumption 5.17 (ii) this implies that:

$$\gamma\delta \quad \leq \quad \varpi\left(\alpha_n |x_n - y_n|^2 + |x_n - y_n|\right).$$

Sending $n$ to infinity, this leads to the desired contradiction of (5.19) and (5.21).
*Step 4:* It remains to prove the claims (5.21). By the upper-semicontinuity of the difference $(u - v)$ and the compactness of $\bar{\mathcal{O}}$, there exists a maximizer $x^*$ of the difference $(u - v)$. Then

$$(u-v)(x^*) \leq M_{\alpha_n} = u(x_n) - v(y_n) - \frac{\alpha_n}{2}|x_n - y_n|^2.$$

Sending $n \to \infty$, this provides

$$\begin{aligned} \bar{\ell} := \frac{1}{2}\limsup_{n\to\infty} \alpha_n |x_n - y_n|^2 \quad &\leq \quad \limsup_{n\to\infty} u(x_{\alpha_n}) - v(y_{\alpha_n}) - (u-v)(x^*) \\ &\leq \quad u(\hat{x}) - v(\hat{y}) - (u-v)(x^*); \end{aligned}$$

in particular, $\bar{\ell} < \infty$ and $\hat{x} = \hat{y}$. Using the definition of $x^*$ as a maximizer of $(u - v)$, we see that:

$$0 \ \leq \ \bar{\ell} \quad \leq \quad (u-v)(\hat{x}) - (u-v)(x^*) \ \leq \ 0.$$

Then $\hat{x}$ is a maximizer of the difference $(u - v)$ and $M_{\alpha_n} \longrightarrow \sup_{\mathcal{O}}(u - v)$. $\diamondsuit$

We list below two interesting examples of operators $F$ which satisfy the conditions of the above theorem:

**Example 5.20.** *Assumption 5.17 is satisfied by the nonlinearity*

$$F(x,r,p,A) \quad = \quad \gamma r + H(p)$$

*for any continuous function $H : \mathbb{R}^d \longrightarrow \mathbb{R}$, and $\gamma > 0$.*

*In this example, the condition $\gamma > 0$ is not needed when $H$ is a convex and $H(D\varphi(x)) \le \alpha < 0$ for some $\varphi \in C^1(\mathcal{O})$. This result can be found in [1].*

**Example 5.21.** *Assumption 5.17 is satisfied by*

$$F(x,r,p,A) \quad = \quad -Tr(\sigma\sigma'(x)A) + \gamma r,$$

*where $\sigma : \mathbb{R}^d \longrightarrow \mathcal{S}_d$ is a Lipschitz function, and $\gamma > 0$. Condition (i) of Assumption 5.17 is obvious. To see that Condition (ii) is satisfied, we consider $(A,B,\alpha) \in \mathcal{S}_d \times \mathcal{S}_d \times \mathbb{R}_+^*$ satisfying* (5.16). *We claim that*

$$\mathrm{Tr}[MM^TA - NN^TB] \quad \le \quad 3\alpha|M-N|^2 \;=\; 3\alpha\sum_{i,j=1}^{d}(M-N)_{ij}^2.$$

*To see this, observe that the matrix*

$$C \quad := \quad \begin{pmatrix} NN^T & NM^T \\ MN^T & MM^T \end{pmatrix}$$

*is a non-negative matrix in $\mathcal{S}_d$. From the right hand-side inequality of* (5.16), *this implies that*

$$\begin{aligned}
\mathrm{Tr}[MM^TA - NN^TB] \quad &= \quad \mathrm{Tr}\left[C\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}\right] \\
&\le \quad 3\alpha\mathrm{Tr}\left[C\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}\right] \\
&= \quad 3\alpha\mathrm{Tr}\left[(M-N)(M-N)^T\right] \;=\; 3\alpha|M-N|^2.
\end{aligned}$$

## 5.5 Comparison in unbounded domains

When the domain $\mathcal{O}$ is unbounded, a growth condition on the functions $u$ and $v$ is needed. Then, by using the growth at infinity, we can build on the proof of Theorem 5.19 to obtain a comparison principle. The following result shows how to handle this question in the case of a sub-quadratic growth. We emphasize that the present argument can be adapted to alternative growth conditions.

The following condition differs from Assumption 5.17 only in its part (ii) where the constant 3 in (5.16) is replaced by 4 in (5.24). Thus the following Assumption 5.22 (ii) is slightly stronger than Assumption 5.17 (ii).

**Assumption 5.22.** (i) *There exists* $\gamma > 0$ *such that*

$$F(x,r,p,A) - F(x,r',p,A) \geq \gamma(r-r') \text{ for all } r \geq r',\ (x,p,A) \in \mathcal{O}\times\mathbb{R}^d\times\mathcal{S}_d.$$

(ii) *There is a function* $\varpi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *with* $\varpi(0+) = 0$, *such that*

$$F(y,r,\alpha(x-y),B) - F(x,r,\alpha(x-y),A) \leq \varpi\left(\alpha|x-y|^2 + |x-y|\right)$$
$$\text{for all } x,y \in \mathcal{O},\ r \in \mathbb{R} \text{ and } A,B \text{ satisfying}$$
$$-4\alpha\begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} \leq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq 4\alpha\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}. \tag{5.24}$$

**Theorem 5.23.** *Let* $F$ *be an elliptic operator satisfying Assumption 5.22 with* $F$ *uniformly continuous in the pair* $(p,A)$. *Let* $u \in USC(\mathcal{O})$ *and* $v \in LSC(\mathcal{O})$ *be viscosity subsolution and supersolution of the equation (E), respectively, with* $|u(x)| + |v(x)| = \circ(|x|)$ *as* $|x| \to \infty$. *Then*

$$u \leq v \text{ on } \partial\mathcal{O} \implies u \leq v \text{ on } cl(\mathcal{O}).$$

*Proof.* We assume to the contrary that

$$\delta := (u-v)(z) > 0 \quad \text{for some} \quad z \in \mathbb{R}^d, \tag{5.25}$$

and we work towards a contradiction. Let

$$M_\alpha := \sup_{x,y\in\mathbb{R}^d} u(x) - v(y) - \phi(x,y),$$

where

$$\phi(x,y) := \frac{1}{2}\left(\alpha|x-y|^2 + \varepsilon|x|^2 + \varepsilon|y|^2\right).$$

**1.** Since $u(x) = \circ(|x|^2)$ and $v(y) = \circ(|y|^2)$ at infinity, there is a maximizer $(x_\alpha, y_\alpha)$ for the previous problem:

$$M_\alpha = u(x_\alpha) - v(y_\alpha) - \phi(x_\alpha, y_\alpha).$$

Moreover, there is a sequence $\alpha_n \to \infty$ such that

$$(x_n, y_n) := (x_{\alpha_n}, y_{\alpha_n}) \longrightarrow (\hat{x}_\varepsilon, \hat{y}_\varepsilon),$$

and, similar to Step 4 of the proof of Theorem 5.19, we can prove that $\hat{x}_\varepsilon = \hat{y}_\varepsilon$,

$$\alpha_n|x_n - y_n|^2 \longrightarrow 0, \text{ and } M_{\alpha_n} \longrightarrow M_\infty := \sup_{x\in\mathbb{R}^d}(u-v)(x) - \varepsilon|x|^2. \tag{5.26}$$

Notice that $M_{\alpha_n} \geq (u-v)(z) - \phi(z,z) \geq \delta - \varepsilon|z|^2 > 0$, by (5.25). then, for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} 0 < \delta - \varepsilon|z|^2 \leq \limsup_{n\to\infty} M_{\alpha_n} &= \limsup_{n\to\infty}\{u(x_n) - v(y_n) - \phi(x_n,y_n)\} \\ &\leq \limsup_{n\to\infty}\{u(x_n) - v(y_n))\} \\ &\leq \limsup_{n\to\infty} u(x_n) - \liminf_{n\to\infty} v(y_n) \\ &\leq (u-v)(\hat{x}_\varepsilon). \end{aligned}$$

Since $u \le v$ on $\partial\mathcal{O}$, we deduce that $\hat{x} \notin \partial\mathcal{O}$ and therefore $(x_n, y_n)$ is a local maximizer of $u - v - \phi$.

**2.** By the Crandall-Ishii Lemma 5.14, there exist $A_n, B_n \in \mathcal{S}_n$, such that

$$\begin{array}{rcl} (D_x\phi(x_n,y_n), A_n) & \in & \bar{\mathcal{J}}^{2,+}_{\mathcal{O}} u(t_n,x_n), \\ (-D_y\phi(x_n,y_n), B_n) & \in & \bar{\mathcal{J}}^{2,-}_{\mathcal{O}} v(t_n,y_n), \end{array} \tag{5.27}$$

and

$$-(\alpha_n + |D^2\phi(x_0,y_0)|)I_{2d} \le \begin{pmatrix} A_n & 0 \\ 0 & -B_n \end{pmatrix} \le D^2\phi(x_n,y_n) + \frac{1}{\alpha_n} D^2\phi(x_n,y_n)^2. \tag{5.28}$$

In the present situation, we immediately calculate that

$$D_x\phi(x_n,y_n) = \alpha_n(x_n - y_n) + \varepsilon x_n, \ \ - D_y\phi(x_n,y_n) = \alpha_n(x_n - y_n) - \varepsilon y_n$$

and

$$D^2\phi(x_n,y_n) \ \ = \ \ \alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + \varepsilon\, I_{2d},$$

which reduces the right hand-side of (5.28) to

$$\begin{pmatrix} A_n & 0 \\ 0 & -B_n \end{pmatrix} \ \le \ (3\alpha_n + 2\varepsilon) \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + \left(\varepsilon + \frac{\varepsilon^2}{\alpha_n}\right) I_{2d}, \tag{5.29}$$

while the left land-side of (5.28) implies:

$$-3\alpha_n I_{2d} \ \le \ \begin{pmatrix} A_n & 0 \\ 0 & -B_n \end{pmatrix} \tag{5.30}$$

**3.** By (5.27) and the viscosity properties of $u$ and $v$, we have

$$\begin{array}{rcl} F(x_n, u(x_n), \alpha_n(x_n - y_n) + \varepsilon x_n, A_n) & \le & 0, \\ F(y_n, v(y_n), \alpha_n(x_n - y_n) - \varepsilon y_n, B_n) & \ge & 0. \end{array}$$

Using Assumption 5.22 (i) together with the uniform continuity of $F$ in $(p, A)$, this implies that:

$$\begin{array}{rcl} \gamma\big(u(x_n) - v(y_n)\big) & \le & F\big(y_n, u(x_n), \alpha_n(x_n - y_n) - \varepsilon y_n, B_n\big) \\ & & -F\big(y_n, v(y_n), \alpha_n(x_n - y_n) - \varepsilon y_n, B_n\big) \\ & \le & F\big(y_n, u(x_n), \alpha_n(x_n - y_n) - \varepsilon y_n, B_n\big) \\ & & -F\big(x_n, u(x_n), \alpha_n(x_n - y_n) + \varepsilon x_n, A_n\big) \\ & \le & F\big(y_n, u(x_n), \alpha_n(x_n - y_n), \tilde{B}_n\big) \\ & & -F\big(x_n, u(x_n), \alpha_n(x_n - y_n), \tilde{A}_n\big) + c(\varepsilon(1 + |x_n| + |y_n|)), \end{array}$$

where $c(.)$ is a modulus of continuity of $F$ in $(p, A)$, and $\tilde{A}_n := A_n - 2\varepsilon I_d$, $\tilde{B}_n := B_n + 2\varepsilon I_d$. By (5.29) and (5.30), we have

$$-4\alpha I_{2d} \ \le \ \begin{pmatrix} \tilde{A}_n & 0 \\ 0 & -\tilde{B}_n \end{pmatrix} \ \le \ 4\alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix},$$

for small $\varepsilon$. Then, it follows from Assumption 5.22 (ii) that

$$\gamma\big(u(x_n) - v(x_n)\big) \quad \leq \quad \varpi\big(\alpha_n|x_n - y_n|^2 + |x_n - y_n|\big) + c(\varepsilon(1 + |x_n| + |y_n|)).$$

By sending $n$ to infinity, it follows from (5.26) that:

$$c(\varepsilon(1 + 2|\hat{x}_\varepsilon|)) \geq \gamma\big(M_\infty + \varepsilon|\hat{x}_\varepsilon|^2\big) \geq \gamma M_\infty \geq \gamma\big(\delta - \varepsilon|z|^2\big),$$

and we get a contradiction of (5.25) by sending $\varepsilon$ to zero as long as

$$\varepsilon|\hat{x}_\varepsilon| \longrightarrow 0 \quad \text{as} \quad \varepsilon \searrow 0,$$

which we now prove. Indeed, we have that $(u-v)(\hat{x}_\varepsilon) - \varepsilon|\hat{x}_\varepsilon|^2 \geq (u-v)z - \varepsilon|z|^2$, and therefore

$$\varepsilon|\hat{x}_\varepsilon| \leq \frac{(u-v)(\hat{x}_\varepsilon)}{|\hat{x}_\varepsilon|} - \frac{\delta}{2|\hat{x}_\varepsilon|} \quad \text{for small} \quad \varepsilon > 0.$$

Using our condition $|u(x)| = \circ(|x|)$ and $|v(x)| = \circ(|x|)$ at infinity, we see that whenever $|\hat{x}_\varepsilon|$ is unbounded in $\varepsilon$, the last inequality implies that $\varepsilon|\hat{x}_\varepsilon| \to 0$. $\diamondsuit$

## 5.6 Useful applications

We conclude this section by two consequences of the above comparison results, which are trivial properties in the context of classical solutions.

**Lemma 5.24.** *Let $\mathcal{O}$ be an open interval of $\mathbb{R}$, and $U : \mathcal{O} \longrightarrow \mathbb{R}$ be a lower semicontinuous viscosity supersolution of the equation $DU \geq 0$ on $\mathcal{O}$. Then $U$ is nondecreasing on $\mathcal{O}$.*

*Proof.* For each $\varepsilon > 0$, define $W(x) := U(x) + \varepsilon x$, $x \in \mathcal{O}$. Then $W$ satisfies in the viscosity sense $DW \geq \varepsilon$ in $\mathcal{O}$, i.e. for all $(x_0, \varphi) \in \mathcal{O} \times C^1(\mathcal{O})$ such that

$$(W - \varphi)(x_0) \quad = \quad \min_{x \in \mathcal{O}} (W - \varphi)(x), \tag{5.31}$$

we have $D\varphi(x_0) \geq \varepsilon$. This proves that $\varphi$ is strictly increasing in a neighborhood $\mathcal{V}$ of $x_0$. Let $(x_1, x_2) \subset \mathcal{V}$ be an open interval containing $x_0$. We intend to prove that

$$W(x_1) \quad < \quad W(x_2), \tag{5.32}$$

which provides the required result from the arbitrariness of $x_0 \in \mathcal{O}$.

To prove (5.32), suppose to the contrary that $W(x_1) \geq W(x_2)$, and the consider the function $v(x) = W(x_2)$ which solves the equation

$$Dv \;=\; 0 \quad \text{on the open interval} \quad (x_1, x_2).$$

together with the boundary conditions $v(x_1) = v(x_2) = W(x_2)$. Observe that $W$ is a lower semicontinuous viscosity supersolution of the above equation. From the comparison result of Example 5.20, this implies that

$$\sup_{[x_1,x_2]} (v - W) \quad = \quad \max\{(v-W)(x_1), (v-W)(x_2)\} \;\leq\; 0.$$

Hence $W(x) \geq v(x) = W(x_2)$ for all $x \in [x_1, x_2]$. Applying this inequality at $x_0 \in (x_1, x_2)$, and recalling that the test function $\varphi$ is strictly increasing on $[x_1, x_2]$, we get :

$$(W - \varphi)(x_0) \quad > \quad (W - \varphi)(x_2),$$

contradicting (5.31). ♢

**Lemma 5.25.** *Let $\mathcal{O}$ be an open interval of $\mathbb{R}$, and $U : \mathcal{O} \longrightarrow \mathbb{R}$ be a lower semicontinuous viscosity supersolution of the equation $-D^2U \geq 0$ on $\mathcal{O}$. Then $U$ is concave on $\mathcal{O}$.*

*Proof.* Let $a < b$ be two arbitrary elements in $\mathcal{O}$, and consider some $\varepsilon > 0$ together with the function

$$v(s) \;:=\; \frac{U(a)\left(e^{\sqrt{\varepsilon}(b-s)} - e^{-\sqrt{\varepsilon}(b-s)}\right) + U(b)\left(e^{\sqrt{\varepsilon}(s-a)} - e^{-\sqrt{\varepsilon}(s-a)}\right)}{e^{\sqrt{\varepsilon}(b-a)} - e^{-\sqrt{\varepsilon}(b-a)}} \quad \text{for} \quad a \leq s \leq b.$$

Clearly, $v$ solves the equation

$$\varepsilon v - D^2 v = 0 \;\text{ on }\; (a,b), \quad v = U \quad \text{on} \;\; \{a,b\}.$$

Since $U$ is lower semicontinuous it is bounded from below on the interval $[a, b]$. Therefore, by possibly adding a constant to $U$, we can assume that $U \geq 0$, so that $U$ is a lower semicontinuous viscosity supersolution of the above equation. It then follows from the comparison theorem 6.6 that :

$$\sup_{[a,b]} (v - U) \quad = \quad \max\{(v-U)(a), (v-U)(b)\} \;=\; 0.$$

Hence,

$$U(s) \quad \geq \quad v(s) \;=\; \frac{U(a)\left(e^{\sqrt{\varepsilon}(b-s)} - e^{-\sqrt{\varepsilon}(b-s)}\right) + U(b)\left(e^{\sqrt{\varepsilon}(s-a)} - e^{-\sqrt{\varepsilon}(s-a)}\right)}{e^{\sqrt{\varepsilon}(b-a)} - e^{-\sqrt{\varepsilon}(b-a)}}$$

and by sending $\varepsilon$ to zero, we see that

$$U(s) \quad \geq \quad (U(b) - U(a))\,\frac{s-a}{b-a} + U(a)$$

for all $s \in [a, b]$. Let $\lambda$ be an arbitrary element of the interval [0,1], and set $s := \lambda a + (1-\lambda)b$. The last inequality takes the form :

$$U\big(\lambda a + (1-\lambda)b\big) \quad \geq \quad \lambda U(a) + (1-\lambda)U(b)\;,$$

proving the concavity of $U$. ♢

## 5.7 Proof of the Crandall-Ishii's lemma

We start with two Lemmas. We say that a function $f$ is $\lambda-$semiconvex if $x \longmapsto f(x) + (\lambda/2)|x|^2$ is convex.

**Lemma 5.26.** *Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a $\lambda-$semiconvex function, for some $\lambda \in \mathbb{R}$, and assume that $f(x) - \frac{1}{2}Bx \cdot x \leq f(0)$ for all $x \in \mathbb{R}^N$. Then there exists $X \in \mathcal{S}_N$ such that*

$$(0, X) \in \overline{J}^{2,+} f(0) \cap \overline{J}^{2,-} f(0) \quad \textit{and} \quad -\lambda I_N \leq X \leq B.$$

Our second lemma requires to introduce the following notion. For a function $v : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $\lambda > 0$, the corresponding $\lambda-$*sup-convolution* is defined by:

$$\hat{v}^\lambda(x) \quad := \quad \sup_{y \in \mathbb{R}^N} \left\{ v(y) - \frac{\lambda}{2}|x - y|^2 \right\}.$$

Observe that

$$\hat{v}^\lambda(x) + \frac{\lambda}{2}|x|^2 \quad = \quad \sup_{y \in \mathbb{R}^N} \left\{ v(y) - \frac{\lambda}{2}|y|^2 + \lambda x \cdot y \right\}$$

is convex, as the supremum of linear functions. Then

$$\hat{v}^\lambda \text{ is } \lambda - \text{semiconvex.} \tag{5.33}$$

In [3], the following property is refered to as the *magical property of the sup-convolution*.

**Lemma 5.27.** *Let $\lambda > 0$, $v$ be a bounded lower-semicontinuous function, $\hat{v}^\lambda$ the corresponding $\lambda-$sup-convolution.*
(i) *If $(p, X) \in J^{2,+}\hat{v}(x)$ for some $x \in \mathbb{R}^N$, then*

$$(p, X) \in J^{2,+} v\Big(x + \frac{p}{\lambda}\Big) \quad \textit{and} \quad \hat{v}^\lambda(x) = v(x + p/\lambda) - \frac{1}{2\lambda}|p|^2.$$

(ii) *For all $x \in \mathbb{R}^N$, we have $(0, X) \in \bar{J}^{2,+}\hat{v}(x)$implies that $(0, X) \in \bar{J}^{2,+}v(x)$.*

Before proving the above lemmas, we show how they imply the Crandall-Ishii's lemma that we reformulate in a more symmetric way.

**Lemma 5.14** *Let $\mathcal{O}$ be an open locally compact subset of $\mathbb{R}^d$ and $u_1, u_2 \in USC(\mathcal{O})$. We denote $w(x_1, x_2) := u_1(x_1) + u_2(x_2)$ and we assume for some $\varphi \in C^2\left(cl(\mathcal{O})^2\right)$ and $x^0 = (x_1^0, x_2^0) \in \mathcal{O} \times \mathcal{O}$ that:*

$$(w - \varphi)(x^0) \quad = \quad \max_{\mathcal{O}^2}(w - \varphi).$$

*Then, for each $\varepsilon > 0$, there exist $X_1, X_2 \in \mathcal{S}_d$ such that*

$$(D_{x_i}\varphi(x^0), X_i) \in \bar{J}^{2,+}_{\mathcal{O}} u_i(x_i^0), \quad i = 1, 2,$$

$$\textit{and} \quad -\left(\varepsilon^{-1} + \left|D^2\varphi(x^0)\right|\right) I_{2d} \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq D^2\varphi(x^0) + \varepsilon D^2\varphi(x^0)^2.$$

*Proof.* <u>Step 1</u>: We first observe that we may reduce the problem to the case

$$\mathcal{O} = \mathbb{R}^d,\ x^0 = 0,\ u_1(0) = u_2(0) = 0, \text{ and } \varphi(x) = \frac{1}{2}Ax \cdot x \text{ for some } A \in \mathcal{S}_d.$$

The reduction to $x^0 = 0$ follows from an immediate change of coordinates. Choose any compact subset of $K \subset \mathcal{O}$ containing the origin and set $\bar{u}_i = u_i$ on $K$ and $-\infty$ otherwise, $i = 1, 2$. Then, the problem can be stated equivalently in terms of the functions $\bar{u}_i$ which are now defined on $\mathbb{R}^d$ and take values on the extended real line. Also by defining

$$\bar{\bar{u}}_i(x_i) := \bar{u}_i(x_i) - u_i(0) - D_{x_i}\varphi(0) \quad \text{and} \quad \bar{\varphi}(x) := \varphi(x) - \varphi(0) - D\varphi(0) \cdot x$$

we may reformulate the problem equivalently with $\bar{\bar{u}}_i(x_i) = 0$ and $\bar{\varphi}(x) = \frac{1}{2}D^2\varphi(0)x \cdot x + \circ(|x|^2)$. Finally, defining $\bar{\bar{\varphi}}(x) := Ax \cdot x$ with $A := D^2\varphi(0) + \eta I_{2d}$ for some $\eta > 0$, it follows that

$$\bar{\bar{u}}_1(x_1) + \bar{\bar{u}}_2(x_2) - \bar{\bar{\varphi}}(x_1, x_2) < \bar{\bar{u}}_1(x_1) + \bar{\bar{u}}_2(x_2) - \bar{\varphi}(x_1, x_2) \leq \bar{\bar{u}}_1(0) + \bar{\bar{u}}_2(0) - \bar{\varphi}(0) = 0.$$

<u>Step 2</u>: From the reduction of the previous step, we have

$$\begin{aligned}
2w(x) &\leq Ax \cdot x \\
&= A(x-y) \cdot (x-y)Ay \cdot y - 2Ay \cdot (y-x) \\
&\leq A(x-y) \cdot (x-y)Ay \cdot y + \varepsilon A^2 y \cdot y + \frac{1}{\varepsilon}|x-y|^2 \\
&= A(x-y) \cdot (x-y) + \frac{1}{\varepsilon}|x-y|^2 + (A + \varepsilon A^2)y \cdot y \\
&\leq (\varepsilon^{-1} + |A|)|x-y|^2 + (A + \varepsilon A^2)y \cdot y.
\end{aligned}$$

Set $\lambda := \varepsilon^{-1} + |A|$ and $B := A + \varepsilon A^2$. The latter inequality implies the following property of the sup-convolution:

$$\hat{w}^\lambda(y) - \frac{1}{2}By \cdot y \quad \leq \quad \hat{w}(0) \ = \ 0.$$

<u>Step 3</u>: Recall from (5.33) that $\hat{w}^\lambda$ is $\lambda-$semiconvex. Then, it follows from Lemma 5.26 that there exist $X \in \mathcal{S}_{2d}$ such that $(0, X) \in \overline{J}^{2,+}\hat{w}^\lambda(0) \cap \overline{J}^{2,-}\hat{w}^\lambda(0)$ and $-\lambda I_{2d} \leq X \leq B$. Moreover, it is immediately checked that $\hat{w}^\lambda(x_1, x_2) = \hat{u}_1^\lambda(x_1) + \hat{u}_2^\lambda(x_2)$, implying that $X$ is bloc-diagonal with blocs $X_1, X_2 \in \mathcal{S}_d$. Hence:

$$-(\varepsilon^{-1} + |A|)I_{2d} \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq A + \varepsilon A^2$$

and $(0, X_i) \in \overline{J}^{2,+}\hat{u}_i^\lambda(0)$ for $i = 1, 2$ which, by Lemma 5.27 implies that $(0, X_i) \in \overline{J}^{2,+}u_i^\lambda(0)$. ♢

We coninue by turning to the proofs of Lemmas 5.26 and 5.27. The main tools which will be used are the following properties of any semiconvex function $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ whose proofs are reported in [3]:

- *Aleksandrov lemma*: $\varphi$ is twice differentiable a.e.
- *Jensen's lemma*: if $x_0$ is a strict maximizer of $\varphi$, then for every $r, \delta > 0$, the set

  $$\left\{\bar{x} \in B(x_0, r) : x \longmapsto \varphi(x) + p \cdot x \text{ has a local maximum at } \bar{x} \text{ for some } p \in B_\delta\right\}$$

  has positive measure in $\mathbb{R}^N$.

**Proof of Lemma 5.26** Notice that $\varphi(x) := f(x) - \frac{1}{2}Bx \cdot x - |x|^4$ has a strict maximum at $x = 0$. Localizing around the origin, we see that $\varphi$ is a semiconvex function. Then, for every $\delta > 0$, by the above Aleksandrov and Jensen lemmas, there exists $q_\delta$ and $x_\delta$ such that

$$q_\delta, x_\delta \in B_\delta, \; D^2\varphi(x_\delta) \text{ exists, and } \varphi(x_\delta) + q_\delta \cdot x_\delta = \text{loc-max}\{\varphi(x) + q_\delta \cdot x\}.$$

We may then write the first and second order optimality conditions to see that:

$$Df(x_\delta) = -q_\delta + Bx_\delta + 4|x_\delta|^3 \quad \text{and} \quad D^2 f(x_\delta) \le B + 12|x_\delta|^2.$$

Together with the $\lambda-$semiconvexity of $f$, this provides:

$$Df(x_\delta) = O(\delta) \quad \text{and} \quad -\lambda I \le D^2 f(x_\delta) \le B + O(\delta^2). \tag{5.34}$$

Clearly $f$ inherits the twice differentiability of $\varphi$ at $x_\delta$. Then

$$\left(Df(x_\delta), D^2 f(x_\delta)\right) \in J^{2,+} f(x_\delta) \cap J^{2,-} f(x_\delta)$$

and, in view of (5.34), we may send $\delta$ to zero along some subsequence and obtain a limit point $(0, X) \in \bar{J}^{2,+} f(0) \cap \bar{J}^{2,-} f(0)$. ♢

**Proof of Lemma 5.27** (i) Since $v$ is bounded, there is a maximizer:

$$\hat{v}^\lambda(x) \quad = \quad v(y) - \frac{\lambda}{2}|x - y|^2. \tag{5.35}$$

By the definition of $\hat{v}^\lambda$ and the fact that $(p, A) \in J^{2,+}\hat{v}(x)$, we have for every $x', y' \in \mathbb{R}^N$:

$$\begin{aligned} v(y') - \frac{\lambda}{2}|x' - y'|^2 &\le \hat{v}(x') \\ &\le \hat{v}(x) + p \cdot (x' - x) + \frac{1}{2}A(x' - x) \cdot (x' - x) + \circ(x' - x) \\ &= v(y) - \frac{\lambda}{2}|x - y|^2 + p \cdot (x' - x) + \frac{1}{2}A(x' - x) \cdot (x' - x) + \circ(x' - x), \end{aligned} \tag{5.36}$$

where we used (5.35) in the last equality.

By first setting $x' = y' + y - x$ in (5.36), we see that:

$$v(y') \leq v(y) + p \cdot (y' - y) + \frac{1}{2}A(y' - y) \cdot (y' - y) + \circ(y' - y) \quad \text{for all} \quad y' \in \mathbb{R}^N,$$

which means that $(p, A) \in J^{2,+}v(y)$.

On the other hand, setting $y' = y$ in (5.36), we deduce that:

$$\lambda(x' - x) \cdot \Big(\frac{x + x'}{2} + \frac{p}{\lambda} - y\Big) \quad \geq \quad O\big(|x - x'|^2\big),$$

which implies that $y = x + \frac{p}{\lambda}$.

(ii) Consider a sequence $(x_n, p_n, A_n)$ with $(x_n, \hat{v}^\lambda(x_n), p_n, A_n) \longrightarrow (x, \hat{v}^\lambda(x), 0, A)$ and $(p_n, A_n) \in J^{2,+}\hat{v}^\lambda(x_n)$. In view of (i) and the definition of $\bar{J}^{2,+}v(x)$, it only remains to prove that

$$v\Big(x_n + \frac{p_n}{\lambda}\Big) \quad \longrightarrow \quad v(x). \tag{5.37}$$

To see this, we use the upper semicontinuity of $v$ together with (i) and the definition of $\hat{v}^\lambda$:

$$\begin{aligned} v(x) \quad &\geq \quad \limsup_n v\Big(x_n + \frac{p_n}{\lambda}\Big) \\ &\geq \quad \liminf_n v\Big(x_n + \frac{p_n}{\lambda}\Big) \\ &= \quad \lim_n \hat{v}^\lambda(x_n) + \frac{1}{2\lambda}|p_n|^2 \;=\; \hat{v}^\lambda(x) \;\geq\; v(x). \end{aligned}$$

$\diamondsuit$

# Chapter 6

# Dynamic Programming Equation in Viscosity Sense

## 6.1 DPE for stochastic control problems

We now turn to the stochastic control problem introduced in Section 2.1. The chief goal of this section is to use the notion of viscosity solutions in order to relax the smoothness condition on the value function $V$ in the statement of Propositions 2.4 and 2.5. Notice that the following proofs are obtained by slight modification of the corresponding proofs in the smooth case.

**Remark 6.1.** Recall that the general theory of viscosity applies for nonlinear partial differential equations on an open domain $\mathcal{O}$. This indeed ensures that the optimizer in the definition of viscosity solutions is an interior point. In the setting of control problems with finite horizon, the time variable moves forward so that the left boundary of the time interval is not relevant. We shall then write the DPE on the domain $\mathbf{S} = [0, T) \times \mathbb{R}^d$. Although this is not an open domain, the general theory of viscosity solutions is still valid.

We first recall the setting of Section 2.1. We shall concentrate on the finite horizon case $T < \infty$, while keeping in mind that the infinite horizon problems are handled by exactly the same arguments. The only reason why we exclude $T = \infty$ is because we do not want to be diverted by issues related to the definition of the set of admissible controls.

Given a subset $U$ of $\mathbb{R}^k$, we denote by $\mathcal{U}$ the set of all progressively measurable processes $\nu = \{\nu_t,\, t < T\}$ valued in $U$ and by $\mathcal{U}_0 := \mathcal{U} \cap \mathbb{H}^2$. The elements of $\mathcal{U}_0$ are called admissible control processes.

The controlled state dynamics is defined by means of the functions

$$b \;:\; (t, x, u) \in \mathbf{S} \times U \quad\longrightarrow\quad b(t, x, u) \in \mathbb{R}^n$$

and

$$\sigma \ : \ (t,x,u) \in \mathbf{S} \times U \quad \longrightarrow \quad \sigma(t,x,u) \in \mathcal{M}_{\mathbb{R}}(n,d)$$

which are assumed to be continuous and to satisfy the conditions

$$|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \quad \leq \quad K\, |x-y|, \tag{6.1}$$

$$|b(t,x,u)| + |\sigma(t,x,u)| \quad \leq \quad K\, (1+|x|+|u|). \tag{6.2}$$

for some constant $K$ independent of $(t,x,y,u)$. For each admissible control process $\nu \in \mathcal{U}_0$, the controlled stochastic differential equation :

$$dX_t \quad = \quad b(t,X_t,\nu_t)dt + \sigma(t,X_t,\nu_t)dW_t \tag{6.3}$$

has a unique solution $X$, for all given initial data $\xi \in \mathbb{L}^2(\mathcal{F}_0,\mathbb{P})$ with

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |X_s^\nu|^2\right] \ < \ C(1+\mathbb{E}[|\xi|^2])e^{Ct} \quad \text{for all} \quad t \in [0,T] \tag{6.4}$$

for some constant $C$. Finally, the gain functional is defined via the functions:

$$f,k \ : \ [0,T) \times \mathbb{R}^d \times U \ \longrightarrow \ \mathbb{R} \quad \text{and} \quad g \ : \ \mathbb{R}^d \ \longrightarrow \ \mathbb{R}$$

which are assumed to be continuous, $\|k^-\|_\infty < \infty$, and:

$$|f(t,x,u)| + |g(x)| \quad \leq \quad K(1+|u|+|x|^2),$$

for some constant $K$ independent of $(t,x,u)$. The cost function $J$ on $[0,T] \times \mathbb{R}^d \times \mathcal{U}$ is:

$$J(t,x,\nu) := \mathbb{E}\left[\int_t^T \beta^\nu(t,s) f(s,X_s^{t,x,\nu},\nu_s)ds + \beta^\nu(t,T)g(X_T^{t,x,\nu})\right], \tag{6.5}$$

when this expression is meaningful, where

$$\beta^\nu(t,s) \quad := \quad \exp\Big(-\int_t^s k(r,X_r^{t,x,\nu},\nu_r)dr\Big),$$

and $\{X_s^{t,x,\nu}, s \geq t\}$ is the solution of (6.3) with control process $\nu$ and initial condition $X_t^{t,x,\nu} = x$. The stochastic control problem is defined by the value function:

$$V(t,x) \ := \ \sup_{\nu\in\mathcal{U}_0} J(t,x,\nu) \quad \text{for} \quad (t,x) \in \mathbf{S}. \tag{6.6}$$

We recall the expression of the Hamiltonian:

$$H(.,r,p,A) := \sup_{u\in U}\left(f(.,u) - k(.,u)r + b(.,u)\cdot p + \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^{\mathrm{T}}(.,u)A\big]\right), \tag{6.7}$$

and the second order operator associated to $X$ and $\beta$:

$$\mathcal{L}^u v \quad := \quad -k(.,u)v + b(.,u)\cdot Dv + \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^{\mathrm{T}}(.,u)D^2 v\big], \tag{6.8}$$

which appears naturally in the following Itô's formula valid for any smooth test function $v$:

$$d\beta^\nu(0,t)v(t,X_t^\nu) = \beta^\nu(0,t)\Big((\partial_t + \mathcal{L}^{\nu_t})v(t,X_t^\nu)dt + Dv(t,X_t^\nu)\cdot\sigma(t,X_t^\nu,\nu_t)dW_t\Big).$$

**Proposition 6.2.** *Assume that $V$ is locally bounded on $[0,T)\times\mathbb{R}^d$. Then, the value function $V$ is a viscosity supersolution of the equation*

$$-\partial_t V(t,x) - H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \quad \geq \quad 0 \tag{6.9}$$

*on $[0,T)\times\mathbb{R}^d$.*

*Proof.* Let $(t,x)\in\mathbf{S}$ and $\varphi\in C^2(\mathbf{S})$ be such that

$$0 \;=\; (V_* - \varphi)(t,x) \quad = \quad \min_{\mathbf{S}}\,(V_* - \varphi). \tag{6.10}$$

Let $(t_n,x_n)_n$ be a sequence in $\mathbf{S}$ such that

$$(t_n,x_n) \;\longrightarrow\; (t,x) \quad \text{and} \quad V(t_n,x_n) \;\longrightarrow\; V_*(t,x).$$

Since $\varphi$ is smooth, notice that

$$\eta_n \;:=\; V(t_n,x_n) - \varphi(t_n,x_n) \quad\longrightarrow\quad 0.$$

Next, let $u\in U$ be fixed, and consider the constant control process $\nu = u$. We shall denote by $X^n := X^{t_n,x_n,u}$ the associated state process with initial data $X^n_{t_n} = x_n$. Finally, for all $n>0$, we define the stopping time :

$$\theta_n \quad := \quad \inf\left\{s > t_n \;:\; (s-t_n, X^n_s - x_n) \notin [0,h_n)\times\alpha B\right\},$$

where $\alpha>0$ is some given constant, $B$ denotes the unit ball of $\mathbb{R}^n$, and

$$h_n \quad := \quad \sqrt{|\eta_n|}\mathbf{1}_{\{\eta_n\neq 0\}} + n^{-1}\mathbf{1}_{\{\eta_n=0\}}.$$

Notice that $\theta_n \longrightarrow t$ as $n\longrightarrow\infty$.
**1.** From the first inequality in the dynamic programming principle of Theorem 2.3, it follows that:

$$0 \quad\leq\quad \mathbb{E}\left[V(t_n,x_n) - \beta(t_n,\theta_n)V_*(\theta_n,X^n_{\theta_n}) - \int_{t_n}^{\theta_n}\beta(t_n,r)f(r,X^n_r,\nu_r)dr\right].$$

Now, in contrast with the proof of Proposition 2.4, the value function is not known to be smooth, and therefore we can not apply Itô's formula to $V$. The main trick of this proof is to use the inequality $V_* \geq \varphi$ on $\mathbf{S}$, implied by (6.10), so that we can apply Itô's formula to the smooth test function $\varphi$:

$$\begin{aligned}
0 &\leq \eta_n + \mathbb{E}\left[\varphi(t_n, x_n) - \beta(t_n, \theta_n)\varphi(\theta_n, X^n_{\theta_n}) - \int_{t_n}^{\theta_n} \beta(t_n, r) f(r, X^n_r, \nu_r) dr\right] \\
&= \eta_n - \mathbb{E}\left[\int_{t_n}^{\theta_n} \beta(t_n, r)(\partial_t \varphi + \mathcal{L}^{\cdot}\varphi - f)(r, X^n_r, u) dr\right] \\
&\quad - \mathbb{E}\left[\int_{t_n}^{\theta_n} \beta(t_n, r) D\varphi(r, X^n_r)\sigma(r, X^n_r, u) dW_r\right],
\end{aligned}$$

where $\partial_t \varphi$ denotes the partial derivative with respect to $t$.

**2.** We now continue exactly along the lines of the proof of Proposition 2.5. Observe that $\beta(t_n, r) D\varphi(r, X^n_r)\sigma(r, X^n_r, u)$ is bounded on the stochastic interval $[t_n, \theta_n]$. Therefore, the second expectation on the right hand-side of the last inequality vanishes, and :

$$\frac{\eta_n}{h_n} - \mathbb{E}\left[\frac{1}{h_n}\int_{t_n}^{\theta_n} \beta(t_n, r)(\partial_t \varphi + \mathcal{L}^{\cdot}\varphi - f)(r, X_r, u) dr\right] \quad \geq \quad 0.$$

We now send $n$ to infinity. The a.s. convergence of the random value inside the expectation is easily obtained by the mean value Theorem; recall that for $n \geq N(\omega)$ sufficiently large, $\theta_n(\omega) = h_n$. Since the random variable $h_n^{-1}\int_t^{\theta_n} \beta(t_n, r)(\mathcal{L}^{\cdot}\varphi - f)(r, X^n_r, u) dr$ is essentially bounded, uniformly in $n$, on the stochastic interval $[t_n, \theta_n]$, it follows from the dominated convergence theorem that :

$$-\partial_t \varphi(t, x) - \mathcal{L}^u \varphi(t, x) - f(t, x, u) \quad \geq \quad 0,$$

which is the required result, since $u \in U$ is arbitrary. ♢

We next wish to show that $V$ satisfies the nonlinear partial differential equation (6.9) with equality, in the viscosity sense. This is also obtained by a slight modification of the proof of Proposition 2.5.

**Proposition 6.3.** *Assume that the value function $V$ is locally bounded on $\mathbf{S}$. Let the function $H$ be finite and upper semicontinuous on $[0, T) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d$, and $\|k^+\|_\infty < \infty$. Then, $V$ is a viscosity subsolution of the equation*

$$-\partial_t V(t, x) - H\left(t, x, V(t, x), DV(t, x), D^2 V(t, x)\right) \quad \leq \quad 0 \qquad (6.11)$$

*on* $[0, T) \times \mathbb{R}^n$.

*Proof.* Let $(t_0, x_0) \in \mathbf{S}$ and $\varphi \in C^2(\mathbf{S})$ be such that

$$0 \;=\; (V^* - \varphi)(t_0, x_0) \;>\; (V^* - \varphi)(t, x) \quad \text{for} \quad (t, x) \in \mathbf{S} \setminus \{(t_0, x_0)\}. \qquad (6.12)$$

In order to prove the required result, we assume to the contrary that

$$h(t_0, x_0) \;:=\; \partial_t\varphi(t_0, x_0) + H\left(t_0, x_0, \varphi(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)\right) \;<\; 0,$$

and work towards a contradiction.

**1.** Since $H$ is upper semicontinuous, there exists an open neighborhood $\mathcal{N}_r := (t_0 - r, t_0 + r) \times rB(t_0, x_0)$ of $(t_0, x_0)$, for some $r > 0$, such that

$$h \;:=\; \partial_t\varphi + H\left(., \varphi, D\varphi, D^2\varphi\right) < 0 \quad \text{on} \quad \mathcal{N}_r. \tag{6.13}$$

Then it follows from (6.12) that

$$-2\eta e^{r\|k^+\|_\infty} \;\;:=\;\; \max_{\partial\mathcal{N}_\eta}\,(V^* - \varphi) \;<\; 0. \tag{6.14}$$

Next, let $(t_n, x_n)_n$ be a sequence in $\mathcal{N}_r$ such that

$$(t_n, x_n) \;\longrightarrow\; (t_0, x_0) \quad \text{and} \quad V(t_n, x_n) \;\longrightarrow\; V^*(t_0, x_0).$$

Since $(V - \varphi)(t_n, x_n) \longrightarrow 0$, we can assume that the sequence $(t_n, x_n)$ also satisfies :

$$|(V - \varphi)(t_n, x_n)| \;\leq\; \eta \quad \text{for all} \quad n \geq 1. \tag{6.15}$$

For an arbitrary control process $\nu \in \mathcal{U}_{t_n}$, we define the stopping time

$$\theta_n^\nu := \inf\{t > t_n : X_t^{t_n, x_n, \nu} \notin \mathcal{N}_r\},$$

and we observe that $\left(\theta_n^\nu, X_{\theta_n^\nu}^{t_n, x_n, \nu}\right) \in \partial\mathcal{N}_r$ by the pathwise continuity of the controlled process. Then, with $\beta_s^\nu := \beta^\nu(t_n, s)$, it follows from (6.14) that:

$$\beta_{\theta_n^\nu}^\nu \varphi\big(\theta_n^\nu, X_{\theta_n^\nu}^{t_n, x_n, \nu}\big) \;\;\geq\;\; 2\eta + \beta_{\theta_n^\nu}^\nu V^*\big(\theta_n^\nu, X_{\theta_n^\nu}^{t_n, x_n, \nu}\big). \tag{6.16}$$

**2.** Since $\beta_{t_n}^\nu = 1$, it follows from (6.15) and Itô's formula that:

$$\begin{aligned}
V(t_n, x_n) \;&\geq\; -\eta + \varphi(t_n, x_n) \\
&=\; -\eta + \mathbb{E}\Big[\beta_{\theta_n^\nu}^\nu \varphi\big(\theta_n^\nu, X_{\theta_n^\nu}^{t_n, x_n, \nu}\big) - \int_{t_n}^{\theta_n^\nu} \beta_s^\nu (\partial_t + \mathcal{L}^{\nu_s})\varphi\big(s, X_s^{t_n, x_n, \nu}\big) ds\Big] \\
&\geq\; -\eta + \mathbb{E}\Big[\beta_{\theta_n^\nu}^\nu \varphi\big(\theta_n^\nu, X_{\theta_n^\nu}^{t_n, x_n, \nu}\big) + \int_{t_n}^{\theta_n^\nu} \beta_s^\nu \big(f(., \nu_s) - h\big)\big(s, X_s^{t_n, x_n, \nu}\big) ds\Big] \\
&\geq\; -\eta + \mathbb{E}\Big[\beta_{\theta_n^\nu}^\nu \varphi\big(\theta_n^\nu, X_{\theta_n^\nu}^{t_n, x_n, \nu}\big) + \int_{t_n}^{\theta_n^\nu} \beta_s^\nu f\big(s, X_s^{t_n, x_n, \nu}, \nu_s\big) ds\Big]
\end{aligned}$$

by (6.13). Using (6.16), this provides:

$$V(t_n, x_n) \;\;\geq\;\; \eta + \mathbb{E}\Big[\beta_{\theta_n^\nu}^\nu V^*\big(\theta_n^\nu, X_{\theta_n^\nu}^{t_n, x_n, \nu}\big) + \int_{t_n}^{\theta_n^\nu} \beta_s^\nu f\big(s, X_s^{t_n, x_n, \nu}, \nu_s\big) ds\Big].$$

Since $\eta > 0$ does not depend on $\nu$, it follows from the arbitrariness of $\nu \in \mathcal{U}_{t_n}$ that latter inequality is in contradiction with the second inequality of the dynamic programming principle of Theorem (2.3). $\diamondsuit$

As a consequence of Propositions 6.3 and 6.2, we have the main result of this section :

**Theorem 6.4.** *Let the conditions of Propositions 6.3 and 6.2 hold. Then, the value function $V$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$-\partial_t V - H\left(\cdot, V, DV, D^2V\right) = 0 \quad on \quad \mathbf{S}. \tag{6.17}$$

The partial differential equation (6.17) has a very simple and specific dependence in the time-derivative term. Because of this, it is usually referred to as a *parabolic* equation.

In order to a obtain a characterization of the value function by means of the dynamic programming equation, the latter viscosity property needs to be complemented by a uniqueness result. This is usually obtained as a consequence of a comparison result.

In the present situation, one may verify the conditions of Theorem 5.23. For completeness, we report a comparison result which is adapted for the class of equations corresponding to stochastic control problems.

Consider the parabolic equation:

$$\partial_t u + G\left(t, x, Du(t,x), D^2u(t,x)\right) \;=\; 0 \quad \text{on} \quad \mathbf{S}, \tag{6.18}$$

where $G$ is elliptic and continuous. For $\gamma > 0$, set

$$\begin{aligned} G^{+\gamma}(t,x,p,A) \quad &:= \quad \sup\left\{G(s,y,p,A) \;:\; (s,y) \in B_{\mathbf{S}}(t,x;\gamma)\right\}, \\ G^{-\gamma}(t,x,p,A) \quad &:= \quad \inf\left\{G(s,y,p,A) \;:\; (s,y) \in B_{\mathbf{S}}(t,x;\gamma)\right\}, \end{aligned}$$

where $B_{\mathbf{S}}(t,x;\gamma)$ is the collection of elements $(s,y)$ in $\mathbf{S}$ such that $|t-s|^2+|x-y|^2 \le \gamma^2$. We report, without proof, the following result from [6] (Theorem V.8.1 and Remark V.8.1).

**Assumption 6.5.** *The above operators satisfy:*

$$\begin{aligned} &\limsup_{\varepsilon\searrow 0}\left\{G^{+\gamma_\varepsilon}(t_\varepsilon, x_\varepsilon, p_\varepsilon, A_\varepsilon) - G^{-\gamma_\varepsilon}(s_\varepsilon, y_\varepsilon, p_\varepsilon, B_\varepsilon)\right\} \\ &\le\; Const\left(|t_0-s_0|+|x_0-y_0|\right)\left[1+|p_0|+\alpha\left(|t_0-s_0|+|x_0-y_0|\right)\right] \end{aligned} \tag{6.19}$$

*for all sequences $(t_\varepsilon, x_\varepsilon)$, $(s_\varepsilon, y_\varepsilon) \in [0,T) \times \mathbb{R}^n$, $p_\varepsilon \in \mathbb{R}^n$, and $\gamma_\varepsilon \ge 0$ with :*

$$((t_\varepsilon, x_\varepsilon), (s_\varepsilon, y_\varepsilon), p_\varepsilon, \gamma_\varepsilon) \;\longrightarrow\; ((t_0, x_0), (s_0, y_0), p_0, 0) \quad as \quad \varepsilon \searrow 0,$$

*and symmetric matrices $(A_\varepsilon, B_\varepsilon)$ with*

$$-KI_{2n} \;\le\; \begin{pmatrix} A_\varepsilon & 0 \\ 0 & -B_\varepsilon \end{pmatrix} \;\le\; 2\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}$$

*for some $\alpha$ independent of $\varepsilon$.*

**Theorem 6.6.** *Let Assumption 6.5 hold true, and let $u \in USC(\bar{\mathbf{S}})$, $v \in LSC(\bar{\mathbf{S}})$ be viscosity subsolution and supersolution of* (6.18)*, respectively. Then*

$$\sup_{\bar{\mathbf{S}}}(u-v) \quad = \quad \sup_{\mathbb{R}^n}(u-v)(T,\cdot).$$

A sufficient condition for (6.19) to hold is that $f(\cdot,\cdot,u)$, $k(\cdot,\cdot,u)$, $b(\cdot,\cdot,u)$, and $\sigma(\cdot,\cdot,u) \in C^1(\overline{\mathbf{S}})$ with

$$\begin{array}{c} \|b_t\|_\infty + \|b_x\|_\infty + \|\sigma_t\|_\infty + \|\sigma_x\|_\infty \; < \; \infty \\ |b(t,x,u)| + |\sigma(t,x,u)| \; \le \; Const(1+|x|+|u|) \; ; \end{array}$$

see [6], Lemma V.8.1.

## 6.2 DPE for optimal stopping problems

We first recall the optimal stopping problem considered in Section 3.1. For $0 \le t \le T \le \infty$, the set $\mathcal{T}_{[t,T]}$ denotes the collection of all $\mathbb{F}-$stopping times with values in $[t,T]$. The state process $X$ is defined by the SDE:

$$dX_t \quad = \quad \mu(t,X_t)dt + \sigma(t,X_t)dW_t, \tag{6.20}$$

where $\mu$ and $\sigma$ are defined on $\bar{\mathbf{S}} := [0,T) \times \mathbb{R}^n$, take values in $\mathbb{R}^n$ and $\mathcal{S}_n$, respectively, and satisfy the usual Lipschitz and linear growth conditions so that the above SDE has a unique strong solution satisfying the integrability of Theorem 1.2.

For a measurable function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, satisfying $\mathbb{E}\left[\sup_{0\le t<T}|g(X_t)|\right] < \infty$, the gain criterion is given by:

$$J(t,x,\tau) := \mathbb{E}\left[g\left(X_\tau^{t,x}\right)\mathbf{1}_{\tau<\infty}\right] \quad \text{for all} \quad (t,x) \in \mathbf{S},\ \tau \in \mathcal{T}_{[t,T]}. \tag{6.21}$$

Here, $X^{t,x}$ denotes the unique strong solution of (3.1) with initial condition $X_t^{t,x} = x$. Then, the optimal stopping problem is defined by:

$$V(t,x) := \sup_{\tau\in\mathcal{T}_{[t,T]}} J(t,x,\tau) \quad \text{for all} \quad (t,x) \in \mathbf{S}. \tag{6.22}$$

The next result derives the dynamic programming equation for the previous optimal stopping problem in the sense of viscosity solution, thus relaxing the $C^{1,2}$ regularity condition in the statement of Theorem 3.4. As usual, the same methodology allows to handle seemingly more general optimal stopping problems:

$$\bar{V}(t,x) \quad := \quad \sup_{\tau\in\mathcal{T}_{[t,T]}} \bar{J}(t,x,\tau), \tag{6.23}$$

where

$$\begin{aligned} \bar{J}(t,x,\tau) \quad &:= \quad \mathbb{E}\left[\int_t^T \beta(t,s)f(s,X_s^{t,x})ds + \beta(t,\tau)g(X_\tau^{t,x})\mathbf{1}_{\{\tau<\infty\}}\right], \\ \beta(t,s) \quad &:= \quad \exp\Big(-\int_t^s k(u,X_u^{t,x})du\Big). \end{aligned}$$

**Theorem 6.7.** *Assume that $V$ is locally bounded, and let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous. Then $V$ is a viscosity solution of the obstacle problem:*

$$\min\left\{-(\partial_t + \mathcal{A})V \ , \ V - g\right\} \;=\; 0 \quad on \quad \mathbf{S}. \tag{6.24}$$

*Proof.* (i) We first show that $V$ is a viscosity supersolution. As in the proof of Theorem 3.4, the inequality $V - g \geq 0$ is obvious, and implies that $V_* \geq g$. Let $(t_0, x_0) \in \mathbf{S}$ and $\varphi \in C^2(\mathbf{S})$ be such that

$$0 = (V_* - \varphi)(t_0, x_0) \quad = \quad \min_{\mathbf{S}}(V_* - \varphi).$$

To prove that $-(\partial_t + \mathcal{A})\varphi(t_0, x_0) \geq 0$, we consider a sequence $(t_n, x_n)_{n\geq 1} \subset [t_0 - h, t_0 + h] \times B$, for some small $h > 0$, such that

$$(t_n, x_n) \longrightarrow (t_0, x_0) \quad \text{and} \quad V(t_n, x_n) \longrightarrow V_*(t_0, x_0).$$

Let $(h_n)_n$ be a sequence of positive scalars converging to zero, to be fixed later, and introduce the stopping times

$$\theta^n_{h_n} := \inf\left\{t > t_n : \ (t, X^{t_n,x_n}_t) \notin [t_n - h_n, t_n + h_n] \times B\right\}.$$

Then $\theta^n_{h_n} \in \mathcal{T}^t_{[t_n,T]}$ for sufficiently small $h_n$, and it follows from (3.10) that:

$$V(t_n, x_n) \quad \geq \quad \mathbb{E}\left[V_*\left(\theta^n_{h_n}, X_{\theta^n_{h_n}}\right)\right].$$

Since $V_* \geq \varphi$, and denoting $\eta_n := (V - \varphi)(t_n, x_n)$, this provides

$$\eta_n + \varphi(t_n, x_n) \geq \mathbb{E}\left[\varphi\left(\theta^n_{h_n}, X_{\theta^n_{h_n}}\right)\right] \quad \text{where} \quad \eta_n \longrightarrow 0.$$

We continue by fixing

$$h_n \quad := \quad \sqrt{|\eta_n|}\mathbf{1}_{\{\eta_n \neq 0\}} + n^{-1}\mathbf{1}_{\{\eta_n = 0\}},$$

as in the proof of Proposition 6.2. Then, the rest of the proof follows exactly the line of argument of the proof of Theorem 3.4 combined with that of Proposition 6.2.

(ii) We next prove that $V$ is a viscosity subsolution of the equation (6.24). Let $(t_0, x_0) \in \mathbf{S}$ and $\varphi \in C^2(\mathbf{S})$ be such that

$$0 \quad = \quad (V^* - \varphi)(t_0, x_0) \;=\; \text{strict} \ \max_{\mathbf{S}} \,(V^* - \varphi),$$

assume to the contrary that

$$(V^* - g)(t_0, x_0) > 0 \quad \text{and} \quad -(\partial_t + \mathcal{A})\varphi(t_0, x_0) > 0,$$

and let us work towards a contradiction of the weak dynamic programming principle.

Since $g$ is continuous, and $V^*(t_0,x_0)=\varphi(t_0,x_0)$, we may finds constants $h>0$ and $\delta>0$ so that

$$\varphi \ge g+\delta \text{ and } -(\partial_t+\mathcal{A})\varphi \ge 0 \quad \text{on} \quad \mathcal{N}_h := [t_0,t_0+h]\times hB, \qquad (6.25)$$

where $B$ is the unit ball centered at $x_0$. Moreover, since $(t_0,x_0)$ is a strict maximizer of the difference $V^*-\varphi$:

$$-\gamma \;:=\; \max_{\partial\mathcal{N}_h}(V^*-\varphi) \;<\; 0. \qquad (6.26)$$

let $(t_n,x_n)$ be a sequence in $\mathbf{S}$ such that

$$(t_n,x_n)\longrightarrow(t_0,x_0) \quad \text{and} \quad V(t_n,x_n)\longrightarrow V^*(t_0,x_0).$$

We next define the stopping times:

$$\theta_n \;:=\; \inf\left\{t>t_n: \;\left(t,X_t^{t_n,x_n}\right)\notin\mathcal{N}_h\right\},$$

and we continue as in Step 2 of the proof of Theorem 3.4. We denote $\eta_n := V(t_n,x_n)-\varphi(t_n,x_n)$, and we compute by Itô's formula that for an arbitrary stopping rule $\tau\in\mathcal{T}^t_{[t,T]}$:

$$\begin{aligned}
V(t_n,x_n) &= \eta_n+\varphi(t_n,x_n)\\
&= \eta_n+\mathbb{E}\left[\varphi\left(\tau\wedge\theta_n, X_{\tau\wedge\theta_n}\right)-\int_{t_n}^{\tau\wedge\theta_n}(\partial_t+\mathcal{A})\varphi(t,X_t)dt\right],
\end{aligned}$$

where the diffusion term has zero expectation because the process $(t,X_t^{t_n,x_n})$ is confined to the compact subset $\mathcal{N}_h$ on the stochastic interval $[t_n,\tau\wedge\theta_n]$. Since $-(\partial_t+\mathcal{A})\varphi\ge 0$ on $\mathcal{N}_h$ by (6.25), this provides:

$$\begin{aligned}
V(t_n,x_n) &\ge \eta_n+\mathbb{E}\left[\varphi\left(\tau,X_\tau\right)\mathbf{1}_{\{\tau<\theta_n\}}+\varphi\left(\theta_n,X_{\theta_n}\right)\mathbf{1}_{\{\tau\ge\theta_n\}}\right]\\
&\ge \eta_n+\mathbb{E}\left[\left(g\left(X_\tau\right)+\delta\right)\mathbf{1}_{\{\tau<\theta_n\}}+\left(V^*\left(\theta_n,X_{\theta_n}\right)+\gamma\right)\mathbf{1}_{\{\theta_n\ge\tau\}}\right]\\
&\ge \eta_n+\gamma\wedge\delta+\mathbb{E}\left[g\left(X_\tau\right)\mathbf{1}_{\{\tau<\theta_n\}}+V^*\left(\theta_n,X_{\theta_n}\right)\mathbf{1}_{\{\theta_n\ge\tau\}}\right],
\end{aligned}$$

where we used the fact that $\varphi\ge g+\delta$ on $\mathcal{N}_h$ by (6.25), and $\varphi\ge V^*+\gamma$ on $\partial\mathcal{N}_h$ by (6.26). Since $\eta_n := (V-\varphi)(t_n,x_n)\longrightarrow 0$ as $n\to\infty$, and $\tau\in\mathcal{T}^t_{[t,T]}$ is arbitrary, this provides the desired contradiction of (3.9). ♢

## 6.3 A comparison result for obstacle problems

In this section, we derive a comparison result for the obstacle problem:

$$\begin{aligned}
&\min\left\{F(.,u,\partial_t u,Du,D^2u),u-g\right\}=0 \;\text{ on }\; [0,T)\times\mathbb{R}^d\\
&u(T,.)=g.
\end{aligned} \qquad (6.27)$$

The dynamic programming equation of the optimal stopping problem (6.23) corresponds to the particular case:

$$F(.,u,\partial_t u,Du,D^2u) \;=\; \partial_t u+b\cdot Du+\frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}D^2u\right]-ku+f.$$

**Theorem 6.8.** *Let $F$ be a uniformly continuous elliptic operator satisfying Assumption 5.22. Let $u \in USC(\mathcal{O})$ and $v \in LSC(\mathcal{O})$ be viscosity subsolution and supersolution of the equation* (6.27)*, respectively, with sub-quadratic growth. Then*

$$u(T,.) \le v(T,.) \quad \Longrightarrow \quad u \le v \ \text{ on } [0,T] \times \mathbb{R}^d.$$

*Proof.* This is an easy adaptation of the proof of Theorem 5.23. We adapt the same notations so that, in the present, $x$ stands for the pair $(t,x)$. The only difference appears at Step 3 which starts from the fact that

$$\begin{aligned} \min\left\{F(x_n, u(x_n), \alpha_n(x_n - y_n) + \varepsilon x_n, A_n), u(x_n) - g(x_n)\right\} &\le 0, \\ \min\left\{F(y_n, v(y_n), \alpha_n(x_n - y_n) - \varepsilon y_n, B_n), v(y_n) - g(y_n)\right\} &\ge 0, \end{aligned}$$

This leads to two cases:
- Either $u(x_n) - g(x_n) \le 0$ along some subsequence. Then the inequality $v(y_n) - g(y_n) \ge 0$ leads to a contradiction of (5.25).
- Or $F(x_n, u(x_n), \alpha_n(x_n - y_y) + \varepsilon x_n, A_n) \le 0$, which can be combined with the supersolution part $F(y_n, v(y_n), \alpha_n(x_n - y_y) - \varepsilon y_n, B_n) \ge 0$ exactly as in the proof of Theorem 5.23, and leads to a contradiction of (5.25). ♢

# Chapter 7

# Stochastic Target Problems

## 7.1 Stochastic target problems

In this section, we study a special class of stochastic target problems which avoids to face some technical difficulties, but reflects in a transparent way the main ideas and arguments to handle this new class of stochastic control problems.

All of the applications that we will be presenting fall into the framework of this section. The interested readers may consult the references at the end of this chapter for the most general classes of stochastic target problems, and their geometric formulation.

### 7.1.1 Formulation

Let $T > 0$ be the finite time horizon and $W = \{W_t,\ 0 \le t \le T\}$ be a $d$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathbb{F} = \{\mathcal{F}_t,\ 0 \le t \le T\}$ the $\mathbb{P}$-augmentation of the filtration generated by $W$.

We assume that the control set $U$ is a convex compact subset of $\mathbb{R}^d$ with non-empty interior, and we denote by $\mathcal{U}$ the set of all progressively measurable processes $\nu = \{\nu_t,\ 0 \le t \le T\}$ with values in $U$.

The state process is defined as follow: given the initial data $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}$, an initial time $t \in [0, T]$, and a control process $\nu \in \mathcal{U}$, let the controlled process $Z^{t,z,\nu} = (X^{t,x,\nu}, Y^{t,z,\nu})$ be the solution of the stochastic differential equation :

$$\begin{aligned} dX_u^{t,x,\nu} &= \mu\left(u, X_u^{t,x,\nu}, \nu_u\right) du + \sigma\left(u, X_u^{t,x,\nu}, \nu_u\right) dW_u, \\ dY_u^{t,z,\nu} &= b\left(u, Z_u^{t,z,\nu}, \nu_u\right) du + \nu_u \cdot dW(u), \quad u \in (t, T)\ , \end{aligned}$$

with initial data

$$X_t^{t,x,\nu} \;=\; x, \quad \text{and} \quad Y_t^{t,x,y,\nu} \;=\; y.$$

Here, $\mu : \mathbf{S}\times U \longrightarrow \mathbb{R}^d$, $\sigma : \mathbf{S}\times U \longrightarrow \mathcal{S}_d$, and $b : \mathbf{S}\times\mathbb{R}\times U \longrightarrow \mathbb{R}$ are continuous functions, Lipschitz in $(x,y)$ uniformly in $(t,u)$. Then, all above processes are well defined for every admissible control process $\nu \in \mathcal{U}_0$ defined by

$$\mathcal{U}_0 \;\;:=\;\; \left\{\nu \in \mathcal{U} :\; \mathbb{E}\left[\int_0^t \left(|\mu_0(s,\nu_s)| + |b_0(s,\nu_s)| + |\sigma_0(s,\nu_s)|^2 + |\nu_s|^2\right) ds\right]\right\}.$$

Throughout this section, we assume that the the function

$$u \;\longmapsto\; \sigma(t,x,u)p$$

has a unique fixed point for every $(t,x) \in \bar{\mathbf{S}} \times \mathbb{R}$ defined by:

$$\sigma(t,x,u)p = u \quad \Longleftrightarrow \quad u = \psi(t,x,p). \tag{7.1}$$

For a measurable function $g : \mathbb{R}^d \longrightarrow \mathbb{R}$, we define the *stochastic target* problem by:

$$V(t,x) := \inf\left\{y \in \mathbb{R} \;:\; Y_T^{t,x,y,\nu} \geq g\left(X_T^{t,x,\nu}\right),\; \mathbb{P}-\text{a.s. for some } \nu \in \mathcal{U}_0\right\}. \tag{7.2}$$

**Remark 7.1.** By introducing the subset of control processes:

$$\mathcal{A}(t,x,y) \;\;:=\;\; \left\{\nu \in \mathcal{U}_0 :\; Y_T^{t,x,y,\nu} \geq g\left(X_T^{t,x,\nu}\right),\; \mathbb{P}-\text{a.s.}\right\},$$

we may re-write the value function of the stochastic target problem into:

$$V(t,x) \;=\; \inf \mathcal{Y}(t,x), \quad \text{where} \quad \mathcal{Y}(t,x) := \{y \in \mathbb{R} \;:\; \mathcal{A}(t,x,y) \neq \emptyset\}.$$

The set $\mathcal{Y}(t,x)$ satisfies the following important property :

$$\text{for all } \; y \in \mathbb{R}, \qquad y \in \mathcal{Y}(t,x) \;\Longrightarrow\; [y,\infty) \subset \mathcal{Y}(t,x).$$

This follows from the fact that the state process $X^{t,x,\nu}$ is independent of $y$ and $Y_T^{t,x,y,\nu}$ is nondecreasing in $y$.

### 7.1.2 Geometric dynamic programming principle

Similar to the standard class of stochastic control and optimal stopping problems studied in the previous chapters, the main tool for the characterization of the value function and the solution of stochastic target problems is the dynamic programming principle. Although the present problem does not fall into the class of problems studied in the previous chapters, the idea of dynamic programming is the same: allow the time origin to move, and deduce a relation between the value function at different points in time.

In these notes, we shall only use the *easy* direction of a more general geometric dynamic programming principle.

**Theorem 7.2.** *Let $(t,x) \in [0,T] \times \mathbb{R}^d$ and $y \in \mathbb{R}$ such that $\mathcal{A}(t,x,y) \neq \emptyset$. Then, for any control process $\nu \in \mathcal{A}(t,x,y)$ and stopping time $\tau \in \mathcal{T}_{[t,T]}$,*

$$Y_\tau^{t,x,y,\nu} \;\geq\; V\left(\tau, X_\tau^{t,x,\nu}\right), \quad P-a.s. \tag{7.3}$$

*Proof.* Let $z = (x,y)$ and $\nu \in \mathcal{A}(t,z)$, and denote $Z_{t,z,\nu} := (X^{t,x,\nu}, Y^{t,z,\nu})$. Then $Y_T^{t,z,\nu} \geq g\left(X_T^{t,x,\nu}\right)$ $\mathbb{P}-$a.s. Notice that

$$Z_T^{t,z,\nu} \quad = \quad Z_T^{\tau, Z_\tau^{t,z,\nu}, \nu}.$$

Then, by the definition of the set $\mathcal{A}$, it follows that $\nu \in \mathcal{A}\left(\tau, Z_\tau^{t,z,\nu}\right)$, and therefore $V\left(\tau, X_\tau^{t,x,\nu}\right) \leq Y_\tau^{t,z,\nu}$, $\mathbb{P}-$a.s. ♢

In the next subsection, we will prove that the value function $V$ is a viscosity supersolution of the corresponding dynamic programming equation which will be obtained as the infinitesimal counterpart of (7.3). The following remark comments on the full geometric dynamic programming principle in the context of stochastic target problems.

**Remark 7.3.** The statement (7.3) in Theorem 7.2 can be strengthened to the following geometric dynamic programming principle:

$$V(t,x) = \inf\left\{y \in \mathbb{R} : Y^{t,x,y,\nu} \geq V\left(\tau, X_\tau^{t,x,\nu}\right),\ \mathbb{P}-\text{a.s. for some } \nu \in \mathcal{U}_0\right\}. \tag{7.4}$$

Let us provide an intuitive justification of this. Denote $\hat{y} := V(t,x)$. In view of (7.3), it is easily seen that (7.4) is implied by

$$\mathbb{P}\left[Y_\tau^{t,x,\hat{y}-\eta,\nu} > V\left(\tau, X_\tau^{t,x,\nu}\right)\right] \;<\; 1 \quad \text{for all} \quad \nu \in \mathcal{U}_0 \text{ and } \eta > 0.$$

In words, there is no control process $\nu$ which allows to reach the value function $V(\tau, X_\tau^{t,x,\nu})$ at time $\tau$, with full probability, starting from an initial data stricly below the value function $V(t,x)$. To see this, suppose to the contrary that there exist $\nu \in \mathcal{U}_0$, $\eta > 0$, and $\tau \in \mathcal{T}_{[t,T]}$ such that:

$$Y_\tau^{t,x,\hat{y}-\eta,\nu} \quad > \quad V\left(\tau, X_\tau^{t,x,\nu}\right), \quad P-\text{a.s.}$$

In view of Remark 7.1, this implies that $Y_\tau^{t,x,\hat{y}-\eta,\nu} \in \mathcal{Y}\left(\tau, X_\tau^{t,x,\nu}\right)$, and therefore, there exists a control $\hat{\nu} \in \mathcal{U}_0$ such that

$$Y_T^{\tau, Z_\tau^{t,x,\hat{y}-\eta,\nu}, \hat{\nu}} \quad \geq \quad g\left(X_T^{\tau, X_\tau^{t,x,\nu}, \hat{\nu}}\right), \quad \mathbb{P}-\text{a.s.}$$

Since the process $\left(X^{\tau, X_\tau^{t,x,\nu}, \hat{\nu}}, Y^{\tau, Z_\tau^{t,x,y^*-\eta,\nu}, \hat{\nu}}\right)$ depends on $\hat{\nu}$ only through its realizations in the stochastic interval $[t,\theta]$, we may chose $\hat{\nu}$ so as $\hat{\nu} = \nu$ on $[t,\tau]$ (this is the difficult part of this proof). Then $Z_T^{\tau, Z_\tau^{t,x,\hat{y}-\eta,\nu}, \hat{\nu}} = Z_T^{t,x,\hat{y}-\eta,\hat{\nu}}$ and therefore $\hat{y} - \eta \in \mathcal{Y}(t,x)$, hence $\hat{y} - \eta \leq V(t,x)$. Recall that by definition $\hat{y} = V(t,x)$ and $\eta > 0$. ♢

### 7.1.3 The dynamic programming equation

In order to have a simpler statement and proof of the main result, we assume in this section that

$$U \quad \text{is a closed convex subset of} \quad \mathbb{R}^d, \ \operatorname{int}(U) \neq \emptyset \text{ and } 0 \in U. \tag{7.5}$$

The formulation of the dynamic programming equation involves the notion of support function from convex analysis.

**a- Dual characterization of closed convex sets** We first introduce the *support function* of the set $U$:

$$\delta_U(\xi) := \sup_{x \in U} x \cdot \xi, \quad \text{for all} \quad \xi \in \mathbb{R}^d.$$

By definition $\delta_U$ is a convex function $\mathbb{R}^d$. Its effective domain

$$\tilde{U} := \operatorname{dom}(\delta_U) \quad = \quad \{\xi \in \mathbb{R}^d \ : \ \delta_U(\xi) < \infty\}$$

is a closed convex cone of $\mathbb{R}^d$. Since $U$ is closed and convex by (7.5), we have the following dual characterization:

$$x \in U \quad \text{if and only if} \quad \delta_U(\xi) - x \cdot \xi \geq 0 \text{ for all } \xi \in \tilde{U}, \tag{7.6}$$

see e.g. Rockafellar [13]. Morover, since $\tilde{U}$ is a cone, we may normalize the dual variables $\xi$ on the right hand-side:

$$x \in U \quad \text{if and only if} \quad \delta_U(\xi) - x \cdot \xi \geq 0 \text{ for all } \xi \in \tilde{U}_1 := \{\xi \in \tilde{U} : |\xi| = 1\}. \tag{7.7}$$

This normalization will be needed in our analysis in order to obtain a dual characterization of $\operatorname{int}(U)$. Indeed, since $U$ has nonempty interior by (7.5), we have:

$$x \in \operatorname{int}(U) \quad \text{if and only if} \quad \inf_{\xi \in \tilde{U}_1} \delta_U(\xi) - x \cdot \xi > 0. \tag{7.8}$$

**b- Formal derivation of the DPE** We start with a formal derivation of the dynamic programming equation which provides the main intuitions.

To simplify the presentation, we suppose that the value function $V$ is smooth and that existence holds, i.e. for all $(t,x) \in \mathbf{S}$, there is a control process $\hat{\nu} \in \mathcal{U}_0$ such that, with $z = (x, V(t,x))$, we have $Y_T^{t,z,\hat{\nu}} \geq g(X_T^{t,x,\hat{\nu}})$, $\mathbb{P}-$a.s. Then it follows from the geometric dynamic programming of Theorem 7.2 that, $\mathbb{P}-$a.s.

$$Y_{t+h}^{t,z,\nu} = v(t,x) + \int_t^{t+h} b\left(s, Z_s^{t,z,\hat{\nu}}, \hat{\nu}_s\right) ds + \int_t^{t+h} \hat{\nu}_s \cdot dW_s \geq V\left(t+h, X_{t+h}^{t,x,\hat{\nu}}\right).$$

By Itô's formula, this implies that

$$\begin{aligned} 0 \quad \leq \quad & \int_t^{t+h} \left(-\partial_t V(s, X_s^{t,x,\hat{\nu}}) + H\left(s, Z_s^{t,z,\hat{\nu}}, DV(s, X_s^{t,x,\hat{\nu}}), D^2V(s, X_s^{t,x,\hat{\nu}}), \hat{\nu}_s\right)\right) ds \\ & + \int_t^{t+h} N^{\nu_s}\left(s, X_s^{t,x,\hat{\nu}}, DV(s, X_s^{t,x,\hat{\nu}})\right) \cdot dW_s, \end{aligned} \tag{7.9}$$

where we introduced the functions:

$$H(t,x,y,p,A,u) \;:=\; b(t,x,y,u) - \mu(t,x,u)\cdot p - \frac{1}{2}\mathrm{Tr}\left[\sigma(t,x,u)^2 A\right] \quad (7.10)$$

$$N^u(t,x,p) \;:=\; u - \sigma(t,x,u)p. \quad (7.11)$$

We continue our intuitive derivation of the dynamic programming equation by assuming that all terms inside the integrals are bounded (we know that this can be achieved by localization). Then the first integral behaves like $Ch$, while the second integral can be viewed as a time changed Brownian motion. By the properties of the Brownian motion, it follows that the integrand of the stochastic integral term must be zero at the origin:

$$N_t^{\nu_t}(t,x,DV(t,x)) = 0 \quad \text{or, equivalently} \quad \nu_t = \psi\big(t,x,DV(t,x)\big), \quad (7.12)$$

where $\psi$ was introduced in (7.1). In particular, this implies that $\psi(t,x,DV(t,x)) \in U$. By (7.7), this is equivalent to:

$$\delta_U(\xi) - \xi\cdot\psi(t,x,DV(t,x)) \geq 0 \quad \text{for all} \quad \xi \in \tilde{U}_1. \quad (7.13)$$

By taking expected values in (7.9), normalizing by $h$, and sending $h$ to zero, we see that:

$$-\partial_t V(t,x) + H\big(t,x,V(t,x),DV(t,x),D^2V(t,x),\psi(t,x,DV(t,x))\big) \geq 0 \quad (7.14)$$

Putting (7.13) and (7.14) together, we obtain

$$\min\left\{-\partial_t V + H\big(.,V,DV,D^2V,\psi(.,DV)\big), \inf_{\xi\in\tilde{U}_1}\big(\delta_U(\xi) - \xi\cdot\psi(.,DV)\big)\right\} \geq 0.$$

By using the second part of the geometric dynamic programming principle, see Remark 7.3, we expect to prove that equality holds in the latter dynamic programming equation.

**c- The dynamic programming equation** We next turn to a rigorous derivation of the dynamic programmin equation.

**Theorem 7.4.** *Assume that $V$ is locally bounded, and let the maps $H$ and $\psi$ be continuous. Then $V$ is a viscosity supersolution of the dynamic programming equation on $\mathbf{S}$:*

$$\min\left\{-\partial_t V + H\big(.,V,DV,D^2V,\psi(.,DV)\big), \inf_{\xi\in\tilde{U}_1}\big(\delta_U(\xi) - \xi\cdot\psi(.,DV)\big)\right\} = 0$$

*Assume further that $\psi$ is locally Lipschitz-continuous, and $U$ has non-empty interior. Then $V$ is a viscosity solution of the above DPE on $\mathbf{S}$.*

*Proof.* As usual, we prove separately the supersolution and the subsolution properties.

**1.** *Supersolution:* Let $(t_0, x_0) \in \mathbf{S}$ and $\varphi \in C^2(\mathbf{S})$ be such that

$$\text{(strict)} \min_{\mathbf{S}}(V_* - \varphi) \;=\; (V_* - \varphi)(t_0, x_0) \;=\; 0,$$

and assume to the contrary that

$$-2\eta \;:=\; \big(-\partial_t V + H\big(., V, DV, D^2V, \psi(., DV)\big)\big)(t_0, x_0) \;<\; 0. \tag{7.15}$$

(1-i) By the continuity of $H$ and $\psi$, we may find $\varepsilon > 0$ such that

$$-\partial_t V(t,x) + H\big(t, x, y, DV(t,x), D^2V(t,x), u\big) \leq -\eta \tag{7.16}$$
$$\text{for } (t,x) \in B_\varepsilon(t_0, x_0),\; |y - \varphi(t,x)| \leq \varepsilon, \text{and } u \in U \text{ s.t. } |N^u(t,x,p)| \leq \varepsilon.$$

Notice that (7.16) is obviously true if $\{u \in U : |N^u(t,x,p)| \leq \varepsilon\} = \emptyset$, so that the subsequent argument holds in this case as well.

Since $(t_0, x_0)$ is a strict minimizer of the difference $V_* - \varphi$, we have

$$\gamma \;:=\; \min_{\partial B_\varepsilon(t_0,x_0)} (V_* - \varphi) \;>\; 0. \tag{7.17}$$

(1-ii) Let $(t_n, x_n)_n \subset B_\varepsilon(t_0, x_0)$ be a sequence such that

$$(t_n, x_n) \longrightarrow (t_0, x_0) \quad \text{and} \quad V(t_n, x_n) \longrightarrow V_*(t_0, x_0), \tag{7.18}$$

and set $y_n := x_n + n^{-1}$ and $z_n := (x_n, y_n)$. By the definition of the problem $V(t_n, x_n)$, there exists a control process $\hat{\nu}^n \in \mathcal{U}_0$ such that the process $Z^n := Z^{t_n, z_n, \hat{\nu}^n}$ satisfies $Y^n_T \geq g(X^n_T)$, $\mathbb{P}-$a.s. Consider the stopping times

$$\begin{aligned}
\theta^0_n &:= \inf\{t > t_n : (t, X^n_t) \notin B_\varepsilon(t_0, x_0)\},\\
\theta_n &:= \theta^0_n \wedge \inf\{t > t_n : |Y^n_t - \varphi(t, X^n_t)| \geq \varepsilon\}
\end{aligned}$$

Then, it follows from the geometric dynamic programming principle that

$$Y^n_{t\wedge\theta_n} \;\geq\; V\left(t \wedge \theta_n, X^n_{t\wedge\theta_n}\right).$$

Since $V \geq V_* \geq \varphi$, and using (7.17) and the definition of $\theta_n$, this implies that

$$\begin{aligned}
Y^n_{t\wedge\theta_n} &\geq \varphi\left(t\wedge\theta_n, X^n_{t\wedge\theta_n}\right) + \mathbf{1}_{\{t=\theta_n\}}\left(\gamma\mathbf{1}_{\{\theta_n=\theta^0_n\}} + \varepsilon\mathbf{1}_{\{\theta_n<\theta^0_n\}}\right)\\
&\geq \varphi\left(t\wedge\theta_n, X^n_{t\wedge\theta_n}\right) + (\gamma\wedge\varepsilon)\mathbf{1}_{\{t=\theta_n\}}.
\end{aligned} \tag{7.19}$$

(1-iii) Denoting $c_n := V(t_n, x_n) - \varphi(t_n, x_n) - n^{-1}$, we write the process $Y^n$ as

$$Y^n_t \;=\; c_n + \varphi(t_n, x_n) + \int_{t_n}^t b(s, Z^n_s, \hat{\nu}^n_s)ds + \int_{t_n}^t \hat{\nu}^n_s \cdot dW_s.$$

Plugging this into (7.19) and applying Itô's formula, we then see that:

$$\begin{aligned}
(\varepsilon\wedge\gamma)\mathbf{1}_{\{t=\theta_n\}} &\leq c_n + \int_{t_n}^{t\wedge\theta_n} \delta^n_s ds + \int_{t_n}^{t\wedge\theta_n} N^{\hat{\nu}^n_s}\left(s, X^n_s, D\varphi(s, X^n_s)\right)\cdot dW_s\\
&\leq M_n := c_n + \int_{t_n}^{t\wedge\theta_n} \delta^n_s \mathbf{1}_{A^n}(s)ds\\
&\qquad + \int_{t_n}^{t\wedge\theta_n} N^{\hat{\nu}^n_s}\left(s, X^n_s, D\varphi(s, X^n_s)\right)\cdot dW_s
\end{aligned} \tag{7.20}$$

where

$$\delta_s^n := -\partial_t\varphi(s, X_s^n) + H\left(s, Z_s^n, D\varphi(s, X_s^n), D^2\varphi(s, X_s^n), \hat{\nu}_s\right)$$

and

$$A^n \quad := \quad \{s \in [t_n, t_n + \theta_n] : \; \delta_s^n > -\eta\}\,.$$

By (7.16), observe that the diffusion term $\zeta_s^n := N^{\hat{\nu}_s^n}(s, X_s^n, D\varphi(s, X_s^n))$ in (7.20) satisfies $|\zeta_s^n| \geq \eta$ for all $s \in A^n$. Then, by introducing the exponential local martingale $L^n$ defined by

$$L_{t_n}^n = 1 \quad \text{and} \quad dL_t^n = L_t^n |\zeta_t^n|^{-2} \zeta_t^n \cdot dW_t,$$

we see that the process $M^n L^n$ is a local martingale which is bounded from below by the constant $\varepsilon \wedge \gamma$. Then $M^n L^n$ is a supermartingale, and it follows from (7.20) that

$$\varepsilon \wedge \gamma \quad \leq \quad \mathbb{E}\left[M_{\theta_n}^n L_{\theta_n}^n\right] \;\leq\; M_{t_n}^n L_{t_n}^n \;=\; c_n,$$

which can not happen because $c_n \longrightarrow 0$. Hence, our starting point (7.22) can not happen, and the proof of the supersolution property is complete.

**2.** *Subsolution:* Let $(t_0, x_0) \in \mathbf{S}$ and $\varphi \in C^2(\mathbf{S})$ be such that

$$\text{(strict)} \max_{\mathbf{S}}(V^* - \varphi) \quad = \quad (V^* - \varphi)(t_0, x_0) \;=\; 0, \tag{7.21}$$

and assume to the contrary that

$$\begin{aligned} 2\eta := \left(-\partial_t\varphi + H\big(., \varphi, D\varphi, D^2\varphi, \psi(., \varphi)\big)\right)(t_0, x_0) > 0, \\ \text{and } \inf_{\xi \in \tilde{U}_1}\left(\delta_U(\xi) - \xi \cdot \psi(., D\varphi)\right)(t_0, x_0) > 0. \end{aligned} \tag{7.22}$$

(2-i) By the continuity of $H$ and $\psi$, and the characterization of $\text{int}(U)$ in (7.8), it follows from (7.22) that

$$\begin{aligned} \left(-\partial_t\varphi + H\big(., y, D\varphi, D^2\varphi, \psi(., D\varphi)\big)\right) \geq \eta \text{ and } \psi(., D\varphi) \in U \\ \text{for } (t, x) \in B_\varepsilon(t_0, x_0) \text{ and } |y - \varphi(t, x)| \leq \varepsilon \end{aligned} \tag{7.23}$$

Also, since $(t_0, x_0)$ is a strict maximizer in (7.21), we have

$$-\zeta \quad := \quad \max_{\partial_p B_\varepsilon(t_0, x_0)} (V^* - \varphi) \;<\; 0, \tag{7.24}$$

where $\partial_p B_\varepsilon(t_0, x_0) := \{t_0 + \varepsilon\} \times \text{cl}\left(B_\varepsilon(t_0, x_0)\right) \cup [t_0, t_0 + \varepsilon) \times \partial B_\varepsilon(x_0)$ denotes the parabolic boundary of $B_\varepsilon(t_0, x_0)$.
(2-ii) Let $(t_n, x_n)_n$ be a sequence in $\mathbf{S}$ which converges to $(t_0, x_0)$ and such that $V(t_n, x_n) \to V^*(t_0, x_0)$. Set $y_n = V(t_n, x_n) - n^{-1}$ and observe that

$$\gamma_n \;:=\; y_n - \varphi(t_n, x_n) \quad \longrightarrow \quad 0. \tag{7.25}$$

Let $Z^n := (X^n, Y^n)$ denote the controlled state process associated to the Markovian control $\hat\nu^n_t = \psi(t, X^n_t, D\varphi(t, X^n_t))$ and the initial condition $Z^n_{t_n} = (x_n, y_n)$. Since $\psi$ is locally Lipschitz-continuous, the process $Z^n$ is well-defined. We next define the stopping times

$$\begin{aligned}
\theta^0_n &:= \inf\left\{s \ge t_n \;:\; (s, X^n_s) \notin B_\varepsilon(t_0, x_0)\right\},\\
\theta_n &:= \theta^0_n \wedge \inf\left\{s \ge t_n \;:\; |Y^n(s) - \varphi(s, X^n_s)| \ge \varepsilon\right\}.
\end{aligned}$$

By the first line in (7.23), (7.25) and a standard comparison theorem, it follows that $Y^n_{\theta_n} - \varphi(\theta_n, X^n_{\theta_n}) \ge \varepsilon$ on $\{|Y^n_{\theta_n} - \varphi(\theta_n, X^n_{\theta_n})| \ge \varepsilon\}$ for $n$ large enough. Since $V \le V^* \le \varphi$, we then deduce from (7.24) and the definition of $\theta_n$ that

$$\begin{aligned}
Y^n_{\theta_n} - V(\theta_n, X^n_{\theta^n}) &\ge \mathbf{1}_{\{\theta_n < \theta^0_n\}}\Big(Y^n_{\theta_n} - \varphi\big(\theta_n, X^n_{\theta_n}\big)\Big)\\
&\qquad + \mathbf{1}_{\{\theta_n = \theta^0_n\}}\Big(Y^n_{\theta^0_n} - V^*\big(\theta^0_n, X^n_{\theta^0_n}\big)\Big)\\
&= \varepsilon \mathbf{1}_{\{\theta_n < \theta^0_n\}} + \mathbf{1}_{\{\theta_n = \theta^o_n\}}\Big(Y^n_{\theta^0_n} - V^*\big(\theta^0_n, X^n_{\theta^0_n}\big)\Big)\\
&\ge \varepsilon \mathbf{1}_{\{\theta_n < \theta^0_n\}} + \mathbf{1}_{\{\theta_n = \theta^0_n\}}\Big(Y^n_{\theta^0_n} + \zeta - \varphi\big(\theta^0_n, X^n_{\theta^0_n}\big)\Big)\\
&\ge \varepsilon \wedge \zeta + \mathbf{1}_{\{\theta_n = \theta^0_n\}}\Big(Y^n_{\theta^0_n} - \varphi\big(\theta^0_n, X^n_{\theta^0_n}\big)\Big).
\end{aligned}$$

We continue by using Itô's formula:

$$Y^n_{\theta_n} - V(\theta_n, X^n_{\theta^n}) \;\ge\; \varepsilon \wedge \zeta + \mathbf{1}_{\{\theta_n = \theta^0_n\}}\left(\gamma_n + \int_{t_n}^{\theta_n} \alpha(s, X^n_s, Y^n_s)ds\right)$$

where the drift term $\alpha(\cdot) \ge \eta$ is defined in (7.23) and the diffusion coefficient vanishes by the definition of the function $\psi$ in (7.1). Since $\varepsilon, \zeta > 0$ and $\gamma_n \to 0$, this implies that

$$Y^n(\theta_n) \;\ge\; V(\theta_n, X^n(\theta^n)) \quad \text{for sufficiently large} \quad n.$$

Recalling that the initial position of the process $Y^n$ is $y_n = V(t_n, x_n) - n^{-1} < V(t_n, x_n)$, this is clearly in contradiction with the second part of the geometric dynamic programming principle discussed in Remark 7.3. $\diamondsuit$

### 7.1.4 Application: hedging under portfolio constraints

As an application of the previous results, we now study the problem of super-hedging under portfolio constraints in the context of the Black-Scholes model.

**a- Formulation** We consider a financial market consisting of $d+1$ assets. The first asset $X^0$ is nonrisky, and is normalized to unity. The $d$ next assets are risky with price process $X = (X^1, \ldots, X^d)^{\mathrm{T}}$ defined by the Blac-Scholes model:

$$dX_t \;=\; X_t \star \sigma dW_t,$$

where $\sigma$ is a constant symmetric nondegenrate matrix in $\mathbb{R}^d$, and $x \star \sigma$ is the square matrix in $\mathbb{R}^d$ with entries $(x \star \sigma)_{i,j} = x_i \sigma_{i,j}$.

**Remark 7.5.** We observe that the normalization of the first asset to unity does not entail any loss of generality as we can always reduce to this case by discounting or, in other words, by taking the price process of this asset as a numéraire.

Also, the formulation of the above process $X$ as a martingale is not a restriction as our subsequent superhedging problem only involves the underlying probability measure through the corresponding zero-measure sets. Therefore, under the no-arbitrage condition (or more precisely, no free-lunch with vanishing risk), we can reduce the model to the above martingale case by an equivalent change of measure. $\diamondsuit$

Under the self-financing condition, the liquidation value of the portfolio is defined by the controlled state process:

$$dY_t^\pi \quad = \quad \sigma\pi_t \cdot dW_t,$$

where $\pi$ is the control process, with $\pi_t^i$ representing the amount invested in the $i-$th risky asset $X^i$ at time $t$.

We introduce portfolio constraints by imposing that the portfolio process $\pi$ must be valued in a subset $U$ of $\mathbb{R}^d$. We shalll assume that

$$U \quad \text{is closed convex subset of } \mathbb{R}^d, \quad \mathrm{int}(U) = \emptyset, \text{ and } 0 \in U. \tag{7.26}$$

We then define the controls set by $\mathcal{U}_o$ as in the previous sections, and we defined the superhedging problem under portfolio constraints by the stochastic target problem:

$$V(t,x) \quad := \quad \inf\left\{y : Y_T^{t,y,\pi} \geq g(X_T^{t,x}),\ \mathbb{P}-\text{a.s. for some } \pi \in \mathcal{U}_0\right\}, \tag{7.27}$$

where $g : \mathbb{R}_+^d \longrightarrow \mathbb{R}_+$ is a non-negative LSC function with linear growth.

We shall provide an explicit solution of this problem by only using the supersolution claim from Theorem 7.4. This will provide a minorant of the superhedging cost $V$. To provide that this minorant is indeed the desired value function, we will use a verification argument.

**b- Deriving a minorant of the superhedging cost** First, since $0 \leq g(x) \geq C(1+|x|)$ for some constant $C > 0$, we deduce that $0 \leq V \geq C(1+|x|)$, the right hand-side inequality is easily justified by the buy-and-hold strategy suggested by the linear upper bound. Then, by a direct application of the first part of Theorem 7.4, we know that the LSC envelope $V_*$ of $V$ is a supersolution of the DPE:

$$-\partial_t V_* - \tfrac{1}{2}\mathrm{Tr}[(x \star \sigma)^2 D^2 V_*] \geq 0 \tag{7.28}$$

$$\delta_U(\xi) - \xi \cdot (x \star DV_*)\big) \geq 0 \text{ for all } \xi \in \tilde{U}. \tag{7.29}$$

Notice that (7.29) is equivalent to:

$$\text{the map} \quad \lambda \longmapsto h(\lambda) := \lambda\delta_U(\xi) - V_*(t, x \star e^{\lambda\xi}) \quad \text{is nondecreasing,} \tag{7.30}$$

where $e^{\lambda\xi}$ is the vector of $\mathbb{R}^d$ with entries $(e^{\lambda\xi})_i = e^{\lambda\xi_i}$. Then $h(1) \geq h(0)$ provides:

$$V_*(t,x) \quad \geq \quad \sup_{\xi\in\tilde{U}} V_*\left(x \star e^{\xi}\right) - \delta_U(\xi).$$

We next observe that $V_*(T,.) \geq g$ (just use the definition of $V$, and send $t \nearrow T$). Then, we deuce from the latter inquality that

$$V_*(T,x) \geq \hat{g}(x) := \sup_{\xi\in\tilde{U}} g\left(x \star e^{\xi}\right) - \delta_U(\xi) \quad \text{for all} \quad x \in \mathbb{R}^d_+. \tag{7.31}$$

In other words, in order to superhedge the derivative security with final payoff $g(X_T)$, the constraints on the portfolio require that one hedges the derivative security with larger payoff $\hat{g}(X_T)$. The function $\hat{g}$ is called the *face-lifted* payoff, and is the smallest majorant of $g$ which satisfies the gradient constraint $x \star Dg(x) \in U$ for all $x \in \mathbb{R}^d_+$.

Combining (7.31) with (7.28), it follows from the comparison result for the linear Black-Scholes PDE that

$$V(t,x) \geq V_*(t,x) \geq v(t,x) := \mathbb{E}\left[\hat{g}(X_T^{t,x})\right] \quad \text{for all} \quad (t,x) \in \mathbf{S}. \tag{7.32}$$

**c- Explicit solution** Our objective is now to prove that $V = v$. To see this consider the Black-Scholes hedging strategy $\hat{\pi}$ of the derivative security $\hat{g}(X_T^{t,x})$:

$$v(t,x) + \int_t^T \hat{\pi}_s \cdot \sigma dW_s \quad = \quad \hat{g}(X_T^{t,x}).$$

Since $\hat{g}$ has linear growth, it follows that $\hat{\pi} \in \mathbb{H}^2$. We also observe that the random variable $\ln X_T^{t,x}$ is gaussian, so that the function $v$ can be written in:

$$v(t,x) \quad = \quad \int \hat{g}(e^w) \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{1}{2}\left(\frac{w-x+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)^2} dw.$$

Under this form, it is clear that $v$ is a smooth function. Then the above hedging portfolio is given by:

$$\hat{\pi}_s \quad := \quad X_s^{t,x} \star DV(s, X_s^{t,x})$$

Notice that, for all $\xi \in \tilde{U}$,

$$\lambda\delta_U(\xi) - v(t, xe^{\lambda\xi}) = \mathbb{E}\left[\lambda\delta_U(\xi) - \hat{g}\left(X_T^{t,xe^{\lambda\xi}}\right)\right]$$

is nondecreasing in $\lambda$ by applying (7.30) to $\hat{g}$ which, by definition satisfies $x \star Dg(x) \in U$ for all $x \in \mathbb{R}^d_+$. Then, $x \star Dg(x) \in U$, and therefore the above replicating portfolio $\hat{\pi}$ takes values in $U$. Since $\hat{g} \geq g$, we deduce from (7.31) that $v \geq V$.

## 7.2 Stochastic target problem with controlled probability of success

In this section, we extend the model presented above to the case where the target has to be reached only with a given probability $p$:

$$\hat{V}(t,x,p) := \inf\left\{y \in \mathbb{R}_+ \ : \ \mathbb{P}\left[Y_T^{t,x,y,\nu} \geq g(X_T^{t,x,\nu})\right] \geq p \text{ for some } \nu \in \mathcal{U}_0\right\}. \tag{7.33}$$

In order to avoid degenerate results, we restrict the analysis to the case where the $Y$ process takes non-negative values, by simply imposing the following conditions on the coefficients driving its dynamics:

$$b(t,x,0,u) \geq 0 \quad \text{for all} (t,x) \in \mathbf{S}, u \in U. \tag{7.34}$$

Notice that the above definition implies that

$$0 \ = \ \hat{V}(.,0) \ \leq \ \hat{V} \ \leq \ \hat{V}(.,1) \ = \ V, \tag{7.35}$$

and

$$\hat{V}(.,p) \ = \ 0 \quad \text{for} \quad p < 0 \text{ and } \hat{V}(.,p) = \infty \ \text{ for } p > 1 \ , \tag{7.36}$$

with the usual convention $\inf \emptyset = \infty$.

### 7.2.1 Reduction to a stochastic target problem

Our first objective is to convert this problem into a (standard) stochastic target problem, so as to apply the geometric dynamic programming arguments of the previous section.

To do this, we introduce an additional controlled state variable:

$$P_s^{t,p,\alpha} := p + \int_t^s \alpha_r \cdot dW_r, \quad \text{for} \quad s \in [t,T], \tag{7.37}$$

where the additional control $\alpha$ is an $\mathbb{F}-$progressively measurable $\mathbb{R}^d-$valued process satisfying the integrability condition $\mathbb{E}[\int_0^T |\alpha_s|^2 ds] < \infty$. We then set $\hat{X} := (X,P)$, $\hat{\mathbf{S}} := [0,T) \times \mathbb{R}^d \times (0,1)$, $\hat{U} := U \times \mathbb{R}^d$, and denote by $\hat{\mathcal{U}}$ the corresponding set of admissible controls. Finally, we introduce the function:

$$\hat{G}(\hat{x},y) := \mathbf{1}_{\{y \geq g(x)\}} - p \quad \text{for} \quad y \in \mathbb{R}, \ \hat{x} := (x,p) \in \mathbb{R}^d \times [0,1].$$

**Proposition 7.6.** *For all $t \in [0,T]$ and $\hat{x} = (x,p) \in \mathbb{R}^d \times [0,1]$, we have*

$$\hat{V}(t,\hat{x}) = \inf\left\{y \in \mathbb{R}_+ \ : \ \hat{G}\left(\hat{X}_T^{t,\hat{x},\hat{\nu}}, Y_T^{t,x,y,\nu}\right) \geq 0 \textit{ for some } \hat{\nu} = (\nu,\alpha) \in \hat{\mathcal{U}}\right\}.$$

*Proof.* We denote by $v(t,x,p)$ the value function apprearing on the right-hand.

We first show that $\hat{V} \geq v$. For $y > \hat{V}(t,x,p)$, we can find $\nu \in \mathcal{U}$ such that $p_0 := \mathbb{P}\left[G\left(X_T^{t,x,\nu}, Y_T^{t,x,y,\nu}\right) \geq 0\right] \geq p$. By the stochastic integral representation theorem, there exists an $\mathbb{F}$-progressively measurable process $\alpha$ such that

$$\mathbf{1}_{\left\{Y_T^{t,x,y,\nu} \geq g\left(X_T^{t,x,\nu}\right)\right\}} = p_0 + \int_t^T \alpha_s \cdot dW_s = P_T^{t,p_0,\alpha} \quad \text{and} \quad \mathbb{E}[\int_t^T |\alpha_s|^2 ds] < \infty.$$

Since $p_0 \geq p$, it follows that $\mathbf{1}_{\left\{Y_T^{t,x,y,\nu} \geq g\left(X_T^{t,x,\nu},\right)\right\}} \geq P_{t,p}^{\alpha}(T)$, and therefore $y \geq v(t,x,p)$ from the definition of the problem $v$.

We next show that $v \geq \hat{V}$. For $y > v(t,x,p)$, we have $\hat{G}\left(\hat{X}_T^{t,\hat{x},\hat{\nu}}, Y_T^{t,x,y,\nu}\right) \geq 0$ for some $\hat{\nu} = (\nu,\alpha) \in \hat{\mathcal{U}}$. Since $P_{t,p}^{\alpha}$ is a martingale, it follows that

$$\mathbb{P}\left[Y_T^{t,x,y,\nu} \geq g\left(X_T^{t,x,\nu}\right)\right] = \mathbb{E}\left[\mathbf{1}_{\left\{Y_T^{t,x,y,\nu} \geq g\left(X_T^{t,x,\nu}\right)\right\}}\right] \geq \mathbb{E}\left[P_T^{t,p,\alpha}\right] = p,$$

which implies that $y \geq \hat{V}(t,x,p)$ by the definition of $\hat{V}$. $\diamondsuit$

**Remark 7.7.** 1. Suppose that the infimum in the definition of $\hat{V}(t,x,p)$ is achieved and there exists a control $\nu \in \mathcal{U}_0$ satisfying $\mathbb{P}\left[Y_T^{t,x,y,\nu} \geq g\left(X_T^{t,x,\nu}\right)\right] = p$, the above argument shows that:

$$P_s^{t,p,\alpha} = \mathbb{P}\left[Y_T^{t,x,y,\nu} \geq g\left(X_T^{t,x,\nu}\right) \Big| \mathcal{F}_s\right] \quad \text{for all} \quad s \in [t,T].$$

2. It is easy to show that one can moreover restrict to controls $\alpha$ such that the process $P^{t,p,\alpha}$ takes values in $[0,1]$. This is rather natural since this process should be interpreted as a conditional probability, and this corresponds to the natural domain $[0,1]$ of the variable $p$. We shall however avoid to introduce this state constraint, and use th efact that the value function $\hat{V}(\cdot,p)$ is constant for $p \leq 0$ and equal $\infty$ for $p > 1$, see (7.36).

### 7.2.2 The dynamic programming equation

The above reduction of the problem $\hat{V}$ to a stochastic target problem allows to apply the geometric dynamic programming principle of the previous section, and to derive the corresponding dynamic programming equation. For $\hat{u} = (u,\alpha) \in \hat{U}$ and $\hat{x} = (x,p) \in \mathbb{R}^d \times [0,1]$, set

$$\hat{\mu}(\hat{x},\hat{u}) := \begin{pmatrix} \mu(x,u) \\ 0 \end{pmatrix}, \qquad \hat{\sigma}(\hat{x},\hat{u}) := \begin{pmatrix} \sigma(x,u) \\ \alpha^{\mathrm{T}} \end{pmatrix}.$$

For $(y,q,A) \in \mathbb{R} \times \mathbb{R}^{d+1} \times \mathcal{S}_{d+1}$ and $\hat{u} = (u,\alpha) \in \hat{U}$,

$$\hat{N}^{\hat{u}}(t,\hat{x},y,q) := u - \hat{\sigma}(t,\hat{x},\hat{u})q = N^u(t,x,q_x) - q_p\alpha \quad \text{for} \quad q = (q_x,q_p) \in \mathbb{R}^d \times \mathbb{R},$$

and we assume that

$$u \longmapsto N^u(t,x,q_x) \text{ is one-to-one, with inverse function } \psi(t,x,q_x,.) \qquad (7.38)$$

Then, by a slight extension of Theorem 7.4, the corresponding dynamic programming equation is given by:

$$\begin{aligned} 0 \;=\; -\partial_t \hat{V} + \sup_{\alpha} \Big\{ & b(., \hat{V}, \psi(., D_x\hat{V}, \alpha D_p\hat{V})) - \mu(., \psi(., D_x\hat{V}, \alpha D_p\hat{V})).D_x\hat{V} \\ & -\frac{1}{2}\mathrm{Tr}\left[\sigma(., \psi(., D_x\hat{V}, \alpha D_p\hat{V}))^2 D_x^2\hat{V}\right] \\ & -\frac{1}{2}\alpha^2 D_p^2\hat{V} - \alpha\sigma(., \psi(., D_x\hat{V}, \alpha D_p\hat{V}))D_{xp}\hat{V}\Big\} \end{aligned}$$

### 7.2.3 Application: quantile hedging in the Black-Scholes model

The problem of quantile hedging was solved by Föllmer and Leukert [7] in the general model of asset prices process (non-necessarilly Markov), by means of the Neyman-Pearson lemma from mathematical statistics. The stochastic control approach developed in the present section allows to solve this type of problems in a wider generality. The objective of this section is to recover the explicit solution of [7] in the context of a complete financial market where the underlying risky assets prices are not affected by the control:

$$\mu(x, u) = \mu(x) \text{ and } \sigma(x, u) = \sigma(x) \quad \text{are independent of } u, \tag{7.39}$$

where $\mu$ and $\sigma$ are Lipschitz-continuous, and $\sigma(x)$ is invertible for all $x$.

Notice that we will be only using the supersolution property from the results of the previous sections.

**a- The financial market** The process $X$, representing the price process of $d$ risky assets, is defined by $X_t^{t,x} = x \in (0, \infty)^d$, and

$$dX_s^{t,x} = X_s^{t,x} \star \sigma(X_s^{t,x})\left(\lambda(X_s^{t,x})ds + dW_s\right) \quad \text{where} \quad \lambda := \sigma^{-1}\mu.$$

We assume that the coefficients $\mu$ and $\sigma$ are such that $X^{t,x} \in (0, \infty)^d$ $\mathbb{P}-$a.s. for all initial conditions $(t, x) \in [0, T] \times (0, \infty)^d$. In order to avoid arbitrage, we also assume that $\sigma$ is invertible and that

$$\sup_{x\in(0,\infty)^d} |\lambda(x)| < \infty \quad \text{where} \quad . \tag{7.40}$$

The drift coefficient of the controlled process $Y$ is given by:

$$b(t, x, y, u) = u \cdot \lambda(x). \tag{7.41}$$

The control process $\nu$ is valued in $U = \mathbb{R}^d$, with components $\nu_s^i$ indicating the dollar investment in the $i-$th security at time $s$. After the usual reduction of the interest rates to zero, it follows from the self-financing condition that the liquidation value of the portfolio is given by

$$Y_s^{t,x,y,\nu} \;=\; y + \int_t^s \nu_r \cdot \sigma(X_s^{t,x})\left(\lambda(X_s^{t,x})ds + dW_s\right), \qquad s \geq t\ ,$$

**b- The quantile hedging problem** The quantile hedging problem of the derivative security $g(X_{t,x}(T))$ is defined by the stochastic target problem with controlled probability of success:

$$\hat{V}(t,x,p) := \inf\left\{y \in \mathbb{R}_+ \ : \ \mathbb{P}\left[Y_T^{t,x,y,\nu} \geq g(X_T^{t,x})\right] \geq p \text{ for some } \nu \in \mathcal{U}_0\right\}.$$

We shall assume throughout that $0 \leq g(x) \leq C(1+|x|)$ for all $x \in \mathbb{R}_+^d$. By the usual buy-and-hold hedging strategies, this implies that $0 \leq V(t,x) \leq C(1+|x|)$.

Under the above assumptions, the corresponding super-hedging cost $V(t,x) := \hat{V}(t,x,1)$ is continuous and is given by

$$V(t,x) \quad = \quad \mathbb{E}^{\mathbb{Q}^{t,x}}\left[g(X_T^{t,x})\right],$$

where $\mathbb{Q}^{t,x}$ is the $\mathbb{P}$-equivalent martingale measure defined by

$$\frac{d\mathbb{Q}^{t,x}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_t^T |\lambda(X_s^{t,x})|^2 ds - \int_t^T \lambda(X_s^{t,x})\cdot dW_s\right).$$

In particular, $V$ is a viscosity solution on $[0,T)\times(0,\infty)^d$ of the linear PDE:

$$0 \quad = \quad -\partial_t V - \frac{1}{2}\mathrm{Tr}\left[\sigma^2 D_x^2 V\right]. \tag{7.42}$$

For later use, let us denote by

$$W^{\mathbb{Q}^{t,x}} := W + \int_t^\cdot \lambda(X_s^{t,x})ds, \quad s \in [t,T],$$

the $\mathbb{Q}^{t,x}$-Brownian motion defined on $[t,T]$.

**c- The viscosity supersolution property** By the results of the previous section, we have $\hat{V}_*$ is a viscosity supersolution on $[0,T)\times\mathbb{R}_+^d\times[0,1]$ of the equation

$$0 \leq -\partial_t\hat{V}_* - \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}D_x^2\hat{V}_*\right] - \inf_{\alpha\in\mathbb{R}^d}\left(-\alpha\lambda D_p\hat{V}_* + \mathrm{Tr}\left[\sigma\alpha D_{xp}\hat{V}_*\right] + \frac{1}{2}|\alpha|^2 D_p^2\hat{V}_*\right). \tag{7.43}$$

The boundary conditions at $p=0$ and $p=1$ are immediate:

$$\hat{V}_*(\cdot,1) = V \quad \text{and} \quad \hat{V}_*(\cdot,0) = 0 \text{ on } [0,T]\times\mathbb{R}_+^d. \tag{7.44}$$

We next determine the boundary condition at the terminal time $t=T$.

**Lemma 7.8.** *For all $x \in \mathbb{R}_+^d$ and $p \in [0,1]$, we have $\hat{V}_*(T,x,p) \geq pg(x)$.*

*Proof.* Let $(t_n,x_n,p_n)_n$ be a sequence in $[0,T)\times\mathbb{R}_+^d\times(0,1)$ converging to $(T,x,p)$ with $\hat{V}(t_n,x_n,p_n) \longrightarrow \hat{V}_*(T,x,p)$, and consider $y_n := \hat{V}(t_n,x_n,p_n)+1/n$. By definition of the quantile hedging problem, there is a sequence $(\nu_n, \alpha_n) \in \hat{\mathcal{U}}_0$ such that

$$\mathbf{1}_{\{Y_T^{t_n,x_n,y_n,\nu_n} - g(X_T^{t_n,x_n}) \geq 0\}} \quad \geq \quad P_T^{t_n,p_n,\alpha_n}.$$

This implies that

$$Y_T^{t_n,x_n,y_n,\nu_n} \quad \geq \quad P_T^{t_n,p_n,\alpha_n} g(X_T^{t_n,x_n}).$$

Taking the expectation under $\mathbb{Q}^{t_n,x_n}$, this provides:

$$\begin{aligned} y_n \;\geq\; \mathbb{E}^{\mathbb{Q}^{t_n,x_n}}\left[Y_T^{t_n,x_n,y_n,\nu_n}\right] \quad &\geq \quad \mathbb{E}^{\mathbb{Q}^{t_n,x_n}}\left[P_T^{t_n,p_n,\alpha_n} g(X_T^{t_n,x_n})\right] \\ &= \quad \mathbb{E}\left[L_T^{t_n,x_n} P_T^{t_n,p_n,\alpha_n} g(X_T^{t_n,x_n})\right] \end{aligned}$$

where we denotes $L_T^{t_n,x_n} := \exp\left(-\int_{t_n}^T \lambda(X_s^{t_n,x_n}) \cdot dW_s - \frac{1}{2}\int_{t_n}^T |\lambda(X_s^{t_n,x_n})|^2 ds\right)$. Then

$$\begin{aligned} y_n \quad &\geq \quad \mathbb{E}\left[P_T^{t_n,p_n,\alpha_n} g(x)\right] + \mathbb{E}\left[P_T^{t_n,p_n,\alpha_n}\left(L_T^{t_n,x_n} g(X_T^{t_n,x_n}) - g(x)\right)\right] \\ &= \quad p_n g(x) + \mathbb{E}\left[P_T^{t_n,p_n,\alpha_n}\left(L_T^{t_n,x_n} g(X_T^{t_n,x_n}) - g(x)\right)\right] \\ &\geq \quad p_n g(x) - \mathbb{E}\left[P_T^{t_n,p_n,\alpha_n}\left|L_T^{t_n,x_n} g(X_T^{t_n,x_n}) - g(x)\right|\right], \end{aligned} \tag{7.45}$$

where we used the fact that $P^{t_n,p_n,\alpha_n}$ is a nonnegative martingale. Now, since this process is also bounded by 1, we have

$$\mathbb{E}\left[P_T^{t_n,p_n,\alpha_n}\left|L_T^{t_n,x_n} g(X_T^{t_n,x_n}) - g(x)\right|\right] \quad \leq \quad \mathbb{E}\left[\left|L_T^{t_n,x_n} g(X_T^{t_n,x_n}) - g(x)\right|\right] \;\longrightarrow 0$$

as $n \to \infty$, by the stability properties of the flow and the dominated convergence theorem. Then, by taking limits in (7.45), we obtain that $\hat{V}_*(T, x, p) = \lim_{n\to\infty} y_n \geq pg(x)$, which is the required inequality. ♢

**d- An explicit minorant of $\hat{V}$** The key idea is to introduce the Legendre-Fenchel dual of $V_*$ with respect to the $p-$variable in order to remove the non-linearity in (7.43):

$$v(t, x, q) := \sup_{p\in\mathbb{R}}\left\{pq - \hat{V}_*(t, x, p)\right\}, \quad (t, x, q) \in [0, T] \times (0, \infty)^d \times \mathbb{R}. \tag{7.46}$$

By the definition of the function $\hat{V}$, we have

$$v(., q) = \infty \text{ for } q < 0 \text{ and } v(., q) = \sup_{p\in[0,1]}\left\{pq - \hat{V}_*(., p)\right\} \text{ for } q > 0. \tag{7.47}$$

Using the above supersolution property of $\hat{V}_*$, we shall prove below that $v$ is an upper-semicontinuous viscosity subsolution on $[0, T) \times (0, \infty)^d \times (0, \infty)$ of

$$-\partial_t v - \frac{1}{2}\mathrm{Tr}\left[\sigma^2 D_x^2 v\right] - \frac{1}{2}|\lambda|^2 q^2 D_q^2 v - \mathrm{Tr}\left[\sigma\lambda D_{xq} v\right] \quad \leq \quad 0 \tag{7.48}$$

with the boundary condition

$$v(T,x,q) \le (q-g(x))^+ . \tag{7.49}$$

Since the above equation is linear, we deduce from the comparison result an explicit upper bound for $v$ given by the Feynman-Kac representation result:

$$v(t,x,q) \le \bar v(t,x,q) := \mathbb{E}^{\mathbb{Q}_{t,x}}\left[\left(Q_T^{t,x,q} - g(X_T^{t,x})\right)^+\right], \tag{7.50}$$

on $[0,T]\times(0,\infty)^d\times(0,\infty)$, where the process $Q^{t,x,q}$ is defined by the dynamics

$$\frac{dQ_s^{t,x,q}}{Q_s^{t,x,q}} = \lambda(X_s^{t,x})\cdot dW_s^{\mathbb{Q}^{t,x}} \quad \text{and} \quad Q^{t,x,q}(t) = q \in (0,\infty). \tag{7.51}$$

Given the explicit representation of $\bar v$, we can now provide a lower bound for the primal function $\hat V$ by using (7.47).

We next deduce from (7.50) a lower bound for the quantile hedging problem $\hat V$. Recall that the convex envelop $\hat V_*^{\mathrm{conv}_p}$ of $\hat V_*$ with respect to $p$ is given by the bi-conjugate function:

$$\hat V_*^{\mathrm{conv}_p}(t,x,p) \;=\; \sup_q \left\{pq - v(t;x;q)\right\},$$

and is the largest convex minorant of $\hat V_*$. Then, since $\hat V \ge \hat V_*$, it follows from (7.50) that:

$$\hat V(t,x,p) \;\ge\; \hat V_*(t,x,p) \;\ge\; \sup_q\left\{pq - \bar v(t,x,q)\right\} \tag{7.52}$$

Clearly the function $\bar v$ is convex in $q$ and there is a unique solution $\bar q(t,x,p)$ to the equation

$$\frac{\partial \bar v}{\partial q}(t,x,\bar q) = \mathbb{E}^{\mathbb{Q}_{t,x}}\left[Q_T^{t,x,1}\mathbf{1}_{\left\{Q^{t,x,\bar q}(T)\ge g(X_T^{t,x})\right\}}\right] = \mathbb{P}\left[Q_T^{t,x,\bar q} \ge g(X_T^{t,x})\right] = p, \tag{7.53}$$

where we have used the fact that $d\mathbb{P}/d\mathbb{Q}^{t,x} = Q_T^{t,x,1}$. Then the maximization on the right hand-side of (7.52) can be solved by the first order condition, and therefore:

$$\begin{aligned}
\hat V(t,x,p) \;&\ge\; p\bar q(t,x,p) - \bar v\left(t,x,\bar q(t,x,p)\right)\\
&=\; \bar q(t,x,p)\left(p - \mathbb{E}^{\mathbb{Q}^{t,x}}\left[Q_T^{t,x,1}\mathbf{1}_{\left\{\bar q(t,x,p)Q_T^{t,x,1}\ge g(X_T^{t,x})\right\}}\right]\right)\\
&\quad + \mathbb{E}^{\mathbb{Q}^{t,x}}\left[g(X_T^{t,x})\mathbf{1}_{\left\{\bar q(t,x,p)Q_T^{t,x,1}\ge g(X_T^{t,x})\right\}}\right]\\
&=\; \mathbb{E}^{\mathbb{Q}^{t,x}}\left[g(X_T^{t,x})\mathbf{1}_{\left\{\bar q(t,x,p)Q_T^{t,x,1}\ge g(X_T^{t,x})\right\}}\right] =: y(t,x,p).
\end{aligned}$$

**e- The explicit solution** We finally show that the above explicit minorant $y(t,x,p)$ is equal to $\hat V(t,x,p)$. By the martingale representation theorem, there exists a control process $\nu \in \mathcal{U}_0$ such that

$$Y_T^{t,x,y(t,x,p),\nu} \ge g\left(X_T^{t,x}\right)\mathbf{1}_{\left\{\bar q(t,x,p)Q_T^{t,x,1}\ge g(X_T^{t,x})\right\}}.$$

Since $\mathbb{P}\left[\bar{q}(t,x,p)Q_T^{t,x,1} \geq g(X_T^{t,x})\right] = p$, by (7.53), this implies that $\hat{V}(t,x,p) = y(t,x,p)$.

**f- Proof of** (7.48)**-**(7.49) First note that the fact that $v$ is upper-semicontinuous on $[0,T]\times(0,\infty)^d\times(0,\infty)$ follows from the lower-semicontinuity of $\hat{V}_*$ and the representation in the right-hand side of (7.47), which allows to reduce the computation of the sup to the compact set $[0,1]$. Moreover, the boundary condition (7.49) is an immediate consequence of the right-hand side inequality in (7.44) and (7.47) again.

We now turn to the supersolution property inside the domain. Let $\varphi$ be a smooth function with bounded derivatives and $(t_0,x_0,q_0)\in[0,T)\times(0,\infty)^d\times(0,\infty)$ be a local maximizer of $v-\varphi$ such that $(v-\varphi)(t_0,x_0,q_0)=0$.
(i) We first show that we can reduce to the case where the map $q\mapsto\varphi(\cdot,q)$ is strictly convex. Indeed, since $v$ is convex, we necessarily have $D_{qq}\varphi(t_0,x_0,q_0)\geq 0$. Given $\varepsilon,\eta>0$, we now define $\varphi_{\varepsilon,\eta}$ by $\varphi_{\varepsilon,\eta}(t,x,q) := \varphi(t,x,q)+\varepsilon|q-q_0|^2+\eta|q-q_0|^2(|q-q_0|^2+|t-t_0|^2+|x-x_0|^2)$. Note that $(t_0,x_0,q_0)$ is still a local maximizer of $U-\varphi_{\varepsilon,\eta}$. Since $D_{qq}\varphi(t_0,x_0,q_0)\geq 0$, we have $D_{qq}\varphi_{\varepsilon,\eta}(t_0,x_0,q_0)\geq 2\varepsilon>0$. Since $\varphi$ has bounded derivatives, we can then choose $\eta$ large enough so that $D_{qq}\varphi_{\varepsilon,\eta}>0$. We next observe that, if $\varphi_{\varepsilon,\eta}$ satisfies (7.48) at $(t_0,x_0,q_0)$ for all $\varepsilon>0$, then (7.48) holds for $\varphi$ at this point too. This is due to the fact that the derivatives up to order two of $\varphi_{\varepsilon,\eta}$ at $(t_0,x_0,q_0)$ converge to the corresponding derivatives of $\varphi$ as $\varepsilon\to 0$.
(ii) From now on, we thus assume that the map $q\mapsto\varphi(\cdot,q)$ is strictly convex. Let $\tilde{\varphi}$ be the Fenchel transform of $\varphi$ with respect to $q$, i.e.

$$\tilde{\varphi}(t,x,p) := \sup_{q\in\mathbb{R}}\{pq-\varphi(t,x,q)\}\,.$$

Since $\varphi$ is strictly convex in $q$ and smooth on its domain, $\tilde{\varphi}$ is strictly convex in $p$ and smooth on its domain. Moreover, we have

$$\varphi(t,x,q) = \sup_{p\in\mathbb{R}}\{pq-\tilde{\varphi}(t,x,p)\} = J(t,x,q)q-\tilde{\varphi}(t,x,J(t,x,q))$$

on $(0,T)\times(0,\infty)^d\times(0,\infty)\subset\operatorname{int}(\operatorname{dom}(\varphi))$, where $q\mapsto J(\cdot,q)$ denotes the inverse of $p\mapsto D_p\tilde{\varphi}(\cdot,p)$, recall that $\tilde{\varphi}$ is strictly convex in $p$.

We now deduce from the assumption $q_0>0$ and (7.47) that we can find $p_0\in[0,1]$ such that $v(t_0,x_0,q_0)=p_0q_0-\hat{V}_*(t_0,x_0,p_0)$ which, by using the very definition of $(t_0,x_0,p_0,q_0)$ and $v$, implies that

$$0 = (\hat{V}_*-\tilde{\varphi})(t_0,x_0,p_0) = (\text{local})\min(\hat{V}_*-\tilde{\varphi}) \tag{7.54}$$

and

$$\begin{aligned}\varphi(t_0,x_0,q_0) &= \sup_{p\in\mathbb{R}}\{pq_0-\tilde{\varphi}(t_0,x_0,p)\} &(7.55)\\ &= p_0q_0-\tilde{\varphi}(t_0,x_0,p_0) \text{ with } p_0=J(t_0,x_0,q_0), &(7.56)\end{aligned}$$

where the last equality follows from (7.54) and the strict convexity of the map $p \mapsto pq_0 - \tilde{\varphi}(t_0, x_0, p)$ in the domain of $\tilde{\varphi}$.

We conclude the proof by discussing three alternative cases depending on the value of $p_0$.

- If $p_0 \in (0,1)$, then (7.54) implies that $\tilde{\varphi}$ satisfies (7.43) at $(t_0, x_0, p_0)$ and the required result follows by exploiting the link between the derivatives of $\tilde{\varphi}$ and the derivatives of its $p$-Fenchel transform $\varphi$, which can be deduced from (7.54).

- If $p_0 = 1$, then the first boundary condition in (7.44) and (7.54) imply that $(t_0, x_0)$ is a local minimizer of $\hat{V}_*(\cdot, 1) - \tilde{\varphi}(\cdot, 1) = V - \tilde{\varphi}(\cdot, 1)$ such that $(V - \tilde{\varphi}(\cdot, 1))(t_0, x_0) = 0$. This implies that $\tilde{\varphi}(\cdot, 1)$ satisfies (7.42) at $(t_0, x_0)$ so that $\tilde{\varphi}$ satisfies (7.43) for $\alpha = 0$ at $(t_0, x_0, p_0)$. We can then conclude as in 1. above.

- If $p_0 = 0$, then the second boundary condition in (7.44) and (7.54) imply that $(t_0, x_0)$ is a local minimizer of $\hat{V}_*(\cdot, 0) - \tilde{\varphi}(\cdot, 0) = 0 - \tilde{\varphi}(\cdot, 0)$ such that $0 - \tilde{\varphi}(\cdot, 0)(t_0, x_0) = 0$. In particular, $(t_0, x_0)$ is a local maximum point for $\tilde{\varphi}(\cdot, 0)$ so that $(\partial_t \tilde{\varphi}, D_x \tilde{\varphi})(t_0, x_0, 0) = 0$ and $D_{xx} \tilde{\varphi}(t_0, x_0, 0)$ is negative semi-definite. This implies that $\tilde{\varphi}(\cdot, 0)$ satisfies (7.42) at $(t_0, x_0)$ so that $\tilde{\varphi}$ satisfies (7.43) at $(t_0, x_0, p_0)$, for $\alpha = 0$. We can then argue as in the first case.

$\diamondsuit$

# Chapter 8

# Backward SDEs and Stochastic Control

In this chapter, we introduce the notion of backward stochastic differential equation (BSDE hereafter) which allows to relate standard stochastic control to stochastic target problems.

More importantly, the general theory in this chapter will be developed in the non-Markov framework. The Markovian framework of the previous chapters and the corresponding PDEs will be obtained under a specific construction. From this viewpoint, BSDEs can be viewed as the counterpart of PDEs in the non-Markov framework.

However, by their very nature, BSDEs can only cover the subclass of standard stochastic control problems with uncontrolled diffusion, with corresponding semilinear DPE. Therefore a further extension is needed in order to cover the more general class of fully nonlinear PDEs, as those obtained as the DPE of standard stochastic control problems. This can be achieved by means of the notion of second order BSDEs which are very connected to second order target problems. We refer to Soner, Zhang and Touzi [**?**] for this extension.

## 8.1 Motivation and examples

The first appearance of BSDEs was in the early work of Bismut [**?**] who was extensding the Pontryagin maximum principle of optimality to the stochastic framework. Similar to the deterministic context, this approach introduces the so-called adjoint process defined by a stochastic differential equation combined with a final condition. In the deterministic framework, the existence of a solution to the adjoint equation follows from the usual theory by obvious time inversion. The main difficulty in the stochastic framework is that the adjoint process is required to be adapted to the given filtration, so that one can not simply solve the existence problem by running the time clock backward.

A systematic study of BSDEs was started by Pardoux and Peng [**?**]. The motivation was also from optimal control which was an important field of interest for Shige Peng. However, the natural connections with problems in financial mathematics was very quickly realized, see Elkaroui, Peng and Quenez [**?**]. Therefore, a large development of the theory was achieved in connection with financial applications and crucially driven by the intuition from finance.

### 8.1.1 The stochastic Pontryagin maximum principle

Our objective in this section is to see how the notion of BSDE appears naturally in the context of the Pontryagin maximum principle. Therefore, we are not intending to develop any general theory about this important question, and we will not make any effort in weakening the conditions for the main statement. We will instead considerably simplify the mathematical framework in order for the main ideas to be as transparent as possible.

Consider the stochastic control problem

$$V_0 := \sup_{\nu\in\mathcal{U}_0} J_0(\nu) \quad \text{where} \quad J_0(\nu) := \mathbb{E}\left[g(X_T^\nu)\right],$$

the set of control processes $\mathcal{U}_0$ is defined as in Section 2.1, and the controlled state process is defined by some initial date $X_0$ and the SDE with random coefficients:

$$dX_t^\nu \quad = \quad b(t, X_t^\nu, \nu_t)dt + \sigma(t, X_t^\nu, \nu_t)dW_t.$$

Observe that we are not emphasizing on the time origin and the position of the state variable $X$ at the time origin. This is a major difference between the dynamic programming approach, developed by the American school, and the Pontryagin maximul principle approach of the Russian school.

For every $u \in U$, we define:

$$L^u(t,x,y,z) \quad := \quad b(t,x,u)\cdot y + \mathrm{Tr}\left[\sigma(t,x,u)^{\mathrm{T}} z\right],$$

so that

$$b(t,x,u) = \frac{\partial L^u(t,x,y,z)}{\partial y} \quad \text{and} \quad \sigma(t,x,u) = \frac{\partial L^u(t,x,y,z)}{\partial z}.$$

We also introduce the function

$$\ell(t,x,y,z) \quad := \quad \sup_{u\in U} L^u(t,x,y,z),$$

and we will denote by $\mathbb{H}^2$ the space of all $\mathbb{F}-$progressively measurable processes with finite $\mathbb{L}^2\left([0,T]\times\Omega, dt\otimes d\mathbb{P}\right)-$norm.

**Theorem 8.1.** *Let $\hat\nu \in \mathcal{U}_0$ be such that:*
(i) *there is a solution $(\hat Y, \hat Z)$ in $\mathbb{H}^2$ of the backward stochastic differential equation:*

$$d\hat Y_t = -\nabla_x L^{\hat\nu_t}(t, \hat X_t, \hat Y_t, \hat Z_t)dt + Z_t dW_t, \quad \textit{and} \quad \hat Y_T = \nabla g(\hat X_T), \qquad (8.1)$$

*where* $\hat{X} := X^{\hat{\nu}}$,

(ii) $\hat{\nu}$ *satisfies the maximum principle:*

$$L^{\hat{\nu}_t}(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t) = \ell(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t). \tag{8.2}$$

(iii) *The functions* $g$ *and* $\ell(t, ., y, z)$ *are concave, for fixed* $t, y, z$, *and*

$$\nabla_x L^{\hat{\nu}_t}(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t) = \nabla_x \ell(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t) \tag{8.3}$$

*Then* $V_0 = J_0(\hat{\nu})$, *i.e.* $\hat{\nu}$ *is an optimal control for the problem* $V_0$.

*Proof.* For an arbitrary $\nu \in \mathcal{U}_0$, we compute that

$$\begin{aligned} J_0(\hat{\nu}) - J_0(\nu) &= \mathbb{E}\left[g(\hat{X}_T) - g(X_T^\nu)\right] \\ &\geq \mathbb{E}\left[(\hat{X}_T - X_T^\nu) \cdot \nabla g(\hat{X}_T)\right] \\ &= \mathbb{E}\left[(\hat{X}_T - X_T^\nu) \cdot \hat{Y}_T\right] \end{aligned}$$

by the concavity assumption on $g$. Using the dynamics of $\hat{X}$ and $\hat{Y}$, this provides:

$$\begin{aligned} J_0(\hat{\nu}) - J_0(\nu) &\geq \mathbb{E}\left[\int_0^T d\big\{(\hat{X}_T - X_T^\nu) \cdot \hat{Y}_T\big\}\right] \\ &= \mathbb{E}\Big[\int_0^T \big(b(t, \hat{X}_t, \hat{\nu}_t) - b(t, X_t^\nu, \nu_t)\big) \cdot \hat{Y}_t dt \\ &\qquad -(\hat{X}_t - X_t^\nu) \cdot \nabla_x L^{\hat{\nu}_t}(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t) dt \\ &\qquad +\mathrm{Tr}\left[\big(\sigma(t, \hat{X}_t, \hat{\nu}_t) - \sigma(t, X_t^\nu, \nu_t)\big)^{\mathrm{T}} \hat{Z}_t\right] dt\Big] \\ &= \mathbb{E}\Big[\int_0^T \big(L^{\hat{\nu}_t}(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t) - L^{\nu_t}(t, X_t, \hat{Y}_t, \hat{Z}_t) \\ &\qquad -(\hat{X}_t - X_t^\nu) \cdot \nabla_x L^{\hat{\nu}_t}(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t)\big) dt\Big], \end{aligned}$$

where the diffusion terms have zero expectations because the processes $\hat{Y}$ and $\hat{Z}$ are in $\mathbb{H}^2$. By Conditions (ii) and (iii), this implies that

$$\begin{aligned} J_0(\hat{\nu}) - J_0(\nu) &\geq \mathbb{E}\Big[\int_0^T \big(\ell(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t) - \ell(t, X_t, \hat{Y}_t, \hat{Z}_t) \\ &\qquad -(\hat{X}_t - X_t^\nu) \cdot \nabla_x \ell(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t)\big) dt\Big] \\ &\geq 0 \end{aligned}$$

by the concavity assumption on $\ell$. ♢

Let us comment on the conditions of the previous theorem.

- Condition (ii) provides a feedback definition to $\hat{\nu}$. In particular, $\hat{\nu}_t$ is a function of $(t, \hat{X}_t, \hat{Y}_t, \hat{Z}_t)$. As a consequence, the forward SDE defining $\hat{X}$ depends on the backward component $(\hat{Y}, \hat{Z})$. This is a situation of forward-backward stochastic differential equation which will not be discussed in these notes.

- Condition (8.3) in (iii) is satisfied under natural smoothness conditions. In the economic literature, this is known as the envelope theorem.

- Condition (i) states the existence of a solution to the BSDE (8.1), which will be the main focus of the subsequent section.

### 8.1.2 BSDEs and stochastic target problems

Let us go back to a subclass of the stochastic target problems studied in Chapter 7 defined by taking the state process $X$ independent of the control $Z$ which is assumed to take values in $\mathbb{R}^d$. For simplicity, let $X = W$. Then the stochastic target problem is defined by

$$V_0 \quad := \quad \inf\left\{Y_0 : \; Y_T^Z \geq g(W_T), \; \mathbb{P}-\text{a.s. for some } Z \in \mathbb{H}^2\right\},$$

where the controlled process $Y$ satisfies the dynamics:

$$dY_t^Z \quad = \quad b(t, W_t, Y_t, Z_t)dt + Z_t \cdot dW_t. \tag{8.4}$$

If existence holds for the latter problem, then there would exist a pair $(Y, Z)$ in $\mathbb{H}^2$ such that

$$Y_0 + \int_0^T \left[b(t, W_t, Y_t, Z_t)dt + Z_t \cdot dW_t\right] \quad \geq \quad g(W_T), \quad \mathbb{P}-\text{a.s.}$$

If in addition equality holds in the latter inequality then $(Y, Z)$ is a solution of the BSDE defined by (8.4) and the terminal condition $Y_T = g(W_T)$, $\mathbb{P}-$a.s.

### 8.1.3 BSDEs and finance

In the Black-scholes model, we know that any derivative security can be perfectly hedged. The corresponding superhedging problem reduces to a hedging problem, and an optimal hedging portfolio exists and is determined by the martingale representation theorem.

In fact, this goes beyond the Markov framework to which the stochastic target problems are restricted. To see this, consider a financial market with interest rate process $\{r_t, t \geq 0\}$, and $s$ risky assets with price process defined by

$$dS_t \quad = \quad S_t \star (\mu_t dt + \sigma_t dW_t).$$

Then, under the self-financing condition, the liquidation value of the portfolio is defined by

$$dY_t^\pi \quad = \quad r_t Y_t^\pi dt + \pi_t \sigma_t \left(dW_t + \lambda_t dt\right), \tag{8.5}$$

where the risk premium process $\lambda_t := \sigma_t^{-1}(\mu_t - r_t\mathbf{1})$ is assumed to be well-defined, and the control process $\pi_t$ denotes the vector of holdings amounts in the $d$ risky assets at each point in time.

Now let $G$ be a random variable indicating the random payoff of a contract. $G$ is called a *contingent claim*. The hedging problem of $G$ consists in searching for a portfolio strategy $\hat{\pi}$ such that

$$Y_T^{\hat{\pi}} \quad = \quad G, \quad \mathbb{P}-\text{a.s.} \tag{8.6}$$

We are then reduced to a problem of solving the BSDE (8.5)-(8.6). This problem can be solved very easily if the process $\lambda$ is so that the process $\{W_t + \int_0^t \lambda_s ds, t \geq 0\}$ is a Brownian motion under the so-called equivalent probability measure $\mathbb{Q}$. Under this condition, it suffices to get rid of the linear term in (8.5) by discounting, then $\hat{\pi}$ is obtained by the martingale representation theorem in the present Brownian filtration under the equivalent measure $\mathbb{Q}$.

We finally provide an example where the dependence of $Y$ in the control variable $Z$ is nonlinear. The easiest example is to consider a financial market with different lending and borrowing rates $\underline{r}_t \leq \overline{r}_t$. Then the dynamics of liquidation value of the portfolio (8.6) is replaced by the following SDE:

$$dY_t \quad = \quad \pi_t \cdot \sigma_t(dW_t + \lambda_t dt)(Y_t - \pi_t \cdot \mathbf{1})^+ \underline{r}_t - (Y_t - \pi_t \cdot \mathbf{1})^- \overline{r}_t \tag{8.7}$$

As a consequence of the general results of the subsequent section, we will obtain the existence of a hedging process $\hat{\pi}$ such that the corresponding liquidation value satisfies (8.7) together with the hedging requirement (8.6).

## 8.2 Wellposedness of BSDEs

Throughout this section, we consider a $d-$dimensional Brownian motion $W$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we denote by $\mathbb{F} = \mathbb{F}^W$ the corresponding augmented filtration.

Given two integers $n, d \in \mathbb{N}$, we consider the mapping

$$f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \quad \longrightarrow \quad \mathbb{R},$$

that we assume to be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{n+nd})-$measurable, where $\mathcal{P}$ denotes the $\sigma-$algebra generated by predictable processes. In other words, for every fixed $(y,z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$, the process $\{f_t(y,z), t \in [0,T]\}$ is $\mathbb{F}-$predictable.

Our interest is on the BSDE:

$$dY_t = -f_t(Y_t, Z_t)dt + Z_t dW_t \quad \text{and} \quad Y_T = \xi, \ \mathbb{P}-\text{a.s.} \tag{8.8}$$

where $\xi$ is some given $\mathcal{F}_T-$measurable r.v. with values in $\mathbb{R}^n$.

We will refer to (8.8) as BSDE$(f, \xi)$. The map $f$ is called the *generator*. We may also re-write the BSDE (8.8) in the integrated form:

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \leq T, \quad , \mathbb{P}-\text{a.s.} \tag{8.9}$$

### 8.2.1 Martingale representation for zero generator

When the generator $f \equiv 0$, the BSDE problem reduces to the martingale representation theorem in the present Brownian filtration. More precisely, for every $\xi \in \mathbb{L}^2(\mathbb{R}^n, \mathcal{F}_T)$, there is a unique pair process $(Y, Z)$ in $\mathbb{H}^2(\mathbb{R}^n \times \mathbb{R}^{n\times d})$ satisfying (8.8):

$$\begin{aligned} Y_t := \mathbb{E}[\xi|\mathcal{F}_t] &= \mathbb{E}[\xi] + \int_0^t Z_s dW_s \\ &= \xi - \int_t^T Z_s dW_s. \end{aligned}$$

Here, for a subset $E$ of $\mathbb{R}^k$, $k \in \mathbb{N}$, we denoted by $\mathbb{H}^2(E)$ the collection of all $\mathbb{F}-$progressively measurable $\mathbb{L}^2([0,T]\times\Omega, \mathrm{Leb}\otimes\mathbb{P})-$processes with values in $E$. We shall frequently simply write $\mathbb{H}^2$ keeping the reference to $E$ implicit.

Let us notice that $Y$ is a uniformly integrable martingale. Moreover, by the Doob's maximal inequality, we have:

$$\|Y\|^2_{\mathcal{S}^2} := \mathbb{E}\left[\sup_{t\le T}|Y_t|^2\right] \le 4\mathbb{E}\left[|Y_T|^2\right] = 4\|Z\|^2_{\mathbb{H}^2}. \qquad (8.10)$$

Hence, the process $Y$ is in the space of continuous processes with finite $\mathcal{S}^2-$norm.

For later use, we report the following necessary and sufficient condition for a martingale to be uniformly integrabile.

**Lemma 8.2.** *Let $M = \{M_t, t \in [0,T)\}$ be a scalar local martingale. Then, $M$ is uniformly integrable if and only if*

$$\lim_{\lambda\to\infty} \lambda\mathbb{P}\left[\sup_{t\le T}|M_t| > \lambda\right] = 0.$$

*Proof.* Denote by $\Theta$ the collection of all $\mathbb{F}-$stopping times, and

$$\Theta(M) := \{\theta \in \Theta : M_{.\wedge\theta} \text{ is a martingale}\}.$$

**1.** We first prove that

$$\begin{gathered}\lim_{n\to\infty}\mathbb{E}|M_{\theta_n}| = \sup_{n\ge 1}\mathbb{E}|M_{\theta_n}| = \sup_{\theta\in\Theta(M)}\mathbb{E}|M_\theta| = \sup_{\theta\in\Theta}\mathbb{E}|M_\theta| \\ \text{for all } (\theta_n)_{n\ge1}\subset\Theta(M) \text{ with } \theta_n \longrightarrow \infty,\ \mathbb{P}-\text{a.s.}\end{gathered} \qquad (8.11)$$

To see this, let $(\theta_n)$ be such a sequence, then it follows from Fatou's lemma that

$$\mathbb{E}|M_\theta| \le \liminf_{n\to\infty}\mathbb{E}|M_{\theta\wedge\theta_n}| \le \liminf_{n\to\infty}\mathbb{E}|M_{\theta_n}| \text{ for all } \theta\in\Theta,$$

by the Jensen inequality. Then

$$\begin{aligned}\mathbb{E}|M_\theta| \le \liminf_{n\to\infty}\mathbb{E}|M_{\theta_n}| &\le \limsup_{n\to\infty}\mathbb{E}|M_{\theta_n}| \\ &\le \sup_{n\ge1}\mathbb{E}|M_{\theta_n}| \le \sup_{\theta\in\Theta(M)}\mathbb{E}|M_\theta| \le \sup_{\theta\in\Theta}\mathbb{E}|M_\theta|.\end{aligned}$$

and the required result follows from the arbitrariness of $\theta \in \Theta$.
**2.** For every $\lambda > 0$, the stopping time $T_\lambda := \inf\{t : |M_t| > \lambda\} \in \Theta(M)$, and

$$\mathbb{E}|M_{T_\lambda}| \quad = \quad \lambda \mathbb{P}\left[\sup_{t<T}|M_t| > \lambda\right] + \mathbb{E}\left[|M_\infty|\mathbf{1}_{\{\sup_{t<T}|M_t|\le\lambda\}}\right].$$

Since $\mathbb{E}|M_\infty| \le \liminf_n \mathbb{E}|M_n| = M_0 < \infty$, the second term on the right hand-side convergence to $\mathbb{E}|M_\infty|$ as $\lambda \to \infty$. Since the left hand-side term is non-decreasing in $\lambda$, we deduce that

$$\lim_{\lambda\to\infty} \mathbb{E}|M_{T_\lambda}| = p + \mathbb{E}|M_\infty| \quad \text{where} \quad p := \lim_{\lambda\to\infty} \lambda\mathbb{P}\left[\sup_{t<T}|M_t| > \lambda\right]. \tag{8.12}$$

**3.** Observe that $T_\lambda \in \Theta(M)$ and $T_\lambda \to \infty$ a.s. when $\lambda \to \infty$. Then, it follows from (8.11) and (8.12) that

$$\lim_{\lambda\to\infty} \mathbb{E}|M_{\theta_n}| = p + \mathbb{E}|M_\infty| \quad \text{for all sequence} \quad (\theta_n)_n \text{ satisfying (8.11).}$$

Then $p = 0$ iff $M_{\theta_n} \longrightarrow M_\infty$ in $\mathbb{L}^1$ for all sequence $(\theta_n)_n$ satisfying (8.11), which is now equivalent to the uniform integrability of $M$. ♢

### 8.2.2 BSDEs with affine generator

We next consider a scalr BSDE ($n = 1$) with generator

$$f_t(y, z) \quad := \quad a_t + b_t y + c_t \cdot z, \tag{8.13}$$

where $a, b, c$ are $\mathbb{F}-$progressively measurable processes with values in $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We also assume that $b, c$ are bounded and $\mathbb{E}[\int_0^T |a_t|^2 dt] < \infty$.

This case is easily handled by reducing to the zero generator case. However, it will play a crucial role for the understanding of BSDEs with generator quadratic in $z$, which will be the focus of the next chapter.

First, by introducing the equivalent probability $\mathbb{Q} \sim \mathbb{P}$ defined by the density

$$\frac{d\mathbb{Q}}{\mathbf{d}\mathbb{P}} \quad = \quad \exp\left(\int_0^T c_t \cdot dW_t - \frac{1}{2}\int_0^T |c_t|^2 dt\right),$$

it follows from the Girsanov theorem that the process $B_t := W_t - \int_0^t c_s ds$ defines a Brownian motion under $\mathbb{Q}$. By formulating the BSDE under $\mathbb{Q}$:

$$dY_t \quad = \quad -(a_t + b_t Y_t)dt + Z_t \cdot dB_t,$$

we have reduced to the case where the generator does not depend on $z$. We next get rid of the linear term in $y$ by introducing:

$$\overline{Y}_t := Y_t e^{\int_0^t b_s ds} \quad \text{so that} \quad d\overline{Y}_t = -a_t e^{\int_0^t b_s ds} dt + Z_t e^{\int_0^t b_s ds} dB_t.$$

Finally, defining

$$\overline{\overline{Y}}_t := \overline{Y}_t + \int_0^t a_u e^{\int_0^u b_s ds} du,$$

we arrive at a BSDE with zero generator for $\overline{\overline{Y}}_t$ which can be solved by the martingale representation theorem under the equivalent probability measure $\mathbb{Q}$.

Of course, one can also express the solution under $\mathbb{P}$:

$$Y_t = \mathbb{E}\left[\Gamma_T^t \xi + \int_t^T \Gamma_s^t a_s ds \Big| \mathcal{F}_t\right], \ t \leq T,$$

where

$$\Gamma_s^t := \exp\left(\int_t^s b_u du - \frac{1}{2}\int_t^s |c_u|^2 du + \int_t^s c_u \cdot dW_u\right), \quad 0 \leq t \leq s \leq T. \quad (8.14)$$

### 8.2.3 The main existence and uniqueness result

The following result was proved by Pardoux and Peng [**?**].

**Theorem 8.3.** *Assume that $\{f_t(0,0), t \in [0,T]\} \in \mathbb{H}^2$ and, for some constant $C > 0$,*

$$|f_t(y,z) - f_t(y',z')| \leq C(|y-y'| + |z-z'|), \quad dt \otimes d\mathbb{P} - a.s.$$

*for all $t \in [0,T]$ and $(y,z),(y',z') \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$. Then, for every $\xi \in \mathbb{L}^2$, there is a unique solution $(Y,Z) \in \mathcal{S}^2 \times \mathbb{H}^2$ to the BSDE$(\xi, f)$.*

*Proof.* Denote $S = (Y,Z)$, and introduce the equivalent norm in the corresponding $\mathbb{H}^2$ space:

$$\|S\|_\alpha := \mathbb{E}\left[\int_0^T e^{\alpha t}(|Y_t|^2 + |Z_t|^2) dt\right].$$

where $\alpha$ will be fixed later. We consider the operator

$$\phi : s = (y,z) \in \mathbb{H}^2 \longmapsto S^s = (Y^s, Z^s)$$

defined by:

$$Y_t^s = \xi + \int_t^T f_u(y_u, z_u) du - \int_t^T Z_u^s \cdot dW_u, \quad t \leq T.$$

**1.** First, since $|f_u(y_u,z_u)| \leq |f_u(0,0)| + C(|y_u| + |z_u|)$, we see that the process $\{f_u(y_u,z_u), u \leq T\}$ is in $\mathbb{H}^2$. Then $S^s$ is well-defined by the martingale representation theorem and $S^s = \phi(s) \in \mathbb{H}^2$.

**2.** For $s, s' \in \mathbb{H}^2$, denote $\delta s := s - s'$, $\delta S := S^s - S^{s'}$ and $\delta f := f_t(S^s) - f_t(S^{s'})$. Since $\delta Y_T = 0$, it follows from Itô's formula that:

$$\begin{aligned}
e^{\alpha t}|\delta Y_t|^2 + \int_t^T e^{\alpha u}|\delta Z_u|^2 du &= \int_t^T e^{\alpha u}\left(2\delta Y_u \cdot \delta f_u - \alpha|\delta Y_u|^2\right) du \\
&\quad -2\int_t^T e^{\alpha u}(\delta Z_u)^{\mathrm{T}}\delta Y_u \cdot dW_u.
\end{aligned}$$

In the remaining part of this step, we prove that

$$M_. := \int_0^. e^{\alpha u}(\delta Z_u)^{\mathrm{T}}\delta Y_u \cdot dW_u \quad \text{is a uniformly integrable martingale.} \tag{8.15}$$

so that we deduce from the previous equality that

$$\mathbb{E}\left[e^{\alpha t}|\delta Y_t|^2 + \int_t^T e^{\alpha u}|\delta Z_u|^2 du\right] = \mathbb{E}\left[\int_t^T e^{\alpha u}\left(2\delta Y_u \cdot \delta f_u - \alpha|\delta Y_u|^2\right) du\right]. \tag{8.16}$$

To prove (8.15), we set $V := \sup_{t \le T}|M_t|$, and we verify the condition of Lemma 8.2:

$$\lambda \mathbb{P}[V > \lambda] \;=\; \lambda \mathbb{E}\left[\mathbf{1}_{\{1 < \lambda^{-1}V\}}\right] \;\le\; \mathbb{E}\left[V\mathbf{1}_{\{V > \lambda\}}\right]$$

which converges to zero, provided that $V \in \mathbb{L}^1$. To check that the latter condition hold true, we estimate by the Burkholder-Davis-Gundy inequality that:

$$\begin{aligned}
\mathbb{E}[V] &\le C\mathbb{E}\left[\left(\int_0^T e^{2\alpha u}|\delta Y_u|^2|\delta Z_u|^2 du\right)^{1/2}\right] \\
&\le C'\mathbb{E}\left[\sup_{u \le T}|\delta Y_u|\left(\int_0^T |\delta Z_u|^2 du\right)^{1/2}\right] \\
&\le \frac{C'}{2}\left(\mathbb{E}\left[\sup_{u \le T}|\delta Y_u|^2\right] + \mathbb{E}\left[\int_0^T |\delta Z_u|^2 du\right]\right) < \infty.
\end{aligned}$$

**3.** We now continue estimating (8.16) by using the Lipschitz property of the generator:

$$\begin{aligned}
\mathbb{E}\Big[e^{\alpha t}|\delta Y_t|^2 &+ \int_t^T e^{\alpha u}|\delta Z_u|^2 du\Big] \\
&\le \mathbb{E}\Big[\int_t^T e^{\alpha u}\left(-\alpha|\delta Y_u|^2 + C2|\delta Y_u|(|\delta y_u| + |\delta z_u|)\right) du\Big] \\
&\le \mathbb{E}\left[\int_t^T e^{\alpha u}\left(-\alpha|\delta Y_u|^2 + C\left(\varepsilon^2|\delta Y_u|^2 + \varepsilon^{-2}(|\delta y_u| + |\delta z_u|)^2\right)\right) du\right]
\end{aligned}$$

for any $\varepsilon > 0$. Choosing $C\varepsilon^2 = \alpha$, we obtain:

$$\begin{aligned}\mathbb{E}\left[e^{\alpha t}|\delta Y_t|^2 + \int_t^T e^{\alpha u}|\delta Z_u|^2 du\right] &\leq \mathbb{E}\left[\int_t^T e^{\alpha u}\frac{C^2}{\alpha}(|\delta y_u| + |\delta z_u|)^2 du\right] \\ &\leq 2\frac{C^2}{\alpha}\|\delta s\|_\alpha^2.\end{aligned}$$

This provides

$$\|\delta Z\|_\alpha^2 \leq 2\frac{C^2}{\alpha}\|\delta s\|_\alpha^2 \quad \text{and} \quad \|\delta Y\|_\alpha^2 dt \leq 2\frac{C^2 T}{\alpha}\|\delta s\|_\alpha^2$$

where we abused notatation by writing $\|\delta Y\|_\alpha$ and $\|\delta Z\|_\alpha$ although these processes do not have the dimension required by the definition. Finally, these two estimates imply that

$$\|\delta S\|_\alpha \leq \sqrt{\frac{2C^2}{\alpha}(1+T)}\|\delta s\|_\alpha.$$

By choosing $\alpha > 2(1+T)C^2$, it follows that the map $\phi$ is a contraction on $\mathbb{H}^2$, and that there is a unique fixed point.

**4.** It remain to prove that $Y \in \mathcal{S}^2$. This is easily obtained by first estimating:

$$\mathbb{E}\left[\sup_{t\leq T}|Y_t|^2\right] \leq C\left(|Y_0|^2 + \mathbb{E}\left[\int_0^T |f_t(Y_t,Z_t)|^2 dt\right] + \mathbb{E}\left[\sup_{t\leq T}\left|\int_0^t Z_s\cdot dW_s\right|^2\right]\right),$$

and then using the Lipschitz property of the generator and the Burkholder-Davis-Gundy inequality. ♢

**Remark 8.4.** Consider the Picard iterations:

$$(Y^0, Z^0) = (0,0), \text{ and}$$
$$Y_t^{k+1} = \xi + \int_t^T f_u(Y_u^k, Z_u^k)du + \int_t^T Z_u^{k+1}\cdot dW_u.$$

Then, $S^k = (Y^k, Z^k) \longrightarrow (Y,Z)$ in $\mathbb{H}^2$. Moreover, since

$$\|S^k\|_\alpha \leq \left(\frac{2C^2}{\alpha}(1+T)\right)^k,$$

it follows that $\sum_k \|S^k\|_\alpha < \infty$, and we conclude by the Borel-Cantelli lemma that the convergence $(Y^k, Z^k) \longrightarrow (Y,Z)$ also holds $dt\otimes d\mathbb{P}-$a.s.

## 8.3 Comparison and stability

**Theorem 8.5.** *Let $n = 1$, and let $(Y^i, Z^i)$ be the solution of BSDE$(f^i, \xi^i)$ for some pair $(\xi^i, f^i)$ satisfying the conditions of Theorem 8.3, $i = 0, 1$. Assume that*

$$\xi^1 \geq \xi^0 \quad \textit{and} \quad f_t^1(Y_t^0, Z_t^0) \geq f_t^0(Y_t^0, Z_t^0), \ dt\otimes d\mathbb{P} - a.s. \tag{8.17}$$

*Then* $Y_t^1 \geq Y_t^0$, $t \in [0,T]$, $\mathbb{P}-a.s.$

*Proof.* We denote

$$\delta Y := Y^1 - Y^0,\ \delta Z := Z^1 - Z^0,\ \delta_0 f := f^1(Y^0, Z^0) - f^0(Y^0, Z^0),$$

and we compute that

$$d(\delta Y_t) \quad = \quad -\left(\alpha_t \delta Y_t + \beta_t \cdot \delta Z_t + \delta_0 f_t\right) dt + \delta Z_t \cdot dW_t, \qquad (8.18)$$

where

$$\alpha_t \quad := \quad \frac{f_t^1(Y_t^1, Z_t^1) - f_t^1(Y_t^0, Z_t^1)}{\delta Y_t}\, \mathbf{1}_{\{\delta Y_t \neq 0\}},$$

and, for $j = 1, \ldots, d$,

$$\beta_t^j \quad := \quad \frac{f_t^1\left(Y_t^0, Z_t^1 \oplus_{j-1} Z_t^0\right) - f_t^1\left(Y_t^0, Z_t^1 \oplus_j Z_t^0\right)}{\delta Z_t^{0,j}}\, \mathbf{1}_{\{\delta Z_t^{0,j} \neq 0\}},$$

where $\delta Z^{0,j}$ denotes the $j-$th component of $\delta Z$, and for every $z^0, z^1 \in \mathbb{R}^d$, $z^1 \oplus_j z^0 := \left(z^{1,1}, \ldots, z^{1,j}, z^{0,j+1}, \ldots, z^{0,d}\right)$ for $0 < j < n$, $z^1 \oplus_0 z^0 := z^0$, $z^1 \oplus_n z^0 := z^1$.

Since $f^1$ is Lipschitz-continuous, the processes $\alpha$ and $\beta$ are bounded. Solving the linear BSDE (8.18) as in subsection 8.2.2, we get:

$$\delta Y_t \quad = \quad \mathbb{E}\left[\Gamma_T^t \delta Y_T + \int_t^T \Gamma_u^t \delta_0 f_u du \Big| \mathcal{F}_t\right],\ t \leq T,$$

where the process $\Gamma^t$ is defined as in (8.14) with $(\delta_0 f, \alpha, \beta)$ substituted to $(a, b, c)$. Then Condition (8.17) implies that $\delta Y \geq 0$, $\mathbb{P}-$a.s. $\diamondsuit$

Our next result compares the difference in absolute value between the solutions of the two BSDEs, and provides a bound which depends on the difference between the corresponding final datum and the genrators. In particular, this bound provide a transparent information about the nature of conditions needed to pass to limits with BSDEs.

**Theorem 8.6.** *Let* $(Y^i, Z^i)$ *be the solution of BSDE*$(f^i, \xi^i)$ *for some pair* $(\xi^i, f^i)$ *satisfying the conditions of Theorem 8.3,* $i = 0, 1$*. Then:*

$$\|Y^1 - Y^0\|_{\mathcal{S}^2}^2 + \|Z^1 - Z^0\|_{\mathbb{H}^2}^2 \quad \leq \quad C\left(\|\xi^1 - \xi^0\|_{\mathbb{L}^2}^2 + \|(f^1 - f^0)(Y^0, Z^0)\|_{\mathbb{H}^2}^2\right),$$

*where* $C$ *is a constant depending only on* $T$ *and the Lipschitz constant of* $f^1$*.*

*Proof.* We denote $\delta\xi := \xi^1 - \xi^0$, $\delta Y := Y^1 - Y^0$, $\delta f := f^1(Y^1, Z^1) - f^0(Y^0, Z^0)$, and $\Delta f := f^1 - f^0$. Given a constant $\beta$ to be fixed later, we compute by Itô's formula that:

$$\begin{aligned} e^{\beta t}|\delta Y_t|^2 \quad &= \quad e^{\beta T}|\delta\xi|^2 + \int_t^T e^{\beta u}\left(2\delta Y_u \cdot \delta f_u - |\delta Z_u|^2 - \beta|\delta Y_u|^2\right) du \\ &\quad +2\int_t^T e^{\beta u} \delta Z_u^{\mathrm{T}} \delta Y_u \cdot dW_u. \end{aligned}$$

By the same argument as in the proof of Theorem 8.3, we see that the stochastic integral term has zero expectation. Then

$$e^{\beta t}|\delta Y_t|^2 = \mathbb{E}_t \left[ e^{\beta T}|\delta\xi|^2 + \int_t^T e^{\beta u} \left( 2\delta Y_u \cdot \delta f_u - |\delta Z_u|^2 - \beta |\delta Y_u|^2 \right) du \right], \quad (8.19)$$

where $\mathbb{E}_t := \mathbb{E}[.|\mathcal{F}_t]$. We now estimate that, for any $\varepsilon > 0$:

$$\begin{aligned} 2\delta Y_u \cdot \delta f_u &\leq \varepsilon^{-2}|\delta Y_u|^2 + \varepsilon^2 |\delta f_u|^2 \\ &\leq \varepsilon^{-2}|\delta Y_u|^2 + \varepsilon^2 \left( C(|\delta Y_u| + |\delta Z_u|) + |\Delta f_u(Y_u^0, Z_u^0)| \right)^2 \\ &\leq \varepsilon^{-2}|\delta Y_u|^2 + 3\varepsilon^2 \left( C^2(|\delta Y_u|^2 + |\delta Z_u|^2) + |\Delta f_u(Y_n^0, Z_u^0)|^2 \right). \end{aligned}$$

We then choose $\varepsilon^2 := C^2/6$ and $\beta := \varepsilon^{-2} + 1/2$, and plug the latter estimate in (8.19). This provides:

$$e^{\beta t}|\delta Y_t|^2 + \mathbb{E}_t \left[ \int_t^T |\delta Z_u|^2 du \right] \quad \leq \quad \mathbb{E}_t \left[ e^{\beta T}|\delta\xi|^2 + \frac{C^2}{2} \int_0^T e^{\beta u} |\delta f_u(Y^1, Z_u^0)|^2 du \right],$$

which implies the required inequality by taking the supremum over $t \in [0,T]$ and using the Doob's maximal inequality for the martingale $\{\mathbb{E}_t[e^{\beta T}|\delta\xi|^2], t \leq T\}$.
$\diamondsuit$

## 8.4 BSDEs and stochastic control

We now turn to the question of controlling the solution of a family of BSDEs in the scalr case $n = 1$. Let $(\xi_\nu, f_\nu)_{\nu\in\mathcal{U}}$ be a family of coefficients, where $\mathcal{U}$ is some given set of controls. We assume that the coefficients $(\xi_\nu, f_\nu)_{\nu\in\mathcal{U}}$ satisfy the conditions of the existence and uniqueness theorem 8.3, and we consider the following stochastic control problem:

$$V_0 \quad := \quad \sup_{\nu\in\mathcal{U}} Y_0^\nu, \quad (8.20)$$

where $(Y^\nu, Z^\nu)$ is the solution of BSDE$(\xi^\nu, f^\nu)$.

The above stochastic control problem boils down to the standard control problems of Section 2.1 when the generators $f^\alpha$ are all zero. When the generators $f^\nu$ are affine in $(y,z)$, the problem (8.20) can also be recasted in the standard framework, by discounting and change of measure.

The following easy result shows that the above maximization problem can be solved by maximizing the coefficients $(\xi^\alpha, f^\alpha)$:

$$f_t(y,z) := \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} f_t^\nu(y,z), \qquad \xi := \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \xi^\nu. \quad (8.21)$$

The notion of essential supremum is recalled in the Appendix of this chapter. We will asume that the coefficients $(f,\xi)$ satisfy the conditions of the existence result of Theorem 8.3, and we will denote by $(Y,Z)$ the corresponding solution.

A careful examination of the statement below shows a great similarity with the verification result in stochastic control. In the present non-Markov framework, this remarkable observation shows that the notion of BSDEs allows to mimic the stochastic control methods developed previous chapters in the Markov case.

**Proposition 8.7.** *Assume that the coefficients $(\xi, f)$ and $(\xi_\nu, f_\nu)$ satisfy the conditions of Theorem 8.3, for all $\nu \in \mathcal{U}$. Assume further that there exists some $\hat\nu \in \mathcal{U}$ such that*

$$f_t(y,z) = f^{\hat\nu}(y,z) \quad \text{and} \quad \xi = \xi^{\hat\nu}.$$

*Then $V_0 = Y_0^{\hat\nu}$ and $Y_t = \operatorname{ess\,sup}_{\nu\in\mathcal{U}} Y_t^\nu$, $t\in[0,T]$, $\mathbb{P}-a.s.$*

*Proof.* The $\mathbb{P}-$a.s. inequality $Y \le Y^\nu$, for all $\nu \in \mathcal{U}$, is a direct consequence of the comparison result of Theorem 8.5. Hence $Y_t \le \sup_{\nu\in\mathcal{U}} Y_t^\nu$, $\mathbb{P}-$a.s. To conclude, we notice that $Y$ and $Y^{\hat\nu}$ are two solutions of the same BSDE, and therefore must coincide, by uniqueness. ♢

The next result characterizes the solution of a standard stochastic control problem in terms of a BSDE. Here, again, we emphasize that, in the present non-Markov framework, the BSDE is playing the role of the dynamic programming equation whose scope is restricted to the Markov case.

Let

$$U_0 \quad := \quad \inf_{\nu\in\mathcal{U}} \mathbb{E}^{\mathbb{P}^\nu}\left[\beta^\nu_{0,T}\xi^\nu + \int_0^T \beta^\nu_{u,T}\ell_u(\nu_u)du\right],$$

where

$$\left.\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\right|_{\mathcal{F}_T} := e^{\int_0^T \lambda_t(\nu_t)\cdot dW_t - \frac12\int_0^T|\lambda_t(\nu_t)|^2dt} \quad \text{and} \quad \beta^\nu_{t,T} := e^{-\int_t^T k_u(\nu_u)du}.$$

We assume that all coefficients involved in the above expression satisfy the required conditions for the problem to be well-defined.

We first notice that for every $\nu\in\mathcal{U}$, defining

$$Y_t^\nu \quad := \quad \mathbb{E}^{\mathbb{P}^\nu}\left[\beta^\nu_{t,T}\xi^\nu + \int_t^T \beta^\nu_{u,T}\ell_u(\nu_u)du\middle|\mathcal{F}_t\right]$$

is the first component of the solution $(Y^\nu, Z^\nu)$ of the affine BSDE:

$$dY_t^\nu = -f_t^\nu(Y_t^\nu, Z_t^\nu)dt + Z_t^\nu dW_t, \qquad Y_T^\nu = \xi^\nu$$

with $f_t^\nu(y,z) := \ell_t(\nu_t) - k_t(\nu_t)y + \lambda_t(\nu_t)z$. In view of this observation, the following result is a direct application of Proposition 8.7.

**Proposition 8.8.** *Assume that the coefficients*

$$\xi := \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} \xi^\nu \quad \text{and} \quad f_t(y,z) := \operatorname*{ess\,sup}_{\nu\in\mathcal{U}} f_t^\nu(y,z)$$

*satisfy the conditions of Theorem 8.3, and let $(Y,Z)$ be the corresponding solution. Then $U_0 = Y_0$.*

## 8.5 BSDEs and semilinear PDEs

In this section, we specialize the discussion to the so-called Markov BSDEs in the one-dimensional case $n = 1$. This class of BSDEs corresponds to the case where

$$f_t(y, z) = F(t, X_t, y, z) \quad \text{and} \quad \xi = g(X_T),$$

where $F : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ are measurable, and $X$ is a Markov diffusion process defined by some initial data $X_0$ and the SDE:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{8.22}$$

Here $\mu$ and $\sigma$ are continuous and satisfy the usual Lipschitz and linear growth condtions on order to ensure existence and uniqueness of a strong solution to the SDE SDE-MarkovBSDE, and

$$f, g \text{ have polynomial growth in x}$$
$$\text{and } f \text{ is uniformly Lipschitz in } (y, z).$$

Then, it follows from Theorem 8.3 that the above Markov BSDE has a unique solution.

We next move the time origin by considering the solution $\{X_s^{t,x}, s \geq t\}$ of (8.22) with initial data $X_t^{t,x} = x$. The corresponding solution of the BSDE

$$dY_s = -F(s, X_s^{t,x}, Y_s, Z_s)ds + Z_s dW_s, \quad Y_T = g\left(X_T^{t,x}\right) \tag{8.23}$$

will be denote by $(Y^{t,x}, Z^{t,x})$.

**Proposition 8.9.** *The process $\{(Y_s^{t,x}, Z_s^{t,x}), s \in [t, T]\}$ is adapted to the filtration*

$$\mathcal{F}_s^t := \sigma\left(W_u - W_t, u \in [t, s]\right), \quad s \in [t, T].$$

*In particular, $u(t, x) := Y_t^{t,x}$ is adeterministic function and*

$$Y_s^{t,x} = Y_s^{s, X_s^{t,x}} = u\left(s, X_s^{t,x}\right), \text{ for all } s \in [t, T], \ \mathbb{P} - a.s.$$

*Proof.* The first claim is obvious, and the second one follows from the fact that $X_r^{t,x} = X_r^{s, X_s^{t,x}}$. ♢

**Proposition 8.10.** *Let $u$ be the function defined in Proposition 8.9, and assume that $u \in C^{1,2}([0, T), \mathbb{R}^d)$. Alors*

$$-\partial_t u - \mathcal{A}u - f(., u, \sigma^{\mathrm{T}} Du) = 0 \text{ on } [0, T) \times \mathbb{R}^d.$$

*Proof.* This an easy application of Itô's lemma together with the usual localization technique. ♢

CONCLUDE WITH NONLINEAR FEYNMAC-KAC

## 8.6 Appendix: essential supremum

The notion of essential supremum has been introduced in probability in order to face maximization problem over an infinite family $\mathcal{Z}$. The problem arises when $\mathcal{Z}$ is not countable because then the supremum is not measurable, in general.

**Theorem 8.11.** *Let $\mathcal{Z}$ be a family of r.v. $Z : \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a unique (a.s.) r.v. $\bar{Z} : \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ such that:*
(a) *$\bar{Z} \geq Z$, a.s. for all $Z \in \mathcal{Z}$,*
(b) *For all r.v. $Z'$ satisfying* (a)*, we have $\bar{Z} \leq Z'$, a.s.*
*Moreover, there exists a sequence $(Z_n)_{n\in\mathbb{N}} \subset \mathcal{Z}$ such that $\bar{Z} = \sup_{n\in\mathbb{N}} Z_n$.*
*The r.v. $\bar{Z}$ is called the essential supremum of the family $\mathcal{Z}$, and denoted by* $\operatorname{ess\,sup} \mathcal{Z}$.

*Proof.* The uniqueness of $\bar{Z}$ is an immediate consequence of (b). To prove existence, we consider the set $\mathcal{D}$ of all countable subsets of $\mathcal{Z}$. For all $D \in \mathcal{D}$, we define $Z_D := \sup\{Z : Z \in D\}$, and we introduce the r.v. $\zeta := \sup\{\mathbb{E}[Z_D] : D \in \mathcal{D}\}$.
**1.** We first prove that there exists $D^* \in \mathcal{D}$ such that $\zeta = \mathbb{E}[Z_{D^*}]$. To see this, let $(D_n)_n \subset \mathcal{D}$ be a maximizing sequence, i.e. $\mathbb{E}[Z_{D_n}] \longrightarrow \zeta$, then $D^* := \cup_n D_n \in \mathcal{D}$ satisfies $\mathbb{E}[Z_{D^*}] = \zeta$. We denote $\bar{Z} := Z_{D^*}$.
**2.** It is clear that the r.v. $\bar{Z}$ satisfies (b). To prove that property (a) holods true, we consider an arbitrary $Z \in \mathcal{Z}$ together with the countable family $D := D^* \cup \{Z\} \subset \mathcal{D}$. Then $Z_D = Z \vee \bar{Z}$, and $\zeta = \mathbb{E}[\bar{Z}] \leq \mathbb{E}[Z \vee \bar{Z}] \leq \zeta$. Consequently, $Z \vee \bar{Z} = \bar{Z}$, and $Z \leq \bar{Z}$, a.s. ♢

# Bibliography


[1] G. Barles. *Solutions de Viscosité des Équations de Hamilton-Jacobi*. Mathématiques & Applications. Springer-Verlag, 1994.

[2] P. Cheridito, M. Soner, and N. Touzi. Small time path behavior of double stochastic integrals, and application to stochastic control. *Annals of Applied Probability*, 15:2472–2495, 2005.

[3] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27:1–67, 1992.

[4] J. Dugundji. *Topology*. Allyn and Bacon series in Advanced Mathematics, Allyn and Bacon edt.

[5] N. ElKaroui and M. Jeanblanc-Picqué. Contrôle de processus de markov. *Séminaire de Probabilités Springer Lecture Notes in Math. 1321,*, XXII:508–541, 1988.

[6] W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, 1993.

[7] H. Foellmer and P. Leukert. Quantile hedging. *Finance and Stochastics*, 3:251–273.

[8] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, New-York Second Edition, 1991.

[9] I. Karatzas and S. Shreve. *Methods of Mathematical Finance*. Springer-Verlag, New-York Second Edition, 1999.

[10] R. C. Merton. Lifetime portfolio selection under uncertainty: the continuous-time model. *Review of Economic Statistics*, 51:247–257, 1969.

[11] R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3:373–413, 1971.

[12] P. J. Reny. On the existence of pure and mixed strategy nash equilibria in discontinuous games. *Econometrica*, 67:1029–1056, 1999.

[13] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.