DETERMINISTIC AND STOCHASTIC CONTROL,

APPLICATION TO FINANCE

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Chapter 1

Overview of static optimization

1.1 Definitions

Let E be a normed vector space, possibly infinite-dimensional, and consider the minimization problem:

$$\phi \;\; := \;\; \inf_{x \in K} \; \varphi(x), \tag{1.1}$$

where $x \in E$ is the *control* variable, $\varphi : E \longrightarrow \mathbb{R}$ is the *objective function*, and $K \subset E$ is the set of *admissible decisions.*

The objective of this section is to provide a quick review of the following issues:

a. The first question addresses the existence of a solution:

$$\exists \; ? \; x^* \in K \;\; : \qquad \phi \; := \; \varphi(x^*). \tag{1.2}$$

If the answer to this question is positive, then the issue of uniqueness arises.

b. If existence holds for the problem (1.1), then one can derive the (Euler) first order conditions. In the unconstrained context ($K = E$), the necessary condition reduces to the standard Lagrange zero gradient at any minimizer.

If the set of admissible decisions K is defined by a finite number of equality and inequality constraints, then the classical duality methods lead to the Kuhn and Tucker first order conditions, which reduces to the Lagrange theorem in the absence of constraints.

Notice that the necessary conditions can be used to solve the existence problem in (1.2). Indeed, there are two alternative cases:

- Either the first order conditions is not satisfied by any point in K, then the problem (1.1) has no solution in K.

- Or the first order conditions allow to isolate a nonempty set of points in K, then one needs to turn to sufficient conditions of optimality to identify those which indeed correspond to the minimum.

Before proceeding to the precise statement of our results, we recall some definitions and classical results:

- A subset K of a metric space is *compact* if and only if every sequence of elements of K contains a converging subsequence.

- A scalar function φ defined on a metric space is said to be *lower semi-continuous* if and only if $\liminf_{x' \to x} \varphi(x') \geq \varphi(x)$ for all $x \in E$.

- A function $\varphi : E \longrightarrow \mathbb{R}$ is *differentiable* at a point $x \in E$ if there exists a linear mapping $D\varphi(x) : E \longrightarrow \mathbb{R}$ such that

$$\varphi(x+h) \quad = \quad \varphi(x) + D\varphi(x) \cdot h + |h| \varepsilon(h),$$

where $\varepsilon : E \longrightarrow \mathbb{R}$ is a function converging to zero as $|h| \to 0$. When E has finite dimension, $D\varphi(x)$ is the gradient of φ at x, and we have :

$$D\varphi(x) \cdot h \quad = \quad \sum_i \frac{\partial \varphi}{\partial x_i} \, h_i,$$

where $(\partial\varphi/\partial x_i)(x)$ is the partial derivative at the point x with respect to the variable x_i.

- A subset K of a vector space is said to be *convex* if $\lambda x + (1-\lambda) y \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$.

- Let K be a convex set. A function $\varphi : K \longrightarrow \mathbb{R}$ is said to be *convex* (resp. *strictly convex*) if $\varphi(\lambda x + (1-\lambda) y) \leq$ (resp. $<$) $\lambda \varphi(x) + (1-\lambda)\varphi(y)$ for all , $x, y \in K$ and $\lambda \in [0,1]$.

1.2 Existence results

The main existence result is the following.

Theorem 1.1. *Suppose that the function φ is lower semicontinuous and K is compact. Then there exists a solution to the minimization problem* (1.1)*, i.e. $\phi = \varphi(x^*)$ for some $x^* \in K$.*

Proof. Let $(x_n)_{n \geq 0}$ be a minimizing sequence for the problem (1.1), i.e.

$$x_n \in K \quad \text{for all } n \text{ and} \quad \varphi(x_n) \longrightarrow \phi.$$

Since K is compact, there exists a subsequence $(x_{n_k})_{k \geq 0}$ which converges towards some $x^* \in K$. Clearly $\varphi(x^*) \geq \phi$. The converse inequality is obtained by using the lower semicontinuity of φ:

$$\phi \quad = \quad \lim_{n \to \infty} \varphi(x_n) \;=\; \lim_{k \to \infty} \varphi(x_{n_k}) \;\geq\; \liminf_{x' \to x^*} \varphi(x') \;\geq\; \varphi(x^*).$$

♢

In the finite dimensional context, compact subsets reduce to bounded closed sets, and the above theorem is frequently used in the following form.

Corollary 1.2. *Let E be a finite dimensional vector space, and K a closed subset of E. Let $\varphi : E \longrightarrow \mathbb{R}$ be a lower semicontinuous function satisfying:*

$$\lim_{|x|\to\infty} \varphi(x) + \chi_K(x) \;=\; +\infty \quad \textit{where} \quad \chi_K(x) \;=\; \begin{cases} 0 \textit{ if } x \in K \\ +\infty \textit{ otherwise.} \end{cases} \tag{1.3}$$

Then the minimization problem (1.1) *has a solution, i.e. $\phi = \varphi(x^*)$ for some $x^* \in K$.*

Proof. Let $(x_n)_{n\geq 0}$ be a minimizing sequence for the problem (1.1), i.e. $\phi = \lim_{n\to\infty}\varphi(x_n)$.
1. We first prove that the sequence $(x_n)_n$ is bounded. If not, then we may extract a subsequence $(x_{n_k})_k$ of elements of K which tends to infinity, and it follows from Condition (1.3) that

$$\phi \;=\; \lim_{k\to\infty} \varphi(x_{n_k}) \;\;=\;\; +\infty,$$

which can not happen.
2. Let $M := \sup_{n\geq 0}|x_n|$, and $K_M := K \cap \{x \in E \;:\; |x| \leq M\}$. Since E has finite dimension and K is closed, the set K_M is compact. Moreover, since $x_n \in K_M$ for all n, we deduce that $\phi = \inf_{x\in K_M}\varphi(x)$. The existence result now follows by direct application of Theorem 1.1. ♢

We conclude this section by the following sufficient condition of uniqueness.

Theorem 1.3. *Assume that K is convex and the function φ is strictly convex. Then the minimization problem* (1.1) *has at most one solution.*

Proof. Let x_1^* and x_2^* be two solutions of the minimization problem (1.1), and let $x^* := (x_1^* + x_2^*)/2$. Since K is convex, $x^* \in K$ and it follows from the strict convexity of φ that $\varphi(x^*) < [\varphi(x_1^*) + \varphi(x_2^*)]/2 = \phi$, contradicting the definition of ϕ. ♢

1.3 Euler necessary condition of optimality

In the unconstrained framework ($K = E$), it is well-known that the objective function must have a zero gradient $D\varphi = 0$ at any extremal point. Notice that this result can be extended to the nonsmooth case, but we will not persue in this direction. In order to account for the restriction of the admissible decision to K, we introduce the following notion.

Definition 1.4. *Let $K \subset E$, and $x \in K$. An element $y \in E$ is said to be tangent to K at the point x (and we denote $y \in T(x, K)$) if there exists a sequence $(h_n)_n$ of non-zero elements of E and a sequence of positive scalars $(\lambda_n)_n$ such that*

$$h_n \;\longrightarrow\; y\;, \quad \lambda_n \;\longrightarrow\; 0\;, \quad \textit{and} \quad x + \lambda_n h_n \;\in\; K.$$

We call $T(K, x)$ the tangent cone to the set K at the point x.

It is easy to check that $T(K, x)$ is a closed convex cone. Apart from its technical aspect, this definition expresses that $T(K, x)$ is the collection of all directions "which point inward to K at the point x". When K is a convex set, we may express this definition in a more natural way...

Theorem 1.5. (Euler condition) *Let* x^* *be a solution of the problem* (1.1). *If* φ *is differentiable at* x^**, then*

$$D\varphi(x^*) \cdot y \;\geq\; 0 \quad \textit{for all} \quad y \in T(K, x). \tag{1.4}$$

Proof. Let $y \in T(K, x^*)$. Since the result is trivial for $y = 0$, we focus on the case $y \neq 0$. By definition of the tangent, there exist sequences $(h_n)_n \subset E$ and $(\lambda_n)_n \subset (0, \infty)$ such that

$$h_n \longrightarrow y, \quad \lambda_n \longrightarrow 0, \quad \text{and} \quad x^* + \lambda_n h_n \;\in\; K.$$

Since x^* is a solution of (1.1), we have

$$|\lambda_n h_n|^{-1} \left[\varphi(x^* + \lambda_n h_n) - \varphi(x^*)\right] \quad \geq \quad 0.$$

Sending n to infinity, it follows from the differentiability of φ at the point x^* that:

$$D\varphi(x^*) \cdot |y|^{-1} y \quad \geq \quad 0.$$

♢

If K is an open subset, then it is easily seen that $T(K, x) = E$ for all $x \in K$, implying the well-known following consequence.

Corollary 1.6. *Let* K *be an open subset of* E*, and* x^* *be a solution of the problem* (1.1). *If* φ *is differentiable at* x^**, then*

$$D\varphi(x^*) \quad = \quad 0.$$

1.4 Sufficient optimality conditions

We say that x^* is a point of *local minimum* for the problem (1.1) if:

$$\varphi(x^*) \;=\; \min_{x \in K \cap B(x^*, r)} \varphi(x) \quad \text{for some} \quad r > 0,$$

where $B(x, r)$ is the open ball centered at x with radius r. we say that x is a point of *global minimum* for the problem (1.1) if $\phi = \varphi(x^*)$.

We now write the sufficient optimality conditions in the context where the objective function φ is twice differentiable at the point x^*. Recall that this means that there exists a bilinear symmetric map $D^2\varphi(x^*) : E \times E \longrightarrow \mathbb{R}$ such that

$$\varphi(x^* + h) \quad = \quad \varphi(x^*) + D\varphi(x^*) \cdot h + \frac{1}{2} D^2\varphi(x^*)(h, h) + |h|^2 \varepsilon(h), \tag{1.5}$$

where $\varepsilon(h) \longrightarrow 0$ as $h \to 0$. If E has a finite dimension d, then:

$$D^2\varphi(x^*)(h,h) \quad = \quad \sum_{i,j=1}^{d} \frac{\partial^2\varphi}{\partial x_i \partial x_j}(x^*)\ h_i h_j.$$

We say that x is a *regular* point for φ if φ is twice differentiable at x. The next result is a direct consequence of (1.5).

Theorem 1.7. *Let $x^* \in K$ be a regular point for φ satisfying the Euler condition* (1.4)*. Suppose that:*

$$D^2\varphi(x^*)(h,h) \ > \ 0 \quad \textit{for all} \quad h \in T(K,x^*),\ h \neq 0.$$

The x^* is a point of local minimium for the problem (1.1)*.*

Corollary 1.8. *Let K and φ be convex. Then any point x^* of differentiability of φ satisfying the Euler condition* (1.4) *is a global solution of the problem* (1.1)*.*

1.5 Equality and inequality constraints

Throughout this section, E is a vector space with finite dimension d, and the set of admissible decisions K is defined by a finite number p of inequality constraints:

$$K \quad = \quad \{x \in E \ : \ b_j(x) \leq 0,\ 1 \leq j \leq p \text{ for all } x \in E\}. \tag{1.6}$$

Here, $b : E \longrightarrow \mathbb{R}^p$ is a given function with components $b_i : E \longrightarrow \mathbb{R}$. Notice that this context includes equality constraints.

In order to apply the Euler condition of Theorem 1.5, we need to determine the tagent cone tangent $T(K,x)$ at any point $x \in K$. To do this, we need to isolated the binding constraints at $x \in K$:

$$J(x) \quad := \quad \{j = 1, \ldots, p \ : \ b_j(x) = 0\}.$$

Then, it is easily shown that whenever b_j is continuously differentiable at x for all $j \in J(x)$:

$$T(K,x) \quad \subset \quad \{y \in \mathbb{R}^n \ : \ y \cdot Db_j(x) \leq 0 \text{ for all } j \in J(x)\}.$$

However, this inclusion is strict, in general. The next result provides a sufficient condition for the equality between these two cones to hold true. Denote

$$J^{conc}(x) \quad := \quad \{j \in J(x) \ : \ b_j \text{ is concave in the neighborhood of } x\}.$$

Proposition 1.9. *Let b be continuous at the point x, and b_j differentiable at x for all $j \in J(x)$. Assume further that there exists $y \in \mathbb{R}^n$ such that*

$$y \cdot Db_j(x) \ \leq \ 0,\ j \in J^{conc}(x) \quad \textit{and} \quad y \cdot Db_j(x) \ < \ 0,\ j \in J(x) \setminus J^{conc}(x).$$

Then $T(K,x) = \{y \in \mathbb{R}^n \ : \ y \cdot Db_j(x) \leq 0$ for all $j \in J(x)\}$.

The proof is left to the reader.

Definition 1.10. *A point $x \in K$ is said to be $K-$qualified if the function b is differentiable at x and*

$$T(K,x) \;=\; \left\{y \in \mathbb{R}^n \;:\; y \cdot Db_j(x) \le 0 \text{ for all } j \in J(x)\right\}.$$

We next introduce the function $L : E \times \mathbb{R}^p \longrightarrow \mathbb{R}$ defined by :

$$L(x,\lambda) \;:=\; \varphi(x) + \lambda \cdot b(x).$$

We call L the *Lagrangian* function associated to the minimization problem (1.1) with inequality constraints (1.6). The scalars $\lambda_1, \ldots, \lambda_p$ are called *Lagrange multipliers*. The following result provides an explicit statement of the Euler condition of Theorem 1.5.

Theorem 1.11. (Kuhn and Tucker) *Let K be defined by the inequality constraints* (1.6)*, and $x^* \in K$ a minimizer of the problem* (1.1)*. Assume that φ is differentiable at x^*, and x^* is a $K-$qualified point. Then, there exists $\lambda \in \mathbb{R}^p_+$ such that:*

$$DL(x^*,\lambda) \;=\; 0 \quad \text{and} \quad \lambda_j\, b_j(x^*) = 0,\; 1 \le j \le p.$$

Proof. Let x^* be a $K-$qualified solution of the minimization problem (1.1). Since φ is differentiable at x^*, it follows from the Euler condition of Theorem 1.5 that:

$$D\varphi(x^*) \cdot y \;\ge\; 0 \quad \text{for all} \quad y \in T(K,x^*).$$

Use the expression of $T(K,x^*)$ for the $K-$qualified point x^*, this provides:

$$\text{for all } y \in \mathbb{R}^n\;, \quad \left\{ \max_{j \in J(x^*)} Db_j(x^*) \cdot y \;\le\; 0 \quad \Longrightarrow \quad -D\varphi(x^*) \cdot y \;\le\; 0 \right\}.$$

By the Farkas separation lemma, this property is equivalent to

$$-D\varphi(x^*) \;=\; \sum_{j \in J(x^*)} \lambda_j Db_j(x^*) \quad \text{for some} \quad \lambda_j \ge 0,\; j \in J(x^*).$$

This defines the multipliers λ_j for all binding constraints $j \in J(x^*)$. For the remaining constraints, we simply set

$$\lambda_j \;:=\; 0 \quad \text{for all} \quad j \notin J(x^*)\;,$$

and the proof of the theorem is complete. ♢

We conclude this section by stating the Lagrange Theorem which provides the first order conditions for the minimization problem (1.1) when the set K is defined by equality constraints:

$$K \;=\; \left\{x \in E \;:\; a_j(x) = 0,\; 1 \le j \le m \text{ for all } x \in E\right\}. \tag{1.7}$$

Similar to the case of inequality constraints, if a_j is differentiable at x for all $j = 1, \ldots, m$, then

$$T(K, x) \quad \subset \quad \{y \in E \; : \; y \cdot Da_j(x) = 0, \; 1 \le j \le m\},$$

and x is called a $K-$qualified point if the above inclusion is an equality.

Proposition 1.12. *Let $K \subset E$ be defined by* (1.7)*, and $x \in K$. Assume that*

(i) *a_j is continuously differentiable in the neighborhood of x for all $j = 1, \ldots, m$,*

(ii) *The family of gradients $\{Da_j(x), \; 1 \le j \le m\}$ is a linearly independent system.*

Then x is a $K-$qualified point.

We refer to Moulin et Fogelman-Soulié [**?**] for the proof of this result. In the context of equality constraints, the Kuhn and Tucker theorem reduces to

Theorem 1.13. (Lagrange) *Let the set of admissible decisions be defined by* (1.7)*, and let x^* be a solution of the minimization problem* (1.1)*. Assume that φ is differentiable at x^*, and x^* is a $K-$qualified point. Then:*

$$DL(x^*, \lambda^*) \; = \; 0 \quad \textit{for some} \quad \lambda^* \in \mathbb{R}^m.$$

Chapter 2

Caculus of variations

2.1 Problem formulation

Let $t_0 < t_1 \in \mathbb{R}$ and $F : [t_0, t_1] \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ C^1-function. In this chapter, we study a first class of dynamic optimization problems:

$$\phi \quad := \quad \inf_{\substack{x \in C^1([t_0,t_1],\mathbb{R}^n) \\ x(t_0) = x_0 \\ x(t_1) - x_1 \in C(I,J,K)}} \varphi(x), \tag{2.1}$$

where

$$\varphi(x) \quad := \quad \int_{t_0}^{t_1} F\left(t, x(t), \dot{x}(t)\right) dt.$$

Here, $x_0, x_1 \in \mathbb{R}^n$ are given, I, J and K are disjoint subsets of indices in $\{1, \ldots, n\}$, and

$$C(I,J,K) \quad := \quad \left\{\xi \in \mathbb{R}^n \ : \ \xi^i \le 0, \ \xi^j \ge 0 \text{ and } \xi^k = 0 \text{ for } (i,j,k) \in I \times J \times K\right\}.$$

The case of an unconstrained terminal value of the state variable is obtained by setting $I = J = K = \emptyset$.

Remark 2.1. Let $G : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^1-function, and consider the objective function with an additional terminal cost:

$$\tilde{\varphi}(x) \quad := \quad \int_{t_0}^{t_1} F\left(t, x(t), \dot{x}(t)\right) dt + G\left(x(t_1)\right),$$

together with the minimization problem

$$\tilde{\phi} \quad := \quad \inf_{\substack{x \in C^1([t_0,t_1],\mathbb{R}^n) \\ x(t_0) = x_0 \\ x(t_1) - x_1 \in C(I,J,K)}} \tilde{\varphi}(x). \tag{2.2}$$

Then, introducing

$$\tilde{F}(t,\xi,v) \quad := \quad F(t,\xi,v) + DG(\xi)\cdot v,$$

we may re-write $\tilde{\varphi}$ as

$$\tilde{\varphi}(x) \quad := \quad \int_{t_0}^{t_1} \tilde{F}\left(t,x(t),\dot{x}(t)\right)dt + G\left(x(t_0)\right),$$

thus reducing the problem (2.2) to the class (2.1) where no terminal cost is involved.

2.2 Necessary conditions of optimality: the equality constraint case

The objective of this section is to prove the local Euler equation for the problem of calculus of variations (2.1) which is the first order condition that any extremum (if it exists) must satisfy.

Theorem 2.2. (local Euler equation) *In the context $I = J = \emptyset$ and $K = \{1,\ldots,n\}$, let $x^* \in C^1\left([t_0,t_1],\mathbb{R}^n\right)$ be a solution of the problem* (2.1). *Then, the function*

$$t \quad \longmapsto \quad F_v\left(t,x^*(t),\dot{x}^*(t)\right)$$

is C^1 on $[t_0,t_1]$ and its differential is given by

$$\frac{d}{dt}F_v\left(t,x^*(t),\dot{x}^*(t)\right) \;=\; F_x\left(t,x^*(t),\dot{x}^*(t)\right) \quad \textit{for all} \quad t\in[t_0,t_1].$$

Proof. Let h be an arbitrary $C^1\left([t_0,t_1],\mathbb{R}^n\right)-$function with $h(t_0) = h(t_1) = 0$. For any scalar $\varepsilon\in\mathbb{R}$, we consider the variation of x^* in the $h-$direction:

$$x^\varepsilon \quad := \quad x^* + \varepsilon h.$$

Since $x^\varepsilon \in C^1\left([t_0,t_1],\mathbb{R}^n\right)$ and $x^\varepsilon(t_0) = x_0$, $x^\varepsilon(t_1) = x_1$, it follows from the optimality of x^* that $\varepsilon = 0$ is a point of minimum of the function $\varepsilon \longmapsto \varphi(x^\varepsilon)$. It is easy to see that this function is differentiable with respect to ε (dominated convergence). Then a necessary condition for the latter optimality is that:

$$\frac{\partial}{\partial\varepsilon}\varphi(x^\varepsilon)\Big|_{\varepsilon=0} \;=\; \frac{\partial}{\partial\varepsilon}\int_{t_0}^{t_1} F\left(t,x^*(t)+\varepsilon h(t),\dot{x}^*(t)+\varepsilon\dot{h}(t)\right)dt\Big|_{\varepsilon=0} \;=\; 0.$$

This leads to

$$\int_{t_0}^{t_1} F_x\left(t,x^*(t),\dot{x}^*(t)\right)\cdot h(t)dt + \int_{t_0}^{t_1} F_v\left(t,x^*(t),\dot{x}^*(t)\right)\cdot\dot{h}(t)dt = 0. \qquad (2.3)$$

Let H be the anti-derivative of the function $t \longmapsto F_x\left(t, x^*(t), \dot{x}^*(t)\right)$

$$H(t) \;:=\; c + \int_{t_0}^{t} F_x\left(t, x^*(t), \dot{x}^*(t)\right) dt,$$

where the constant vector $c \in \mathbb{R}^n$ is chosen so that

$$\int_{t_0}^{t_1} H(t)dt \;=\; \int_{t_0}^{t_1} F_v\left(t, x^*(t), \dot{x}^*(t)\right) dt \tag{2.4}$$

Integrating by part the first integral in (2.3), and recalling that $h(t_0) = h(t_1) = 0$, we obtain:

$$\int_{t_0}^{t_1} \left\{-H(t) + F_v\left(t, x^*(t), \dot{x}^*(t)\right)\right\} \cdot \dot{h}(t)dt \;=\; 0. \tag{2.5}$$

We now observe that the function

$$\bar{h}(t) \;:=\; \int_{t_0}^{t} \left\{-H(s) + F_v\left(s, x^*(s), \dot{x}^*(s)\right)\right\} ds$$

is $C^1\left([t_0, t_1], \mathbb{R}^n\right)$ and satisfies $\bar{h}(t_0) = \bar{h}(t_1) = 0$ by (2.4). Then, substituting this function in (2.5), we see that:

$$\int_{t_0}^{t_1} \left|-H(t) + F_v\left(t, x^*(t), \dot{x}^*(t)\right)\right|^2 dt \;=\; 0.$$

By the continuity of the functions H, F_v and $\dot{x}^*$, this shows that:

$$H(t) \;=\; F_v\left(t, x^*(t), \dot{x}^*(t)\right) \quad \text{for all} \quad t \in [t_0, t_1].$$

Since H is C^1, the proof is completed by differentiating the latter expression.
♢

Remark 2.3. Consider the problem of calculus of variations with terminal cost (2.2). Then the local Euler equation is:

$$\frac{d}{dt}\tilde{F}_v\left(t, x^*(t), \dot{x}^*(t)\right) \;=\; \tilde{F}_x\left(t, x^*(t), \dot{x}^*(t)\right) \quad \text{for all} \quad t \in [t_0, t_1].$$

Since $\tilde{F}(t, \xi, v) = F(t, \xi, v) + G'(\xi) \cdot v$, this provides:

$$\frac{d}{dt}F_v\left(t, x^*(t), \dot{x}^*(t)\right) \;=\; F_x\left(t, x^*(t), \dot{x}^*(t)\right) \quad \text{for all} \quad t \in [t_0, t_1].$$

Hence, the local Euler equation for the problem (2.2) does not involve the terminal cost function G.

We next extend the previous result to the case where the minimization is performed over the larger set of piecewise C^1-functions.

Definition 2.4. *A function* $x\ :[t_0,t_1]\longrightarrow\mathbb{R}^n$ *is said to be piecewise continuously differentiable, denoted as* $x\in C^1_{\rm pm}\left([t_0,t_1],\mathbb{R}^n\right)$*, if there exists a partition* $t_0=s_0<\ldots<s_m=t_1$ *of the interval* $[t_0,t_1]$ *such that*
- $x\in C^0\left([t_0,t_1],\mathbb{R}^n\right)$,
- $x\in C^1\left(]s_{i-1},s_i[,\mathbb{R}^n\right)$ *for all* $i=1,\ldots,m$,
- $\dot{x}$ *has right and left limits at the endpoints* s_i *for all* $i=0,\ldots,m$.

The relaxed minimization problem is defined by:

$$\phi_{\rm pm}\ :=\ \inf_{\substack{x\in C^1_{\rm pm}\left([t_0,t_1],\mathbb{R}^n\right)\\ x(t_0)=x_0\\ x(t_1)=x_1}}\ \varphi(x) \tag{2.6}$$

where

$$\varphi(x)\ :=\ \int_{t_0}^{t_1}F\left(t,x(t),\dot{x}(t)\right)dt.$$

The following example illustrates a case where the relaxation of the minimization problem to the larger set $C^1_{\rm pm}\left([t_0,t_1],\mathbb{R}^n\right)$ is relevant.

Example 2.5. Consider the objective function

$$\varphi(x):=\int_{-1}^{1}x(t)^2[1-\dot{x}(t)]^2dt,$$

and let $x_0=0$, $x_1=1$.

Clearly there exists a solution $x^*\in C^1_{\rm pm}\left([-1,1],\mathbb{R}\right)$ for the problem (2.6) given by

$$x^*(t)\ :=\ t^+\ =\ \max\{0,t\}\quad\text{for all}\quad t\in[-1,1],$$

inducing a zero minimum $\phi_{\rm pm}=0$. Notice that the value function of the minimization problem (2.1) on $C^1\left([-1,1],\mathbb{R}\right)$ is also zero, as one can find an approximating sequence $(x_n)_n\subset C^1\left([-1,1],\mathbb{R}\right)$ of x^* such that $\varphi(x_n)\longrightarrow 0$. However the problem has no solution in $C^1\left([-1,1],\mathbb{R}\right)$.

Reviewing the previous proof, we see that the following extension of the local Euler equation holds true.

Theorem 2.6. (Integral Euler equation) *Let* $x^*\in C^1_{\rm pm}\left([t_0,t_1],\mathbb{R}^n\right)$ *be a solution of the problem* (2.6)*. then, there exists a constant* $K\in\mathbb{R}$ *such that*

$$F_v\left(t,x^*(t),\dot{x}^*(t)\right)\ =\ K+\int_{t_0}^{t}F_x\left(t,x^*(t),\dot{x}^*(t)\right)dt.$$

In particular, we have:
(i) *At any point* $\bar{t}$ *where the function* $t\longmapsto F_x\left(t,x^*(t),\dot{x}^*(t)\right)$ *is continuous, the local Euler equation holds:*

$$\frac{d}{dt}F_v\left(\bar{t},x^*(\bar{t}),\dot{x}^*(\bar{t})\right)\ =\ F_x\left(\bar{t},x^*(\bar{t}),\dot{x}^*(\bar{t})\right),$$

(ii) *Although, in general,* $\dot{x}^*(s_i^-) \neq \dot{x}^*(s_i^+)$ *for some* $0 \leq i \leq m$*, we have:*

$$F_v\left(s_i, x^*(s_i), \dot{x}^*(s_i^-)\right) \quad = \quad F_v\left(s_i, x^*(s_i), \dot{x}^*(s_i^+)\right).$$

2.3 Unconstrained terminal state : transversality conditions

In this section, we study the problem of calculus of variations when the final state is not subject to equality constraints. Let

$$I,\ J \text{ and } K\ \subset\ \{1, \ldots, n\}$$

be given disjoint subsets of indices. Notice that the union $I \cup J \cup K$ is in general a strict subset of $\{1, \ldots, n\}$ so that $(I \cup J \cup K)^c := \{1, \ldots, n\} \setminus (I \cup J \cup K)$ is in general not empty.

For an arbitrary $x_1 \in \mathbb{R}^n$, we consider the problem:

$$\phi \quad := \quad \inf_{\substack{x \in C^1([t_0,t_1],\mathbb{R}^n) \\ x(t_0) = x_0 \\ x(t_1) - x_1 \in C(I,J,K)}} \quad \varphi(x) \tag{2.7}$$

where

$$\varphi(x) \quad := \quad \int_{t_0}^{t_1} F\left(t, x(t), \dot{x}(t)\right) dt,$$

and

$$C(I,J,K) := \left\{\xi \in \mathbb{R}^n\ :\ \xi^i \leq 0,\ \xi^j \geq 0 \text{ et } \xi^k = 0 \text{ pour } (i,j,k) \in I \times J \times K\right\}.$$

Notice that the final state is subject to
- inequality constraints for the indices in $I \cup J$,
- equality constraints for the indices in $k \in K$,
- no constraint for the indices $\ell \in (I \cup J \cup K)^c$.

Finally, for all function x satisfying the constraints of the problem (2.7), we define:

$$\begin{aligned} I(x) \quad &:= \quad \left\{i \in I\ :\ x^i(t_1) = x_1^i\right\} \\ J(x) \quad &:= \quad \left\{j \in J\ :\ x^j(t_1) = x_1^j\right\}, \end{aligned}$$

and we denote

$$L(x) \quad := \quad [I(x) \cup J(x) \cup K]^c.$$

Theorem 2.7. *Assume that* $x^* \in C^1([t_0,t_1],\mathbb{R}^n)$ *is a solution of the problem* (2.7)*. Then:*

(i) (local Euler equation) *the function* $t \longmapsto F_v\left(t, x^*(t), \dot{x}^*(t)\right)$ *is of class* C^1 *on* $[t_0, t_1]$ *with differential*

$$\frac{d}{dt} F_v\left(t, x^*(t), \dot{x}^*(t)\right) \;=\; F_x\left(t, x^*(t), \dot{x}^*(t)\right) \quad \textit{for all} \quad t \in [t_0, t_1],$$

(ii) (transversality conditions) *for all* $i \in I(x^*)$, $j \in J(x^*)$ *and* $\ell \in L(x^*)$,

$$\begin{gathered} F_{v^i}\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) \;\leq\; 0, \;\; F_{v^j}\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) \;\geq\; 0, \\ F_{v^\ell}\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) \;=\; 0. \end{gathered}$$

Proof. We first observe that x^* is also a solution of the problem of calculus of variations with equality constraint on the final state:

$$\inf_{\substack{x \in C^1([t_0,t_1],\mathbb{R}^n) \\ x(t_0) = x_0 \\ x(t_1) = x^*(t_1)}} \varphi(x).$$

Then Part (i) of the theorem is a consequence of Theorem 2.2.

To see that x^* satisfies the transversality conditions, we introduce the scalars $(\lambda_i)_{i \in I(x^*)}$, $(\mu_j)_{j \in J(x^*)}$ and $(\gamma_\ell)_{\ell \in L(x^*)}$ such that

$$\lambda_i \;\geq\; 0, \quad \mu_j \;\geq\; 0, \quad \gamma_\ell \in \mathbb{R} \quad \text{for all} \quad i \in I(x^*), \; j \in J(x^*) \text{ and } \ell \in L(x^*).$$

Let h be a $C^1\left([t_0, t_1], \mathbb{R}^n\right)$ −function with $h(t_0) = 0$,

$$\begin{gathered} h^i(t_1) = -\lambda_i, \;\; h^j(t_1) = \mu_j, \;\; h^k(t_1) = 0 \text{ and } h^\ell(t_1) = \gamma_\ell \\ \text{for all} \;\; i \in I(x^*), \;\; j \in J(x^*), \;\; k \in K \text{ et } \ell \in L(x^*). \end{gathered} \tag{2.8}$$

then, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in [0, \bar{\varepsilon}]$, the function $x^\varepsilon := x^* + \varepsilon h$ satisfies all the constraints of (2.7). By the optimality of x^*, it follows that the function $\varepsilon \longmapsto J(x^\varepsilon)$, defined on $[0, \bar{\varepsilon}]$, is minimal at $\varepsilon = 0$, and therefore:

$$\left.\frac{\partial}{\partial \varepsilon}\, \varphi(x^\varepsilon)\right|_{\varepsilon = 0} \quad \geq \quad 0. \tag{2.9}$$

Differentiating inside the integral, as in the proof of Theorem 2.2, we obtain that:

$$\int_{t_0}^{t_1} F_x\left(t, x^*(t), \dot{x}^*(t)\right) \cdot h(t) dt + \int_{t_0}^{t_1} F_v\left(t, x^*(t), \dot{x}^*(t)\right) \cdot \dot{h}(t) dt \quad \geq \quad 0.$$

By Part (i) of the theorem, the function $t \longmapsto F_v\left(t, x^*(t), \dot{x}^*(t)\right)$ is C^1. Integrating by parts, we see that:

$$\begin{aligned} 0 \quad \leq \quad & \int_{t_0}^{t_1} \left\{ F_x\left(t, x^*(t), \dot{x}^*(t)\right) - \frac{d}{dt} F_v\left(t, x^*(t), \dot{x}^*(t)\right) \right\} \cdot h(t) dt \\ & + \Big[F_v\left(t, x^*(t), \dot{x}^*(t)\right) \cdot h(t) \Big]_{t_0}^{t_1}. \end{aligned}$$

Since x^* satisfies the local Euler equation, again by Part (i), and recalling that $h(t_0) = 0$, this leads to:

$$0 \ \le \ F_v\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) \cdot h(t_1).$$

Finally, using (2.8), we obtain:

$$\begin{aligned} 0 \ \le \ & - \sum_{i \in I(x^*)} \lambda_i F_{v^i}\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) \\ & + \sum_{j \in J(x^*)} \mu_j F_{v^j}\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) \\ & + \sum_{\ell \in L(x^*)} \gamma_\ell F_{v^\ell}\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) \ , \end{aligned}$$

and the required transversality constraints follows from the arbitrariness of $\lambda_i \ge 0$, $\mu_j \ge 0$ and $\gamma_\ell \in \mathbb{R}$. ♢

Remark 2.8. In the context of a maximization problem, i.e. the infimum in (2.7) is replaced by a supremum, the inequality (2.9) is reversed. Consequently, **all inequalities in the transversality conditions are reversed.**

Remark 2.9. Suppose that the objective function contains a terminal cost as in Remark 2.1:

$$\inf_{\substack{x \in C^1([t_0,t_1],\mathbb{R}^n) \\ x(t_0) = x_0 \\ x(t_1) - x_1 \in C(I,J,K)}} \int_{t_0}^{t_1} F\left(t, x(t), \dot{x}(t)\right) dt + G\left(x(t_1)\right). \tag{2.10}$$

Then, the conclusions of Theoram 2.7 in terms of the functions F and G are as follows:
(i) the local Euler equation does not involve G, as already observed in Remark 2.3,
(ii) the transversality conditions are:

$$\begin{aligned} F_{v^i}\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) + G_{x^i}\left(x^*(t_1)\right) \ &\le \ 0 \quad \text{for} \quad i \in I(x^*), \\ F_{v^j}\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) + G_{x^j}\left(x^*(t_1)\right) \ &\ge \ 0 \quad \text{for} \quad j \in J(x^*), \\ F_{v^\ell}\left(t_1, x^*(t_1), \dot{x}^*(t_1)\right) + G_{x^\ell}\left(x^*(t_1)\right) \ &= \ 0 \quad \text{for} \quad \ell \in L(x^*). \end{aligned}$$

2.4 A sufficient condition of optimality

In this section, we consider the problem of calculus of variations:

$$\phi \ := \ \inf_{\substack{x \in C^1([t_0,t_1],\mathbb{R}^n) \\ x(t_0) = x_0 \\ x(t_1) - x_1 \in C(I,J,K)}} \int_{t_0}^{t_1} F\left(t, x(t), \dot{x}(t)\right) dt \tag{2.11}$$

where F is C^1, and I, J, K are disjoint subsets of indices in $\{1, \dots, n\}$. As usual, we denote

$$\varphi(x) \quad = \quad \int_{t_0}^{t_1} F\left(t, x(t), \dot{x}(t)\right) dt.$$

Theorem 2.10. *Assume that the function* $(\xi, v) \longmapsto F(t, \xi, v)$ *is convex for all* $t \in [t_0, t_1]$. *Let* $\bar{x}$ *be a* $C^1\left([t_0, t_1], \mathbb{R}^n\right)$ *function satisfying the constraints of the problem* (2.11)*, the local Euler equation, and the corresponding transversality conditions. Then* $\bar{x}$ *is a solution of the problem* (2.11).

Proof. Let $x \in C^1\left([t_0, t_1], \mathbb{R}^n\right)$ be a function satisfying the constraints in (2.11), let us show that $\varphi(x) - \varphi(\bar{x}) \geq 0$.
1. By the convexity of $F(t, \cdot, \cdot)$, together with the local Euler equation satisfied by $\bar{x}$, we see that:

$$\begin{aligned} F\left(t, x(t), \dot{x}(t)\right) - F\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \quad &\geq \quad F_x\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \cdot [x(t) - \bar{x}(t)] \\ &\qquad + F_v\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \cdot [\dot{x}(t) - \dot{\bar{x}}(t)] \\ &= \quad \frac{d}{dt}\left\{F_v\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \cdot [x(t) - \bar{x}(t)]\right\}. \end{aligned}$$

Integrating between t_0 and t_1, this shows that:

$$\begin{aligned} \varphi(x) - \varphi(\bar{x}) \quad &\geq \quad \Big[F_v\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \cdot \left(x(t) - \bar{x}(t)\right)\Big]_{t_0}^{t_1} \\ &= \quad F_v\left(t_1, \bar{x}(t_1), \dot{\bar{x}}(t_1)\right) \cdot \left(x(t_1) - \bar{x}(t_1)\right), \end{aligned}$$

since $x(t_0) = \bar{x}(t_0) = x_0$.
2. We next observe that $\bar{x}^k(t_1) = x^k(t_1) = x_1^k$ for all $k \in K$, and that $F_{v^\ell}\left(t_1, \bar{x}(t_1), \dot{\bar{x}}(t_1)\right) = 0$ for all $\ell \in L(\bar{x})$ by the transversality conditions. This allows to write the previous inequatility into:

$$\begin{aligned} \varphi(x) - \varphi(\bar{x}) \quad &\geq \quad \sum_{i \in I(\bar{x})} F_{v^i}\left(t_1, \bar{x}(t_1), \dot{\bar{x}}(t_1)\right)\left(x^i(t_1) - \bar{x}^i(t_1)\right) \\ &\qquad + \sum_{j \in J(\bar{x})} F_{v^j}\left(t_1, \bar{x}(t_1), \dot{\bar{x}}(t_1)\right)\left(x^j(t_1) - \bar{x}^j(t_1)\right) \\ &= \quad \sum_{i \in I(\bar{x})} F_{v^i}\left(t_1, \bar{x}(t_1), \dot{\bar{x}}(t_1)\right)\left(x^i(t_1) - x_1^i\right) \\ &\qquad + \sum_{j \in J(\bar{x})} F_{v^j}\left(t_1, \bar{x}(t_1), \dot{\bar{x}}(t_1)\right)\left(x^j(t_1) - x_1^j\right). \end{aligned}$$

Finally, since x satisfies the constraints of the problem (2.11) and $I(\bar{x}) \subset I$, $J(\bar{x}) \subset J$, we have $x^i(t_1) - x_1^i \leq 0$ and $x^j(t_1) - x_1^j \geq 0$ for all $i \in I(x)$ and $j \in J(x)$. We then deduce from the transversality conditions that:

$$\begin{aligned} F_{v^i}\left(t_1, \bar{x}(t_1), \dot{\bar{x}}(t_1)\right)\left(x^i(t_1) - x_1^i\right) \;&\geq\; 0 \quad \text{for all} \quad i \in I(\bar{x}), \\ F_{v^j}\left(t_1, \bar{x}(t_1), \dot{\bar{x}}(t_1)\right)\left(x^j(t_1) - x_1^j\right) \;&\geq\; 0 \quad \text{for all} \quad j \in J(\bar{x}), \end{aligned}$$

which provides the required inequality $\varphi(x) - \varphi(\bar{x}) \geq 0$. ♢

2.5 Examples

2.5.1 One-dimensional quadratic problem

Consider the problem

$$\inf_{\substack{x \in C^1([0,1],\mathbb{R}) \\ x(0)=1}} \int_0^1 \left[x(t)^2 + \dot{x}(t)^2\right] dt.$$

Then $F(t,\xi,v) = \xi^2 + v^2$, $F_x(t,\xi,v) = 2\xi$, $F_v(t,\xi,v) = 2v$, and the local Euler equation is:

$$\ddot{x}(t) \;=\; x(t) \quad \text{for} \quad t \in [0,1].$$

This provides the function x up to two scalar constants α and β:

$$x(t) \quad = \quad \alpha e^t + \beta e^{-t}.$$

Since the final state is unconstrained, we have the transversality condition:

$$F_v\left(1, x(1), \dot{x}(1)\right) \quad = \quad 2\,\dot{x}(1) \;=\; 0.$$

By combining this equation with the initial condition $x(0) = 1$, we determine the constants:

$$\alpha \;=\; \frac{e^{-1}}{e+e^{-1}} \quad \text{and} \quad \beta \;=\; \frac{e}{e+e^{-1}}.$$

Finally, observe that we are in the context of Theorem 2.10. Then the above determined function x is a solution of our problem. By strict convexity of $F(t,x,v)$ in (x,v), this is in fact the unique solution of our problem.

2.5.2 Optimal consumption model

La wealth of an economic agent is governed by the dynamics:

$$\dot{x}(t) \;=\; -c(t) \quad \text{and} \quad x(0) \;=\; x_0, \tag{2.12}$$

where $c(t)$ is the rate of consumption at time t. The preferences of the agent are defined by the utility function:

$$U(c) \quad = \quad \int_0^T e^{-\beta t} u\left(c(t)\right) dt,$$

where $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is an increasing function, strictly concave and satisfies the so-called *Inada condition*

$$u'(0+) \quad = \quad +\infty. \tag{2.13}$$

The latter condition allows to ignore the positivity constraint on the consumption rate function.

The problem of optimal consumption is defined by:

$$\sup_{\substack{x \in C^1([0,1],\mathbb{R}) \\ x(0)=x_0 \\ x(T)\geq 0}} \int_0^T e^{-\beta t} u\left(-\dot{x}(t)\right)\, dt.$$

In the present example, we have $F(t,x,v) = e^{-\beta t}u(-v)$. The local Euler condition is given by:

$$0 \;=\; -\frac{d}{dt}\left\{e^{-\beta t}u'\left(-\dot{x}\right)\right\} \;=\; e^{-\beta t}\big(\beta u'\left(-\dot{x}\right) + u\text{"}\left(-\dot{x}\right)\ddot{x}(t)\big).$$

It is clear that the final wealth of the agent must be zero (because objective function does not compensate any remaining final wealth). Consequently, there is no transversality condition.

In order to obtain more explicit calculations, we specialize the discussion to the power utility:

$$u(\xi) \;=\; \frac{\xi^\gamma}{\gamma}$$

where $\gamma < 1$ is a given parameter. The local Euler condition is then:

$$-\beta\left(-\dot{x}\right)^{\gamma-1} + (1-\gamma)\left(-\dot{x}\right)^{\gamma-2}\ddot{x}(t) \;=\; 0.$$

Multiplying by $\left(-\dot{x}\right)^{-\gamma+2}$, this provides:

$$(1-\gamma)\ddot{x}(t) + \beta\dot{x}(t) \;=\; 0.$$

Combined with the endpoints conditions $x(0) = x_0$ and $x(T) = 0$, this provides the unique solution:

$$x(t) \;=\; x_0\left[1 - \frac{1-e^{-\beta t/(1-\gamma)}}{1-e^{-\beta T/(1-\gamma)}}\right].$$

Finally, observe that this example also fits in the context of Theorem 2.10. Then the above determined function x is a solution of our problem, and even the unique solution by the strict convexity of the utility function.

2.5.3 Optimal growth with non-renewable resource

We now add an extra component to the previous problem by allowing the agent to manage his capital denoted by k in the present example. Given a parameter $\alpha \in (0,1)$, the dynamics of the capital is defined by the equation:

$$\dot{k}(t) \;=\; ak(t)^{1-\alpha}r(t)^\alpha - c(t), \tag{2.14}$$

where $c(t)$ is the rate of consumption at time t, and $r(t)$ is a rate of resource needed for the production of capital, and taken from a non-renewable stock of resources $y(t)$ governed by:

$$\dot{y}(t) \;=\; -r(t). \tag{2.15}$$

Denote $x := (y, k)$ the controlled state of the system. The control variable (c, r) takes values in $U = \mathbb{R}^2_+$. Our objective is to solve the problem

$$\sup_{\substack{(c,r)\in\mathcal{U}\\ x(0)=x_0\\ x(T)=0}} \int_0^T \ln c(t)dt.$$

where $\mathcal{U}$ is the collection of all pairs (c, r) of piecewise continuous functions from $[t_0, t_1]$ to U.

Notice that the positivity constraint on the control variables can be ignored by definition of the objective function and the final objective. In order to reduce this problem to the setting of a calculus of variations problem, we substitute the expressions of $c(t)$ and $r(t)$ in terms of the state variables (y, k) and $(\dot{y}, \dot{k})$:

$$c(t) \quad = \quad h(t, y(t), k(t), \dot{y}(t), \dot{k}(t)).$$

This leads to

$$\sup_{\substack{(c,r)\in\mathcal{U}\\ x(0)=x_0\\ x(T)=0}} \int_0^T \ln\left[ak(t)^{1-\alpha}\left(-\dot{y}(t)\right)^{\alpha} - \dot{k}(t)\right]dt.$$

In the present example, $F(t, x, v) = \ln h(t, x, v)$ with

$$h(t, x_1, x_2, v_1, v_2) \quad := \quad ax_2^{1-\alpha}(-v_1)^{\alpha} - v_2.$$

The local Euler condition is:

$$\frac{d}{dt}\begin{pmatrix} -c(t)^{-1}a\alpha z(t)^{\alpha-1} \\ -c(t)^{-1} \end{pmatrix} \;=\; c(t)^{-1}\begin{pmatrix} 0 \\ a(1-\alpha)z(t)^{\alpha} \end{pmatrix} \quad \text{where} \quad z(t) \;:=\; \frac{-\dot{y}(t)}{k(t)}.$$

The first equation of this system shows that:

$$c(t)^{-1}z(t)^{\alpha-1} \;=\; b_1 \qquad \text{is a constant function.} \tag{2.16}$$

Plugging this information in the second equation, we see that:

$$z(t)^{-1-\alpha}\dot{z}(t) \quad = \quad -a,$$

which shows existence of another constant b_2 such that

$$az(t)^{\alpha} \quad = \quad \frac{1}{b_2 + \alpha t}.$$

Combining this equation with (2.14) with (2.16), we obtain an ordinay differential equation for k:

$$\begin{aligned} \dot{k}(t) &= az(t)^{\alpha}k(t) - c(t) \\ &= (b_2 + \alpha t)^{-1}k(t) - b_1^{-1}a^{(1-\alpha)/\alpha}(b_2 + \alpha t)^{(1-\alpha)/\alpha}, \end{aligned}$$

which can be solved explicitly. Since $k(T) = 0$, the expression of $k(t)$ (up to two scalar constants) is:

$$k(t) = b_1^{-1}a^{(1-\alpha)/\alpha}(b_2 + \alpha t)^{1/\alpha} \ln\left(\frac{b_2 + \alpha T}{b_2 + \alpha t}\right)^{1/\alpha}.$$

We can now determine the state variable y:

$$\dot{y}^*(t) = -z(t)k(t) = -\frac{1}{ab_1} \ln\left(\frac{b_2 + \alpha T}{b_2 + \alpha t}\right)^{1/\alpha}.$$

Given the final constraint $y(T) = 0$, we obtain the expression of y, up to two constants:

$$y(t) = \frac{1}{ab_1} \int_t^T \ln\left(\frac{b_2 + \alpha T}{b_2 + \alpha s}\right)^{1/\alpha} ds.$$

Finally, we determine the constants b_1 and b_2 by writing:

$$k(0) = k_0 \quad \text{and} \quad y(0) = y_0.$$

Chapter 3

Pontryagin maximum Principle

3.1 Lagrange formulation

In this chapter, we study a larger class of dynamic optimization problems. We consider a dynamic system whose evolution is governed by the differential equation:

$$\dot{x}(t) \;=\; f\left(t, x(t), u(t)\right) \quad \text{for} \quad t_0 \leq t \leq t_1, \tag{3.1}$$

called *state equation*. Here, $u(.)$ is a map from $[t_0, t_1]$ to $U \subset \mathbb{R}^k$ which stands for the control variable of the system. For technical reasons, we assume that u lies in the set of admissible controls

$$\mathcal{U} \;\; := \;\; C^0_{\mathrm{pm}}\left([t_0, t_1], U\right)$$

of piecewise continuous functions from $[t_0, t_1]$ to U.

Definition 3.1. *A function $u : \; [t_0, t_1] \longrightarrow \mathbb{R}^k$ is said to be piecewise continuous if*
- u has finite left and right limits at every point of the open interval (t_0, t_1), a finite right limit at the left endpoint t_0, and a finite left limit at the right endpoint t_1,
- the set of discontinuity points in (t_0, t_1) is finite.

In Section 3.3, we shall state conditions on the function f which ensure that for all control variale $u \in \mathcal{U}$ and all initial condition $x(t_0) = x_0$, there exists a unique solution $x(.) = x^u(.)$ of the state equation (3.1), called *controlled state of the system*, or *controlled trajectory of the system.*

Given a cost function $F : [t_0, t_1] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$, the minimization problem is defined by:

$$\inf_{\substack{u \in \mathcal{U} \\ x^u(t_0) = x_0}} \quad \int_{t_0}^{t_1} F\left(t, x^u(t), u(t)\right)\, dt, \tag{3.2}$$

where $x_0 \in \mathbb{R}^n$ is a given initial condition.

The dependence of the state variable x^u on the control variable u will be frequently omitted. We will be simply writing x, except when there is a potential risk of confusion.

Finally, we observe that the problem of calculus of variations studied in Chapter 2 falls in the general class of this chapter by setting $f(t, \xi, u) = u$.

3.2 Equivalent formulations

The optimization problem (3.2) is known under the name of *Lagrange formulation*. In this section, we present two alternative formulations which we prove to be equivalent to the Largrange one.

The Mayer formulation. Here, the cost functional depends only on the final value of the terminal controlled state. Given a function $G : \mathbb{R}^n \longrightarrow \mathbb{R}$, the dynamic optimization problem is defined by:

$$\inf_{\substack{u \in \mathcal{U} \\ x^u(t_0) = x_0}} \quad G\left(x^u(t_1)\right). \tag{3.3}$$

Assume G is C^1, then:

$$\begin{aligned} G\left(x^u(t_1)\right) &= G\left(x^u(t_0)\right) + \int_{t_0}^{t_1} DG\left(x^u(t)\right) \cdot \dot{x}^u(t)\, dt \\ &= G\left(x_0\right) + \int_{t_0}^{t_1} DG\left(x^u(t)\right) \cdot f\left(t, x^u(t), u(t)\right)\, dt, \end{aligned}$$

where we used the ODE satisfied by the controlled state. Introducing

$$F(t, \xi, \nu) \quad := \quad DG(\xi) \cdot f(t, \xi, \nu),$$

we can reduce the above problem to the Lagrange formulation.

Conversely, the Lagrange problem can be recasted under the Mayer formulation by augmenting the controlled state:

$$y^u(t) \;:=\; \begin{pmatrix} x^u(t) \\ x^u_{n+1}(t) \end{pmatrix} \quad \text{with} \quad x^u_{n+1}(t) \;:=\; \int_{t_0}^{t} F\left(s, x^u(s), u(s)\right)\, ds.$$

The augmented state system is governed by the ordinary differential equation:

$$\dot{y}(t) \;=\; g\left(t, y(t), u(t)\right) \quad \text{with} \quad g(t, \xi, \zeta, \nu) \;:=\; \begin{pmatrix} f(t, \xi, \nu) \\ F(t, \xi, \nu) \end{pmatrix},$$

so that the Lagrange problem can be written into

$$\inf_{\substack{u \in \mathcal{U} \\ y^u(t_0) = (x_0, 0)}} \quad y^u_{n+1}(t_1),$$

and is thus reduced to the Mayer formulation.

The Boza Formulation. Let $F : [t_0, t_1] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$ and $G : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two given functions, and consider the dynamic optimization problem:

$$\inf_{\substack{u \in \mathcal{U} \\ x^u(t_0) = x_0}} \quad \int_{t_0}^{t_1} F\left(t, x^u(t), u(t)\right) \, dt \; + \; G\left(x^u(t_1)\right). \tag{3.4}$$

Both Lagrange and Mayer formulations are particular examples of the Bolza formulation. Conversely, introducing

$$\tilde{F}(t, \xi, \nu) \quad := \quad F(t, \xi, \nu) + DG(\xi) \cdot f(t, \xi, \nu),$$

allows to recast the Bolza problem into the Lagrange formulation.

3.3 Controlled differential equations: existence and uniqueness

For $u \in \mathcal{U}$, we consider the differential equation (3.1) where f is a continuous function. To avoid problems of non-differentiability of x at the discontinuity points of the control u, we write (3.1) in the integral form

$$x(t) \quad = \quad x_0 + \int_{t_0}^{t} f\left(s, x(s), u(s)\right) ds, \quad t_0 \le t \le t_1. \tag{3.5}$$

Definition 3.2. *We say that* $x(.)$ *is a solution of* (3.1) *with initial condition* $x(t_0) = x_0$ *if* x *satisfies* (3.5).

Notice that a solution x of (3.1) is necessarilly continuous. It also has left and right derivatives at every point in $[t_0, t_1]$ with

$$\dot{x}(t-) \; = \; f\left(t, x(t), u(t-)\right) \quad \text{et} \quad \dot{x}(t+) \; = \; f\left(t, x(t), u(t+)\right),$$

in particular, the state x is differentiable at every continuity point of the control u.

The next theorem provides an existence and uniqueness result for the state equation (3.1).

Theorem 3.3. *Let* $f : [t_0, t_1] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$ *be a continuous function satisfying the Lipschitz and linear growth conditions: there exists* $c > 0$ *such that*

$$\begin{array}{rcl} |f(t,\xi_1,\nu) - f(t,\xi_2,\nu)| & \le & c\ |\xi_1 - \xi_2|\ ,\quad \xi_1,\xi_2 \in \mathbb{R}^n,\ (t,\nu) \in [t_0,t_1] \times U, \\ |f(t,\xi,\nu)| & \le & c\ (1 + |\xi| + |\nu|)\,,\quad (t,\xi,\nu) \in [t_0,t_1] \times \mathbb{R}^n \times U. \end{array} \tag{3.6}$$

Then for every integrable function $u : [t_0, t_1] \longrightarrow U$*, and every initial condition* $x(t_0) = x_0$*, there is a unique absolutely continuous solution of* (3.5)*.*

Proof. **1.** We first prove the uniqueness statement. Let x and y be two absolutely continuous solutions of (3.5). Then, it follows from the Lipschitz condition on f that:

$$\begin{aligned} |x(t) - y(t)| &= \left| \int_{t_0}^t \left[f\left(s, x(s), u(s)\right) - f\left(s, y(s), u(s)\right) \right]\, ds \right| \\ &\le \int_{t_0}^t \left| f\left(s, x(s), u(s)\right) - f\left(s, y(s), u(s)\right) \right|\, ds \\ &\le c \int_{t_0}^t |x(s) - y(s)|\, ds, \end{aligned}$$

this implies that $|x(t) - y(t)| = 0$ for all $t \in [t_0, t_1]$ by the Gronwall Lemma 3.4 below.
2. We next prove a local existence result. More precisely, we prove the existence of a solution of (3.5) on the interval $[t_0, t_0 + \alpha]$ for all $\alpha < \min\{c^{-1}, t_1 - t_0\}$. To do this, we consider the operator

$$T \ :\ (C([t_0, t_0 + \alpha], \mathbb{R}^n), \|.\|_\infty) \quad\longrightarrow\quad (C([t_0, t_0 + \alpha], \mathbb{R}^n), \|.\|_\infty)$$

defined by:

$$Tx(t) \ :=\ x_0 + \int_{t_0}^t f\left(s, x(s), u(s)\right) ds\ ; \qquad t_0 \le t \le t_0 + \alpha,$$

and we compute that:

$$\begin{aligned} \|Tx - Ty\|_\infty &\le \int_{t_0}^t \left| f\left(s, x(s), u(s)\right) - f\left(s, y(s), u(s)\right) \right|\, ds \\ &\le K\alpha \|x - y\|_\infty. \end{aligned}$$

Since $\alpha < K^{-1}$, it follows that T is a contraction, and therefore has a unique fixed point.
3. Finally, we prove the existence of a maximal solution on the interval $[t_0, t_1]$ by using the linear growth of f. This is obtained by the same argument as in the case u is continuous, see e.g. Schwartz [**?**], Theorem 4.2.10. ♢

In the previous proof, we used the following result.

Lemma 3.4. (Gronwall) *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a piecewise continuous function satisfying*

$$f(t) \ \le \ \alpha + \beta \int_a^t f(s)ds \quad \textit{for all} \quad t \in [a,b], \tag{3.7}$$

for some scalars $\alpha, \beta > 0$ independent of t. Then

$$f(t) \ \le \ \alpha e^{\beta(t-a)} \quad \textit{for all} \quad t \in [a,b].$$

Proof. Multiplying (3.7) by $e^{-\beta(t-a)}$, we see that:

$$\frac{d}{dt}\left\{e^{-\beta(t-a)} \int_a^t f(s)ds\right\} \ \le \ \alpha e^{-\beta(t-a)} \quad \text{for all} \quad t \in [a,b].$$

Integrating by parts, this provides:

$$e^{-\beta(t-a)} \int_a^t f(s)ds \ \le \ \frac{\alpha}{\beta}\left[1 - e^{-\beta(t-a)}\right] \quad \text{for all} \quad t \in [a,b].$$

The required estimate is obtained by plugging this inequality in (3.7). ♢

3.4 Statement of the Pontryagin maximum principle

We introduce the *Hamiltonian* corresponding the Lagrange optimization problem (3.2):

$$H(t,\xi,\nu,\pi) \quad := \quad F(t,\xi,\nu) \ + \ \pi \cdot f(t,\xi,\nu) \tag{3.8}$$

defined on $[t_0,t_1] \times \mathbb{R}^n \times U \times \mathbb{R}$ with values in $\mathbb{R}$.

Theorem 3.5. *Let f and F be continuous and satisfy the conditions* (3.6)*. Assume further that the partial gradients f_x et F_x exist, are continuous, and for some $\alpha > 0$,*

$$\xi \longmapsto (f_x, F_x)(t,\xi,\nu) \quad \textit{is } \alpha\textit{−Hölder-continuous for all} \quad (t,\nu) \in [t_0,t_1] \times U. \tag{3.9}$$

Let u^* be an optimal control for the problem (3.2)*, and $x^* := x^{u^*}$ the corresponding controlled state. Then, there exists a C^1−function $p : [t_0,t_1] \longrightarrow \mathbb{R}^n$ such that for all $t \in [t_0,t_1]$:*
(i) $H\left(t, x^*(t), u^*(t), p(t)\right) = \min_{\nu \in U} H\left(t, x^*(t), \nu, p(t)\right)$,
(ii) $\dot{p}(t) = -H_x\left(t, x^*(t), u^*(t), p(t)\right)$ *and* $p(t_1) = 0$.

Let us observe that Condition (3.9) in the above statement can be weakened, and is here assumed in order to simplify the presentation.

We conclude this section with the terminology attached to the previous statement.

- The function p introduced in the previous statement is called the *adjoint state of the system.*
- The differential equation governing the dynamics of p is called the *adjoint state equation.*
- The final condition on p is called *transversality condition.*
- The state equation and the adjoint state equation are usually collected into the so-called *Hamiltonien system*:

$$\left\{ \begin{array}{rcll} \dot{x}(t) & = & \frac{\partial H}{\partial p}\left(t, x^*(t), u^*(t), p(t)\right), & x(t_0) = x_0, \\ \dot{p}(t) & = & -\frac{\partial H}{\partial x}\left(t, x^*(t), u^*(t), p(t)\right), & p(t_1) = 0. \end{array} \right. \tag{3.10}$$

3.5 Proof of the Pontryagin maximum principle

In this section, we consider the Mayer formulation

$$\inf_{\substack{u \in \mathcal{U} \\ x^u(t_0) = x_0}} \quad G\left(x^u(t_1)\right). \tag{3.11}$$

Obviously, the statement of Theorem 3.5 can be immediately deduced by using the equivalence between the Lagrange, Mayer and Bolza formulations.

Step 1: Perturbation of the control problem. Let $\nu \in U$, $0 < \varepsilon < t_1 - t_0$ and $t_0 + \varepsilon < \tau \leq t_1$. Consider the control variable

$$u_\varepsilon := \left\{ \begin{array}{l} \nu \text{ sur } (\tau - \varepsilon, \tau] \\ u^* \text{ sur } [t_0, \tau - \varepsilon] \cup (\tau, t_1]. \end{array} \right.$$

Clearly $u_\varepsilon \in \mathcal{U}$. We denote

$$x_\varepsilon := x^{u_\varepsilon} \quad y_\varepsilon := x_\varepsilon - x^* \quad \text{and} \quad z_\varepsilon := \frac{1}{\varepsilon}\, y_\varepsilon.$$

Step 2: Effect of the perturbation on the state system. Obviously:

$$y_\varepsilon = 0 \quad \text{on} \quad [0, \tau - \varepsilon].$$

The following result shows that y_ε converges to 0 uniformly on $[t_0, t_1]$, with rate of convergence ε, thus preparing the analysis of z_ε.

Lemma 3.6. *Under Conditions* (3.6) *on f, there exists a constant c such that:*

$$\sup_{t_0 \leq t \leq t_1} |y_\varepsilon(t)| \leq \varepsilon\, e^{c(t_1 - t_0)}.$$

Proof. (i) For $t \in (\tau - \varepsilon, \tau]$, we have

$$\dot{y}_\varepsilon(t) = f\left(t, x_\varepsilon(t), \nu\right) - f\left(t, x^*(t), u^*(t)\right).$$

By (3.6), this implies that:

$$\begin{aligned} |y_\varepsilon(t)| &= \left| \int_{\tau-\varepsilon}^{t} [f(s, x_\varepsilon(s), \nu) - f(t, x^*(s), \nu)]\, ds \right. \\ &\quad \left. + \int_{\tau-\varepsilon}^{t} [f(s, x^*(s), \nu) - f(t, x^*(s), u^*(s))]\, ds \right| \\ &\leq c \int_{\tau-\varepsilon}^{t} |y_\varepsilon(s)| ds \;+\; \varepsilon c\, (2 + |\nu| + \|x^*\|_\infty + \|u^*\|_\infty) \\ &\leq c' \left(\varepsilon + \int_{\tau-\varepsilon}^{t} |y_\varepsilon(s)| ds \right), \end{aligned}$$

where $\|\varphi\|_\infty = \max_{[t_0,t_1]} |\varphi|$ and c' is a positive constant. The function y_ε is piecewise continuous as the difference of two piecewise continuous functions. Then, it follows from the Gronwall Lemma that

$$|y_\varepsilon(t)| \;\leq\; c'\varepsilon\, e^{c'\varepsilon(t-\tau+\varepsilon)} \;\leq\; 2K'\varepsilon \quad \text{for} \quad t \in\, (\tau - \varepsilon, \tau] \tag{3.12}$$

and ε sufficiently small.
(ii) For $t \in\, (\tau, t_1]$, we have

$$\dot{y}_\varepsilon(t) \;=\; f(t, x_\varepsilon(t), u^*(t)) - f(t, x^*(t), u^*(t)),$$

and therefore, it follows from (3.6) that:

$$\begin{aligned} |y_\varepsilon(t)| &\leq |y_\varepsilon(\tau)| \;+\; c \int_{\tau}^{t} |y_\varepsilon(s)| ds \\ &\leq c' \left(2\varepsilon + \int_{\tau}^{t} |y_\varepsilon(s)| ds \right), \end{aligned}$$

by (3.12). We then obtain by the Gronwall Lemma:

$$|y_\varepsilon(t)| \;\leq\; 2c'\varepsilon\, e^{c'(t-\tau)} \;\leq\; 2K'\varepsilon e^{K'(t_1-t_0)} \quad \text{for} \quad t \in\, (\tau, t_1]. \tag{3.13}$$

The required result follows from (3.12) and (3.13). ♢

We are now ready for the asymptotic analysis of z_ε.

Lemma 3.7. *Assume that f satisfies Conditions (3.6), the partial gradient f_x exists, is continuous, and satisfies (3.9). Then, the function z_ε converges pointwise on $[t_0, t_1]$ towards the function z defined by:*

$$\begin{aligned} z(t) &= 0\,; \quad t \in [0, \tau) \\ z(\tau) &= f(\tau, x^*(\tau), \nu) - f(\tau, x^*(\tau), u^*(\tau)) \\ \dot{z}(t) &= f_x(t, x^*(t), u^*(t))\; z(t)\,; \quad t \in [\tau, t_1). \end{aligned} \tag{3.14}$$

Proof. (i) The convergence of $z(t)$ towards zero for $t < \tau$ is obvious. We start by studying the convergence of $z_\varepsilon(\tau)$. For $t \in (\tau - \varepsilon, \tau]$, we have

$$\begin{aligned} \dot{y}_\varepsilon(t) &= f\left(t, x_\varepsilon(t), \nu\right) - f\left(t, x^*, u^*(t)\right) \\ &= f\left(t, x^*(t), \nu\right) - f\left(t, x^*(t), u^*(t)\right) + f_x\left(t, x^*(t), \nu\right)\ y^\varepsilon(t) + \varepsilon \eta_\varepsilon(t), \end{aligned}$$

where $\varepsilon\eta_\varepsilon(t) = \circ\left(|y^\varepsilon(t)|\right)$. Using Condition (3.9) on f_x, we obtain a better estimate of η_ε. Indeed, there exists a convex combination $\bar{x}(t)$ of $x_\varepsilon(t)$ and $x^*(t)$, such that:

$$\begin{aligned} |\varepsilon\eta_\varepsilon| &= \left|f_x\left(t, \bar{x}(t), u^*(t)\right) - f_x\left(t, x^*(t), u^*(t)\right)\right| \cdot |y_\varepsilon(t)| \\ &\leq C\ |\bar{x}(t) - x^*(t)|^\alpha |y_\varepsilon(t)|\ \leq\ C\ |y_\varepsilon|^{1+\alpha}\ \leq\ C'\ \varepsilon^{1+\alpha}, \\ &\quad \text{for}\ \ \tau - \varepsilon < t \leq \tau, \end{aligned} \tag{3.15}$$

by Lemma 3.6. To prove the convergence of $z_\varepsilon(\tau)$ towards $z(\tau)$ (defined in the statement of the lemma), we compute

$$\begin{aligned} z_\varepsilon(\tau) &= \int_{\tau-\varepsilon}^{\tau} \dot{z}_\varepsilon(t) dt \\ &= \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} \left[f\left(t, x^*(t), \nu\right) - f\left(t, x^*(t), u^*(t)\right)\right] dt \\ &\quad + \int_{\tau-\varepsilon}^{\tau} \eta_\varepsilon(t) dt + \int_{\tau-\varepsilon}^{\tau} f_x\left(t, x^*(t), \nu\right) z^\varepsilon(t) dt. \end{aligned}$$

By (3.15), we see that

$$\int_{\tau-\varepsilon}^{\tau} \eta_\varepsilon(t) dt \ \longrightarrow\ 0 \quad \text{quand} \quad \varepsilon \ \longrightarrow\ 0.$$

Since z^ε and $t \longmapsto f_x\left(t, x^*(t), \nu\right) z^\varepsilon(t)$ are bounded on $[t_0, t_1]$, we also see that

$$\int_{\tau-\varepsilon}^{\tau} f_x\left(t, x^*(t), \nu\right) z^\varepsilon(t) dt \ \longrightarrow\ 0 \quad \text{quand} \quad \varepsilon \ \longrightarrow\ 0.$$

Finally, by the mean value theorem:

$$\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} \left[f\left(t, x^*(t), \nu\right) - f\left(t, x^*(t), u^*(t)\right)\right] dt \ \longrightarrow\ z(\tau) \quad \text{quand} \quad \varepsilon \ \longrightarrow\ 0.$$

(ii) For every point $t \in (\tau, t_1]$ of continuity of u^*, we have:

$$\begin{aligned} \dot{y}_\varepsilon(t) &= f\left(t, x_\varepsilon(t), u^*(t)\right) - f\left(t, x^*, u^*(t)\right) \\ &= f_x\left(t, x^*(t), u^*(t)\right)\ y_\varepsilon(t) + \varepsilon\eta_\varepsilon(t), \end{aligned}$$

where $\varepsilon\eta_\varepsilon(t) = \circ\left(|y^\varepsilon(t)|\right)$. Using Condition (3.9) as in the first step of this proof, we obtain the following estimate for η_ε:

$$|\eta_\varepsilon(t)| \quad \leq \quad C'\ \varepsilon^\alpha\ , \quad \tau \leq t \leq t_1. \tag{3.16}$$

To prove the convergence of z_ε towards z on $(\tau, t_1]$, we compute that at every point of continuity of u^*:

$$\dot{z}_\varepsilon(t) - \dot{z}(t) \quad = \quad f_x\left(t, x^*(t), u^*(t)\right)\left[z_\varepsilon(t) - z(t)\right] + \eta_\varepsilon(t).$$

Since the set of points of discontinuity of u^* is finite, this implies that:

$$\begin{aligned} |z_\varepsilon(t) - z(t)| \quad &\leq \quad |z_\varepsilon(\tau) - z(\tau)| + \int_\tau^t |\eta_\varepsilon(s)| ds \\ &\qquad + \int_\tau^t |f_x\left(s, x^*(s), u^*(s)\right)| \cdot |z_\varepsilon(s) - z(s)|\, ds. \end{aligned}$$

Since f_x is continuous, it follows that $f_x\left(s, x^*(s), u^*(s)\right)$ is bounded on the interval $[\tau, t_1]$. Using (3.16), this shows the existence of a constant C such that

$$|z_\varepsilon(t) - z(t)| \quad \leq \quad |z_\varepsilon(\tau) - z(\tau)| + C\left(\varepsilon^\alpha + \int_\tau^t |z_\varepsilon(s) - z(s)| ds\right).$$

By the Gronwall Lemma, this ensures that

$$|z_\varepsilon(t) - z(t)| \;\leq\; \left(|z_\varepsilon(\tau) - z(\tau)| + C\varepsilon^\alpha\right) e^{C(t-\tau)} \quad \text{for} \quad t \in [\tau, t_1],$$

which proves the pointwise convergence of z_ε towards z, as a consequence of the convergence of $z_\varepsilon(\tau)$ towards $z(\tau)$ established in the first part of this proof. ♢

Step 3: Effect of the perturbation on the objective function. Since x^* is a solution of the minimization problem (3.3), the function $\varepsilon \longmapsto G\left(x_\varepsilon(t_1)\right)$ defined on $[0, \tau - t_0]$ is minimized by $\varepsilon = 0$. Since G is differentiable, the first order condition is given by:

$$0 \quad \leq \quad \left.\frac{\partial}{\partial \varepsilon} G\left(x_\varepsilon(t_1)\right)\right|_{\varepsilon=0} \quad = \quad DG\left(x^*(t_1)\right) \cdot z(t_1). \tag{3.17}$$

hence, for all $\nu \in U$ and $\tau \in (t_0, t_1)$, inequality (3.17) holds true, where $z(.)$ is the function depending on (ν, τ) as defined in (3.14).

This necessary condition is not convenient for practical use. We shall therefore express it equivalently in a more suitable form. Let $p(.)$ be the function defined by:

$$\dot{p}(t) \;=\; -f_x\left(t, x^*(t), u^*(t)\right)^T p(t) \quad \text{et} \quad p(t_1) \;=\; DG\left(x^*(t_1)\right),$$

where ^{T} denotes transposition. Then, for all point $t \in [\tau, t_0]$ of continuity of u^*:

$$\begin{aligned} \frac{d}{dt}\big[p(t) \cdot z(t)\big] \quad &= \quad \dot{p}(t) \cdot z(t) + p(t) \cdot \dot{z}(t) \\ &= \quad -f_x\left(t, x^*(t), u^*(t)\right)^T p(t) \cdot z(t) + p(t) \cdot f_x\left(t, x^*(t), u^*(t)\right) z(t) \\ &= \quad 0. \end{aligned}$$

Since p and z are continuous, this proves that the function $t \longmapsto p(t) \cdot z(t)$ is constant on $[\tau, t_1]$, and therefore $p(t_1) \cdot z(t_1) = p(\tau) \cdot z(\tau)$. By the expression of $z(\tau)$ in (3.14) together with Condition (3.17), this provides:

$$p(\tau) \cdot f\left(\tau, x^*(\tau), \nu\right) \quad \geq \quad p(\tau) \cdot f\left(\tau, x^*(\tau), u^*(\tau)\right)$$

for all $\tau \in]t_0, t_1[$ and $\nu \in U$, completing the proof of Theorem 3.5. ♢

Step 4: Back to the Lagrange formulation. We now apply the result obtained in the previous step to the Mayer problem

$$\inf_{\substack{u \in \mathcal{U} \\ y^u(t_0) - y_0 = 0}} \qquad G\left(y(t_1)\right),$$

where y is the augmented state

$$y \ := \ \begin{pmatrix} x \\ y^{n+1} \end{pmatrix} \quad \text{with} \quad y^{n+1}(t) \ := \ \int_{t_0}^t F\left(t, x^u(t), u(t)\right) dt.$$

The dynamics of the controlled state y is governed by the differential equation

$$\dot{y}(t) \ = \ g\left(t, y(t), u(t)\right) \quad \text{where} \quad g(t, \xi, \zeta, \nu) \ := \ \begin{pmatrix} f \\ F \end{pmatrix}(t, \xi, \nu)$$

for all $(t, \xi, \zeta, \nu) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R} \times U$. Finally, the objective function is defined by

$$G(\xi, \zeta) \ = \ \zeta \quad \text{for all} \quad (\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}. \tag{3.18}$$

Denote by $q(t) := \begin{pmatrix} p(t) \\ p_0(t) \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}$ the adjoint state of the system defined by the adjoint state equation and the transversality condition:

$$\dot{q}(t) = -g_y^T\left(t, y^*(t), u^*(t)\right), \quad q(t_1) = DG\left(y^*(t_1)\right).$$

The Pontryagin maximum principle applied to the above Mayer problem says that

$$q(t) \cdot g\left(t, y^*(t), u^*(t)\right) \ = \ \min_{\nu \in U} q(t) \cdot g\left(t, y^*(t), \nu\right) \quad \text{for} \quad t \in [t_0, t_1]. \tag{3.19}$$

By the expression of g and G, this provides $\dot{p_0}(t) = 0$, and therefore

$$p_0(t) \ = \ 1 \quad \text{for all} \quad t \in [t_0, t_1],$$

and

$$\dot{p}(t) \ = \ F_x\left(t, x^*(t), u^*(t)\right) + p(t) \cdot f_x^T\left(t, x^*(t), u^*(t)\right), \qquad p(t_1) \ = \ 0.$$

Finally, Condition (3.19) can be written as:

$$H\left(t, x^*(t), u^*(t), p(t)\right) \ = \ \min_{\nu \in U} H\left(t, x^*(t), \nu, p(t)\right) \quad \text{for all} \quad t \in [t_0, t_1],$$

where

$$H(t, \xi, \nu, \pi) \quad := \quad F(t, \xi, \nu) + \pi \cdot f(t, \xi, \nu).$$

3.6 Constrained final state

In this section, we consider optimal control problems with constraints on the final state of the system:

$$\inf_{\substack{u \in \mathcal{U} \\ x^u(t_0)-x_0=0 \\ x^u(t_1)-x_1 \in C(I,J,K)}} \int_{t_0}^{t_1} F\left(t, x^u(t), u(t)\right) dt, \tag{3.20}$$

where $x_0, x_1 \in \mathbb{R}^n$ are given, I, J, K are disjoint subsets of indices in $\{1, \ldots, n\}$, and

$$C(I,J,K) \;:=\; \left\{\xi \in \mathbb{R}^n \;:\; \xi^i \leq 0,\ \xi^j \geq 0 \text{ and } \xi^k = 0 \text{ for } (i,j,k) \in I \times J \times K\right\}.$$

Among all inequality constraints, we isolate those which are binding by introducing for all $x = x^u(.)$:

$$I(x) \;=\; \left\{i \in I \;:\; x^i(t_1) - x_1^i = 0\right\} \quad \text{and} \quad J(x) \;=\; \left\{j \in J \;:\; x^j(t_1) - x_1^j = 0\right\}.$$

We also denote

$$L(x) \;:=\; \left[I(x) \cup J(x) \cup K\right]^c.$$

In order to simplify the presentation, we use the Mayer formulation:

$$\inf_{\substack{u \in \mathcal{U} \\ y^u(t_0)-y_0=0 \\ y^u(t_1)-y_1 \in D(I,J,K)}} G\left(y(t_1)\right), \tag{3.21}$$

where y is the augmented state

$$y \;:=\; \begin{pmatrix} x \\ y^{n+1} \end{pmatrix} \quad \text{with} \quad y^{n+1}(t) \;:=\; \int_{t_0}^{t} F\left(s, x^u(s), u(s)\right) ds,$$

$y_0 := (x_0^T, 0)^T$, $y_1 := (x_1^T, 0)^T$, $G(\xi, \zeta) := \zeta$ for all $(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}$, and

$$D(I,J,K) \;:=\; C(I,J,K) \times \mathbb{R}.$$

The dynamics of the augmented system is governed by the differential equation

$$\dot{y}(t) \;=\; g\left(t, y(t), u(t)\right) \quad \text{where} \quad g(t, \xi, \zeta, \nu) \;:=\; \begin{pmatrix} f \\ F \end{pmatrix}(t, \xi, \nu)$$

for all $(t, \xi, \zeta, \nu) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R} \times U$.

Let $\lambda \in \mathbb{R}^n$ be a vector of Lagrange multipliers associated to the constraints $C(I,J,K)$. By the Kuhn and Tucker Theorem, the components of λ satisfy

$$\lambda_i \;\geq\; 0, \quad \lambda_j \;\leq\; 0, \quad \lambda_\ell \;=\; 0 \quad \text{for} \quad i \in I,\ j \in J,\ \ell \in [I \cup J \cup K]^c. \tag{3.22}$$

Then, assuming that G is convex, it follows that the problem (3.21) is reduced to the unconstrained Mayer problem

$$\inf_{\substack{u \in \mathcal{U} \\ y^u(t_0) - y_0 = 0}} L\left(y(t_1), \lambda^*\right), \quad \text{for some } \lambda^* \text{ satisfying} \quad (3.22), \tag{3.23}$$

where L is the Lagrangian:

$$L(\xi, \lambda) := G(\xi) + \lambda \cdot \xi. \quad \text{for all} \quad \xi \in \mathbb{R}^{n+1}.$$

Remark 3.8. In the context of a Lagrange formulation, the function G is given by (3.18), and is obviously convex.

We are now in the context of application of Theorem 3.5: suppose that u^* is an optimal control variable in $\mathcal{U}$ for the problem (3.23), Let $x^* := x^{u^*}$ be the corresponding state, then there exists an adjoint state $q : [t_0, t_1] \longrightarrow \mathbb{R}^n$ defined by the adjoint equation

$$\dot{q}(t) = -g_y^T\left(t, y^*(t), u^*(t)\right), \tag{3.24}$$

together with the transversality condition:

$$q(t_1) = L_y\left(y(t_1), \lambda\right),$$

such that

$$q(t) \cdot g\left(t, y^*(t), u^*(t)\right) = \min_{\nu \in U} q(t) \cdot g\left(t, y^*(t), \nu\right). \tag{3.25}$$

Using the Kuhn and Tucker conditions (3.22) on the multiplier λ, we re-write the transversality condition as:

$$q^i(t_1) \geq G_y^i\left(y^*(t_1)\right), \ p^j(t_1) \leq G_y^j\left(y^*(t_1)\right), \ \text{and } p^\ell(t_1) = G_y^\ell\left(y^*(t_1)\right)$$
$$\text{for all } (i, j, \ell) \in I(y^*) \times J(y^*) \times L(y^*). \tag{3.26}$$

In conclusion, the Pontryagin maximum principle in the optimal control problem with constrained final state (3.21) states the existence of an adjoint state system defined by the adjoint equation (3.24) and the transversality condition (3.26), such that the triple (y^*, u^*, q) satisfies (3.25).

Returning back to the initial variables, we now state the Pontryagin maximum principle for the Largrange formulation (3.20) with constrained final state.

Theorem 3.9. *Let the conditions of theorem 3.5 hold true. Suppose that u^* is an optimal control for the problem* (3.20)*, and let $x^* := x^{u^*}$ be the corresponding state. Then, there exists a C^1-function $p : [t_0, t_1] \longrightarrow \mathbb{R}^n$ such that for all $t \in [t_0, t_1]$:*

(i) $H\left(t, x^*(t), u^*(t), p(t)\right) = \min_{\nu \in U} H\left(t, x^*(t), \nu, p(t)\right)$,

(ii) $\dot{p}(t) = -H_x\left(t, x^*(t), u^*(t), p(t)\right)$,

(iii) *p satisfies the transversality conditions:*

$$p^i(t_1) \geq 0, \ p^j(t_1) \leq 0, \ p^\ell(t_1) = 0$$
$$\textit{for all } (i, j, \ell) \in I(x^*) \times J(x^*) \times L(x^*).$$

Remark 3.10. In the context of a maximization problem, i.e. supremum instead of infimum in (3.20). Then it is easily seen that the Hamiltonian is defined similarly with a supremum substituted to the infimum, and **all inequalities in the transversality conditions are reversed.**

3.7 Formal reduction to a calculus of variations problem

In this section, we consider a control problem in the Lagrange formulation:

$$\inf_{\substack{u \in \mathcal{U} \\ x^u(t_0) = x_0}} \int_{t_0}^{t_1} F\left(t, x^u(t), u(t)\right) \, dt, \tag{3.27}$$

with (unconstrained) state dynamics governed by the state equation:

$$\dot{x}(t) \;=\; f\left(t, x(t), u(t)\right) \quad \text{pour} \quad t_0 \le t \le t_1.$$

Our main objective is to derive the Pontryagin maximum principle as a consequence of the Lagrange Theorem 1.13 and the local Euler equation of Theorem 2.2.

The subsequent discussion relies on formal arguments, and it is only intended to strengthen the intuition of the reader. We also observe that our arguments extend with no further difficulties to the case where the final state is subject to constraints.

In order the reduce the control problem (3.27) to a problem of calculus of variations, we consider the state equation as an equality constraint. We then introduce the corresponding Lagrange multiplier $p(t) \in \mathbb{R}^n$ for all $t \in [t_0, t_1]$, and we define the Lagrangian:

$$\begin{aligned} L\left(x, \dot{x}, u, p\right) \;&:=\; \int_{t_0}^{t_1} \left[F\left(t, x(t), u(t)\right) - p(t) \cdot \dot{x}(t) + p(t) \cdot f\left(t, x(t), u(t)\right)\right] dt \\ &=\; \int_{t_0}^{t_1} \left[H\left(t, x(t), u(t), p(t)\right) - p(t) \cdot \dot{x}(t)\right] dt, \end{aligned}$$

where H is the Hamiltonian of the system. Since $x(t_1)$ is not constrained we impose

$$p(t_1) \;=\; 0. \tag{3.28}$$

Given the Lagrange multiplier $p(.)$, we minimize the Lagrangian with respect to the variables x, $\dot{x}$, u, which are now unconstrained. The minimization with respect to the control variable u implies that the optimal control u^* satisfies:

$$H^*\left(t, x(t), p(t)\right) \;:=\; H\left(t, x(t), u^*(t), p(t)\right) \;=\; \min_{\nu \in U} H\left(t, x(t), \nu, p(t)\right)$$

for all $t \in [t_0, t_1]$. We are then reduced to the problem of calculus of variations:

$$\min \int_{t_0}^{t_1} \left[H^*\left(t, x(t), p(t)\right) - p(t) \cdot \dot{x}(t)\right] dt.$$

By Theorem 2.2, we have the first order condition:

$$\dot{p}(t) \quad = \quad -H_x^*\left(t, x^*(t), p(t)\right),$$

which together with (3.28) is exactly the adjoint equation.

3.8 A sufficient condition of optimality

In this section, we consider the optimal control problem with constrained final state

$$\inf_{\substack{u \in \mathcal{U} \\ x^u(t_0) - x_0 = 0 \\ x^u(t_1) - x_1 \in C(I,J,K)}} \int_{t_0}^{t_1} F\left(t, x^u(t), u(t)\right) dt, \tag{3.29}$$

where we use the same notations as in Section 3.6.

Theorem 3.11. *Let $u^* \in \mathcal{U}$ be such that the corresponding controlled state $x^* = x^{u^*}$ satisfies the constraints of the problem* (3.29). *Assume further that (u^*, x^*) satisfy the necessary conditions of Theorem 3.9 and that the function*

$$H^*(t, \xi, \pi) \quad := \quad \min_{\nu \in U} H(t, \xi, \nu, \pi)$$

- *is convex in ξ for all $(t, \pi) \in [t_0, t_1] \times \mathbb{R}^n$,*
- *and $H_x^*\big(t, x^*(t), u^*(t)\big) = H_x\big(t, x^*(t), u^*(t), p(t)\big)$ for all $t \in [t_0, t_1]$.*

Then u^* is an optimal control for the problem (3.29).

Proof. Let $u \in \mathcal{U}$ be such that the corresponding state x, with initial condition $x(t_0) = x_0$, satisfies the constraints $x(t_1) - x_1 \in C(I, J, K)$. In order to prove the required result, we have to verify that

$$\delta \ := \int_{t_0}^{t_1} F\left(t, x^*(t), u^*(t)\right) dt - \int_{t_0}^{t_1} F\left(t, x(t), u(t)\right) dt \quad \leq \quad 0. \tag{3.30}$$

By definition of the Hamiltonian system, we have

$$\begin{aligned} \delta \quad = \quad & \int_{t_0}^{t_1} \left[H\left(t, x^*(t), u^*(t), p(t)\right) - H\left(t, x(t), u(t), p(t)\right)\right] dt \\ & + \int_{t_0}^{t_1} p(t) \cdot \left[f\left(t, x(t), u(t)\right) - f\left(t, x^*(t), u^*(t)\right)\right] dt, \end{aligned}$$

where $p(t)$ is the adjoint state of the systeme. Since u^* satisfies

$$H\left(t, x^*(t), u^*(t), p(t)\right) \quad = \quad \min_{\nu \in U} H\left(t, x^*(t), \nu, p(t)\right) \;=\; H^*\left(t, x^*(t), p(t)\right) \ ,$$

we deduce from the state equation that:

$$\begin{aligned} \delta \quad &= \quad \int_{t_0}^{t_1} \left[H^*\left(t, x^*(t), p(t)\right) - H\left(t, x(t), u(t), p(t)\right)\right] dt \\ &\quad\quad + \int_{t_0}^{t_1} p(t) \cdot \left[\dot{x}(t) - \dot{x}^*(t)\right] dt \\ &\leq \quad \int_{t_0}^{t_1} \left[H^*\left(t, x^*(t), p(t)\right) - H^*\left(t, x(t), p(t)\right)\right] dt \\ &\quad\quad + \int_{t_0}^{t_1} p(t) \cdot \left[\dot{x}(t) - \dot{x}^*(t)\right] dt. \end{aligned}$$

By the convexity of $H^*(t, \xi, \pi)$ in ξ, we have:

$$H^*\left(t, x(t), p(t)\right) \quad \geq \quad H^*\left(t, x^*(t), p(t)\right) + H_x^*\left(t, x^*(t), p(t)\right) \cdot \left[x(t) - x^*(t)\right] \ .$$

Using the adjoint equation governing the dynamics of the adjoint state, this provides:

$$\begin{aligned} \delta \quad &\leq \quad -\int_{t_0}^{t_1} H_x^*\left(t, x^*(t), p(t)\right) \cdot \left[x(t) - x^*(t)\right] dt + \int_{t_0}^{t_1} p(t) \cdot \left[\dot{x}(t) - \dot{x}^*(t)\right] dt \\ &= \quad \int_{t_0}^{t_1} \left[\dot{p}(t) \cdot \left(x(t) - x^*(t)\right) + p(t) \cdot \left(\dot{x}(t) - \dot{x}^*(t)\right)\right] dt \\ &= \quad \int_{t_0}^{t_1} \frac{d}{dt} \left\{p(t) \cdot \left(x(t) - x^*(t)\right)\right\} dt \\ &= \quad p(t_1) \cdot \left(x(t_1) - x^*(t_1)\right) \end{aligned}$$

since $x(t_0) = x^*(t_0) = x_0$. We decompose the latter scalar product so as to distinguish between those constraints which are binding from the others:

$$\begin{aligned} \delta \quad \leq \quad & \sum_{i \in I(x^*)} p^i(t_1) \left(x^i(t_1) - x_1^i\right) \\ & + \sum_{j \in J(x^*)} p^j(t_1) \left(x^j(t_1) - x_1^j\right) \\ & + \sum_{\ell \in L(x^*)} p^\ell(t_1) \left(x^\ell(t_1) - x^{*\ell}(t_1)\right), \end{aligned}$$

where we used the fact that $x^k(t_1) = x^{*k}(t_1) = x_1^k$ for all $k \in K$. Finally, we observe that $x^i(t_1) - x_1^i \leq 0$ for $i \in I(x^*)$ and $x^j(t_1) - x_1^j \geq 0$ for $j \in J(x^*)$. Inequlity (3.30) follows from the transversality condition of Theorem 3.9:

$$\begin{gathered} p^i(t_1) \geq 0,\ p^j(t_1) \leq 0 \text{ et } p^\ell(t_1) = 0 \\ \text{for all } (i, j, \ell) \in I(x^*) \times J(x^*) \times L(x^*). \end{gathered}$$

♢

3.9 Examples

3.9.1 Linear quadratic regulator

In this classical example, the control variable u takes values in $U = \mathbb{R}^p$ and the state equation is linear in (x, u):

$$\dot{x} \;=\; A(t)x(t) + B(t)u(t),$$

where A and B are two functions defined on $[t_0, t_1]$ and taking values respectively in $\mathcal{M}_{\mathbb{R}}(n, n)$ and $\mathcal{M}_{\mathbb{R}}(n, p)$.

We consider the optimal control problem in the Lagrange form:

$$\inf_{\substack{u \in \mathcal{U} \\ x^u(t_0) = x_0}} \int_{t_0}^{t_1} F\left(t, x^u(t), u(t)\right) dt,$$

where the instantaneous cost function F is linear in x and u:

$$F(t, \xi, \nu) \;:=\; \xi \cdot M(t)\xi + \nu \cdot N(t)\nu,$$

and M, N are functions defined on $[t_0, t_1]$ with values respectively in $\mathcal{S}_{\mathbb{R}}^{++}(n)$ and $\mathcal{S}_{\mathbb{R}}^{++}(p)$ (set of symmetric semidefinite positive matrices of size n and p).

We first write the Hamiltonian of the system:

$$H(t, \xi, \nu, \pi) \;:=\; \xi \cdot M(t)\xi + \nu \cdot N(t)\nu + \pi \cdot \left[A(t)\xi + B(t)\nu\right].$$

Notice that H is convex in ν, because $N(t)$ is a positive matrix. The candidate optimal control is obtained by minimizing the Hamiltonian with respect to the control:

$$\min_{\nu \in U} H(t, \xi, \nu, \pi) \;=\; H(t, \xi, u^*(t), \pi) \quad \text{with} \quad u^*(t) \;:=\; -\frac{1}{2} N(t)^{-1} B(t)^T \pi.$$

The state associated to this optimal control is then defined by:

$$\dot{x}(t) \;=\; A(t)x(t) - \frac{1}{2} B(t)N(t)^{-1}B(t)^T p(t), \qquad x(0) = x_0.$$

Finally, the adjoint state equation is:

$$\dot{p}(t) \;=\; A(t)^T p(t) + 2M(t)x(t),$$

with transversality condition:

$$p(t_1) \;=\; 0.$$

The pair (x, p) is then defined by the first order linear system:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{p}(t) \end{pmatrix} \;=\; \begin{pmatrix} A(t) & -\frac{1}{2}B(t)N(t)^{-1}B(t)^T \\ 2M(t) & A(t)^T \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix},$$

with boundary conditions

$$x(t_0) \;=\; x_0, \qquad p(t_1) \;=\; 0.$$

3.9.2 A two-consumption goods model

Consider an agent facing two consumption goods. The main ingredients of the model are:

- the relative price of the consumption good 2 with respect to the consumption good 1, denoted by $y(t)$,

- the external revenue of the agent expressed in terms of the consumption good 1, with instantaneous rate denoted by $s(t)$,

- the instantaneous interest rate $r(t)$,

- the total wealth of the agent expressed in terms of the consumption good 1, denoted by $x(t)$.

Assuming that the agent's capital invested in the bank produces the return corresponding to the interest rate r, we see that the dynamics of the wealth is:

$$\dot{x}(t) \quad = \quad r(t)x(t) + s(t) - c_1(t) - y(t)c_2(t), \tag{3.31}$$

where $c_i(t)$ is the rate of consumption in the consumption good i at time t. Hence, the control variable is the pair (c_1, c_2), a function from $[0, T]$ with values in $U = \mathbb{R}^2_+$.

Let $x_0 > 0$ be some given initial capital. The agent's problem is defined by:

$$\sup_{\substack{(c_1,c_2)\in\mathcal{U} \\ x(0)=x_0 \\ x(T)\geq 0}} \quad \int_0^T e^{-\delta t} U\left(c_1(t), c_2(t)\right)\, dt\ ,$$

where $\delta > 0$ is a discount facteur, and

$$(\sigma_1, \sigma_2) \in \mathbb{R}^2_+ \quad \longmapsto \quad U(\sigma_1, \sigma_2)$$

is a C^1-function concave in (σ_1, σ_2), and increasing with respect to both arguments. To simplify the analysis, we assume that

$$\frac{\partial U}{\partial \sigma_i}(\sigma_1, \sigma_2) \; = \; +\infty \quad \text{for} \quad (\sigma_1, \sigma_2) \; \in \; \partial\mathbb{R}^2_+. \tag{3.32}$$

1. We first write the first order conditions. The dynamics of the adjoint state is given by

$$\dot{p}(t) \quad = \quad -r(t)p(t)dt.$$

We also notice that terminal state constraint $x(T) \geq 0$ is necessarilly binding for the optimal trajectory, i.e.

$$x^*(T) \quad = \quad 0. \tag{3.33}$$

Then, the transversality condition for our **maximisation** problem is

$$p_T \; := \; p(T) \quad \geq \quad 0.$$

Then, the adjoint state is given by:

$$p(t) \;=\; p_T e^{\int_t^T r(s)ds} \quad \text{pour tout} \quad t \in [0,T].$$

The Hamiltonian of the system is given by:

$$H\left(t, \xi, \sigma_1, \sigma_2, \pi\right) \;:=\; e^{-\delta t} U(\sigma_1, \sigma_2) + \pi\left[r(t)\xi + s(t) - \sigma_1 - y(t)\sigma_2\right].$$

Since H is concave with respect to the pair (c_1, c_2), the maximization of the Hamiltonian is characterized by the first order condition:

$$\begin{cases} e^{-\delta t}\dfrac{\partial U}{\partial \sigma_1}\left(c_1^*(t), c_2^*(t)\right) \;=\; p(t) \\ e^{-\delta t}\dfrac{\partial U}{\partial \sigma_2}\left(c_1^*(t), c_2^*(t)\right) \;=\; p(t)y(t), \end{cases} \tag{3.34}$$

ignoring the positivity restriction on the consumption rates c_1 et c_2, by virtue of (3.32).

2. We continue in the setting of

$$U(\sigma_1, \sigma_2) \;=\; \ln\left[V(\sigma_1, \sigma_2)\right],$$

where V is a concave function, homogeneous of degree 1, and increasing in both arguments. Then, the system (3.34) can be written in:

$$\begin{cases} V\left(c_1^*(t), c_2^*(t)\right) - c_2^*(t)V_{\sigma_2}\left(1, \frac{c_2^*(t)}{c_1^*(t)}\right) \;=\; e^{+\delta t}p(t)c_1^*(t)V\left(c_1^*(t), c_2^*(t)\right) \\ c_2^*(t)V_{\sigma_2}\left(1, \frac{c_2^*(t)}{c_1^*(t)}\right) \;=\; e^{+\delta t}p(t)y(t)c_2^*(t)V\left(c_1^*(t), c_2^*(t)\right). \end{cases} \tag{3.35}$$

Adding up the two equations, we get:

$$\hat{c}(t) \;:=\; c_1^*(t) + y(t)c_2^*(t) \;=\; p_T^{-1}e^{-\delta t - \int_t^T r(s)ds}.$$

Returning to the state equation, we see that we can express it in terms of this variable:

$$\dot{x}^*(t) \;=\; r(t)x^*(t) + s(t) - \hat{c}(t),$$

which provides the explicit solution in terms of the initial condition $x^*(0) = x_0$:

$$x^*(t) \;=\; x_0 e^{\int_0^t r(s)ds} - \frac{1 - e^{-\delta t}}{\delta p_T}e^{-\int_t^T r(s)ds} + \int_0^t s(u)e^{\int_u^t r(v)dv}du.$$

Finally, the value of the constant is determined by writing the condition (3.33):

$$p_T \;=\; \frac{\left(1 - e^{-\delta t}\right)/\delta}{x_0 e^{\int_0^T r(s)ds} + \int_0^T s(u)e^{\int_u^T r(v)dv}du},$$

thus identifying completely the optimal state and adjoint state. The optimal consumption rates are obtained by solving the systeme (3.35).

3. In order to push further the explicit calculations, we now specify the function V as

$$V(\sigma_1, \sigma_2) \quad := \quad \sigma_1^{\alpha} \sigma_2^{1-\alpha},$$

where α is a parameter in the interval $(0,1)$. Direct calculation leads to the optimal controls:

$$c_1^*(t) \;=\; \frac{\alpha}{p(t)} e^{-\delta t} \quad \text{and} \quad c_2^*(t) \;=\; \alpha y(t), \quad t \in [0,T].$$

3.9.3 Optimal growth with non-renewable resources

This example was already considered in Section 2.5.3 and solved as a problem of calculus of variations. We recall the optimal control problem:

$$\sup_{\substack{(c,r) \in \mathcal{U} \\ x(0) = x_0 \\ x(T) = 0}} \int_0^T \ln c(t) dt$$

with controlled state variable $x := (y,k)$ defined by the state equation:

$$\dot{y}(t) \;=\; -r(t) \quad \text{and} \quad \dot{k}(t) \;=\; ak(t)^{1-\alpha} r(t)^{\alpha} - c(t).$$

The Hamiltonian of the system is:

$$H(y,k,c,r,\pi,\mu) \quad := \quad \ln c - \pi r + \mu \left[ak^{1-\alpha} r^{\alpha} - c\right].$$

Notice that H is strictly concave in (c,r). Then the maximum is obtained by the first order condition:

$$\begin{cases} \frac{1}{c^*(t)} - q(t) \;=\; 0 \\ -p(t) + \alpha a q(t) k^*(t)^{1-\alpha} r^*(t)^{\alpha-1} \;=\; 0. \end{cases} \tag{3.36}$$

The dynamics of the adjoint state is governed by the adjoint equation:

$$\dot{p}(t) \;=\; 0 \quad \text{and} \quad \dot{q}(t) \;=\; -a(1-\alpha) \left(\frac{r^*(t)}{k^*(t)}\right)^{\alpha} q(t). \tag{3.37}$$

We introduce the variable $z(t) = r^*(t)/k^*(t)$. From the first equation in (3.37), we see that $p(t) = \pi$ for all $t \in [0,T]$. Then, differentiating the second equation of (3.36) with respect to t, we get:

$$(1-\alpha)\frac{\dot{z}(t)}{z(t)} \quad = \quad \frac{\dot{q}(t)}{q(t)},$$

so that the second equation in (3.37) reduces to:

$$z(t)^{-(1+\alpha)}\dot{z}(t) \quad = \quad -a.$$

Then, there exists a constant b such that

$$a\left(\frac{r(t)}{k(t)}\right)^{\alpha} \quad = \quad az(t)^{\alpha} \;=\; \frac{1}{b+\alpha t}.$$

We have then determined the expression of the adjoint variables up to two constants:

$$p(t) \;=\; \pi \quad \text{and} \quad q(t) \;=\; \frac{\pi}{\alpha}a^{-1/\alpha}(b+\alpha t)^{1-\frac{1}{\alpha}}.$$

This allows to determine the optimal consumption (up to two constants) by the first equation in (3.36):

$$c^*(t) \quad = \quad \frac{\alpha}{\pi}a^{1/\alpha}(b+\alpha t)^{-1+\frac{1}{\alpha}},$$

and the dynamics of the state variable k^* is then given by:

$$\begin{aligned} \dot{k}^*(t) \quad &= \quad az(t)^{\alpha}k^*(t) - c^*(t) \\ &= \quad (b+\alpha t)^{-1}k^*(t) - \frac{\alpha}{\pi}a^{1/\alpha}(b+\alpha t)^{-1+\frac{1}{\alpha}}. \end{aligned}$$

Given the boundary condition $k^*(T) = 0$, this provides:

$$k^*(t) \quad = \quad \frac{\alpha}{\pi}a^{1/\alpha}(b_2+\alpha t)^{1/\alpha}\ln\left(\frac{b_2+\alpha T}{b_2+\alpha t}\right)^{1/\alpha},$$

The state equation for the variable y is:

$$\dot{y}^*(t) \quad = \quad -z(t)k^*(t).$$

Together with the boundary condition $y^*(T) = 0$, this provides:

$$y(t) \quad = \quad \frac{\alpha}{\pi}\int_t^T \ln\left(\frac{b_2+\alpha T}{b_2+\alpha s}\right)^{1/\alpha} ds.$$

Finally, we determine the constants π and b by writing:

$$k^*(0) \;=\; k_0 \quad \text{and} \quad y^*(0) \;=\; y_0.$$

Chapter 4

The dynamic programming approach

As in the previous chapters, we are concerned with the optimal control problems

$$\inf_{u\in\mathcal{U}} \int_{t_0}^{t_1} F\left(t, x(t), u(t)\right) dt + G\left(x(t_1)\right) \tag{4.1}$$

where the controlled state is defined by the dynamics

$$x(t_0) = x_0, \quad \text{and} \quad \dot{x}(t) \;=\; f\left(t, x(t), u(t)\right). \tag{4.2}$$

Here, $\mathcal{U}$ denotes the set of all piecewise continuous functions $u :\; [t_0, t_1] \longrightarrow U$, a closed subset of $\mathbb{R}^k$.

The function $f :\; [t_0, t_1] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$ satisfies the Lipschitz and linear growth conditions of Theorem 3.3, which ensure existence of a unique solution to the state equation (4.2) for every control variable $u \in \mathcal{U}$.

We assume that the function $F :\; [t_0, t_1] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$ is continuous.

4.1 The dynamic value function

The dynamic programming approach to the control problem (4.1) exploits the dynamic feature of the system, and introduces *the dynamic version* of the problem by placing the time origin at any time $t \in [t_0, t_1]$.

Admissible controls : in order to define the evolution of the system on $[t, t_1]$, we only need the restriction of the control variable to $[t, t_1]$. We then introduce

$$\mathcal{U}_t \quad := \quad \left\{u \;:\; u = u^0|_{[t,t_1]} \text{ for some } u^0 \in \mathcal{U}\right\}$$

the set of piecewise continuous maps from $[t, t_1]$ to U.

The state equation of the system is characterized by an initial condition at time t and a control variable $u \in \mathcal{U}_t$:

$$x(t) = \xi, \quad \text{and} \quad \dot{x}(s) = f\left(s, x(s), u(s)\right).$$

The cost function relative to the remaining time period $[t, t_1]$:

$$J\left(t, \xi, u\right) \quad := \quad \int_t^{t_1} F\left(s, x(s), u(s)\right) ds + G\left(x(t_1)\right),$$

where the dependence of the state on the control variable has been omitted.

The dynamic value function associated to the problem (4.1) is defined by:

$$V(t, \xi) \quad := \quad \inf_{u \in \mathcal{U}_t} J(t, \xi, u), \tag{4.3}$$

so that the problem (4.1) corresponding to the time origin t_0 is given by $V(t_0, x_0)$. The dynamic programming approach solves the problem $V(0, x_0)$ by analyzing the dependence of V in the variables t and ξ.

4.2 The dynamic programming principle

Theorem 4.1. *Let $t \in [t_0, t_1[$ and $\xi \in \mathbb{R}^n$ be given. Then, for all $s \in [t, t_1]$, we have:*

$$V(t, \xi) \quad = \quad \inf_{u \in \mathcal{U}_t} \left\{ \int_t^s F\left(r, x(r), u(r)\right) dr + V\left(s, x(s)\right) \right\}.$$

Proof. For $t \in [t_0, t_1[$, $s \in [t, t_1]$ and $\xi \in \mathbb{R}^n$ fixed, we denote

$$W(t, \xi) \quad := \quad \inf_{u \in \mathcal{U}_t} \left\{ \int_t^s F\left(r, x(r), u(r)\right) dr + V\left(s, x(s)\right) \right\}.$$

1. To prove that $V \leq W$, we consider two arbitrary control variables $u \in \mathcal{U}_t$ and $v \in \mathcal{U}_s$, and we observe that

$$w \quad := \quad u\mathbf{1}_{[t,s[} + v\mathbf{1}_{[s,t_1[} \tag{4.4}$$

defines a control variable in $\mathcal{U}_t$. Then, by definition de $V(t, \xi)$, we have:

$$\begin{aligned}
V(t, \xi) \quad \leq \quad J(t, \xi, w) \;&=\; \int_t^{t_1} F\left(r, x(r), w(r)\right) dr + G\left(x(t_1)\right) \\
&= \int_t^s F\left(r, x(r), u(r)\right) dr + \int_s^{t_1} F\left(r, x(r), v(r)\right) dr + G\left(x(t_1)\right) \\
&= \int_t^s F\left(r, x(r), u(r)\right) dr + J\left(s, x(s), v\right).
\end{aligned}$$

By minimizing over $u \in \mathcal{U}_t$ and $v \in \mathcal{U}_s$, this provides the inequality $V \leq W$.

2. We now prove the converse inequality $V \geq W$. For $\varepsilon > 0$, let $u_\varepsilon \in \mathcal{U}_t$ be an $\varepsilon-$optimal control variable for the problem $V(t,\xi)$:

$$V(t,\xi) \quad \leq \quad J(t,\xi,u_\varepsilon) \;\leq\; V(t,\xi)+\varepsilon.$$

Notice that the function $\tilde{u}_\varepsilon := u_\varepsilon|_{[s,t_1]}$ is a control variable in $\mathcal{U}_s$. Then it follows from the definition of J that:

$$\begin{aligned} W(t,\xi) &\leq \int_t^s F\left(r,x(r),u_\varepsilon(r)\right)dr + V\left(s,x(s)\right) \\ &\leq \int_t^s F\left(r,x(r),u_\varepsilon(r)\right)dr + J\left(s,x(s),\tilde{u}_\varepsilon\right) \\ &= J\left(t,x_t,u_\varepsilon\right) \\ &\leq V(t,\xi)+\varepsilon, \end{aligned}$$

and the required inequality follows from the arbitrariness of $\varepsilon > 0$. ♢

Remark 4.2. The main argument in the previous proof is the concatenation of control variable in(4.4). Here, the definition of the set of admissible controls is important. For instance if the controls choice was restricted to continuous maps from $[t_0,t_1]$ to U, then the concatenation property (4.4) is not true. Still, this does not mean that the dynamic programming principle would not be true in such a context, but one needs to involve some approximation argument...

Remark 4.3. The dynamic programming principle says in particular that
(i) the function

$$s \quad \longmapsto \quad \int_t^s F\left(r,x(r),u(r)\right)dr + V\left(s,x(s)\right)$$

is non-decreasing, for all control variable $u \in \mathcal{U}_t$,
(ii) If an optimal control $u^* \in \mathcal{U}_t$ exists for the problem (4.3), i.e. $V(t,\xi) = J(t,\xi,u^*)$, then the function

$$s \quad \longmapsto \quad \int_t^s F\left(r,x^*(r),u^*(r)\right)dr + V\left(s,x^*(s)\right)$$

is constant, where we denoted as usual $x^* := x^{u^*}$. Indeed, from the decrease property in (i) and the fact that $V(t_1,x_{t_1}) = G(x_{t_1})$, we see that:

$$\begin{aligned} V(t,\xi) &\leq \int_t^{t_1} F\left(r,x(r),u^*(r)\right)dr + V\left(t_1,x^*(t_1)\right) \\ &= \int_t^{t_1} F\left(r,x(r),u^*(r)\right)dr + G\left(x^*(t_1)\right) \\ &= J(t,\xi,u^*) \;=\; V(t,\xi). \end{aligned}$$

Remark 4.4. From the previous remark, it follows that if $u^* \in \mathcal{U}_t$ is an optimal control for the problem $V(t,\xi)$, then the restriction of u^* to the interval $[s,t_1]$ is an optimal control for the problem $V(s,x^*(s))$ for all $s \in [t,t_1]$.

4.3 The dynamic programming equation

Recall the definition of the Hamiltonian of the system:

$$H(t,\xi,\nu,\pi) \quad := \quad F(t,\xi,\nu)+\pi\cdot f(t,\xi,\nu)$$

for all $(t,\xi,\nu)\in[t_0,t_1]\times\mathbb{R}^n\times U$. As in the Pontryagin maximum principle approach, the optimal control is related to the minimization of the Hamiltonian. We then define:

$$H^*(t,\xi,\pi) \quad := \quad \inf_{\nu\in U} H(t,\xi,\nu,\pi).$$

Theorem 4.5. *Suppose that the function V is $C^1\left([t_0,t_1]\times\mathbb{R}^n\right)$. Then:*
(i) *V is a supersolution of the dynamic programming equation:*

$$\frac{\partial V}{\partial t}(t,\xi)+H^*\left(t,\xi,D_xV(t,\xi)\right) \;\geq\; 0 \quad \text{for} \quad (t,\xi)\in[t_0,t_1[\times\mathbb{R}^n.$$

(ii) *Assume in addition that the function H^* is continuous, then V is a solution of the dynamic programming equation:*

$$\frac{\partial V}{\partial t}(t,\xi)+H^*\left(t,\xi,D_xV(t,\xi)\right) \;=\; 0 \quad \text{for} \quad (t,\xi)\in[t_0,t_1[\times\mathbb{R}^n.$$

Proof. (i) By the dynamic programming principle,

$$V(t,\xi) \quad \leq \quad \int_t^{t+h} F\left(r,x(r),u(r)\right)dr+V\left(t+h,x(t+h)\right)$$

for all $h\in\,]0,t_1-t]$ et $u\in\mathcal{U}_t$. Consider a constant control variable $u(r)=\nu$ for all $r\in[t,t_1]$ for some arbitrary $\nu\in U$. Since V is C^1, we can rewrite the previous inequality as:

$$0 \quad \leq \quad \frac{1}{h}\int_t^{t+h}\left\{\frac{\partial V}{\partial t}(r,x(r))+H\left(r,x(r),\nu,D_xV(r,x(r))\right)\right\}dr.$$

We next observe that the function inside the integral is continuous in the time variable. By sending h to zero, we then deduce from the mean value theorem that

$$0 \quad \leq \quad \frac{\partial V}{\partial t}(t,\xi)+H\left(t,\xi,\nu,D_xV(t,\xi)\right),$$

and the required result follows from the arbitrariness of $\nu\in U$.

(ii) To prove the second part of the theorem, we assume to the contrary the existence of $(t^*,\xi^*)\in[t_0,t_1[\times\mathbb{R}^n$ such that

$$\frac{\partial V}{\partial t}(t^*,\xi^*)+H^*\left(t^*,\xi^*,D_xV(t^*,\xi^*)\right) \quad > \quad 0,$$

and we work towards a contradiction. Let

$$\varphi(t,\xi) \;:=\; V(t,\xi) - |t-t^*|^2 - |\xi-\xi^*|^2 \quad \text{for} \quad (t,\xi) \in [t_0,t_1]\times\mathbb{R}^n.$$

Since $DV(t^*,\xi^*) = D\varphi(t^*,\xi^*)$, we see that

$$\frac{\partial\varphi}{\partial t}(t^*,\xi^*) + H^*\left(t^*,\xi^*,D_x\varphi(t^*,\xi^*)\right) \quad > \quad 0$$

and, by the continuity of H^*, there exists $\delta > 0$ such that

$$\frac{\partial\varphi}{\partial t}(t,\xi) + H^*\left(t,\xi,D_x\varphi(t,\xi)\right) \;\geq\; 0 \tag{4.5}$$

$$\text{for all } \; (t,\xi)\in Q_\delta := [t^*,t^*+\delta]\times\overline{B}_\delta(\xi^*) \tag{4.6}$$

where $\overline{B}_\delta(\xi^*)$ is the closed ball with radius δ centered at ξ^*. Since (t^*,ξ^*) is a point of strict minimum for the difference $V-\varphi$, we have:

$$2\varepsilon \;:=\; \min_{\partial Q_\delta}(V-\varphi) \quad > \quad 0. \tag{4.7}$$

Finally, let $u_\varepsilon \in \mathcal{U}_{t^*}$ be an $\varepsilon-$optimal control for the problem $V(t^*,\xi^*)$, $x_\varepsilon := x^{u_\varepsilon}$ the corresponding state, and $h_\varepsilon > 0$ the time duration defined by

$$t^* + h_\varepsilon \quad := \quad \inf\left\{t > t_0 \;:\; (t,x_\varepsilon(t)) \notin Q_\delta\right\}.$$

By continuity of x_ε, we have $(t^*+h_\varepsilon, x_\varepsilon(t^*+h_\varepsilon)) \in \partial Q_\delta$, and therefore:

$$V\left(t^*+h_\varepsilon, x_\varepsilon(t^*+h_\varepsilon)\right) \quad \geq \quad 2\varepsilon + \varphi\left(t^*+h_\varepsilon, x_\varepsilon(t^*+h_\varepsilon)\right), \tag{4.8}$$

by definition of ε in (4.7).

Since u_ε is an $\varepsilon-$optimal control, this provides:

$$\begin{aligned} V(t^*,\xi^*) + \varepsilon \quad &\geq \quad J(t^*,\xi^*,u_\varepsilon) \\ &= \quad \int_{t^*}^{t^*+h_\varepsilon} F\left(r,x_\varepsilon(r),u_\varepsilon(r)\right)dr + J\left(t^*+h_\varepsilon, x_\varepsilon(t^*+h_\varepsilon),\tilde{u}_\varepsilon\right) \\ &\geq \quad \int_{t^*}^{t^*+h_\varepsilon} F\left(r,x_\varepsilon(r),u_\varepsilon(r)\right)dr + V\left(t^*+h_\varepsilon, x_\varepsilon(t^*+h_\varepsilon)\right), \end{aligned}$$

where $\tilde{u}_\varepsilon := u_\varepsilon|_{[t^*+h_\varepsilon,t_1]}$. Recalling that $V(t^*,\xi^*) = \varphi(t^*,\xi^*)$ and using (4.8), we see that:

$$\varphi(t^*,\xi^*) + \varepsilon \quad \geq \quad \int_{t^*}^{t^*+h_\varepsilon} F\left(r,x_\varepsilon(r),u_\varepsilon(r)\right)dr + 2\varepsilon + \varphi\left(t^*+h_\varepsilon, x_\varepsilon(t^*+h_\varepsilon)\right),$$

and therefore:

$$\begin{aligned} -\varepsilon \quad &\geq \quad \int_{t^*}^{t^*+h_\varepsilon} \left\{\frac{\partial\varphi}{\partial t}(r,x_\varepsilon(r)) + H\left(r,x_\varepsilon(r),u_\varepsilon(r),D_x\varphi(r,x_\varepsilon(r))\right)\right\}dr \\ &\geq \quad \int_{t^*}^{t^*+h_\varepsilon} \left\{\frac{\partial\varphi}{\partial t}(r,x_\varepsilon(r)) + H^*\left(r,x_\varepsilon(r),D_x\varphi(r,x_\varepsilon(r))\right)\right\}dr \\ &\geq \quad 0, \end{aligned}$$

since $(r, x_\varepsilon(r)) \in Q_\delta$ for $t^* \leq r \leq t^* + h_\varepsilon$. Notice that this inequality is in contradiction with (4.7), thus completing the proof. ♢

Before closing this section, we observe that the C^1 regularity assumption on the value function is too strong. Indeed, it is easy to construct examples of control problems with nonsmooth value function (see example below). This is the main motivation to interprete the dynamic programming equation in some weak sense. There are two alternative routes which exist in the literature:

- either, rewrite the proof of the theorem using the notion of generalized derivatives; this approach requires to prove that the value function has this weak regularity, see Fleming and Rishel [**?**],
- or use the theory of viscosity solutions which only requires the value function to be locally bounded; this approach will be developed later in the more general context of stochastic control problems.

Example (*Optimal control problem with nonsmooth value function*) Let $f(t, \xi, \nu) = \nu$, $U = [-1, 1]$, and $n = 1$. The controlled state is defined by:

$$x(t) \;=\; x + \int_{t_0}^{t} u(s)ds \quad \text{for} \quad t_0 \leq t \leq t_1,$$

and the control problem is:

$$V(t,x) \quad := \quad \sup_{u \in \mathcal{U}} |x(t_1)|^2 \;=\; \sup_{u \in \mathcal{U}} \left(x + \int_t^{t_1} u(s)ds \right)^2 .$$

It is easily seen that:

$$V(t,x) \quad = \quad \begin{cases} (x + t_1 - t)^2 & \text{for } x \geq 0 \quad \text{with optimal control } \hat{\nu} = 1, \\ (x - t_1 + t)^2 & \text{for } x \leq 0 \quad \text{with optimal control } \hat{\nu} = -1. \end{cases}$$

This function is continuous, but is not differentiable at the point $x = 0$.

4.4 The verification argument

In this section, we provide sufficient conditions so that a solution of the dynamic programming equation can be identified to the dynamic value function of the problem (4.1).

Theorem 4.6. *Let* $W \; : [t_0, t_1] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a* C^1*−function.*
(i) *If*

$$W(t_1, \xi) \;\leq\; G(\xi) \quad \text{and} \quad -\frac{\partial W}{\partial t}(t, \xi) - H^*\left(t, \xi, D_x W(t, \xi)\right) \;\leq\; 0,$$

then $W \leq v$.

(ii) *If*

$$W(t_1,\xi) \;=\; G(\xi), \qquad -\frac{\partial W}{\partial t}(t,\xi) - H^*\left(t,\xi,D_xW(t,\xi)\right) \;=\; 0,$$

and there exists a control variable $u^* \in \mathcal{U}_t$ *such that for all* $s \in [t,T]$*:*

$$H^*\left(s,x^*(s),D_xW(s,x^*(s))\right) = H\left(s,x^*(s),u^*(s),D_xW(s,x^*(s))\right),$$

then $V = W$*.*

Proof. Let u be a control variable in $\mathcal{U}_t$ and $x := x^u$ the corresponding state with initial condition $x(t) = \xi$. Since W is C^1, we have:

$$\begin{aligned} W(t_1,x(t_1)) &= W(t,\xi) + \int_t^{t_1} \left\{ \frac{\partial W}{\partial t}(r,x(r)) + D_xW(r,x(r)) \cdot \dot{x}(r) \right\} dr \\ &= W(t,\xi) + \int_t^{t_1} \left\{ \frac{\partial W}{\partial t}(r,x(r)) + D_xW(r,x(r)) \cdot f(r,x(r),u(r)) \right\} dr \\ &= W(t,\xi) - \int_t^{t_1} F(r,x(r),u(r))dr \\ &\quad + \int_t^{t_1} \left\{ \frac{\partial W}{\partial t}(r,x(r)) + H\left(r,x(r),u(r),D_xW(r,x(r))\right) \right\} dr. \end{aligned}$$

By the definition of H^* together with the differential inequality satisfied by W, this provides:

$$\begin{aligned} W(t_1,x(t_1)) &\geq W(t,\xi) - \int_t^{t_1} F(r,x(r),u(r))dr \\ &\quad + \int_t^{t_1} \left\{ \frac{\partial W}{\partial t}(r,x(r)) + H^*\left(r,x(r),D_xW(r,x(r))\right) \right\} dr \\ &\geq W(t,\xi) - \int_t^{t_1} F(r,x(r),u(r))dr. \end{aligned}$$

Finally, since $W(t_1,.) \leq G$ and the control variable u is arbitrary in $\mathcal{U}_t$, this implies that:

$$\begin{aligned} W(t,\xi) &\leq \inf_{u\in\mathcal{U}_t} \int_t^{t_1} F(r,x(r),u(r))dr + W(t_1,x(t_1)) \\ &\leq \inf_{u\in\mathcal{U}_t} \int_t^{t_1} F(r,x(r),u(r))dr + G(x(t_1)) \;=\; V(t,\xi). \end{aligned}$$

(ii) We follow the above argument with the control variable u^* introduced in Part (ii) of the theorem, and we observe that all inequalities turn into equalities.
♢

Remark 4.7. The control variable u^* introduced in Theorem 8.1 (ii) is obtained by minimizing the function $H\left(t, x^*(t), \nu, D_x W(t, x^*(t))\right)$. Consequently, $u^*(t) = \hat{\nu}\left[t, x^*(t), D_x W(t, x^*(t))\right]$ for some function $\hat{\nu}$, and the state equation is given by:

$$\dot{x}^*(t) \quad = \quad g\left(t, x^*(t)\right) \;:=\; f\left(t, x^*(t), \hat{\nu}\left[t, x^*(t), \nu, D_x W(t, x^*(t))\right]\right).$$

In order to guarantee that u^* is an admissible control, i.e. $u^* \in \mathcal{U}_t$, we have to check that the obove ordinary differential equation has a unique solution...

4.5 Examples

4.5.1 Linear quadratic regulator

We start by the example developed in Section 3.9.1. We recall that the control variable u takes values in $U = \mathbb{R}^p$ and the state equation is linear in (x, u):

$$\dot{x} \quad = \quad A(t)x(t) + B(t)u(t),$$

where A and B are two functions defined on $[t_0, t_1]$ with values in $\mathcal{M}_{\mathbb{R}}(n, n)$ and $\mathcal{M}_{\mathbb{R}}(n, p)$, respectively. The control problem is:

$$\inf_{\substack{u \in \mathcal{U} \\ x^u(t_0) = x_0}} \quad \int_{t_0}^{t_1} F\left(t, x^u(t), u(t)\right) \, dt \;+\; G\left(x^u(t_1)\right),$$

where the instantaneous cost function F is quadratic in x and u:

$$F(t, \xi, \nu) \;:=\; \xi \cdot M(t)\xi + \nu \cdot N(t)\nu \quad \text{and} \quad G(\xi) \;:=\; \xi \cdot Q\xi,$$

M, N are two functions defined on $[t_0, t_1]$ and valued respectively in $\mathcal{S}_{\mathbb{R}}^{++}(n)$ and $\mathcal{S}_{\mathbb{R}}^{++}(p)$, and $Q \in \mathcal{S}_{\mathbb{R}}^{+}(p)$.

Since $N(t)$ is a positive matrix, the Hamiltonian of the system

$$H(t, \xi, \nu, \pi) \quad := \quad \xi \cdot M(t)\xi + \nu \cdot N(t)\nu + \pi \cdot \left[A(t)\xi + B(t)\nu\right]$$

is a convex function of ν. The candidate optimal control is obtained by minimizing the Hamiltonian with respect to the control

$$H^*(t, \xi, \pi) \;=\; H(t, \xi, u^*(t), \pi) \quad \text{with} \quad u^*(t) \;:=\; -\frac{1}{2} N(t)^{-1} B(t)^T \pi,$$

and

$$H^*(t, \xi, \pi) \quad = \quad \xi \cdot M(t)\xi + \pi \cdot A(t)\xi - \frac{1}{4}\pi \cdot B(t)N(t)^{-1}B(t)^T\pi.$$

The dynamc programming equation is given by:

$$\begin{aligned} 0 \quad &= \quad \frac{\partial V}{\partial t}(t, \xi) + H^*\left(t, \xi, D_x V(t, \xi)\right) \\ &= \quad \frac{\partial V}{\partial t}(t, \xi) + \xi \cdot M(t)\xi + D_x V(t, \xi) \cdot A(t)\xi \\ &\qquad -\frac{1}{4} D_x V(t, \xi) \cdot B(t)N(t)^{-1}B(t)^T D_x V(t, \xi). \end{aligned}$$

We search for a solution of the form

$$V(t,\xi) \;=\; \xi\cdot K(t)\xi, \quad \text{for} \quad t_0\le t\le t_1, \tag{4.9}$$

for some function $K:[t_0,t_1]\longrightarrow \S_{\mathbb{R}}^{+}(n)$. Notice that the boundary condition $V(t_1,\xi)=\xi\cdot Q\xi$ is compatible with this form and imposes

$$K(t_1) \;=\; Q.$$

Injecting this form in the dynamic programming equation, it follows from arbitrariness of $\xi\in\mathbb{R}^n$ that:

$$\dot K(t) \;=\; K(t)\cdot B(t)N(t)^{-1}B(t)^TK(t)-2K(t)\cdot A(t)-M(t).$$

The latter is a Riccati equation which can be solved explicitly in some cases. Finally, the solution of the problem is completely characterized by verifying that the candidate value function $V(t,\xi)$ satisfies the conditions of the verification theorem.

4.5.2 An optimal consumption model

The state variable is governed by the dynamics

$$\dot x(t) \;=\; -c(t) \quad \text{and} \quad x(0) \;=\; x_0,$$

where $c(t)$ is the consumption rate at time t. The preferences of the agent are defined by the utility function:

$$U(c) \;=\; \int_0^T e^{-\beta t}u\left(c(t)\right)dt+e^{-\beta T}u\left(x(T)\right)$$

where

$$u(\xi) \;:=\; \frac{\xi^\gamma}{\gamma},$$

and $0<\gamma<1$ is a given parameter. we recall that the positivity constraint on the consumption can be ignored. The problem of optimal consumption is defined by:

$$\sup_{c\in\mathcal{U}} U(c),$$

where $\mathcal{U}$ is the set of piecewise continuous functions from $[t_0,t_1]$ to $\mathbb{R}^+$.

In the context of this example, the Hamiltonian is given by

$$H(t,\xi,\sigma,\pi) \;:=\; e^{-\beta t}u(\sigma)-\pi\sigma.$$

Since H is concave with respect to the control σ, we directly calculate that

$$H^*(t,\xi,\pi) \;=\; H(t,\xi,\sigma^*(t),\pi) \quad \text{with} \quad \sigma^*(t) \;:=\; \left(\pi e^{\beta t}\right)^{-1/(1-\gamma)},$$

which leads to:

$$H^*(t,\xi,\pi) \quad = \quad \frac{1-\gamma}{\gamma} e^{-\beta t} \left(\pi e^{\beta t}\right)^{-\gamma/(1-\gamma)} .$$

The dynamic programming equation is:

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t}(t,\xi) + H^*\left(t,\xi,D_x V(t,\xi)\right) \\ &= \frac{\partial V}{\partial t}(t,\xi) + \frac{1-\gamma}{\gamma} e^{-\beta t} \left(e^{\beta t} D_x V(t,\xi)\right)^{-\gamma/(1-\gamma)} . \end{aligned}$$

Let us seak for a solution of the form

$$V(t,\xi) \;=\; e^{-\beta t} A(t) u(\xi), \quad \text{for} \quad t_0 \le t \le t_1.$$

Since $V(T,\xi) = e^{-\beta T} u(\xi)$, the function A must satisfy the boundary condition

$$A(T) \quad = \quad 1.$$

Substituting in the dynamic programming equation, we see that $A(.)$ must satisfy the ordinary differential equation:

$$\dot{A}(t) + (1-\gamma) A(t)^{-\gamma/(1-\gamma)} - \beta A(t) \quad = \quad 0,$$

or equivalent:

$$\frac{d}{dt}\left\{A(t)^{1/(1-\gamma)}\right\} \quad = \quad \frac{\beta}{1-\gamma} A(t)^{1/(1-\gamma)} - 1.$$

In view of the boundary condition $A(T) = 1$, this provides the unique solution

$$A(t) \quad = \quad \left(\frac{1-\gamma}{\beta} + \left(1 - \frac{1-\gamma}{\beta}\right) e^{-\frac{\beta}{1-\gamma}(T-t)}\right)^{1-\gamma} .$$

To conclude that the candidate function $V(t,\xi)$ found above coincides with the dynamic value function, it only remains to check that V satisfies the conditions of the verification theorem...

4.5.3 Nonsmooth value function at isolated points

Reviewing the proof of Theorem 8.1, we see that *the statement of the theorem remains true if the candidate value function W is C^1 except at a set of isolated points.*

To illustrate this, we consider the example of Section 4.3 where the value function V is defined by

$$V(t,x) \quad := \quad \sup_{u \in \mathcal{U}} |x(t_1)|^2,$$

with state equation

$$\dot{x}(t) \;=\; u(t)dt \quad \text{and} \quad x(t_0) \;=\; x,$$

and controls u taking values in $U = [-1, 1]$.

Then the value function is given by

$$V(t,x) \quad = \quad \begin{cases} (x+T-t)^2 & \text{pour } x \geq 0 \\ (x-T+t)^2 & \text{pour } x \leq 0. \end{cases}$$

Notice that V is C^1 on $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$. Despite the nonsmoothness on the axis $x = 0$, we now verify that V solves the dynamic programming equation at any point of smoothness.

The Hamiltonian of the problem is:

$$H(t, \xi, \nu, \pi) \quad := \quad \nu\pi,$$

and can be maximized explicitly:

$$H^*(t, \xi, \pi) \quad := \quad \sup_{|\nu| \leq 1} H(t, \xi, \nu, \pi) \;=\; |\pi|.$$

By direct calculation, we verify that V solves the dynamic programming equation

$$\frac{\partial V}{\partial t} + |D_x V| \quad = \quad 0$$

at any point $(t, x) \in [t_0, t_1] \times (\mathbb{R} \setminus \{0\})$.

4.6 Pontryaging maximum principle and dynamic programming

Recall that the Pontryagin maximum principle leads to a system of ordinary differential equations for the optimal state (subject to an initial condition) and the corresponding adjoint state (subject to a final condition).

The dynamic programming approach leads instead to a partial differential equation on the value function with given boundary condition at the final date.

In this section, we show the connection between the two approaches. The following arguments are purely formal and ignore all difficulties related to the regularity of the value function.

Given the dynamic value function $V(t, x)$, we introduce the function

$$p(t) \;:=\; D_x V\left(t, x^*(t)\right); \qquad t_0 \leq t \leq t_1. \tag{4.10}$$

Since $V(t_1, .) = G$, the function p satisfies the transversality condition

$$p(t_1) \quad = \quad D_x G\left(x(t_1)\right).$$

We now verify that p satisfies the adjoint state equation:

$$\dot{p}(t) \quad = \quad -H_x^* \left(t, x^*(t), p(t)\right), \tag{4.11}$$

so that p is indeed the adjoint state introduced in the Pontryagin maximum principle. To obtain (4.11), we differentiate (4.10) with respect to t:

$$\begin{aligned} \dot{p}(t) &= \frac{d}{dt}\left\{D_x V\left(t, x^*(t)\right)\right\} \\ &= \frac{\partial}{\partial x}\left\{\frac{\partial V}{\partial t}\left(t, x^*(t)\right)\right\} + D_{xx} V\left(t, x^*(t)\right) \dot{x}^*(t). \end{aligned}$$

Since $\dot{x}^*(t) = f\left(t, x^*(t), u^*(t)\right) = \left(\partial H^*/\partial p\right)\left(t, x^*(t), D_x V(t, x^*(t))\right)$, we see that:

$$\begin{aligned} \dot{p}(t) &= \frac{\partial}{\partial x}\left\{\frac{\partial V}{\partial t}\left(t, x^*(t)\right)\right\} \\ &\quad + D_{xx} V\left(t, x^*(t)\right) \frac{\partial H^*}{\partial p}\left(t, x^*(t), D_x V(t, x^*(t))\right). \end{aligned} \tag{4.12}$$

Finally, using the dynamic programming equation, we obtain:

$$\begin{aligned} \frac{\partial}{\partial x}\left\{\frac{\partial V}{\partial t}\left(t, x^*(t)\right)\right\} &= -\frac{\partial}{\partial x}\left\{H^*\left(t, x^*(t), D_x V(t, x^*(t))\right)\right\} \\ &= -\frac{\partial H^*}{\partial x}\left(t, x^*(t), D_x V(t, x^*(t))\right) \\ &\quad - D_{xx} V\left(t, x^*(t)\right) \frac{\partial H^*}{\partial p}\left(t, x^*(t), D_x V(t, x^*(t))\right), \end{aligned}$$

and (4.11) follows by substitution in (4.12).

Chapter 5

CONDITIONAL EXPECTATION AND LINEAR PARABOLIC PDEs

Throughout this chapter, $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t,\, t \geq 0\}$ satisfying the usual conditions. Let $W = \{W_t,\, t \geq 0\}$ be a Brownian motion valued in $\mathbb{R}^d$, defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Throughout this chapter, a maturity $T > 0$ will be fixed. By $\mathbb{H}^2$, we denote the collection of all progressively measurble processes ϕ with appropriate (finite) dimension such that $\mathbb{E}\left[\int_0^T |\phi_t|^2 dt\right] < \infty$.

5.1 Stochastic differential equations with random coefficients

In this section, we recall the basic tools from stochastic differential equations

$$dX_t \quad = \quad b_t(X_t)dt + \sigma_t(X_t)dW_t, \;\; t \in [0,T], \tag{5.1}$$

where $T > 0$ is a given maturity date. Here, b and σ are $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}^n)$-progressively measurable functions from $[0,T] \times \Omega \times \mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathcal{M}_{\mathbb{R}}(n,d)$, respectively. In particular, for every fixed $x \in \mathbb{R}^n$, the processes $\{b_t(x), \sigma_t(x), t \in [0,T]\}$ are $\mathbb{F}-$progressively measurable.

Definition 5.1. *A strong solution of* (5.1) *is an* $\mathbb{F}-$*progressively measurable process* X *such that* $\int_0^T (|b_t(X_t)| + |\sigma_t(X_t)|^2)dt < \infty$*, a.s. and*

$$X_t \quad = \quad X_0 + \int_0^t b_s(X_s)ds + \int_0^t \sigma_s(X_s)dW_s, \quad t \in [0,T].$$

Let us mention that there is a notion of weak solutions which relaxes some conditions from the above definition in order to allow for more general stochastic differential equations. Weak solutions, as opposed to strong solutions, are defined on some probabilistic structure (which becomes part of the solution), and not necessarily on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$. Thus, for a weak solution we search for a probability structure $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W})$ and a process $\tilde{X}$ such that the requirement of the above definition holds true. Obviously, any strong solution is a weak solution, but the opposite claim is false.

The main existence and uniqueness result is the following.

Theorem 5.2. *Let $X_0 \in \mathbb{L}^2$ be a r.v. independent of W. Assume that the processes $b_.(0)$ and $\sigma_.(0)$ are in $\mathbb{H}^2$, and that for some $K > 0$:*

$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \le K|x - y| \quad \text{for all} \quad t \in [0, T],\ x, y \in \mathbb{R}^n.$$

Then, for all $T > 0$, there exists a unique strong solution of (5.1) in $\mathbb{H}^2$. Moreover,

$$\mathbb{E}\left[\sup_{t \le T} |X_t|^2\right] \;\le\; C\left(1 + \mathbb{E}|X_0|^2\right) e^{CT}, \tag{5.2}$$

for some constant $C = C(T, K)$ depending on T and K.

Proof. We first establish the existence and uniqueness result, then we prove the estimate (5.2).

<u>*Step 1*</u> For a constant $c > 0$, to be fixed later, we introduce the norm

$$\|\phi\|_{\mathbb{H}^2_c} := \mathbb{E}\left[\int_0^T e^{-ct} |\phi_t|^2 dt\right]^{1/2} \qquad \text{for every} \quad \phi \in \mathbb{H}^2.$$

Clearly , the norms $\|.\|_{\mathbb{H}^2}$ and $\|.\|_{\mathbb{H}^2_c}$ on the Hilbert space $\mathbb{H}^2$ are equivalent. Consider the map U on $\mathbb{H}^2$ by:

$$U(X)_t \;:=\; X_0 + \int_0^t b_s(X_s) ds + \int_0^t \sigma_s(X_s) dW_s, \qquad 0 \le t \le T.$$

By the Lipschitz property of b and σ in the $x-$variable and the fact that $b_.(0), \sigma_.(0) \in \mathbb{H}^2$, it follows that this map is well defined on $\mathbb{H}^2$. In order to prove existence and uniqueness of a solution for (5.1), we shall prove that $U(X) \in \mathbb{H}^2$ for all $X \in \mathbb{H}^2$ and that U is a contracting mapping with respect to the norm $\|.\|_{\mathbb{H}^2_c}$ for a convenient choice of the constant $c > 0$.

1- We first prove that $U(X) \in \mathbb{H}^2$ for all $X \in \mathbb{H}^2$. To see this, we decompose:

$$\begin{aligned} \|U(X)\|^2_{\mathbb{H}^2} \;\le\;& 3T\|X_0\|^2_{\mathbb{L}^2} + 3T\mathbb{E}\left[\int_0^T \left|\int_0^t b_s(X_s) ds\right|^2 dt\right] \\ &+ 3\mathbb{E}\left[\int_0^T \left|\int_0^t \sigma_s(X_s) dW_s\right|^2 dt\right] \end{aligned}$$

By the Lipschitz-continuity of b and σ in x, uniformly in t, we have $|b_t(x)|^2 \leq K(1+|b_t(0)|^2+|x|^2)$ for some constant K. We then estimate the second term by:

$$\mathbb{E}\left[\int_0^T \left|\int_0^t b_s(X_s)ds\right|^2 dt\right] \quad \leq \quad KT\mathbb{E}\left[\int_0^T (1+|b_t(0)|^2+|X_s|^2)ds\right] < \infty,$$

since $X \in \mathbb{H}^2$, and $b(.,0) \in \mathbb{L}^2([0,T])$.

As, for the third term, we use the Doob maximal inequality together with the fact that $|\sigma_t(x)|^2 \leq K(1+|\sigma_t(0)|^2+|x|^2)$, a consequence of the Lipschitz property on σ:

$$\begin{aligned} \mathbb{E}\left[\int_0^T \left|\int_0^t \sigma_s(X_s)dW_s\right|^2 dt\right] &\leq T\mathbb{E}\left[\max_{t\leq T}\left|\int_0^t \sigma_s(X_s)dW_s\right|^2 dt\right] \\ &\leq 4T\mathbb{E}\left[\int_0^T |\sigma_s(X_s)|^2 ds\right] \\ &\leq 4TK\mathbb{E}\left[\int_0^T (1+|\sigma_s(0)|^2+|X_s|^2)ds\right] < \infty. \end{aligned}$$

2- To see that U is a contracting mapping for the norm $\|.\|_{\mathbb{H}^2_c}$, for some convenient choice of $c>0$, we consider two process $X, Y \in \mathbb{H}^2$ with $X_0 = Y_0$, and we estimate that:

$$\begin{aligned} & \mathbb{E}\left|U(X)_t - U(Y)_t\right|^2 \\ \leq\ & 2\mathbb{E}\left|\int_0^t (b_s(X_s)-b_s(Y_s))\,ds\right|^2 + 2\mathbb{E}\left|\int_0^t (\sigma_s(X_s)-\sigma_s(Y_s))\,dW_s\right|^2 \\ =\ & 2\mathbb{E}\left|\int_0^t (b_s(X_s)-b_s(Y_s))\,ds\right|^2 + 2\mathbb{E}\int_0^t |\sigma_s(X_s)-\sigma_s(Y_s)|^2\,ds \\ =\ & 2t\mathbb{E}\int_0^t |b_s(X_s)-b_s(Y_s)|^2\,ds + 2\mathbb{E}\int_0^t |\sigma_s(X_s)-\sigma_s(Y_s)|^2\,ds \\ \leq\ & 2(T+1)K \int_0^t \mathbb{E}\left|X_s - Y_s\right|^2 ds. \end{aligned}$$

Hence, $\|U(X)-U(Y)\|_c \leq \dfrac{2K(T+1)}{c}\|X-Y\|_c$, and therefore U is a contracting mapping for sufficiently large c.

Step 2 We next prove the estimate (5.2). We shall alleviate the notation writing $b_s := b_s(X_s)$ and $\sigma_s := \sigma_s(X_s)$. We directly estimate:

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t}|X_u|^2\right] &= \mathbb{E}\left[\sup_{u\le t}\left|X_0+\int_0^u b_s ds+\int_0^u \sigma_s dW_s\right|^2\right]\\ &\le 3\left(\mathbb{E}|X_0|^2+t\mathbb{E}\left[\int_0^t |b_s|^2 ds\right]+\mathbb{E}\left[\sup_{u\le t}\left|\int_0^u \sigma_s dW_s\right|^2\right]\right)\\ &\le 3\left(\mathbb{E}|X_0|^2+t\mathbb{E}\left[\int_0^t |b_s|^2 ds\right]+4\mathbb{E}\left[\int_0^t |\sigma_s|^2 ds\right]\right)\end{aligned}$$

where we used the Doob's maximal inequality. Since b and σ are Lipschitz-continuous in x, uniformly in t and ω, this provides:

$$\mathbb{E}\left[\sup_{u\le t}|X_u|^2\right] \le C(K,T)\left(1+\mathbb{E}|X_0|^2+\int_0^t \mathbb{E}\left[\sup_{u\le s}|X_u|^2\right] ds\right)$$

and we conclude by using the Gronwall lemma. ♢

The following exercise shows that the Lipschitz-continuity condition on the coefficients b and σ can be relaxed. We observe that further relaxation of this assumption is possible in the one-dimensional case, see e.g. Karatzas and Shreve [**?**].

Exercise 5.3. *In the context of this section, assume that the coefficients μ and σ are locally Lipschitz and linearly growing in x, uniformly in (t,ω). By a localization argument, prove that strong existence and uniqueness holds for the stochastic differential equation* (5.1).

In addition to the estimate (5.2) of Theorem 5.2, we have the following flow continuity results of the solution of the SDE.

Theorem 5.4. *Let the conditions of Theorem 5.2 hold true, and consider some $(t,x)\in[0,T)\times\mathbb{R}^n$ with $t\le t'\le T$.*
(i) *There is a constant C such that:*

$$\mathbb{E}\left[\sup_{t\le s\le t'}\left|X_s^{t,x}-X_s^{t,x'}|^2\right|\right] \le Ce^{Ct'}|x-x'|^2. \tag{5.3}$$

(ii) *Assume further that $B := \sup_{t<t'\le T}(t'-t)^{-1}\mathbb{E}\int_t^{t'}\left(|b_r(0)|^2+|\sigma_r(0)|^2\right)dr < \infty$. Then for all $t'\in[t,T]$:*

$$\mathbb{E}\left[\sup_{t'\le s\le T}\left|X_s^{t,x}-X_s^{t',x}|^2\right|\right] \le Ce^{CT}(B+|x|^2)|t'-t|. \tag{5.4}$$

Proof. (i) To simplify the notations, we set $X_s := X_s^{t,x}$ and $X'_s := X_s^{t,x'}$ for all $s\in[t,T]$. We also denote $\delta x := x-x'$, $\delta X := X-X'$, $\delta b := b(X)-b(X')$ and

$\delta\sigma := \sigma(X) - \sigma(X')$. We first decompose:

$$\begin{aligned}|\delta X_s|^2 &\leq 3\left(|\delta x|^2 + \Big|\int_t^s \delta b_u du\Big|^2 + \Big|\int_t^s \delta\sigma_u dW_u\Big|^2\right)\\ &\leq 3\left(|\delta x|^2 + (s-t)\int_t^s \big|\delta b_u\big|^2 du + \int_t^s \delta\sigma_u dW_u\Big|^2\right).\end{aligned}$$

Then, it follows from the Doob maximal inequality and the Lipschitz property of the coefficients b and σ that:

$$\begin{aligned}h(t') := \mathbb{E}\left[\sup_{t\leq s\leq t'}|\delta X_s|^2\right] &\leq 3\left(|\delta x|^2 + (s-t)\int_t^s \mathbb{E}\big|\delta b_u\big|^2 du + 4\int_t^s \mathbb{E}\big|\delta\sigma_u\big|^2 d_u\right)\\ &\leq 3\left(|\delta x|^2 + K^2(t'+4)\int_t^s \mathbb{E}|\delta X_u|^2 du\right)\\ &\leq 3\left(|\delta x|^2 + K^2(t'+4)\int_t^s h(u)du\right).\end{aligned}$$

Then the required estimate follows from the Gronwall inequality.

2. We next prove (5.4). We again simplify the notation by setting $X_s := X_s^{t,x}$, $s \in [t,T]$, and $X'_s := X_s^{t',x}$, $s \in [t',T]$. We also denote $\delta t := t'-t$, $\delta X := X - X'$, $\delta b := b(X) - b(X')$ and $\delta\sigma := \sigma(X) - \sigma(X')$. Then following the same arguments as in the previous step, we obtain for all $u \in [t',T]$:

$$\begin{aligned}h(u) := \mathbb{E}\left[\sup_{t'\leq s\leq u}|\delta X_s|^2\right] &\leq 3\left(\mathbb{E}|X_{t'} - x|^2 + K^2(T+4)\int_{t'}^u \mathbb{E}|\delta X_r|^2 dr\right)\\ &\leq 3\left(\mathbb{E}|X_{t'} - x|^2 + K^2(T+4)\int_{t'}^u h(r)dr\right) \quad (5.5)\end{aligned}$$

Observe that

$$\begin{aligned}\mathbb{E}|X_{t'} - x|^2 &\leq 2\left(\mathbb{E}\Big|\int_t^{t'} b_r(X_r)dr\Big|^2 + \mathbb{E}\Big|\int_t^{t'} \sigma_r(X_r)dr\Big|^2\right)\\ &\leq 2\left(T\int_t^{t'} \mathbb{E}|b_r(X_r)|^2 dr + \int_t^{t'} \mathbb{E}|\sigma_r(X_r)|^2 dr\right)\\ &\leq 6(T+1)\int_t^{t'} \big(K^2\mathbb{E}|X_r - x|^2 + |x|^2 + \mathbb{E}|b_r(0)|^2\big)dr\\ &\leq 6(T+1)\Big((t'-t)(|x|^2 + B) + K^2\int_t^{t'} \mathbb{E}|X_r - x|^2 dr\Big).\end{aligned}$$

By the Gronwall inequality, this shows that

$$\mathbb{E}|X_{t'} - x|^2 \quad \leq \quad C(|x|^2 + B)|t'-t|e^{C(t'-t)}.$$

Plugging this estimate in (5.5), we see that:

$$h(u) \quad \leq \quad 3\left(C(|x|^2 + B)|t'-t|e^C(t'-t) + K^2(T+4)\int_{t'}^u h(r)dr\right), \quad (5.6)$$

and the required estimate follows from the Gronwall inequality. ◇

5.2 Markov solutions of SDEs

In this section, we restrict the coefficients b and σ to be deterministic functions of (t,x). In this context, we write

$$b_t(x) = b(t,x),\ \sigma_t(x) = \sigma(t,x) \quad \text{for} \quad t \in [0,T],\ x \in \mathbb{R}^n,$$

where b and σ are continuous functions, Lipschitz in x uniformly in t. Let $X^{t,x}_{.}$ denote the solution of the stochastic differential equation

$$X^{t,x}_s = x + \int_t^s b\big(u, X^{t,x}_u\big)du + \int_t^s \sigma\big(u, X^{t,x}_u\big)dW_u \ \ s \geq t$$

The two following properties are obvious:

- Clearly, $X^{t,x}_s = F\left(t,x,s,(W_. - W_t)_{t\leq u\leq s}\right)$ for some deterministic function F.
- For $t \leq u \leq s$: $X^{t,x}_s = X^{u,X^{t,x}_u}_s$. This follows from the pathwise uniqueness, and holds also when u is a stopping time.

With these observations, we have the following Markov property for the solutions of stochastic differential equations.

Proposition 5.5. *(Markov property) For all $0 \leq t \leq s$:*

$$\mathbb{E}\left[\Phi\left(X_u, t \leq u \leq s\right)|\mathcal{F}_t\right] \quad = \quad \mathbb{E}\left[\Phi\left(X_u, t \leq u \leq s\right)|X_t\right]$$

for all bounded function $\Phi :\ C([t,s]) \longrightarrow \mathbb{R}$.

5.3 Connection with linear partial differential equations

5.3.1 Generator

Let $\{X^{t,x}_s, s \geq t\}$ be the unique strong solution of

$$X^{t,x}_s \quad = \quad x + \int_t^s \mu(u, X^{t,x}_u)du + \int_t^s \sigma(u, X^{t,x}_u)dW_u,\ \ s \geq t,$$

where μ and σ satisfy the required condition for existence and uniqueness of a strong solution.

For a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we define the function $\mathcal{A}f$ by

$$\mathcal{A}f(t,x) = \lim_{h\to 0} \frac{\mathbb{E}[f(X^{t,x}_{t+h})] - f(x)}{h} \quad \text{if the limit exists.}$$

Clearly, $\mathcal{A}f$ is well-defined for all bounded C^2- function with bounded derivatives and

$$\mathcal{A}f(t,x) \quad = \quad \mu(t,x)\cdot f(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}(t,x)\frac{\partial^2 f}{\partial x\partial x^{\mathrm{T}}}\right], \tag{5.7}$$

(Exercise !). The linear differential operator $\mathcal{A}$ is called the *generator* of X. It turns out that the process X can be completely characterized by its generator or, more precisely, by the generator and the corresponding domain of definition...

As the following result shows, the generator provides an intimate connection between conditional expectations and linear partial differential equations.

Proposition 5.6. *Assume that the function* $(t,x) \longmapsto v(t,x) := \mathbb{E}\left[g(X_T^{t,x}\right]$ *is* $C^{1,2}\left([0,T)\times\mathbb{R}^n\right)$*. Then* v *solves the partial differential equation:*

$$\frac{\partial v}{\partial t} + \mathcal{A}v = 0 \quad \textit{and} \quad v(T,.) = g.$$

Proof. Given (t,x), let $\tau_1 := T \wedge \inf\{s > t : |X_s^{t,x} - x| \geq 1\}$. By the law of iterated expectation together with the Markov property of the process X, it follows that

$$v(t,x) \quad = \quad \mathbb{E}\left[v\left(s\wedge\tau_1, X_{s\wedge\tau_1}^{t,x}\right)\right].$$

Since $v \in C^{1,2}([0,T),\mathbb{R}^n)$, we may apply Itô's formula, and we obtain by taking expectations:

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_t^{s\wedge\tau_1}\left(\frac{\partial v}{\partial t}+\mathcal{A}v\right)(u,X_u^{t,x})du\right] \\ &\quad +\mathbb{E}\left[\int_t^{s\wedge\tau_1}\frac{\partial v}{\partial x}(u,X_s^{t,x})\cdot\sigma(u,X_u^{t,x})dW_u\right] \\ &= \mathbb{E}\left[\int_t^{s\wedge\tau_1}\left(\frac{\partial v}{\partial t}+\mathcal{A}v\right)(u,X_u^{t,x})du\right], \end{aligned}$$

where the last equality follows from the boundedness of $(u, X_u^{t,x})$ on $[t, s\wedge\tau_1]$. We now send $s \searrow t$, and the required result follows from the dominated convergence theorem. ♢

5.3.2 Cauchy problem and the Feynman-Kac representation

In this section, we consider the following linear partial differential equation

$$\begin{array}{l} \frac{\partial v}{\partial t} + \mathcal{A}v - k(t,x)v + f(t,x) = 0, \quad (t,x)\in[0,T)\times\mathbb{R}^d \\ v(T,.) = g \end{array} \tag{5.8}$$

where $\mathcal{A}$ is the generator (5.7), g is a given function from $\mathbb{R}^d$ to $\mathbb{R}$, k and f are functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}$, b and σ are functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}^d$ and and $\mathcal{M}_{\mathbb{R}}(d,d)$, respectively. This is the so-called Cauchy problem.

For example, when $k = f \equiv 0$, $b \equiv 0$, and σ is the identity matrix, the above partial differential equation reduces to the heat equation.

Our objective is to provide a representation of this purely deterministic problem by means of stochastic differential equations. We then assume that μ and

σ satisfy the conditions of Theorem 5.2, namely that

$$\mu, \sigma \quad \text{Lipschitz in } x \text{ uniformly in } t, \quad \int_0^T \left(|\mu(t,0)|^2 + |\sigma(t,0)|^2\right) dt < \infty \tag{5.9}$$

Theorem 5.7. *Let the coefficients* μ, σ *be continuous and satisfy* (5.9). *Assume further that the function* k *is uniformly bounded from below, and* f *has quadratic growth in* x *uniformly in* t. *Let* v *be a* $C^{1,2}\left([0,T), \mathbb{R}^d\right)$ *solution of* (5.8) *with quadratic growth in* x *uniformly in* t. *Then*

$$v(t,x) \;=\; \mathbb{E}\left[\int_t^T \beta_s^{t,x} f(s, X_s^{t,x}) ds + \beta_T^{t,x} g\left(X_T^{t,x}\right)\right], \qquad t \le T,\; x \in \mathbb{R}^d,$$

where $X_s^{t,x} := x + \int_t^s \mu(u, X_u^{t,x}) du + \int_t^s \sigma(u, X_u^{t,x}) dW_u$ *and* $\beta_s^{t,x} := e^{-\int_t^s k(u, X_u^{t,x}) du}$ *for* $t \le s \le T$.

Proof. We first introduce the sequence of stopping times

$$\tau_n \;:=\; T \wedge \inf\left\{s > t \;:\; \left|X_s^{t,x} - x\right| \ge n\right\},$$

and we oberve that $\tau_n \longrightarrow T$ $\mathbb{P}-$a.s. Since v is smooth, it follows from Itô's formula that for $t \le s < T$:

$$\begin{aligned} d\left(\beta_s^{t,x} v\left(s, X_s^{t,x}\right)\right) &= \beta_s^{t,x}\left(-kv + \frac{\partial v}{\partial t} + \mathcal{A}v\right)\left(s, X_s^{t,x}\right) ds \\ &\quad + \beta_s^{t,x} \frac{\partial v}{\partial x}\left(s, X_s^{t,x}\right) \cdot \sigma\left(s, X_s^{t,x}\right) dW_s \\ &= \beta_s^{t,x}\left(-f(s, X_s^{t,x}) ds + \frac{\partial v}{\partial x}\left(s, X_s^{t,x}\right) \cdot \sigma\left(s, X_s^{t,x}\right) dW_s\right), \end{aligned}$$

by the PDE satisfied by v in (5.8). Then:

$$\begin{aligned} &\mathbb{E}\left[\beta_{\tau_n}^{t,x} v\left(\tau_n, X_{\tau_n}^{t,x}\right)\right] - v(t,x) \\ &= \mathbb{E}\left[\int_t^{\tau_n} \beta_s^{t,x}\left(-f(s, X_s) ds + \frac{\partial v}{\partial x}\left(s, X_s^{t,x}\right) \cdot \sigma\left(s, X_s^{t,x}\right) dW_s\right)\right]. \end{aligned}$$

Now observe that the integrands in the stochastic integral is bounded by definition of the stopping time τ_n, the smoothness of v, and the continuity of σ. Then the stochastic integral has zero mean, and we deduce that

$$v(t,x) \;=\; \mathbb{E}\left[\int_t^{\tau_n} \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_{\tau_n}^{t,x} v\left(\tau_n, X_{\tau_n}^{t,x}\right)\right]. \tag{5.10}$$

Since $\tau_n \longrightarrow T$ and the Brownian motion has continuous sample paths $\mathbb{P}-$a.s. it follows from the continuity of v that, $\mathbb{P}-$a.s.

$$\begin{aligned} &\int_t^{\tau_n} \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_{\tau_n}^{t,x} v\left(\tau_n, X_{\tau_n}^{t,x}\right) \\ &\quad \overset{n \to \infty}{\longrightarrow} \int_t^T \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_T^{t,x} v\left(T, X_T^{t,x}\right) \qquad (5.11) \\ &\quad = \int_t^T \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_T^{t,x} g\left(X_T^{t,x}\right) \end{aligned}$$

by the terminal condition satisfied by v in (5.8). Moreover, since k is bounded from below and the functions f and v have quadratic growth in x uniformly in t, we have

$$\left|\int_t^{\tau_n} \beta_s^{t,x} f\left(s, X_s^{t,x}\right) ds + \beta_{\tau_n}^{t,x} v\left(\tau_n, X_{\tau_n}^{t,x}\right)\right| \quad \leq \quad C\left(1 + \max_{t \leq T} |X_t|^2\right) .$$

By the estimate stated in the existence and uniqueness theorem 5.2, the latter bound is integrable, and we deduce from the dominated convergence theorem that the convergence in (5.11) holds in $\mathbb{L}^1(\mathbb{P})$, proving the required result by taking limits in (5.10). ♢

The above Feynman-Kac representation formula has an important numerical implication. Indeed it opens the door to the use of Monte Carlo methods in order to obtain a numerical approximation of the solution of the partial differential equation (5.8). For sake of simplicity, we provide the main idea in the case $f = k = 0$. Let $\left(X^{(1)}, \ldots, X^{(k)}\right)$ be an iid sample drawn in the distribution of $X_T^{t,x}$, and compute the mean:

$$\hat{v}_k(t,x) \quad := \quad \frac{1}{k} \sum_{i=1}^{k} g\left(X^{(i)}\right) .$$

By the Law of Large Numbers, it follows that $\hat{v}_k(t,x) \longrightarrow v(t,x)$ $\mathbb{P}-$a.s. Moreover the error estimate is provided by the Central Limit Theorem:

$$\sqrt{k}\left(\hat{v}_k(t,x) - v(t,x)\right) \overset{k\to\infty}{\longrightarrow} \mathcal{N}\left(0, \mathbb{V}ar\left[g\left(X_T^{t,x}\right)\right]\right) \qquad \text{in distribution,}$$

and is remarkably independent of the dimension d of the variable X !

5.3.3 Representation of the Dirichlet problem

Let D be an open subset of $\mathbb{R}^d$. The *Dirichlet problem* is to find a function u solving:

$$\mathcal{A}u - ku + f = 0 \text{ on } D \quad \text{and} \quad u = g \text{ on } \partial D, \tag{5.12}$$

where ∂D denotes the boundary of D, and $\mathcal{A}$ is the generator of the process X^{0,X_0} defined as the unique strong solution of the stochastic differential equation

$$X_t^{0,X_0} = X_0 + \int_0^t \mu(s, X_s^{0,X_0}) ds + \int_0^t \sigma(s, X_s^{0,X_0}) dW_s, \ t \geq 0.$$

Similarly to the the representation result of the Cauchy problem obtained in Theorem 5.7, we have the following representation result for the Dirichlet problem.

Theorem 5.8. *Let u be a C^2-solution of the Dirichlet problem* (5.12)*. Assume that k is nonnegative, and*

$$\mathbb{E}[\tau_D^x] < \infty, \; x \in \mathbb{R}^d, \quad \textit{where} \quad \tau_D^x := \inf\left\{t \geq 0 : \; X_t^{0,x} \notin D\right\}.$$

Then, we have the representation:

$$u(x) \;=\; \mathbb{E}\left[g\left(X_{\tau_D^x}^{0,x}\right) e^{-\int_0^{\tau_D^x} k(X_s)ds} + \int_0^{\tau_D^x} f\left(X_t^{0,x}\right) e^{-\int_0^t k(X_s)ds} dt\right].$$

Exercise 5.9. *Provide a proof of Theorem 5.8 by imitating the arguments in the proof of Theorem 5.7.*

5.4 The stochastic control approach to the Black-Scholes model

5.4.1 The continuous-time financial market

Let T be a finite horizon, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting a Brownian motion $W = \{(W_t^1, \ldots, W_t^d), \, 0 \leq t \leq T\}$ with values in $\mathbb{R}^d$. We denote by $\mathbb{F} = \mathbb{F}^W = \{\mathcal{F}_t, \, 0 \leq t \leq T\}$ the canonical augmented filtration of W, i.e. the canonical filtration augmented by zero measure sets of $\mathcal{F}_T$.

We consider a financial market consisting of $d+1$ assets :

(i) The first asset S^0 is non-risky, and is defined by

$$S_t^0 \;=\; \exp\left(\int_0^t r_u du\right), \qquad 0 \leq t \leq T,$$

where $\{r_t, t \in [0,T]\}$ is a non-negative adapted processes with $\int_0^T r_t dt < \infty$ a.s., and represents the instantaneous interest rate.

(ii) The d remaining assets S^i, $i = 1, \ldots, d$, are risky assets with price processes defined by the dynamics

$$\frac{dS_t^i}{S_t^i} \;=\; \mu_t^i dt + \sum_{j=1}^d \sigma_t^{i,j} dW_t^j, \; t \in [0,T],$$

for $1 \leq i \leq d$, where μ, σ are $\mathbb{F}-$adapted processes with $\int_0^T |\mu_t^i| dt + \int_0^T |\sigma_t^{i,j}|^2 dt < \infty$ for all $i, j = 1, \ldots, d$. It is convenient to use the matrix notations to represent the dynamics of the price vector $S = (S^1, \ldots, S^d)$:

$$dS_t \;=\; S_t \star (\mu_t dt + \sigma_t dW_t), \; t \in [0,T],$$

where, for two vectors $x, y \in \mathbb{R}^d$, we denote $x \star y$ the vector of $\mathbb{R}^d$ with components $(x \star y)_i = x_i y_i$, $i = 1, \ldots, d$, and μ, σ are the $\mathbb{R}^d-$vector with components μ^i's, and the $\mathcal{M}_{\mathbb{R}}(d,d)-$matrix with entries $\sigma^{i,j}$.

We assume that the $\mathcal{M}_{\mathbb{R}}(d,d)-$matrix σ_t is invertible for every $t \in [0,T]$ a.s., and we introduce the process

$$\lambda_t := \sigma_t^{-1}\left(\mu_t - r_t\mathbf{1}\right), \qquad 0 \le t \le T,$$

called the *risk premium process*. Here $\mathbf{1}$ is the vector of ones in $\mathbb{R}^d$. We shall frequently make use of the discounted processes

$$\tilde{S}_t \ := \ \frac{S_t}{S_t^0} \ = \ S_t \exp\left(-\int_0^t r_u du\right),$$

Using the above matrix notations, the dynamics of the process $\tilde{S}$ are given by

$$d\tilde{S}_t \ = \ \tilde{S}_t \star \big((\mu_t - r_t\mathbf{1})dt + \sigma_t dW_t\big) \ = \ \tilde{S}_t \star \sigma_t\left(\lambda_t dt + dW_t\right).$$

5.4.2 Portfolio and wealth process

A portfolio strategy is an $\mathbb{F}-$adapted process $\pi = \{\pi_t, 0 \le t \le T\}$ with values in $\mathbb{R}^d$. For $1 \le i \le n$ and $0 \le t \le T$, π_t^i is the amount (in Euros) invested in the risky asset S^i.

We next recall the self-financing condition in the present framework. Let X_t^π denote the portfolio value, or wealth, process at time t induced by the portfolio strategy π. Then, the amount invested in the non-risky asset is $X_t^\pi - \sum_{i=1}^n \pi_t^i$ $= X_t^\pi - \pi_t \cdot \mathbf{1}$.

Under the self-financing condition, the dynamics of the wealth process is given by

$$dX_t^\pi \ = \ \sum_{i=1}^n \frac{\pi_t^i}{S_t^i}\, dS_t^i \ + \ \frac{X_t^\pi - \pi_t \cdot \mathbf{1}}{S_t^0}\, dS_t^0.$$

Let $\tilde{X}^\pi$ be the discounted wealth process

$$\tilde{X}_t^\pi \ := \ X_t^\pi \exp\left(-\int_0^t r(u)du\right), \qquad 0 \le t \le T.$$

Then, by an immediate application of Itô's formula, we see that

$$d\tilde{X}_t \ = \ \tilde{\pi}_t \cdot \sigma_t\left(\lambda_t dt + dW_t\right),\ 0 \le t \le T, \tag{5.13}$$

where $\tilde{\pi}_t := e^{-rt}\pi_t$. We still need to place further technical conditions on π, at least in order for the above wealth process to be well-defined as a stochastic integral.

Before this, let us observe that, assuming that the risk premium process satisfies the Novikov condition:

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T |\lambda_t|^2 dt}\right] \ < \ \infty,$$

it follows from the Girsanov theorem that the process

$$B_t \;:=\; W_t + \int_0^t \lambda_u du\,, \qquad 0 \le t \le T\,, \tag{5.14}$$

is a Brownian motion under the equivalent probability measure

$$\mathbb{Q} := Z_T \cdot \mathbb{P} \text{ on } \mathcal{F}_T \quad \text{where} \quad Z_T := \exp\left(-\int_0^T \lambda_u \cdot dW_u - \frac{1}{2}\int_0^T |\lambda_u|^2 du\right).$$

In terms of the $\mathbb{Q}$ Brownian motion B, the discounted price process satisfies

$$d\tilde{S}_t \;=\; \tilde{S}_t \star \sigma_t dB_t, \;\; t \in [0,T],$$

and the discounted wealth process induced by an initial capital X_0 and a portfolio strategy π can be written in

$$\tilde{X}^\pi_t \;=\; \tilde{X}_0 + \int_0^t \tilde{\pi}_u \cdot \sigma_u dB_u, \quad \text{for} \quad 0 \le t \le T. \tag{5.15}$$

Definition 5.10. *An admissible portfolio process $\pi = \{\theta_t, t \in [0,T]\}$ is an $\mathbb{F}-$progressively measurable process such that $\int_0^T |\sigma_t^{\mathrm{T}} \pi_t|^2 dt < \infty$, a.s. and the corresponding discounted wealth process is bounded from below by a $\mathbb{Q}-$martingale*

$$\tilde{X}^\pi_t \ge M^\pi_t, \;\; 0 \le t \le T, \quad \textit{for some } \mathbb{Q}-\textit{martingale} \quad M^\pi.$$

The collection of all admissible portfolio processes will be denoted by $\mathcal{A}$.

The lower bound M^π, which may depend on the portfolio π, has the interpretation of a finite credit line imposed on the investor. This natural generalization of the more usual constant credit line corresponds to the situation where the total credit available to an investor is indexed by some financial holding, such as the physical assets of the company or the personal home of the investor, used as collateral. From the mathematical viewpoint, this condition is needed in order to exclude any arbitrage opportunity, and will be justified in the subsequent subsection.

5.4.3 Admissible portfolios and no-arbitrage

We first define precisely the notion of no-arbitrage.

Definition 5.11. *We say that the financial market contains no arbitrage opportunities if for any admissible portfolio process $\theta \in \mathcal{A}$,*

$$X_0 = 0 \textit{ and } X^\theta_T \ge 0 \; \mathbb{P}-a.s. \quad \textit{implies} \quad X^\theta_T \;=\; 0 \; \mathbb{P}-a.s.$$

The purpose of this section is to show that the financial market described above contains no arbitrage opportunities. Our first observation is that, by the very definition of the probability measure $\mathbb{Q}$, the discounted price process $\tilde{S}$ satisfies:

$$\text{the process} \quad \left\{\tilde{S}_t,\ 0 \le t \le T\right\} \quad \text{is a } \mathbb{Q}-\text{local martingale.} \tag{5.16}$$

For this reason, $\mathbb{Q}$ is called a *risk neutral measure*, or an *equivalent local martingale measure*, for the price process S.

We also observe that the discounted wealth process satisfies:

$$\tilde{X}^\pi \quad \text{is a } \mathbb{Q}-\text{local martingale for every} \quad \pi \in \mathcal{A}, \tag{5.17}$$

as a stochastic integral with respect to the $\mathbb{Q}-$Brownian motion B.

Theorem 5.12. *The continuous-time financial market described above contains no arbitrage opportunities, i.e. for every $\pi \in \mathcal{A}$:*

$$X_0 = 0 \ \textit{and} \ X_T^\pi \ge 0 \ \mathbb{P} - a.s. \quad \Longrightarrow \quad X_T^\pi = 0 \ \mathbb{P} - a.s.$$

Proof. For $\pi \in \mathcal{A}$, the discounted wealth process $\tilde{X}^\pi$ is a $\mathbb{Q}-$local martingale bounded from below by a $\mathbb{Q}-$martingale. Then $\tilde{X}^\pi$ is a $\mathbb{Q}-$super-martingale. In particular, $\mathbb{E}^{\mathbb{Q}}\left[\tilde{X}_T^\pi\right] \le \tilde{X}_0 = X_0$. Recall that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and S^0 is strictly positive. Then, this inequality shows that, whenever $X_0^\pi = 0$ and $X_T^\pi \ge 0$ $\mathbb{P}-$a.s. (or equivalently $\mathbb{Q}-$a.s.), we have $\tilde{X}_T^\pi = 0$ $\mathbb{Q}-$a.s. and therefore $X_T^\pi = 0$ $\mathbb{P}-$a.s. ♢

5.4.4 Super-hedging and no-arbitrage bounds

Let G be an $\mathcal{F}_T-$measurable random variable representing the payoff of a derivative security with given maturity $T > 0$. The *super-hedging* problem consists in finding the minimal initial cost so as to be able to face the payment G without risk at the maturity of the contract T:

$$V(G) \quad := \quad \inf\left\{X_0 \in \mathbb{R} \ : \ X_T^\pi \ge G \ \mathbb{P} - \text{a.s. for some } \pi \in \mathcal{A}\right\} .$$

Remark 5.13. Notice that $V(G)$ depends on the reference measure $\mathbb{P}$ only by means of the corresponding null sets. Therefore, the super-hedging problem is not changed if $\mathbb{P}$ is replaced by any equivalent probability measure.

We now show that, under the no-arbitrage condition, the super-hedging problem provides *no-arbitrage bounds* on the market price of the derivative security.

Assume that the buyer of the contingent claim G has the same access to the financial market than the seller. Then $V(G)$ *is the maximal amount that the buyer of the contingent claim contract is willing to pay.* Indeed, if the seller requires a premium of $V(G) + 2\varepsilon$, for some $\varepsilon > 0$, then the buyer would not accept to pay this amount as he can obtain at least G by trading on the financial market with initial capital $V(G) + \varepsilon$.

Now, since selling of the contingent claim G is the same as buying the contingent claim $-G$, we deduce from the previous argument that

$$-V(-G) \quad \le \ \text{market price of } G \ \le \quad V(G). \tag{5.18}$$

5.4.5 The no-arbitrage valuation formula

We denote by $p(G)$ the market price of a derivative security G.

Theorem 5.14. *Let G be an $\mathcal{F}_T-$measurabel random variable representing the payoff of a derivative security at the maturity $T > 0$, and recall the notation $\tilde{G} := G \exp\left(-\int_0^T r_t dt\right)$. Assume that $\mathbb{E}^{\mathbb{Q}}[|\tilde{G}|] < \infty$. Then*

$$p(G) \;=\; V(G) \;\;=\;\; \mathbb{E}^{\mathbb{Q}}[\tilde{G}].$$

Moreover, there exists a portfolio $\pi^* \in \mathcal{A}$ such that $X_0^{\pi^*} = p(G)$ and $X_T^{\pi^*} = G$, a.s., that is π^* is a perfect replication strategy.

Proof. 1- We first prove that $V(G) \geq \mathbb{E}^{\mathbb{Q}}[\tilde{G}]$. Let X_0 and $\pi \in \mathcal{A}$ be such that $X_T^{\pi} \geq G$, a.s. or, equivalently, $\tilde{X}_T^{\pi} \geq \tilde{G}$ a.s. Notice that $\tilde{X}^{\pi}$ is a $\mathbb{Q}-$super-martingale, as a $\mathbb{Q}-$local martingale bounded from below by a $\mathbb{Q}-$martingale. Then $X_0 = \tilde{X}_0 \geq \mathbb{E}^{\mathbb{Q}}[\tilde{X}_T^{\pi}] \geq \mathbb{E}^{\mathbb{Q}}[\tilde{G}]$.
2- We next prove that $V(G) \leq \mathbb{E}^{\mathbb{Q}}[\tilde{G}]$. Define the $\mathbb{Q}-$martingale $Y_t := \mathbb{E}^{\mathbb{Q}}[\tilde{G}|\mathcal{F}_t]$ and observe that $\mathbb{F}^W = \mathbb{F}^B$. Then, it follows from the martingale representation theorem that $Y_t = Y_0 + \int_0^T \phi_t \cdot dB_t$ for some $\mathbb{F}-$adapted process ϕ with $\int_0^T |\phi_t|^2 dt < \infty$ a.s. Setting $\tilde{\pi}^* := (\sigma^{\mathrm{T}})^{-1}\phi$, we see that

$$\pi^* \in \mathcal{A} \quad \text{and} \quad Y_0 + \int_0^T \tilde{\pi}^* \cdot \sigma_t dB_t = \tilde{G} \ \mathbb{P} - \text{a.s.}$$

which implies that $Y_0 \geq V(G)$ and π^* is a perfect hedging stratgey for G, starting from the initial capital Y_0.
3- From the previous steps, we have $V(G) = \mathbb{E}^{\mathbb{Q}}[\tilde{G}]$. Applying this result to $-G$, we see that $V(-G) = -V(G)$, so that the no-arbitrage bounds (5.18) imply that the no-arbitrage market price of G is given by $V(G)$. ♢

5.4.6 PDE characterization of the Black-Scholes price

In this subsection, we specialize further the model to the case where the risky securities price processes are Markov diffusions defined by the stochastic differential equations:

$$dS_t \;=\; S_t \star \big(r(t,S_t)dt + \sigma(t,S_t)dB_t\big).$$

Here $(t,s) \longmapsto s \star r(t,s)$ and $(t,s) \longmapsto s \star \sigma(t,s)$ are Lipschitz-continuous functions from $\mathbb{R}_+ \times [0,\infty)^d$ to $\mathbb{R}^d$ and $\mathcal{S}_d$, successively. We also consider a *Vanilla* derivative security defined by the payoff

$$G \;=\; g(S_T),$$

where $g : [0,\infty)^d \to \mathbb{R}$ is a measurable function bounded from below. From the previous subsection, the no-arbitrage price at time t of this derivative security is given by

$$V(t,S_t) \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u,S_u)du} g(S_T)|\mathcal{F}_t\right] \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u,S_u)du} g(S_T)|S_t\right],$$

where the last equality follows from the Markov property of the process S. Assuming further that g has linear growth, it follows that V has linear growth in s uniformly in t. Since V is defined by a conditional expectation, it is expected to satisfy the linear PDE:

$$-\partial_t V - rs \star DV - \frac{1}{2}\mathrm{Tr}\left[(s \star \sigma)^2 D^2 V\right] - rV \quad = \quad 0. \tag{5.19}$$

More precisely, if $V \in C^{1,2}(\mathbb{R}_+, \mathbb{R}^d)$, the V is a classical solution of (5.19) and satisfies the final condition $V(T,.) = g$. Coversely, if the PDE (5.19) combined with the final condition $v(T,.) = g$ has a classical solution v with linear growth, then v coincides with the derivative security price V.

Chapter 6

STOCHASTIC CONTROL AND DYNAMIC PROGRAMMING

In this chapter, we assume that the filtration $\mathbb{F}$ is the $\mathbb{P}-$augmentation of the canonical filtration of the Brownian motion W. This restriction is only needed in order to simplify the presentation of the proof of the dynamic programming principle. We will also denote by

$$\mathbf{S} \;:=\; [0,T)\times\mathbb{R}^n \quad \text{where} \quad T\in[0,\infty].$$

The set $\mathbf{S}$ is called the *parabolic* interior of the state space. We will denote by $\bar{\mathbf{S}} := \mathrm{cl}(\mathbf{S})$ its closure, i.e. $\bar{\mathbf{S}} = [0,T]\times\mathbb{R}^n$ for finite T, and $\bar{\mathbf{S}} = \mathbf{S}$ for $T=\infty$.

6.1 Stochastic control problems in standard form

Control processes. Given a subset U of $\mathbb{R}^k$, we denote by $\mathcal{U}$ the set of all progressively measurable processes $\nu = \{\nu_t,\, t<T\}$ valued in U. The elements of $\mathcal{U}$ are called control processes.

Controlled Process. Let

$$b \;:\; (t,x,u)\in\mathbf{S}\times U \quad\longrightarrow\quad b(t,x,u)\in\mathbb{R}^n$$

and

$$\sigma \;:\; (t,x,u)\in\mathbf{S}\times U \quad\longrightarrow\quad \sigma(t,x,u)\in\mathcal{M}_{\mathbb{R}}(n,d)$$

be two continuous functions satisfying the conditions

$$\begin{aligned} |b(t,x,u)-b(t,y,u)|+|\sigma(t,x,u)-\sigma(t,y,u)| &\leq K\,|x-y|, & (6.1)\\ |b(t,x,u)|+|\sigma(t,x,u)| &\leq K\,(1+|x|+|u|). & (6.2)\end{aligned}$$

for some constant K independent of (t,x,y,u). For each control process $\nu \in \mathcal{U}$, we consider the controlled stochastic differential equation :

$$dX_t \quad = \quad b(t,X_t,\nu_t)dt + \sigma(t,X_t,\nu_t)dW_t. \tag{6.3}$$

If the above equation has a unique solution X, for a given initial data, then the process X is called the controlled process, as its dynamics is driven by the action of the control process ν.

We shall be working with the following subclass of control processes :

$$\mathcal{U}_0 \quad := \quad \mathcal{U} \cap \mathbb{H}^2, \tag{6.4}$$

where $\mathbb{H}^2$ is the collection of all progressively measurable processes with finite $\mathbb{L}^2(\Omega \times [0,T))-$norm. Then, for every finite maturity $T' \le T$, it follows from the above uniform Lipschitz condition on the coefficients b and σ that

$$\mathbb{E}\left[\int_0^{T'} \left(|b|+|\sigma|^2\right)(s,x,\nu_s)ds\right] < \infty \quad \text{for all} \quad \nu \in \mathcal{U}_0,\ x \in \mathbb{R}^n,$$

which guarantees the existence of a controlled process on the time interval $[0,T']$ for each given initial condition and control. The following result is an immediate consequence of Theorem 5.2.

Theorem 6.1. *Let $\nu \in \mathcal{U}_0$ be a control process, and $\xi \in \mathbb{L}^2(\mathbb{P})$ be an $\mathcal{F}_0-$measurable random variable. Then, there exists a unique $\mathbb{F}-$adapted process X^ν satisfying (6.3) together with the initial condition $X_0^\nu = \xi$. Moreover for every $T > 0$, there is a constant $C > 0$ such that*

$$\mathbb{E}\left[\sup_{0\le s\le t} |X_s^\nu|^2\right] \;<\; C(1+\mathbb{E}[|\xi|^2])e^{Ct} \quad \textit{for all} \quad t \in \mathrm{cl}([0,T)). \tag{6.5}$$

<u>*Cost functional*</u>. Let

$$f,k \;:\; [0,T)\times\mathbb{R}^n\times U \longrightarrow \mathbb{R} \quad \text{and} \quad g \;:\; \mathbb{R}^n \longrightarrow \mathbb{R}$$

be given functions. We assume that f,k are continuous and $\|k^-\|_\infty < \infty$ (i.e. $\max(-k,0)$ is uniformly bounded). Moreover, we assume that f and g satisfy the quadratic growth condition :

$$|f(t,x,u)| + |g(x)| \quad \le \quad K(1+|u|+|x|^2),$$

for some constant K independent of (t,x,u). We define the cost function J on $[0,T]\times\mathbb{R}^n\times\mathcal{U}$ by :

$$J(t,x,\nu) \quad := \quad \mathbb{E}\left[\int_t^T \beta^\nu(t,s)f(s,X_s^{t,x,\nu},\nu_s)ds + \beta^\nu(t,T)g(X_T^{t,x,\nu})\mathbf{1}_{T<\infty}\right],$$

when this expression is meaningful, where

$$\beta^\nu(t,s) \quad := \quad e^{-\int_t^s k(r,X_r^{t,x,\nu},\nu_r)dr},$$

and $\{X_s^{t,x,\nu}, s \geq t\}$ is the solution of (6.3) with control process ν and initial condition $X_t^{t,x,\nu} = x$.

Admissible control processes. In the finite horizon case $T < \infty$, the quadratic growth condition on f and g together with the bound on k^- ensure that $J(t,x,\nu)$ is well-defined for all control process $\nu \in \mathcal{U}_0$. We then define the set of admissible controls in this case by $\mathcal{U}_0$.

More attention is needed for the infinite horizon case. In particular, the discount term k needs to play a role to ensure the finiteness of the integral. In this setting the largest set of admissible control processes is given by

$$\mathcal{U}_0 := \Big\{\nu \in \mathcal{U} : \mathbb{E}\big[\int_0^\infty \beta^\nu(t,s)\big(1+|X_s^{t,x,\nu}|^2+|\nu_s)|\big)ds\big] < \infty \text{ for all } x\Big\} \text{ when } T = \infty.$$

The stochastic control problem. The purpose of this section is to study the minimization problem

$$V(t,x) := \sup_{\nu \in \mathcal{U}_0} J(t,x,\nu) \quad \text{for} \quad (t,x) \in \mathbf{S}.$$

Our main concern is to describe the local behavior of the value function V by means of the so-called *dynamic programming equation*, or *Hamilton-Jacobi-Bellman* equation. We continue with some remarks.

Remark 6.2. (i) If $V(t,x) = J(t,x,\hat{\nu}_{t,x})$, we call $\hat{\nu}_{t,x}$ an *optimal control* for the problem $V(t,x)$.

(ii) The following are some interesting subsets of controls :

- a process $\nu \in \mathcal{U}_0$ which is adapted to the natural filtration $\mathbb{F}^X$ of the associated state process is called *feedback control*,

- a process $\nu \in \mathcal{U}_0$ which can be written in the form $\nu_s = \tilde{u}(s, X_s)$ for some measurable map $\tilde{u}$ from $[0,T] \times \mathbb{R}^n$ into U, is called *Markovian control*; notice that any Markovian control is a feedback control,

- the deterministic processes of $\mathcal{U}_0$ are called *open loop controls*.

(iii) Suppose that $T < \infty$, and let (Y, Z) be the controlled processes defined by

$$dY_s = Z_s f(s, X_s, \nu_s)ds \quad \text{and} \quad dZ_s = -Z_s k(s, X_s, \nu_s)ds ,$$

and define the augmented state process $\bar{X} := (X, Y, Z)$. Then, the above value function V can be written in the form :

$$V(t,x) = \bar{V}(t,x,0,1) ,$$

where $\bar{x} = (x,y,z)$ is some initial data for the augmented state process $\bar{X}$,

$$\bar{V}(t,\bar{x}) := \mathbb{E}_{t,\bar{x}}\left[\bar{g}(\bar{X}_T)\right] \quad \text{and} \quad \bar{g}(x,y,z) := y + g(x)z .$$

Hence the stochastic control problem V can be reduced without loss of generality to the case where $f = k \equiv 0$. We shall appeal to this reduced form whenever convenient for the exposition.

(iv) For notational simplicity we consider the case $T < \infty$ and $f = k = 0$. The previous remark shows how to immediately adapt the following argument so that the present remark holds true without the restriction $f = k = 0$. The extension to the infinite horizon case is also immediate.

Consider the value function

$$\tilde{V}(t,x) \quad := \quad \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[g(X_T^{t,x,\nu})\right], \tag{6.6}$$

differing from V by the restriction of the control processes to

$$\mathcal{U}_t \quad := \quad \{\nu \in \mathcal{U}_0 : \nu \text{ independent of } \mathcal{F}_t\}. \tag{6.7}$$

Since $\mathcal{U}_t \subset \mathcal{U}_0$, it is obvious that $\tilde{V} \le V$. We claim that

$$\tilde{V} \quad = \quad V, \tag{6.8}$$

so that both problems are indeed equivalent. To see this, fix $(t,x) \in \mathbf{S}$ and $\nu \in \mathcal{U}_0$. Then, ν can be written as a measurable function of the canonical process $\nu((\omega_s)_{0\le s\le t}, (\omega_s - \omega_t)_{t\le s\le T})$, where, for fixed $(\omega_s)_{0\le s\le t}$, the map $\nu_{(\omega_s)_{0\le s\le t}} : (\omega_s - \omega_t)_{t\le s\le T} \mapsto \nu((\omega_s)_{0\le s\le t}, (\omega_s - \omega_t)_{t\le s\le T})$ can be viewed as a control independent on $\mathcal{F}_t$. Using the independence of the increments of the Brownian motion, together with Fubini's Lemma, it thus follows that

$$\begin{aligned} J(t,x;\nu) \quad &= \quad \int \mathbb{E}\left[g(X_T^{t,x,\nu_{(\omega_s)_{0\le s\le t}}})\right] d\mathbb{P}((\omega_s)_{0\le s\le t}) \\ &\le \quad \int \tilde{V}(t,x) d\mathbb{P}((\omega_s)_{0\le s\le t}) \;=\; \tilde{V}(t,x). \end{aligned}$$

By arbitrariness of $\nu \in \mathcal{U}_0$, this implies that $\tilde{V}(t,x) \ge V(t,x)$.

6.2 The dynamic programming principle

6.2.1 A weak dynamic programming principle

The dynamic programming principle is the main tool in the theory of stochastic control. In these notes, we shall prove rigorously a weak version of the dynamic programming which will be sufficient for the derivation of the dynamic programming equation. We denote:

$$V_*(t,x) := \liminf_{(t',x')\to(t,x)} V(t',x') \quad \text{and} \quad V^*(t,x) := \limsup_{(t',x')\to(t,x)} V(t',x'),$$

for all $(t,x) \in \bar{\mathbf{S}}$. We also recall the subset of controls $\mathcal{U}_t$ introduced in (6.7) above.

Theorem 6.3. *Assume that V is locally bounded and fix $(t,x)\in\mathbf{S}$. Let $\{\theta^\nu,\ \nu\in\mathcal{U}_t\}$ be a family of finite stopping times independent of $\mathcal{F}_t$ with values in $[t,T]$. Then:*

$$V(t,x) \quad \geq \quad \sup_{\nu\in\mathcal{U}_t}\mathbb{E}\left[\int_t^{\theta^\nu}\beta^\nu(t,s)f(s,X_s^{t,x,\nu},\nu_s)ds+\beta^\nu(t,\theta^\nu)V_*(\theta^\nu,X_{\theta^\nu}^{t,x,\nu})\right].$$

Assume further that g is lower-semicontinuous and $X_{t,x}^\nu\mathbf{1}_{[t,\theta^\nu]}$ is $\mathbb{L}^\infty-$bounded for all $\nu\in\mathcal{U}_t$. Then

$$V(t,x) \quad \leq \quad \sup_{\nu\in\mathcal{U}_t}\mathbb{E}\left[\int_t^{\theta^\nu}\beta^\nu(t,s)f(s,X_s^{t,x,\nu},\nu_s)ds+\beta^\nu(t,\theta^\nu)V^*(\theta^\nu,X_{\theta^\nu}^{t,x,\nu})\right].$$

We shall provide an intuitive justification of this result after the following comments. A rigorous proof is reported in Section 6.2.2 below.

(i) If V is continuous, then $V=V_*=V^*$, and the above weak dynamic programming principle reduces to the classical dynamic programming principle:

$$V(t,x) \quad = \quad \sup_{\nu\in\mathcal{U}}\mathbb{E}_{t,x}\left[\int_t^\theta\beta(t,s)f(s,X_s,\nu_s)ds+\beta(t,\theta)V(\theta,X_\theta)\right] \qquad (6.9)$$

(ii) In the discrete-time framework, the dynamic programming principle (6.9) can be stated as follows :

$$V(t,x) \quad = \quad \sup_{u\in U}\mathbb{E}_{t,x}\left[f(t,X_t,u)+e^{-k(t+1,X_{t+1},u)}V(t+1,X_{t+1})\right].$$

Observe that the supremum is now taken over the subset U of the finite dimensional space R^k. Hence, the dynamic programming principle allows to reduce the initial maximization problem, over the subset $\mathcal{U}$ of the infinite dimensional set of $\mathbb{R}^k-$valued processes, into a finite dimensional maximization problem. However, we are still facing an infinite dimensional problem since the dynamic programming principle relates the value function at time t to the value function at time $t+1$.

(iii) In the context of the above discrete-time framework with finite horizon $T<\infty$, notice that the dynamic programming principle suggests the following backward algorithm to compute V as well as the associated optimal strategy (when it exists). Since $V(T,\cdot)=g$ is known, the above dynamic programming principle can be applied recursively in order to deduce the value function $V(t,x)$ for every t.

(iv) In the continuous time setting, there is no obvious counterpart to the above backward algorithm. But, as the stopping time θ approaches t, the above dynamic programming principle implies a special local behavior for the value function V. When V is known to be smooth, this will be obtained by means of Itô's formula.

(v) It is usually very difficult to determine *a priori* the regularity of V. The situation is even worse since there are many counter-examples showing that the value function V can not be expected to be smooth in general; see Section 6.4. This problem is solved by appealing to the notion of viscosity solutions, which provides a weak local characterization of the value function V.

(vi) Once the local behavior of the value function is characterized, we are faced to the important uniqueness issue, which implies that V is completely characterized by its local behavior together with some convenient boundary condition.

Intuitive justification of (6.9). Let us assume that V is continuous. In particular, V is measurable and $V = V_* = V^*$. Let $\tilde{V}(t,x)$ denote the right hand-side of (6.9).

By the tower Property of the conditional expectation operator, it is easily checked that

$$J(t,x,\nu) \;=\; \mathbb{E}_{t,x}\left[\int_t^\theta \beta(t,s)f(s,X_s,\nu_s)ds + \beta(t,\theta)J(\theta,X_\theta,\nu)\right].$$

Since $J(\theta,X_\theta,\nu) \le V(\theta,X_\theta)$, this proves that $V \le \tilde{V}$. To prove the reverse inequality, let $\mu \in \mathcal{U}$ and $\varepsilon > 0$ be fixed, and consider an $\varepsilon-$optimal control ν^ε for the problem $V(\theta,X_\theta)$, i.e.

$$J(\theta,X_\theta,\nu^\varepsilon) \;\ge\; V(\theta,X_\theta) - \varepsilon.$$

Clearly, one can choose $\nu^\varepsilon = \mu$ on the stochastic interval $[t,\theta]$. Then

$$\begin{aligned} V(t,x) \;&\ge\; J(t,x,\nu^\varepsilon) \;=\; \mathbb{E}_{t,x}\left[\int_t^\theta \beta(t,s)f(s,X_s,\mu_s)ds + \beta(t,\theta)J(\theta,X_\theta,\nu^\varepsilon)\right] \\ &\ge\; \mathbb{E}_{t,x}\left[\int_t^\theta \beta(t,s)f(s,X_s,\mu_s)ds + \beta(t,\theta)V(\theta,X_\theta)\right] - \varepsilon\,\mathbb{E}_{t,x}[\beta(t,\theta)]\,. \end{aligned}$$

This provides the required inequality by the arbitrariness of $\mu \in \mathcal{U}$ and $\varepsilon > 0$.
♢

Exercise. Where is the gap in the above sketch of the proof ?

6.2.2 Dynamic programming without measurable selection

In this section, we provide a rigorous proof of Theorem 6.3. Notice that, we have no information on whether V is measurable or not. Because of this, the right-hand side of the classical dynamic programming principle (6.9) is not even known to be well-defined.

The formulation of Theorem 6.3 avoids this measurability problem since V_* and V^* are lower- and upper-semicontinuous, respectively, and therefore measurable. In addition, it allows to avoid the typically heavy technicalities related to measurable selection arguments needed for the proof of the classical (6.9) after a convenient relaxation of the control problem, see e.g. El Karoui and Jeanblanc [**?**].

Proof of Theorem 6.3 For simplicity, we consider the finite horizon case $T < \infty$, so that, without loss of generality, we assume $f = k = 0$, See Remark 6.2 (iii). The extension to the infinite horizon framework is immediate.

1. Let $\nu \in \mathcal{U}_t$ be arbitrary and set $\theta := \theta^\nu$. Then:

$$\mathbb{E}\left[g\left(X_T^{t,x,\nu}\right)|\mathcal{F}_\theta\right](\omega) = J(\theta(\omega), X_\theta^{t,x,\nu}(\omega); \tilde{\nu}_\omega),$$

where $\tilde{\nu}_\omega$ is obtained from ν by freezing its trajectory up to the stopping time θ. Since, by definition, $J(\theta(\omega), X_\theta^{t,x,\nu}(\omega); \tilde{\nu}_\omega) \le V^*(\theta(\omega), X_\theta^{t,x,\nu}(\omega))$, it follows from the tower property of conditional expectations that

$$\mathbb{E}\left[g\left(X_T^{t,x,\nu}\right)\right] \;=\; \mathbb{E}\left[\mathbb{E}\left[g\left(X_T^{t,x,\nu}\right)|\mathcal{F}_\theta\right]\right] \;\le\; \mathbb{E}\left[V^*\left(\theta, X_\theta^{t,x,\nu}\right)\right],$$

which provides the second inequality of Theorem 6.3 by the arbitrariness of $\nu \in \mathcal{U}_t$.

2. Let $\varepsilon > 0$ be given, and consider an arbitrary function

$$\varphi : \mathbf{S} \longrightarrow \mathbb{R} \quad \text{such that} \quad \varphi \text{ upper-semicontinuous and } V \ge \varphi.$$

2.a. There is a family $(\nu^{(s,y),\varepsilon})_{(s,y)\in\mathbf{S}} \subset \mathcal{U}_0$ such that:

$$\nu^{(s,y),\varepsilon} \in \mathcal{U}_s \text{ and } J(s,y;\nu^{(s,y),\varepsilon}) \ge V(s,y) - \varepsilon, \quad \text{for every} \quad (s,y) \in \mathbf{S} \tag{6.10}$$

Since g is lower-semicontinuous and has quadratic growth, it follows from Theorem 6.1 that the function $(t',x') \mapsto J(t',x';\nu^{(s,y),\varepsilon})$ is lower-semicontinuous, for fixed $(s,y) \in \mathbf{S}$. Together with the upper-semicontinuity of φ, this implies that we may find a family $(r_{(s,y)})_{(s,y)\in\mathbf{S}}$ of positive scalars so that, for any $(s,y) \in \mathbf{S}$,

$$\begin{gathered}\varphi(s,y) - \varphi(t',x') \ge -\varepsilon \text{ and } J(s,y;\nu^{(s,y),\varepsilon}) - J(t',x';\nu^{(s,y),\varepsilon}) \le \varepsilon \\ \text{for } (t',x') \in B(s,y;r_{(s,y)}),\end{gathered} \tag{6.11}$$

where, for $r > 0$ and $(s,y) \in \mathbf{S}$,

$$B(s,y;r) := \{(t',x') \in \mathbf{S} \;:\; t' \in (s-r,s),\ |x'-y| < r\}\ .$$

Clearly, $\left\{B(s,y;r) : (s,y) \in \mathbf{S},\ 0 < r \le r_{(s,y)}\right\}$ forms an open covering of $[0,T) \times \mathbb{R}^d$. It then follows from the Lindelöf covering Theorem, see e.g. [**?**] Theorem 6.3 Chap. VIII, that we can find a countable sequence $(t_i,x_i,r_i)_{i\ge1}$ of elements of $\mathbf{S} \times \mathbb{R}$, with $0 < r_i \le r_{(t_i,x_i)}$ for all $i \ge 1$, such that $\mathbf{S} \subset$

$\{T\} \times \mathbb{R}^d \cup (\cup_{i\geq 1} B(t_i, x_i; r_i))$. Set $A_0 := \{T\} \times \mathbb{R}^d$, $C_{-1} := \emptyset$, and define the sequence

$$A_{i+1} := B(t_{i+1}, x_{i+1}; r_{i+1}) \setminus C_i \quad \text{where} \quad C_i := C_{i-1} \cup A_i,\ i \geq 0.$$

With this construction, it follows from (6.10), (6.11), together with the fact that $V \geq \varphi$, that the countable family $(A_i)_{i\geq 0}$ satisfies

$$\begin{array}{c}(\theta, X_\theta^{t,x,\nu}) \in \cup_{i\geq 0} A_i \ \mathbb{P}-\text{a.s.},\ A_i \cap A_j = \emptyset \text{ for } i \neq j \in \mathbb{N}, \\ \text{and } J(\cdot; \nu^{i,\varepsilon}) \geq \varphi - 3\varepsilon \text{ on } A_i \text{ for } i \geq 1,\end{array} \tag{6.12}$$

where $\nu^{i,\varepsilon} := \nu^{(t_i,x_i),\varepsilon}$ for $i \geq 1$.

2.b. We now prove the first inequality in Theorem 6.3. We fix $\nu \in \mathcal{U}_t$ and $\theta \in \mathcal{T}^t_{[t,T]}$. Set $A^n := \cup_{0\leq i\leq n} A_i$, $n \geq 1$. Given $\nu \in \mathcal{U}_t$, we define for $s \in [t, T]$:

$$\nu_s^{\varepsilon,n} \quad := \quad \mathbf{1}_{[t,\theta]}(s)\nu_s + \mathbf{1}_{(\theta,T]}(s)\Big(\nu_s \mathbf{1}_{(A^n)^c}(\theta, X_\theta^{t,x,\nu}) + \sum_{i=1}^n \mathbf{1}_{A_i}(\theta, X_\theta^{t,x,\nu})\nu_s^{i,\varepsilon}\Big).$$

Notice that $\{(\theta, X_\theta^{t,x,\nu}) \in A_i\} \in \mathcal{F}^t_\theta$. Then, it follows that $\nu^{\varepsilon,n} \in \mathcal{U}_t$. Then, it follows from (6.12) that:

$$\begin{aligned}
\mathbb{E}\left[g\left(X_T^{t,x,\nu^{\varepsilon,n}}\right)|\mathcal{F}_\theta\right]\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right) \quad &= \quad V\left(T, X_T^{t,x,\nu^{\varepsilon,n}}\right)\mathbf{1}_{A_0}\left(\theta, X_\theta^{t,x,\nu}\right) \\
&+ \quad \sum_{i=1}^n J(\theta, X_\theta^{t,x,\nu}, \nu^{i,\varepsilon})\mathbf{1}_{A_i}\left(\theta, X_\theta^{t,x,\nu}\right) \\
&\geq \quad \sum_{i=0}^n \left(\varphi(\theta, X_\theta^{t,x,\nu} - 3\varepsilon\right)\mathbf{1}_{A_i}\left(\theta, X_\theta^{t,x,\nu}\right) \\
&= \quad \left(\varphi(\theta, X_\theta^{t,x,\nu}) - 3\varepsilon\right)\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right),
\end{aligned}$$

which, by definition of V and the tower property of conditional expectations, implies

$$\begin{aligned}
V(t,x) \quad &\geq \quad J(t, x, \nu^{\varepsilon,n}) \\
&= \quad \mathbb{E}\Big[\mathbb{E}\Big[g\left(X_T^{t,x,\nu^{\varepsilon,n}}\right)|\mathcal{F}_\theta\Big]\Big] \\
&\geq \quad \mathbb{E}\left[\left(\varphi\left(\theta, X_\theta^{t,x,\nu}\right) - 3\varepsilon\right)\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right)\right] \\
&\qquad + \mathbb{E}\left[g\left(X_T^{t,x,\nu}\right)\mathbf{1}_{(A^n)^c}\left(\theta, X_\theta^{t,x,\nu}\right)\right].
\end{aligned}$$

Since $g\left(X_T^{t,x,\nu}\right) \in \mathbb{L}^1$, it follows from the dominated convergence theorem that:

$$\begin{aligned}
V(t,x) \quad &\geq \quad -3\varepsilon + \liminf_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right)\right] \\
&= \quad -3\varepsilon + \lim_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})^+\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right)\right] \\
&\qquad\qquad - \lim_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})^-\mathbf{1}_{A^n}\left(\theta, X_\theta^{t,x,\nu}\right)\right] \\
&= \quad -3\varepsilon + \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})\right],
\end{aligned}$$

where the last equality follows from the left-hand side of (6.12) and from the monotone convergence theorem, due to the fact that either $\mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})^+\right] < \infty$ or $\mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})^-\right] < \infty$. By the arbitrariness of $\nu \in \mathcal{U}_t$ and $\varepsilon > 0$, this shows that:

$$V(t,x) \quad \geq \quad \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[\varphi(\theta, X_\theta^{t,x,\nu})\right]. \tag{6.13}$$

3. It remains to deduce the first inequality of Theorem 6.3 from (6.13). Fix $r > 0$. It follows from standard arguments, see e.g. Lemma 3.5 in [**?**], that we can find a sequence of continuous functions $(\varphi_n)_n$ such that $\varphi_n \leq V_* \leq V$ for all $n \geq 1$ and such that φ_n converges pointwise to V_* on $[0,T] \times B_r(0)$. Set $\phi_N := \min_{n\geq N} \varphi_n$ for $N \geq 1$ and observe that the sequence $(\phi_N)_N$ is non-decreasing and converges pointwise to V_* on $[0,T] \times B_r(0)$. By (6.13) and the monotone convergence Theorem, we then obtain:

$$V(t,x) \quad \geq \quad \lim_{N\to\infty} \mathbb{E}\left[\phi_N(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right] = \mathbb{E}\left[V_*(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right].$$

♢

6.3 The dynamic programming equation

The dynamic programming equation is the infinitesimal counterpart of the dynamic programming principle. It is also widely called the *Hamilton-Jacobi-Bellman* equation. In this section, we shall derive it under strong smoothness assumptions on the value function. Let $\mathcal{S}^d$ be the set of all $d \times d$ symmetric matrices with real coefficients, and define the map $H : \mathbf{S} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^d$ by :

$$\begin{aligned} &H(t,x,r,p,\gamma) \\ &:= \sup_{u\in U} \left\{ -k(t,x,u)r + b(t,x,u)\cdot p + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}}(t,x,u)\gamma] + f(t,x,u) \right\}. \end{aligned}$$

We also need to introduce the linear second order operator $\mathcal{L}^u$ associated to the controlled process $\{\beta(0,t)X_t^u,\ t \geq 0\}$ controlled by the constant control process u :

$$\begin{aligned} \mathcal{L}^u\varphi(t,x) \quad := \quad & -k(t,x,u)\varphi(t,x) + b(t,x,u)\cdot D\varphi(t,x) \\ & + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}(t,x,u)D^2\varphi(t,x)\right], \end{aligned}$$

where D and D^2 denote the gradient and the Hessian operators with respect to the x variable. With this notation, we have by Itô's formula:

$$\begin{aligned} \beta^\nu(0,s)\varphi(s,X_s^\nu) - \beta^\nu(0,t)\varphi(t,X_t^\nu) \quad = \quad & \int_t^s \beta^\nu(0,r)\left(\partial_t + \mathcal{L}^{\nu_r}\right)\varphi(r,X_r^\nu)dr \\ & + \int_t^s \beta^\nu(0,r)D\varphi(r,X_r^\nu)\cdot\sigma(r,X_r^\nu,\nu_r)dW_r \end{aligned}$$

for every $s \geq t$ and smooth function $\varphi \in C^{1,2}([t,s],\mathbb{R}^n)$ and each admissible control process $\nu \in \mathcal{U}_0$.

Proposition 6.4. *Assume the value function $V \in C^{1,2}([0,T),\mathbb{R}^n)$, and let the coefficients $k(\cdot,\cdot,u)$ and $f(\cdot,\cdot,u)$ be continuous in (t,x) for all fixed $u \in U$. Then, for all $(t,x) \in \mathbf{S}$:*

$$-\partial_t V(t,x) - H\big(t,x,V(t,x),DV(t,x),D^2V(t,x)\big) \quad \geq \quad 0. \tag{6.14}$$

Proof. Let $(t,x) \in \mathbf{S}$ and $u \in U$ be fixed and consider the constant control process $\nu = u$, together with the associated state process X with initial data $X_t = x$. For all $h > 0$, Define the stopping time :

$$\theta_h \quad := \quad \inf\left\{s > t \ : \ (s-t, X_s - x) \notin [0,h) \times \alpha B\right\} \ ,$$

where $\alpha > 0$ is some given constant, and B denotes the unit ball of $\mathbb{R}^n$. Notice that $\theta_h \longrightarrow t$, $\mathbb{P}-$a.s. when $h \searrow 0$, and $\theta_h = h$ for $h \leq \bar{h}(\omega)$ sufficiently small.

1. From the first inequality of the dynamic programming principle, it follows that :

$$\begin{aligned} 0 \quad &\leq \quad \mathbb{E}_{t,x}\left[\beta(0,t)V(t,x) - \beta(0,\theta_h)V(\theta_h, X_{\theta_h}) - \int_t^{\theta_h} \beta(0,r)f(r,X_r,u)dr\right] \\ &= \quad -\mathbb{E}_{t,x}\left[\int_t^{\theta_h} \beta(0,r)(\partial_t V + \mathcal{L}^{\cdot} V + f)(r,X_r,u)dr\right] \\ &\qquad -\mathbb{E}_{t,x}\left[\int_t^{\theta_h} \beta(0,r)DV(r,X_r)\cdot\sigma(r,X_r,u)dW_r\right], \end{aligned}$$

the last equality follows from Itô's formula and uses the crucial smoothness assumption on V.

2. Observe that $\beta(0,r)DV(r,X_r)\cdot\sigma(r,X_r,u)$ is bounded on the stochastic interval $[t,\theta_h]$. Therefore, the second expectation on the right hand-side of the last inequality vanishes, and we obtain :

$$-\mathbb{E}_{t,x}\left[\frac{1}{h}\int_t^{\theta_h} \beta(0,r)(\partial_t V + \mathcal{L}^{\cdot} V + f)(r,X_r,u)dr\right] \quad \geq \quad 0$$

We now send h to zero. The a.s. convergence of the random value inside the expectation is easily obtained by the mean value Theorem; recall that $\theta_h = h$ for sufficiently small $h > 0$. Since the random variable $h^{-1}\int_t^{\theta_h} \beta(0,r)(\mathcal{L}^{\cdot} V + f)(r,X_r,u)dr$ is essentially bounded, uniformly in h, on the stochastic interval $[t,\theta_h]$, it follows from the dominated convergence theorem that :

$$-\partial_t V(t,x) - \mathcal{L}^u V(t,x) - f(t,x,u) \quad \geq \quad 0.$$

By the arbitrariness of $u \in U$, this provides the required claim. ♢

We next wish to show that V satisfies the nonlinear partial differential equation (6.15) with equality. This is a more technical result which can be proved by different methods. We shall report a proof, based on a contradiction argument, which provides more intuition on this result, although it might be slightly longer than the usual proof reported in standard textbooks.

Proposition 6.5. *Assume the value function* $V \in C^{1,2}([0,T),\mathbb{R}^n)$*, and let the function* H *be continuous, and* $\|k^+\|_\infty < \infty$*. Then, for all* $(t,x) \in \mathbf{S}$*:*

$$-\partial_t V(t,x) - H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \quad \le \quad 0. \tag{6.15}$$

Proof. Let $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$ be fixed, assume to the contrary that

$$\partial_t V(t_0,x_0) + H\left(t_0,x_0,V(t_0,x_0),DV(t_0,x_0),D^2V(t_0,x_0)\right) \quad < \quad 0, \tag{6.16}$$

and let us work towards a contradiction.
1. For a given parameter $\varepsilon > 0$, define the smooth function $\varphi \ge V$ by

$$\varphi(t,x) \quad := \quad V(t,x) + \varepsilon\left(|t-t_0|^2 + |x-x_0|^4\right).$$

Then

$$(V-\varphi)(t_0,x_0) = 0, \ \ (DV - D\varphi)(t_0,x_0) = 0, \ \ (\partial_t V - \partial_t\varphi)(t_0,x_0) = 0, \\ \text{and} \ \ (D^2V - D^2\varphi)(t_0,x_0) = 0,$$

and (6.16) says that:

$$h(t_0,x_0) \ := \ \partial_t\varphi(t_0,x_0) + H\left(t_0,x_0,\varphi(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0)\right) \quad < \quad 0$$

for a sufficiently small $\varepsilon > 0$.
2. By continuity of H, we have:

$$h(t,x) < 0 \ \text{ on } \ \mathcal{N}_\eta := (-\eta,\eta) \times \eta B \quad \text{for } \eta > 0 \text{ sufficiently small,}$$

where B denotes the unit balla centered at x_0. We next observe that the parmeter γ defined by the following is positive:

$$-2\gamma e^{\eta\|k^+\|_\infty} \quad := \quad \max_{\partial\mathcal{N}_\eta}(V-\varphi) \ < \ 0. \tag{6.17}$$

Next, let $\tilde\nu$ be a $\gamma-$optimal control for the problem $V(t_0,x_0)$, i.e.

$$J(t_0,x_0,\tilde\nu) \quad \ge \quad V(t_0,x_0) - \gamma. \tag{6.18}$$

We shall denote by $\tilde X$ and $\tilde\beta$ the controlled process and the discount factor defined by $\tilde\nu$ and the initial data $\tilde X_{t_0} = x_0$.
3. Consider the stopping time

$$\theta \quad := \quad \inf\left\{s > t \ : \ (s,\tilde X_s) \notin \mathcal{N}_\eta\right\},$$

and observe that, by continuity of the state process, $(\theta,\tilde X_\theta) \in \partial\mathcal{N}_\eta$, so that :

$$(V-\varphi)(\theta,\tilde X_\theta) \quad \le \quad -2\gamma e^{\eta\|k^+\|_\infty}$$

by (6.17). Recalling that $\tilde{\beta}(t_0,t_0)=1$, we now compute that:

$$\begin{aligned}
\tilde{\beta}(t_0,\theta)V(\theta,\tilde{X}_\theta)-V(t_0,x_0) &\le \int_{t_0}^{\theta} d[\tilde{\beta}(t_0,r)\varphi(r,\tilde{X}_r)] - 2\gamma e^{\eta\|k^+\|_\infty}\tilde{\beta}(t_0,\theta) \\
&\le \int_{t_0}^{\theta} d[\tilde{\beta}(t_0,r)\varphi(r,\tilde{X}_r)] - 2\gamma.
\end{aligned}$$

By Itô's formula, this provides :

$$V(t_0,x_0) \;\ge\; \mathbb{E}_{t_0,x_0}\left[\tilde{\beta}(t_0,\theta)V(\theta,\tilde{X}_\theta) - \int_{t_0}^{\theta}(\partial_t\varphi+\mathcal{L}^{\tilde{\nu}_r}\varphi)(r,\tilde{X}_r)dr\right] + 2\gamma\ ,$$

where the "dW" integral term has zero mean, as its integrand is bounded on the stochastic interval $[t_0,\theta]$. Observe also that $(\partial_t\varphi+\mathcal{L}^{\tilde{\nu}_r}\varphi)(r,\tilde{X}_r)+f(r,\tilde{X}_r,\tilde{\nu}_r)$ $\le h(r,\tilde{X}_r)\le 0$ on the stochastic interval $[t_0,\theta]$. We therefore deduce that :

$$\begin{aligned}
V(t_0,x_0) &\ge 2\gamma+\mathbb{E}_{t_0,x_0}\left[\int_{t_0}^{\theta}\tilde{\beta}(t_0,r)f(r,\tilde{X}_r,\tilde{\nu}_r)dr+\tilde{\beta}(t_0,\theta)V(\theta,\tilde{X}_\theta)\right] \\
&\ge 2\gamma + J(t_0,x_0,\tilde{\nu}) \\
&\ge V(t_0,x_0)+\gamma,
\end{aligned}$$

where the last inequality follows by (6.18). This completes the proof. ♢

As a consequence of Propositions 6.4 and 6.5, we have the main result of this section :

Theorem 6.6. *Let the conditions of Propositions 6.5 and 6.4 hold. Then, the value function V solves the Hamilton-Jacobi-Bellman equation*

$$-\partial_t V - H\left(.,V,DV,D^2V\right) \;=\; 0 \quad on \quad \mathbf{S}. \tag{6.19}$$

6.4 On the regularity of the value function

The purpose of this paragraph is to show that the value function should not be expected to be smooth in general. We start by proving the continuity of the value function under strong conditions; in particular, we require the set U in which the controls take values to be bounded. We then give a simple example in the deterministic framework where the value function is not smooth. Since it is well known that stochastic problems are "more regular" than deterministic ones, we also give an example of stochastic control problem whose value function is not smooth.

6.4.1 Continuity of the value function for bounded controls

For notational simplicity, we reduce the stochastic control problem to the case $f = k \equiv 0$, see Remark 6.2 (iii). Our main concern, in this section, is to show the standard argument for proving the continuity of the value function. Therefore, the following results assume strong conditions on the coefficients of the model in order to simplify the proofs. We first start by examining the value function $V(t,\cdot)$ for fixed $t \in [0,T]$.

Proposition 6.7. *Let $f = k \equiv 0$, $T < \infty$, and assume that g is Lipschitz continuous. Then:*
(i) *V is Lipschitz in x, uniformly in t.*
(ii) *Assume further that U is bounded. Then V is $\frac{1}{2}-$Hölder-continuous in t, and there is a constant $C > 0$ such that:*

$$\left|V(t,x) - V(t',x)\right| \;\; \leq \;\; C(1+|x|)\sqrt{|t-t'|}; \quad t,t' \in [0,T],\ x \in \mathbb{R}^n.$$

Proof. (i) For $x, x' \in \mathbb{R}^n$ and $t \in [0,T)$, we first estimate that:

$$\begin{aligned} |V(t,x) - V(t,x')| &\leq \sup_{\nu \in \mathcal{U}_0} \mathbb{E}\left| g\left(X_T^{t,x,\nu}\right) - g\left(X_T^{t,x',\nu}\right)\right| \\ &\leq Const \sup_{\nu \in \mathcal{U}_0} \mathbb{E}\left|X_T^{t,x,\nu} - X_T^{t,x',\nu}\right| \\ &\leq Const\,|x - x'|, \end{aligned}$$

where we used the Lipschitz-continuity of g together with the flow estimates of Theorem 5.4, and the fact that the coefficients b and σ are Lipschitz in x uniformly in (t,u). This compltes the proof of the Lipschitz property of the value function V.
(ii) To prove the Hölder continuity in t, we shall use the dynamic programming principle.

(ii-1) We first make the following important observation. A careful review of the proof of Theorem 6.3 reveals that, whenever the stopping times θ^ν are constant (i.e. deterministic), the dynamic programming principle holds true with the semicontinuous envelopes taken only with respect to the $x-$variable. Since V was shown to be continuous in the first part of this proof, we deduce that:

$$V(t,x) \;\; = \;\; \sup_{\nu \in \mathcal{U}_0} \mathbb{E}\left[V\big(t', X_{t'}^{t,x,\nu}\big)\right] \tag{6.20}$$

for all $x \in \mathbb{R}^n$, $t < t' \in [0,T]$.

(ii-2) Fix $x \in \mathbb{R}^n$, $t < t' \in [0,T]$. By the dynamic programming principle (6.20), we have :

$$\begin{aligned} |V(t,x) - V(t',x)| &= \left| \sup_{\nu \in \mathcal{U}_0} \mathbb{E}\left[V\big(t', X_{t'}^{t,x,\nu}\big)\right] - V(t',x)\right| \\ &\leq \sup_{\nu \in \mathcal{U}_0} \mathbb{E}\left|V\big(t', X_{t'}^{t,x,\nu}\big) - V(t',x)\right|. \end{aligned}$$

By the Lipschitz-continuity of $V(s,\cdot)$ established in the first part of this proof, we see that :

$$|V(t,x)-V(t',x)| \quad \le \quad Const \sup_{\nu\in\mathcal{U}_0} \mathbb{E}\left|X_{t'}^{t,x,\nu}-x\right|. \tag{6.21}$$

We shall now prove that

$$\sup_{\nu\in\mathcal{U}} \mathbb{E}\left|X_{t'}^{t,x,\nu}-x\right| \quad \le \quad Const\,(1+|x|)|t-t'|^{1/2}, \tag{6.22}$$

which provides the required $(1/2)-$Hölder continuity in view of (6.21). By definition of the process X, and assuming $t<t'$, we have

$$\begin{aligned}\mathbb{E}\left|X_{t'}^{t,x,\nu}-x\right|^2 &= \mathbb{E}\left|\int_t^{t'} b(r,X_r,\nu_r)dr+\int_t^{t'}\sigma(r,X_r,\nu_r)dW_r\right|^2\\ &\le Const\,\mathbb{E}\left[\int_t^{t'}|h(r,X_r,\nu_r)|^2\,dr\right]\end{aligned}$$

where $h:=[b^2+\sigma^2]^{1/2}$. Since h is Lipschitz-continuous in (t,x,u) and has quadratic growth in x and u, this provides:

$$\mathbb{E}\left|X_{t'}^{t,x,\nu}-x\right|^2 \quad \le \quad Const\left(\int_t^{t'}(1+|x|^2+|\nu_r|^2)dr+\int_t^{t'}\mathbb{E}\left|X_r^{t,x,\nu}-x\right|^2dr\right).$$

Since the control process ν is uniformly bounded, we obtain by the Gronwall lemma the estimate:

$$\mathbb{E}\left|X_{t'}^{t,x,\nu}-x\right|^2 \quad \le \quad Const\,(1+|x|)|t'-t|, \tag{6.23}$$

where the constant does not depend on the control ν. ♢

Remark 6.8. When f and/or k are non-zero, the conditions required on f and k in order to obtain the $(1/2)-$Hölder continuity of the value function can be deduced from the reduction of Remark 6.2 (iii).

Remark 6.9. Further regularity results can be proved for the value function under convenient conditions. Typically, one can prove that $\mathcal{L}^u V$ exists in the generalized sense, for all $u\in U$. This implies immediately that the result of Proposition 6.5 holds in the generalized sense. More technicalities are needed in order to derive the result of Proposition 6.4 in the generalized sense. We refer to [**?**], §IV.10, for a discussion of this issue.

6.4.2 A deterministic control problem with non-smooth value function

Let $\sigma\equiv 0$, $b(x,u)=u$, $U=[-1,1]$, and $n=1$. The controlled state is then the one-dimensional deterministic process defined by :

$$X_s \;=\; X_t+\int_t^s \nu_t dt \quad \text{for} \quad 0\le t\le s\le T\,.$$

Consider the deterministic control problem

$$V(t,x) \quad := \quad \sup_{\nu \in \mathcal{U}} (X_T)^2.$$

The value function of this problem is easily seen to be given by :

$$V(t,x) \quad = \quad \begin{cases} (x+T-t)^2 & \text{for } x \geq 0 \quad \text{with optimal control } \hat{u} = 1 \ , \\ (x-T+t)^2 & \text{for } x \leq 0 \quad \text{with optimal control } \hat{u} = -1 \ . \end{cases}$$

This function is continuous. However, a direct computation shows that it is not differentiable at $x = 0$.

6.4.3 A stochastic control problem with non-smooth value function

Let $U = \mathbb{R}$, and the controlled process X be the scalar process defined by the dynamics:

$$dX_t \quad = \quad \nu_t dW_t,$$

where W is a scalar Brownian motion. Let g be a bounded lower semicontinuous mapping on $\mathbb{R}$, and consider the stochastic control problem

$$V(t,x) \quad := \quad \sup_{\nu \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[g(X_T^\nu)\right].$$

Let us assume that V is smooth, and work towards a contradiction.

1. If V is $C^{1,2}([0,T),\mathbb{R})$, then it follows from Proposition 6.4 that V satisfies

$$-\partial_t V - \frac{1}{2} u^2 D^2 V \geq 0 \quad \text{for all} \quad u \in \mathbb{R},$$

and all $(t,x) \in [0,T) \times \mathbb{R}$. By sending u to infinity, it follows that

$$V(t,\cdot) \quad \text{is concave for all} \quad t \in [0,T). \tag{6.24}$$

2. Notice that $V(t,x) \geq \mathbb{E}_{t,x}\left[g(X_T^0)\right] = g(x)$. Then, it follows from (6.24) that:

$$V(t,x) \ \geq \ g^{\mathrm{conc}}(x) \quad \text{for all} \quad (t,x) \in [0,T) \times \mathbb{R}, \tag{6.25}$$

where g^{conc} is the concave envelope of g, i.e. the smallest concave majorant of g.

3. Since $g \leq g^{\mathrm{conc}}$, we see that

$$V(t,x) \ := \ \sup_{\nu \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[g(X_T^\nu)\right] \ \leq \ \sup_{\nu \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[g^{\mathrm{conc}}(X_T^\nu)\right].$$

Now, observe that X^ν is a local martingale for every $\nu \in \mathcal{U}_0$. Since g^{conc} is concave, the process $g^{\mathrm{conc}}(X^\nu)$ is a local supermartingale. Moreover g^{conc}

inherits the boundedness of g. Then the process $g^{\mathrm{conc}}(X^\nu)$ is a supermartingale. In particular, $\mathbb{E}_{t,x}\left[g^{\mathrm{conc}}(X^\nu_T)\right] \leq g^{\mathrm{conc}}(x)$, and

$$V(t,x) \quad \leq \quad g^{\mathrm{conc}}(x).$$

In view of (6.25), we have then proved that

$$\begin{aligned} & V \;\in\; C^{1,2}([0,T),\mathbb{R}) \\ \Longrightarrow \quad & V(t,x) \;=\; g^{\mathrm{conc}}(x) \text{ for all } (t,x) \in [0,T) \times \mathbb{R}. \end{aligned}$$

Now recall that this implication holds for any arbitrary non-negative lower semi-continuous function g. We then obtain a contradiction whenever the function g^{conc} is not $C^2(\mathbb{R})$. Hence

$$g^{\mathrm{conc}} \;\notin\; C^2(\mathbb{R}) \quad \Longrightarrow \quad V \;\notin\; C^{1,2}([0,T),\mathbb{R}^2).$$

Chapter 7

OPTIMAL STOPPING AND DYNAMIC PROGRAMMING

As in the previous chapter, we assume here that the filtration $\mathbb{F}$ is defined as the $\mathbb{P}-$augmentation of the canonical filtration of the Brownian motion W defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Our objective is to derive similar results, as those obtained in the previous chapter for standard stochastic control problems, in the context of optimal stopping problems. We will then first start by the formulation of optimal stopping problems, then the corresponding dynamic programming principle, and dynamic programming equation.

7.1 Optimal stopping problems

For $0 \le t \le T \le \infty$, we denote by $\mathcal{T}_{[t,T]}$ the collection of all $\mathbb{F}-$stopping times with values in $[t,T]$. We also recall the notation $\mathbf{S} := [0,T) \times \mathbb{R}^n$ for the parabolic state space of the underlying state process X defined by the stochastic differential equation:

$$dX_t \quad = \quad \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{7.1}$$

where μ and σ are defined on $\bar{\mathbf{S}}$ and take values in $\mathbb{R}^n$ and $\mathcal{S}_n$, respectively. We assume that μ and σ satisfies the usual Lipschitz and linear growth conditions so that the above SDE has a unique strong solution satisfying the integrability proved in Theorem 5.2.

The infinitesimal generator of the Markov diffusion process X is denoted by

$$\mathcal{A}\varphi := \mu \cdot D\varphi + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}} D^2\varphi\right].$$

Let g be a measurable function from $\mathbb{R}^n$ to $\mathbb{R}$, and assume that:

$$\mathbb{E}\left[\sup_{0\le t<T} |g(X_t)|\right] \quad < \quad \infty. \tag{7.2}$$

For instance, if g has polynomial growth, the latter integrability condition is automatically satisfied. Under this condition, the following criterion:

$$J(t,x,\tau) := \mathbb{E}\left[g\left(X_\tau^{t,x}\right)\mathbf{1}_{\tau<\infty}\right] \tag{7.3}$$

is well-defined for all $(t,x) \in \mathbf{S}$ and $\tau \in \mathcal{T}_{[t,T]}$. Here, $X^{t,x}$ denotes the unique strong solution of (7.1) with initial condition $X_t^{t,x} = x$.

The optimal stopping problem is now defined by:

$$V(t,x) := \sup_{\tau\in\mathcal{T}_{[t,T]}} J(t,x,\tau) \quad \text{for all} \quad (t,x)\in\mathbf{S}. \tag{7.4}$$

A stopping time $\hat{\tau} \in \mathcal{T}_{[t,T]}$ is called an optimal stopping rule if $V(t,x) = J(t,x,\hat{\tau})$.

The set

$$\mathcal{S} \quad := \quad \{(t,x) : \; V(t,x) = g(x)\} \tag{7.5}$$

is called the *stopping region* and is of particular interest: whenever the state is in this region, it is optimal to stop immediately. Its complement $\mathcal{S}^c$ is called the *continuation region*.

Remark 7.1. As in the previous chapter, we could have considered an appearently more general criterion

$$V(t,x) \quad := \quad \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}\left[\int_t^\tau \beta(t,s)f(s,X_s)ds + \beta(t,\tau)g\left(X_\tau^{t,x}\right)\mathbf{1}_{\tau<\infty}\right],$$

with

$$\beta(t,s) := e^{-\int_t^s k(s,X_s)ds} \quad \text{for} \quad 0\le t\le s<T.$$

However by introducing the additional state

$$\begin{aligned} Y_t &:= Y_0 + \int_0^t \beta_s f(s,X_s)ds, \\ Z_t &:= Z_0 + \int_0^t Z_s k(s,X_s)ds, \end{aligned}$$

we see immediately that we may reduce this problem to the context of (7.4).

Remark 7.2. Consider the subset of stopping rules:

$$\mathcal{T}_{[t,T]}^t \quad := \quad \left\{\tau\in\mathcal{T}_{[t,T]} : \; \tau \text{ independent of } \mathcal{F}_t\right\}. \tag{7.6}$$

By a similar argument as in Remark 6.2 (iv), we can see that the maximization in the optimal stopping problem (7.4) can be restricted to this subset, i.e.

$$V(t,x) := \sup_{\tau\in\mathcal{T}_{[t,T]}^t} J(t,x,\tau) \quad \text{for all} \quad (t,x)\in\mathbf{S}. \tag{7.7}$$

7.2 The dynamic programming principle

In the context of optimal stopping problems, the proof of the dynamic programming principle is easier than in the context of stochastic control problems of the previous chapter. The reader may consult the excellent exposition in the book of Karatzas and Shreve [**?**], Appendix D, where the following dynamic programming principle is proved:

$$V(t,x) \quad = \quad \sup_{\tau\in\mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<\theta\}} g(X^{t,x}_\tau) + \mathbf{1}_{\{\tau\geq\theta\}} V(\theta, X^{t,x}_\theta)\right], \tag{7.8}$$

for all $(t,x)\in\mathbf{S}$ and $\tau\in\mathcal{T}_{[t,T]}$. In particular, the proof in the latter reference does not require any heavy measurable selection, and is essentially based on the supermartingale nature of the so-called Snell envelope process. Moreover, we observe that it does not require any Markov property of the underlying state process.

We report here a different proof in the sprit of the weak dynamic programming principle for stochastic control problems proved in the previous chapter. The subsequent argument is specific to our Markovian framework and, in this sense, is weaker than the classical dynamic programming principle. However, the combination of the arguments of this chapter with those of the previous chapter allow to derive a dynamic programming principle for mixed stochastic control and stopping problem.

The following claim will be making using of the subset $\mathcal{T}^t_{[t,T]}$, introduced in (7.6), of all stopping times in $\mathcal{T}_{[t,T]}$ which are independent of $\mathcal{F}_t$, and the notations:

$$V_*(t,x) := \liminf_{(t',x')\to(t,x)} V(t',x') \quad \text{and} \quad V^*(t,x) := \limsup_{(t',x')\to(t,x)} V(t',x')$$

for all $(t,x)\in\bar{\mathbf{S}}$. We recall that V_* and V^* are the lower and upper semicontinuous envelopes of V, and that $V_* = V^* = V$ whenever V is continuous.

Theorem 7.3. *Assume that V is locally bounded. For $(t,x)\in\mathbf{S}$, let $\theta\in\bar{\mathcal{T}}^t_{[t,T]}$ be a stopping time such that $X^{t,x}_\theta$ is bounded. Then:*

$$V(t,x) \quad \leq \quad \sup_{\tau\in\mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<\theta\}} g(X^{t,x}_\tau) + \mathbf{1}_{\{\tau\geq\theta\}} V^*(\theta, X^{t,x}_\theta)\right], \tag{7.9}$$

$$V(t,x) \quad \geq \quad \sup_{\tau\in\mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<\theta\}} g(X^{t,x}_\tau) + \mathbf{1}_{\{\tau\geq\theta\}} V_*(\theta, X^{t,x}_\theta))\right]. \tag{7.10}$$

Proof. Inequality (7.9) follows immediately from the tower property and the fact that $J\leq V^*$.

We next prove inequality (7.10) with V_* replaced by an arbitrary function

$$\varphi:\mathbf{S}\longrightarrow\mathbb{R} \quad \text{such} \quad \varphi \text{ is upper-semicontinuous and } V\geq\varphi,$$

which implies (7.10) by the same argument as in Step 3 of the proof of Theorem 6.3.

Arguying as in Step 2 of the proof of Theorem 6.3, we first observe that, for every $\varepsilon > 0$, we can find a countable family $\bar{A}_i \subset (t_i - r_i, t_i] \times A_i \subset \mathbf{S}$, together with a sequence of stopping times $\tau^{i,\varepsilon}$ in $\mathcal{T}^{t_i}_{[t_i,T]}$, $i \geq 1$, satisfying $\bar{A}_0 = \{T\} \times \mathbb{R}^d$ and

$$\cup_{i\geq 0}\bar{A}_i = \mathbf{S},\ \ \bar{A}_i \cap \bar{A}_j = \emptyset \text{ for } i \neq j \in \mathbb{N},\ \ \bar{J}(\cdot;\tau^{i,\varepsilon}) \geq \varphi - 3\varepsilon \text{ on } \bar{A}_i \text{ for } i \geq 1. \tag{7.11}$$

Set $\bar{A}^n := \cup_{i\leq n}\bar{A}_i$, $n \geq 1$. Given two stopping times $\theta, \tau \in \mathcal{T}^t_{[t,T]}$, it is clear that

$$\tau^{n,\varepsilon} := \tau \mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\geq\theta\}} \left(T\mathbf{1}_{(\bar{A}^n)^c}\left(\theta, X^{t,x}_\theta\right) + \sum_{i=1}^n \tau^{i,\varepsilon}\mathbf{1}_{\bar{A}_i}\left(\theta, X^{t,x}_\theta\right) \right)$$

defines a stopping time in $\mathcal{T}^t_{[t,T]}$. We then deduce from the tower property and (7.11) that

$$\begin{array}{rcl} \bar{V}(t,x) & \geq & \bar{J}(t,x;\tau^{n,\varepsilon}) \\ & \geq & \mathbb{E}\left[g\left(X^{t,x}_\tau\right)\mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\geq\theta\}}\left(\varphi(\theta, X^{t,x}_\theta) - 3\varepsilon\right)\mathbf{1}_{\bar{A}^n}(\theta, X^{t,x}_\theta)\right] \\ & & +\mathbb{E}\left[\mathbf{1}_{\{\tau\geq\theta\}} g(X^{t,x}_T)\mathbf{1}_{(\bar{A}^n)^c}(\theta, X^{t,x}_\theta)\right]. \end{array}$$

By sending $n \to \infty$ and arguing as in the end of Step 2 of the proof of Theorem 6.3, we deduce that

$$\bar{V}(t,x) \ \ \geq \ \ \mathbb{E}\left[g\left(X^{t,x}_\tau\right)\mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\geq\theta\}}\varphi(\theta, X^{t,x}_\theta)\right] - 3\varepsilon,$$

and the result follows from the arbitrariness of $\varepsilon > 0$ and $\tau \in \mathcal{T}^t_{[t,T]}$. ♢

7.3 The dynamic programming equation

In this section, we explore the infinitesimal counterpart of the dynamic programming principle of Theorem 7.3, when the value function V is a priori known to be smooth. The smoothness that will be required in this chapter must be so that we can apply Itô's formula to V. In particular, V is continuous, and the dynamic programming principle of Theorem 7.3 reduces to the classical dynamic programming principle (7.8).

Loosely speaking, the following dynamic programming equation says the following:

- In the stopping region $\mathcal{S}$ defined in (7.5), continuation is sub-optimal, and therefore the linear PDE must hold with inequality in such a way that the value function is a submartingale.

- In the continuation region $\mathcal{S}^c$, it is optimal to delay the stopping decision after some small moment, and therefore the value function must solve a linear PDE as in Chapter 5.

Theorem 7.4. *Assume that* $V \in C^{1,2}([0,T),\mathbb{R}^n)$*, and let* $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be continuous. Then* V *solves the obstacle problem:*

$$\min\left\{-(\partial_t + \mathcal{A})V \ , \ V - g\right\} \ = \ 0 \quad \textit{on} \quad \mathbf{S}. \tag{7.12}$$

Proof. We organize the proof into two steps.
1. We first show that:

$$\min\left\{-(\partial_t + \mathcal{A})V \ , \ V - g\right\} \ \geq \ 0 \quad \text{on} \quad \mathbf{S}. \tag{7.13}$$

The inequality $V - g \geq 0$ is obvious as the constant stopping rule $\tau = t \in \mathcal{T}_{[t,T]}$ is admissible. Next, for $(t_0, x_0) \in \mathbf{S}$, consider the stopping times

$$\theta_h := \inf\left\{t > t_0 : \ (t, X_t^{t_0,x_0}) \notin [t_0, t_0 + h] \times B\right\}, h > 0,$$

where B is the unit ball of $\mathbb{R}^n$ centered at x_0. Then $\theta_h \in \mathcal{T}^t_{[t,T]}$ for sufficiently small h, and it follows from (7.10)that:

$$V(t_0, x_0) \quad \geq \quad \mathbb{E}\left[V\left(\theta_h, X_{\theta_h}\right)\right].$$

We next apply Itô's formula, and observe that the expected value of the diffusion term vanishes because (t, X_t) lies in the compact subset $[t_0, t_0 + h] \times B$ for $t \in [t_0, \theta_h]$. Then:

$$\mathbb{E}\left[\frac{-1}{h}\int_{t_0}^{\theta_h} (\partial_t + \mathcal{A})V(t, X_t^{t_0,x_0})dt\right] \quad \geq \quad 0.$$

Clearly, there exists $\hat{h}_\omega > 0$, depending on ω, $\theta_h = h$ for $h \leq \hat{h}_\omega$. Then, it follows from the mean value theorem that the expression inside the expectation converges $\mathbb{P}-$a.s. to $-(\partial_t + \mathcal{A})V(t_0, x_0)$, and we conclude by dominated convergence that $-(\partial_t + \mathcal{A})V(t_0, x_0) \geq 0$.
2. In order to complete the proof, we use a contradiction argument, assuming that

$$V(t_0, x_0) > 0 \text{ and } -(\partial_t + \mathcal{A})V(t_0, x_0) > 0 \quad \text{at some} \quad (t_0, x_0) \in \mathbf{S} \tag{7.14}$$

and we work towards a contradiction of (7.9). Introduce the function

$$\varphi(t, x) := V(t, x) + \frac{\varepsilon}{2}|x - x_0|^2 \quad \text{for} \quad (t, x) \in \mathbf{S}.$$

Then, it follows from (7.14) that for a sufficiently small $\varepsilon > 0$, we may find $h > 0$ and $\delta > 0$ such that

$$V \geq g + \delta \text{ and } -(\partial_t + \mathcal{A})\varphi \geq 0 \quad \text{on} \quad \mathcal{N}_h := [t_0, t_0 + h] \times hB. \tag{7.15}$$

Moreover:

$$-\gamma \quad := \quad \max_{\partial\mathcal{N}_h}(V - \varphi) \ < \ 0. \tag{7.16}$$

Next, let

$$\theta \quad := \quad \inf\left\{t > t_0 : \ \left(t, X_t^{t_0,x_0}\right) \not\in \mathcal{N}_h\right\}.$$

For an arbitrary stopping rule $\tau \in \mathcal{T}_{[t,T]}^t$, we compute by Itô's formula that:

$$\begin{aligned} \mathbb{E}\left[V\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right) - V(t_0,x_0)\right] &= \mathbb{E}\left[(V-\varphi)\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right)\right] \\ &\quad + \mathbb{E}\left[\varphi\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right) - \varphi(t_0,x_0)\right] \\ &= \mathbb{E}\left[(V-\varphi)\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right)\right] \\ &\quad + \mathbb{E}\left[\int_{t_0}^{\tau\wedge\theta} (\partial_t + \mathcal{A})\varphi(t, X_t^{t_0,x_0})dt\right], \end{aligned}$$

where the diffusion term has zero expectation because the process $(t, X_t^{t_0,x_0})$ is confined to the compact subset $\mathcal{N}_h$ on the stochastic interval $[t_0, \tau\wedge\theta]$. Since $-\mathcal{L}\varphi \geq 0$ on $\mathcal{N}_h$ by (7.15), this provides:

$$\begin{aligned} \mathbb{E}\left[V\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right) - V(t_0,x_0)\right] &\leq \mathbb{E}\left[(V-\varphi)\left(\tau\wedge\theta, X_{\tau\wedge\theta}\right)\right] \\ &\leq -\gamma\mathbb{P}[\tau \geq \theta], \end{aligned}$$

by (7.16). Then, since $V \geq g + \delta$ on $\mathcal{N}_h$ by (7.15):

$$\begin{aligned} V(t_0,x_0) &\geq \gamma\mathbb{P}[\tau\geq\theta] + \mathbb{E}\left[\left(g(X_\tau^{t_0,x_0}) + \delta\right)\mathbf{1}_{\{\tau<\theta\}} + V\left(\theta, X_\theta^{t_0,x_0}\right)\mathbf{1}_{\{\tau\geq\theta\}}\right] \\ &\geq (\gamma\wedge\delta) + \mathbb{E}\left[g(X_\tau^{t_0,x_0})\mathbf{1}_{\{\tau<\theta\}} + V\left(\theta, X_\theta^{t_0,x_0}\right)\mathbf{1}_{\{\tau\geq\theta\}}\right]. \end{aligned}$$

By the arbitrariness of $\tau \in \mathcal{T}_{[t,T]}^t$, this provides the desired contradiction of (7.9). ♢

7.4 Regularity of the value function

7.4.1 Finite horizon optimal stopping

In this subsection, we consider the case $T < \infty$. Similar to the continuity result of Proposition 6.7 for the stochastic control framework, the following continuity result is obtained as a consequence of the flow continuiy of Theorem 5.4 together with the dynamic programming principle.

Proposition 7.5. *Assume g is Lipschitz-continuous, and let $T < \infty$. Then, there is a constant C such that:*

$$\left|V(t,x) - V(t',x')\right| \ \leq \ C\left(|x-x'| + \sqrt{|t-t'|}\right) \quad \textit{for all} \quad (t,x),(t',x') \in \mathbf{S}.$$

Proof. (i) For $t \in [0,T]$ and $x, x' \in \mathbb{R}^n$, it follows from the Lipschitz property of g that:

$$\begin{aligned} |V(t,x) - V(t,x')| &\leq Const \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}\left|X_\tau^{t,x} - X_\tau^{t,x'}\right| \\ &\leq Const\, \mathbb{E}\sup_{t\leq s\leq T}\left|X_\tau^{t,x} - X_\tau^{t,x'}\right| \\ &\leq Const\,|x-x'| \end{aligned}$$

by the flow continuity result of Theorem 5.4.
ii) To rpove the H"older continuity result in t, we argue as in the proof of Proposition 6.7 using the dynamic programming principle of Theorem 7.3.

(ii-1) We first observe that, whenever the stopping time $\theta = t' > t$ is constant (i.e. deterministic), the dynamic programming principle (7.9)-(7.10) holds true if the semicontinuous envelopes are taken with respect to the variable x, with fixed time variable. Since V is continuous in x by the first part of this proof, we deduce that

$$V(t,x) \quad = \quad \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<t'\}} g\left(X^{t,x}_\tau\right) + \mathbf{1}_{\{\tau \geq t'\}} V\left(t', X^{t,x}_{t'}\right)\right] \tag{7.17}$$

(ii) We then estimate that

$$\begin{aligned} 0 \;\leq\; V(t,x) - \mathbb{E}\left[V\left(t', X^{t,x}_{t'}\right)\right] \quad &\leq \quad \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<t'\}}\left(g\left(X^{t,x}_\tau\right) - V\left(t', X^{t,x}_{t'}\right)\right)\right] \\ &\leq \quad \sup_{\tau \in \mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<t'\}}\left(g\left(X^{t,x}_\tau\right) - g\left(X^{t,x}_{t'}\right)\right)\right], \end{aligned}$$

where the last inequality follows from the fact that $V \geq g$. Using the Lipschitz property of g, this provides:

$$\begin{aligned} 0 \;\leq\; V(t,x) - \mathbb{E}\left[V\left(t', X^{t,x}_{t'}\right)\right] \quad &\leq \quad Const\,\mathbb{E}\left[\sup_{t \leq s \leq t'} \left|X^{t,x}_s - X^{t,x}_{t'}\right|\right] \\ &\leq \quad Const\,(1+|x|)\sqrt{t'-t} \end{aligned}$$

by the flow continuity result of Theorem 5.4. Using this estimate together with the Lipschitz property proved in (i) above, this provides:

$$\begin{aligned} |V(t,x) - V(t',x)| \quad &\leq \quad \left|V(t,x) - \mathbb{E}\left[V\left(t', X^{t,x}_{t'}\right)\right]\right| + \left|\mathbb{E}\left[V\left(t', X^{t,x}_{t'}\right)\right] - V(t',x)\right| \\ &\leq \quad Const\left((1+|x|)\sqrt{t'-t} + \mathbb{E}\left|X^{t,x}_{t'} - x\right|\right) \\ &\leq \quad Const\,(1+|x|)\sqrt{t'-t}, \end{aligned}$$

by using again Theorem 5.4. ♢

7.4.2 Infinite horizon optimal stopping

In this section, the state process X is defined by a homogeneous scalar diffusion:

$$dX_t \quad = \quad \mu(X_t)dt + \sigma(X_t)dW_t. \tag{7.18}$$

We introduce the hitting times:

$$H^x_b := \inf\left\{t > 0 :\; X^{0,x} = b\right\},$$

and we assume that the process X is regular, i.e.

$$\mathbb{P}\left[H^x_b < \infty\right] > 0 \quad \text{for all} \quad x, b \in \mathbb{R}, \tag{7.19}$$

which means that there is no subinterval of $\mathbb{R}$ from which the process X can not exit.

We consider the infinite horizon optimal stopping problem:

$$V(x) \quad := \quad \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\beta\tau} g\left(X_\tau^{0,x}\right) \mathbf{1}_{\{\tau<\infty\}}\right], \tag{7.20}$$

where $\mathcal{T} := \mathcal{T}_{[0,\infty]}$, and $\beta > 0$ is the discount rate parameter.

According to Theorem 7.3, the dynamic programming equation corresponding to this optimal stopping problem is the obstacle problem:

$$\min\left\{\beta v - \mathcal{A}v, v - g\right\} \quad = \quad 0,$$

where the differential operator in the present homogeneous context is given by the generator of the diffusion:

$$\mathcal{A}v \quad := \quad \mu v' + \frac{1}{2}\sigma^2 v''. \tag{7.21}$$

The ordinary differential equation

$$\mathcal{A}v - \beta v \quad = \quad 0 \tag{7.22}$$

has two positive linearly independent solutions

$$\psi, \phi \geq 0 \quad \text{such that} \quad \psi \text{ strictly increasing, } \phi \text{ strictly decreasing.} \tag{7.23}$$

Clearly ψ and ϕ are uniquely determined up to a positive constant, and all other solution of (7.22) can be expressed as a linear combination of ψ and ϕ.

The following result follows from an immediate application of Itô's formula.

Lemma 7.6. *For any $b_1 < b_2$, we have:*

$$\begin{aligned}
\mathbb{E}\left[e^{-\beta H_{b_1}^x} \mathbf{1}_{\{H_{b_1}^x \leq H_{b_2}^x\}}\right] &= \frac{\psi(x)\phi(b_2) - \psi(b_2)\phi(x)}{\psi(b_1)\phi(b_2) - \psi(b_2)\phi(b_1)}, \\
\mathbb{E}\left[e^{-\beta H_{b_2}^x} \mathbf{1}_{\{H_{b_1}^x \geq H_{b_2}^x\}}\right] &= \frac{\psi(b_1)\phi(x) - \psi(x)\phi(b_1)}{\psi(b_1)\phi(b_2) - \psi(b_2)\phi(b_1)}.
\end{aligned}$$

We now show that the value function V is concave up to some change of variable, and provides conditions under which V is C^1 across the exercise boundary, i.e. the boundary between the exercise and the continuation regions. For the next result, we observe that the fnction (ψ/ϕ) is continuous and strictly increasing by (7.23), and therefore invertible.

Theorem 7.7. (i) *The function $(V/\phi) \circ (\psi/\phi)^{-1}$ is concave. In particular, V is continuous on $\mathbb{R}$.*
(ii) *Let x_0 be such that $V(x_0) = g(x_0)$, and assume that g, ψ and ϕ are differentiable at x_0. Then V is differentiable at x_0, and $V'(x_0) = g'(x_0)$.*

Proof. For (i), it is sufficient to prove that:

$$\frac{\frac{V}{\phi}(x) - \frac{V}{\phi}(b_1)}{F(x) - F(b_1)} \leq \frac{\frac{V}{\phi}(b_2) - \frac{V}{\phi}(x)}{F(b_2) - F(x)} \quad \text{for all} \quad b_1 < x < b_2. \tag{7.24}$$

For $\varepsilon > 0$, consider the $\varepsilon-$optimal stopping rules $\tau_1, \tau_2 \in \mathcal{T}$ for the problems $V(b_1)$ and $V(b_2)$:

$$\mathbb{E}\left[e^{-\beta\tau_i} g\left(X^{0,x}_{\tau_i}\right)\right] \geq V(b_i) - \varepsilon \quad \text{for} \quad i = 1, 2.$$

We next define the stopping time

$$\tau^\varepsilon \quad := \quad \left(H^x_{b_1} + \tau_1 \circ \theta_{H^x_{b_1}}\right) \mathbf{1}_{\{H^x_{b_1} < H^x_{b_2}\}} + \left(H^x_{b_2} + \tau_2 \circ \theta_{H^x_{b_2}}\right) \mathbf{1}_{\{H^x_{b_2} < H^x_{b_1}\}},$$

where θ denotes the shift operator on the canonical space. In words, the stopping rule τ^ε uses the $\varepsilon-$optimal stopping rule τ_1 if the level b_1 is reached before the level b_2, and the $\varepsilon-$optimal stopping rule τ_2 otherwise. Then, it follows from the strong Markov property that

$$\begin{aligned}
V(x) \quad &\geq \quad \mathbb{E}\left[e^{-\beta\tau^\varepsilon} g\left(X^{0,x}_{\tau^\varepsilon}\right)\right] \\
&= \quad \mathbb{E}\left[e^{-\beta H^x_{b_1}} \mathbb{E}\left[e^{-\beta\tau_1} g\left(X^{0,b_1}_{\tau_1}\right)\right] \mathbf{1}_{\{H^x_{b_1} < H^x_{b_2}\}}\right] \\
&\qquad + \mathbb{E}\left[e^{-\beta H^x_{b_2 1}} \mathbb{E}\left[e^{-\beta\tau_2} g\left(X^{0,b_2}_{\tau_2}\right)\right] \mathbf{1}_{\{H^x_{b_2} < H^x_{b_1}\}}\right] \\
&\geq \quad \left(V(b_1) - \varepsilon\right) \mathbb{E}\left[e^{-\beta H^x_{b_1}} \mathbf{1}_{\{H^x_{b_1} < H^x_{b_2}\}}\right] \\
&\qquad + \left(V(b_2) - \varepsilon\right) \mathbb{E}\left[e^{-\beta H^x_{b_2}} \mathbf{1}_{\{H^x_{b_2} < H^x_{b_1}\}}\right].
\end{aligned}$$

Sending $\varepsilon \searrow 0$, this provides

$$V(x) \quad \geq \quad V(b_1)\mathbb{E}\left[e^{-\beta H^x_{b_1}} \mathbf{1}_{\{H^x_{b_1} < H^x_{b_2}\}}\right] + V(b_2)\mathbb{E}\left[e^{-\beta H^x_{b_2}} \mathbf{1}_{\{H^x_{b_2} < H^x_{b_1}\}}\right].$$

By using the explicit expressions of Lemma 7.6 above, this provides:

$$\frac{V(x)}{\phi(x)} \quad \geq \quad \frac{V(b_1)}{\phi(b_1)} \frac{\frac{\psi}{\phi}(b_2) - \frac{\psi}{\phi}(x)}{\frac{\psi}{\phi}(b_2) - \frac{\psi}{\phi}(b_1)} + \frac{V(b_2)}{\phi(b_2)} \frac{\frac{\psi}{\phi}(x) - \frac{\psi}{\phi}(b_1)}{\frac{\psi}{\phi}(b_2) - \frac{\psi}{\phi}(b_1)},$$

which implies (7.24).

(ii) We next prove the smoothfit result. Let x_0 be such that $V(x_0) = g(x_0)$. Then, since $V \geq g$, ψ is strictly increasing, $\phi \geq 0$ is strictly decreasing, it follows from (7.24) that:

$$\begin{aligned}
\frac{\frac{g}{\phi}(x_0 + \varepsilon) - \frac{g}{\phi}(x_0)}{\frac{\psi}{\phi}(x_0 + \varepsilon) - \frac{\psi}{\phi}(x_0)} \quad &\leq \quad \frac{\frac{V}{\phi}(x_0 + \varepsilon) - \frac{V}{\phi}(x_0)}{\frac{\psi}{\phi}(x_0 + \varepsilon) - \frac{\psi}{\phi}(x_0)} \qquad (7.25) \\
&\leq \quad \frac{\frac{V}{\phi}(x_0 - \delta) - \frac{V}{\phi}(x_0)}{\frac{\psi}{\phi}(x_0 - \delta) - \frac{\psi}{\phi}(x_0)} \leq \frac{\frac{g}{\phi}(x_0 - \delta) - \frac{g}{\phi}(x_0)}{\frac{\psi}{\phi}(x_0 - \delta) - \frac{\psi}{\phi}(x_0)}
\end{aligned}$$

for all $\varepsilon > 0$, $\delta > 0$. Multiplying by $((\psi/\phi)(x_0+\varepsilon) - (\psi/\phi)(x_0))/\varepsilon$, this implies that:

$$\frac{\frac{g}{\phi}(x_0+\varepsilon) - \frac{g}{\phi}(x_0)}{\varepsilon} \le \frac{\frac{V}{\phi}(x_0+\varepsilon) - \frac{V}{\phi}(x_0)}{\varepsilon} \le \frac{\Delta^+(\varepsilon)}{\Delta^-(\delta)} \frac{\frac{g}{\phi}(x_0-\delta) - \frac{g}{\phi}(x_0)}{\delta}, \tag{7.26}$$

where

$$\Delta^+(\varepsilon) := \frac{\frac{\psi}{\phi}(x_0+\varepsilon) - \frac{\psi}{\phi}(x_0)}{\varepsilon} \quad \text{and} \quad \Delta^-(\delta) := \frac{\frac{\psi}{\phi}(x_0-\delta) - \frac{\psi}{\phi}(x_0)}{\delta}.$$

We next consider two cases:

- If $(\psi/\phi)'(x_0) \neq 0$, then we may take $\varepsilon = \delta$ and send $\varepsilon \searrow 0$ in (7.26) to obtain:

  $$\frac{d^+(\frac{V}{\phi})}{dx}(x_0) \quad = \quad \left(\frac{g}{\phi}\right)'(x_0). \tag{7.27}$$

- If $(\psi/\phi)'(x_0) = 0$, then, we use the fact that for every sequence $\varepsilon_n \searrow 0$, there is a subsequence $\varepsilon_{n_k} \searrow 0$ and $\delta_k \searrow 0$ such that $\Delta^+(\varepsilon_{n_k}) = \Delta^-(\delta_k)$. Then (7.26) reduces to:

  $$\frac{\frac{g}{\phi}(x_0+\varepsilon_{n_k}) - \frac{g}{\phi}(x_0)}{\varepsilon_{n_k}} \le \frac{\frac{V}{\phi}(x_0+\varepsilon_{n_k}) - \frac{V}{\phi}(x_0)}{\varepsilon_{n_k}} \le \frac{\frac{g}{\phi}(x_0-\delta_k) - \frac{g}{\phi}(x_0)}{\delta_k},$$

  and therefore

  $$\frac{\frac{V}{\phi}(x_0+\varepsilon_{n_k}) - \frac{V}{\phi}(x_0)}{\varepsilon_{n_k}} \quad \longrightarrow \quad \left(\frac{g}{\phi}\right)'(x_0).$$

  By the arbitrariness of the sequence $(\varepsilon_n)_n$, this provides (7.27).

Similarly, multiplying (7.25) by $((\psi/\phi)(x_0) - (\psi/\phi)(x_0-\delta))/\delta$, and arguying as above, we obtain:

$$\frac{d^-(\frac{V}{\phi})}{dx}(x_0) \quad = \quad \left(\frac{g}{\phi}\right)'(x_0),$$

thus completing the proof. ♢

7.4.3 An optimal stopping problem with nonsmooth value

We consider the example

$$X_s^{t,x} := x + (W_t - W_s) \quad \text{for} \quad s \ge t.$$

Let $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a measurable nonnegative function with $\liminf_{x\to\infty} g(x) = 0$, and consider the infinite horizon optimal stopping problem:

$$\begin{aligned} V(t,x) &:= \sup_{\tau\in\mathcal{T}_{[t,\infty]}} \mathbb{E}\left[g\left(X_\tau^{t,x}\right)\mathbf{1}_{\{\tau<\infty\}}\right] \\ &= \sup_{\tau\in\mathcal{T}_{[t,\infty)}} \mathbb{E}\left[g\left(X_\tau^{t,x}\right)\right]. \end{aligned}$$

Let us assume that $V \in C^{1,2}(\mathbf{S})$, and work towards a contradiction. We first observe by the homogeneity of the problem that $V(t,x) = V(x)$ is independent of t. Moreover, it follows from Theorem 7.4 that V is concave in x and $V \geq g$. Then

$$V \geq g^{\text{conc}}, \tag{7.28}$$

where g^{conc} is the concave envelope of g. If $g^{\text{conc}} = \infty$, then $V = \infty$. We then continue in the more inetersting case where $g^{\text{conc}} < \infty$.

By the jensen inequality and the non-negativity of g, the process $\{g\left(X_s^{t,x}\right), s \geq t\}$ is a supermartingale, and:

$$V(t,x) \;\leq\; \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}\left[g^{\text{conc}}\left(X_\tau^{t,x}\right)\right] \;\leq\; g^{\text{conc}}(x).$$

Hence, $V = g^{\text{conc}}$, and we obtain the required contradiction whenever g^{conc} is not differentiable at some point of $\mathbb{R}$.

Chapter 8

Solving Control Problems by Verification

In this chapter, we present a general argument, based on Itô's formula, which allows to show that some "guess" of the value function is indeed equal to the unknown value function. Namely, given a smooth solution v of the dynamic programming equation, we give sufficient conditions which allow to conclude that v coincides with the value function V. This is the so-called *verification argument.* The statement of this result is heavy, but its proof is simple and relies essentially on Itô's formula. However, depending on the problem in hand, the verification of the conditions which must be satisfied by the candidate solution can be difficult.

The verification argument will be provided in the contexts of stochastic control and optimal stopping problems. We conclude the chapter by some examples of application of the verification theorem.

8.1 The verification argument for stochastic control problems

We recall the stochastic control problem formulation of Section 6.1. The set of admissible control processes $\mathcal{U}_0 \subset \mathcal{U}$ is the collection of all progressively measurable processes with values in the subset $U \subset \mathbb{R}^k$. For every admissible control process $\nu \in \mathcal{U}_0$, the controlled process is defined by the stochastic differential equation:

$$dX_t^\nu \;=\; b(t, X_t^\nu, \nu_t)dt + \sigma(t, X_t^\nu, \nu_t)dW_t.$$

The gain criterion is given by

$$J(t,x,\nu) := \mathbb{E}\left[\int_t^T \beta^\nu(t,s) f(s, X_s^{t,x,\nu}, \nu_s)ds + \beta^\nu(t,T) g(X_T^{t,x,\nu})\right],$$

with

$$\beta^\nu(t,s) \quad := \quad e^{-\int_t^s k(r,X_r^{t,x,\nu},\nu_r)dr}.$$

The stochastic control problem is defined by the value function:

$$V(t,x) := \sup_{\nu\in\mathcal{U}_0} J(t,x,\nu), \quad \text{for} \quad (t,x)\in\mathbf{S}.$$

We follow the notations of Section 6.3. We recall the Hamiltonian $H:\mathbf{S}\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}_d$ defined by :

$$\begin{aligned} & H(t,x,r,p,\gamma) \\ := & \sup_{u\in U}\left\{-k(t,x,u)r+b(t,x,u)\cdot p+\frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}}(t,x,u)\gamma]+f(t,x,u)\right\}, \end{aligned}$$

where b and σ satisfy the conditions (6.1)-(6.2), and the coefficients f and k are measurable. The linear second order operator

$$\begin{aligned} \mathcal{L}^u\varphi(t,x) \quad := \quad & -k(t,x,u)\varphi(t,x)+b(t,x,u)\cdot D\varphi(t,x) \\ & +\frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}(t,x,u)D^2\varphi(t,x)\right], \end{aligned}$$

corresponds to the controlled process $\{\beta^u(0,t)X_t^u,\ t\ge 0\}$ controlled by the constant control process u. By Itô's formula:

$$\begin{aligned} \beta^\nu(0,s)\varphi(s,X_s^\nu)-\beta^\nu(0,t)\varphi(t,X_t^\nu) \quad = \quad & \int_t^s \beta^\nu(0,r)\left(\partial_t+\mathcal{L}^{\nu_r}\right)\varphi(r,X_r^\nu)dr \\ & +\int_t^s \beta^\nu(0,r)D\varphi(r,X_r^\nu)\cdot\sigma(r,X_r^\nu,\nu_r)dW_r \end{aligned}$$

for every $t\le s$ and smooth function $\varphi\in C^{1,2}([t,s],\mathbb{R}^d)$ and each admissible control process $\nu\in\mathcal{U}_0$.

Theorem 8.1. *Let $T<\infty$, and $v\in C^{1,2}([0,T),\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)$. Assume that $\|k^-\|_\infty<\infty$ and v and f have quadratic growth, i.e. there is a constant C such that*

$$|f(t,x,u)|+|v(t,x)| \;\le\; C(1+|x|^2) \quad \textit{for all} \quad (t,x,u)\in[0,T)\times\mathbb{R}^d\times U.$$

(i) *Suppose that $v(T,\cdot)\ge g$ and*

$$-\partial_t v(t,x)-H\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) \quad \ge \quad 0$$

on $[0,T)\times\mathbb{R}^d$. Then $v\ge V$ on $[0,T]\times\mathbb{R}^d$.
(ii) *Assume further that $v(T,\cdot)=g$, and there exists a minimizer $\hat{u}(t,x)$ of $u \longmapsto \mathcal{L}^u v(t,x)+f(t,x,u)$ such that*

- $0=\partial_t v(t,x)+\mathcal{L}^{\hat{u}(t,x)}v(t,x)+f(t,x,u)$,

- *the stochastic differential equation*

$$dX_s \quad = \quad b\left(s, X_s, \hat{u}(s, X_s)\right) ds + \sigma\left(s, X_s, \hat{u}(s, X_s)\right) dW_s$$

  defines a unique solution X for each given initial data $X_t = x$,

- *the process $\hat{\nu}_s := \hat{u}(s, X_s)$ is a well-defined control process in $\mathcal{U}_0$.*

Then $v = V$, and $\hat{\nu}$ is an optimal Markov control process.

Proof. Let $\nu \in \mathcal{U}_0$ be an arbitrary control process, X the associated state process with initial data $X_t = x$, and define the stopping time

$$\theta_n \quad := \quad T \wedge \inf\left\{s > t \ : \ |X_s - x| \geq n\right\}.$$

By Itô's formula, we have

$$\begin{aligned} v(t,x) \quad = \quad & \beta(t,\theta_n) v\left(\theta_n, X_{\theta_n}\right) - \int_t^{\theta_n} \beta(t,r)(\partial_t + \mathcal{L}^{\nu(r)}) v(r, X_r) dr \\ & - \int_t^{\theta_n} \beta(t,r) Dv(r, X_r) \cdot \sigma(r, X_r, \nu_r) dW_r \end{aligned}$$

Observe that $(\partial_t + \mathcal{L}^{\nu(r)})v + f(\cdot,\cdot,u) \leq \partial_t v + H(\cdot,\cdot,v,Dv,D^2 v) \leq 0$, and that the integrand in the stochastic integral is bounded on $[t,\theta_n]$, a consequence of the continuity of Dv, σ and the condition $\|k^-\|_\infty < \infty$. Then :

$$v(t,x) \quad \geq \quad \mathbb{E}\left[\beta(t,\theta_n) v\left(\theta_n, X_{\theta_n}\right) + \int_t^{\theta_n} \beta(t,r) f(r, X_r, \nu_r) dr\right]. \tag{8.1}$$

We now take the limit as n increases to infinity. Since $\theta_n \longrightarrow T$ a.s. and

$$\begin{aligned} & \left|\beta(t,\theta_n) v\left(\theta_n, X_{\theta_n}\right) + \int_t^{\theta_n} \beta(t,r) f(r, X_r, \nu_r) dr\right| \\ & \qquad \leq \ C e^{T\|k^-\|_\infty} (1 + |X_{\theta_n}|^2 + T + \textstyle\int_t^T |X_s|^2 ds) \\ & \qquad \leq \ C e^{T\|k^-\|_\infty} (1+T)(1 + \sup_{t \leq s \leq T} \ |X_s|^2) \ \in \ \mathbb{L}^1, \end{aligned}$$

by the estimate (6.5) of Theorem 6.1, it follows from the dominated convergence that

$$\begin{aligned} v(t,x) \quad & \geq \quad \mathbb{E}\left[\beta(t,T) v(T, X_T) + \int_t^T \beta(t,r) f(r, X_r, \nu_r) dr\right] \\ & \geq \quad \mathbb{E}\left[\beta(t,T) g(X_T) + \int_t^T \beta(t,r) f(r, X_r, \nu_r) dr\right], \end{aligned}$$

where the last inequality uses the condition $v(T,\cdot) \geq g$. Since the control $\nu \in \mathcal{U}_0$ is arbitrary, this completes the proof of (i).

Statement (ii) is proved by repeating the above argument and observing that the control $\hat{\nu}$ achieves equality at the crucial step (8.1). ♢

Remark 8.2. When U is reduced to a singleton, the optimization problem V is degenerate. In this case, the DPE is linear, and the verification theorem reduces to the so-called *Feynman-Kac formula.*

Notice that the verification theorem assumes the existence of such a solution, and is by no means an existence result. However, it provides uniqueness in the class of function with quadratic growth.

We now state without proof an existence result for the DPE together with the terminal condition $V(T,\cdot) = g$ (see [**?**] and the references therein). The main assumption is the so-called *uniform parabolicity* condition :

$$\begin{array}{c}\text{there is a constant } c > 0 \text{ such that} \\ \xi' \; \sigma\sigma'(t,x,u) \; \xi \;\; \geq \;\; c|\xi|^2 \text{ for all } (t,x,u) \in [0,T] \times \mathbb{R}^d \times U \; .\end{array} \tag{8.2}$$

In the following statement, we denote by $C_b^k(\mathbb{R}^d)$ the space of bounded functions whose partial derivatives of orders $\leq k$ exist and are bounded continuous. We similarly denote by $C_b^{p,k}([0,T],\mathbb{R}^d)$ the space of bounded functions whose partial derivatives with respect to t, of orders $\leq p$, and with respect to x, of order $\leq k$, exist and are bounded continuous.

Theorem 8.3. *Let Condition 8.2 hold, and assume further that :*
- *U is compact;*
- *b, σ and f are in $C_b^{1,2}([0,T],\mathbb{R}^d)$;*
- *$g \in C_b^3(\mathbb{R}^d)$.*

Then the DPE (6.19) *with the terminal data $V(T,\cdot) = g$ has a unique solution $V \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$.*

8.2 Examples of control problems with explicit solution

8.2.1 Optimal portfolio allocation

We now apply the verification theorem to a classical example in finance, which was introduced by Merton [**?**, **?**], and generated a huge literature since then.

Consider a financial market consisting of a non-risky asset S^0 and a risky one S. The dynamics of the price processes are given by

$$dS_t^0 \;=\; S_t^0 r dt \quad \text{and} \quad dS_t \;=\; S_t[\mu dt + \sigma dW_t] \; .$$

Here, r, μ and σ are some given positive constants, and W is a one-dimensional Brownian motion.

The investment policy is defined by an $\mathbb{F}-$adapted process $\pi = \{\pi_t,\; t \in [0,T]\}$, where π_t represents the amount invested in the risky asset at time t; The remaining wealth $(X_t - \pi_t)$ is invested in the risky asset. Therefore, the wealth process satisfies

$$\begin{aligned} dX_t^\pi &= \pi_t \frac{dS_t}{S_t} + (X_t^\pi - \pi_t)\frac{dS_t^0}{S_t^0} \\ &= \left(rX_t^\pi + (\mu - r)\pi_t\right) dt + \sigma\pi_t dW_t. \end{aligned} \tag{8.3}$$

Such a process π is said to be admissible if

$$\mathbb{E}\left[\int_0^T |\pi_t|^2 dt\right] \quad < \quad \infty \, .$$

We denote by $\mathcal{U}_0$ the set of all admissible portfolios. Observe that, in view of the particular form of our controlled process X^π, this definition agrees with (6.4).

Let γ be an arbitrary parameter in $(0,1)$ and define the *power utility function* :

$$U(x) \; := \; x^\gamma \quad \text{for} \quad x \geq 0 \, .$$

The parameter γ is called the relative risk aversion coefficient.

The objective of the investor is to choose an allocation of his wealth so as to maximize the expected utility of his terminal wealth, i.e.

$$V(t,x) \quad := \quad \sup_{\pi \in \mathcal{U}_0} \; \mathbb{E}\left[U(X_T^{t,x})\right] ,$$

where $X^{t,x}$ is the solution of (8.3) with initial condition $X_t^{t,x} = x$.

The dynamic programming equation corresponding to this problem is :

$$\frac{\partial w}{\partial t}(t,x) + \sup_{u \in \mathbb{R}} \; \mathcal{A}^u w(t,x) \quad = \quad 0, \tag{8.4}$$

where $\mathcal{A}^u$ is the second order linear operator :

$$\mathcal{A}^u w(t,x) \quad := \quad (rx + (\mu - r)u) \; \frac{\partial w}{\partial x}(t,x) + \frac{1}{2}\sigma^2 u^2 \; \frac{\partial^2 w}{\partial x^2}(t,x).$$

We next search for a solution of the dynamic programming equation of the form $w(t,x) = x^\gamma h(t)$. Plugging this form of solution into the PDE (8.4), we get the following ordinary differential equation on h :

$$\begin{aligned} 0 &= h' + \gamma h \sup_{u \in \mathbb{R}} \left\{ r + (\mu - r)\frac{u}{x} + \frac{1}{2}(\gamma - 1)\sigma^2 \frac{u^2}{x^2} \right\} && (8.5) \\ &= h' + \gamma h \sup_{\delta \in \mathbb{R}} \left\{ r + (\mu - r)\delta + \frac{1}{2}(\gamma - 1)\sigma^2 \delta^2 \right\} && (8.6) \\ &= h' + \gamma h \left[r + \frac{1}{2}\frac{(\mu - r)^2}{(1-\gamma)\sigma^2} \right], && (8.7) \end{aligned}$$

where the maximizer is :

$$\hat{u} \quad := \quad \frac{\mu - r}{(1-\gamma)\sigma^2} \, x.$$

Since $V(T,\cdot) = U(x)$, we seek for a function h satisfying the above ordinary differential equation together with the boundary condition $h(T) = 1$. This induces the unique candidate:

$$h(t) \;:=\; e^{a(T-t)} \quad \text{with} \quad a \;:=\; \gamma\left[r + \frac{1}{2}\frac{(\mu-r)^2}{(1-\gamma)\sigma^2}\right].$$

Hence, the function $(t,x) \longmapsto x^\gamma h(t)$ is a classical solution of the HJB equation (8.4). It is easily checked that the conditions of Theorem 8.1 are all satisfied in this context. Then $V(t,x) = x^\gamma h(t)$, and the optimal portfolio allocation policy is given by the linear control process:

$$\hat{\pi}_t \;=\; \frac{\mu - r}{(1-\gamma)\sigma^2}\, X_t^{\hat{\pi}}.$$

8.2.2 Law of iterated logarithm for double stochastic integrals

The main object of this paragraph is Theorem 8.5 below, reported from [**?**], which describes the local behavior of double stochastic integrals near the starting point zero. This result will be needed in the problem of hedging under gamma constraints which will be discussed later in these notes. An interesting feature of the proof of Theorem 8.5 is that it relies on a verification argument. However, the problem does not fit exactly in the setting of Theorem 8.1. Therefore, this is an interesting exercise on the verification concept.

Given a bounded predictable process b, we define the processes

$$Y_t^b \;:=\; Y_0 + \int_0^t b_r dW_r \quad \text{and} \quad Z_t^b \;:=\; Z_0 + \int_0^t Y_r^b dW_r\,, \quad t \geq 0\,,$$

where Y_0 and Z_0 are some given initial data in $\mathbb{R}$.

Lemma 8.4. *Let λ and T be two positive parameters with $2\lambda T < 1$. Then :*

$$E\left[e^{2\lambda Z_T^b}\right] \;\leq\; E\left[e^{2\lambda Z_T^1}\right] \quad \textit{for each predictable process } b \textit{ with } \quad \|b\|_\infty \;\leq\; 1\,.$$

Proof. We split the argument into three steps.
1. We first directly compute that

$$E\left[e^{2\lambda Z_T^1}\middle|\mathcal{F}_t\right] \;=\; v(t, Y_t^1, Z_t^1)\,,$$

where, for $t \in [0,T]$, and $y, z \in \mathbb{R}$, the function v is given by :

$$\begin{aligned} v(t,y,z) \;&:=\; E\left[\exp\left(2\lambda\left\{z + \int_t^T (y + W_u - W_t)\, dW_u\right\}\right)\right] \\ &=\; e^{2\lambda z} E\left[\exp\left(\lambda\{2yW_{T-t} + W_{T-t}^2 - (T-t)\}\right)\right] \\ &=\; \mu \exp\left[2\lambda z - \lambda(T-t) + 2\mu^2\lambda^2(T-t)y^2\right]\,, \end{aligned}$$

where $\mu := [1 - 2\lambda(T-t)]^{-1/2}$. Observe that

$$\text{the function } v \text{ is strictly convex in } y, \tag{8.8}$$

and

$$yD^2_{yz}v(t,y,z) \;=\; 8\mu^2\lambda^3(T-t)\,v(t,y,z)\,y^2 \;\geq\; 0\,. \tag{8.9}$$

2. For an arbitrary real parameter β, we denote by $\mathcal{L}^\beta$ the Dynkin operator associated to the process $\left(Y^b, Z^b\right)$:

$$\mathcal{L}^\beta \;:=\; D_t + \frac{1}{2}\beta^2 D^2_{yy} + \frac{1}{2}y^2 D^2_{zz} + \beta y D^2_{yz}\,.$$

In this step, we intend to prove that for all $t \in [0,T]$ and $y, z \in \mathbb{R}$:

$$\max_{|\beta|\leq 1} \mathcal{L}^\beta v(t,y,z) \;=\; \mathcal{L}^1 v(t,y,z) \;=\; 0\,. \tag{8.10}$$

The second equality follows from the fact that $\{v(t, Y^1_t, Z^1_t),\, t \leq T\}$ is a martingale . As for the first equality, we see from (8.8) and (8.9) that 1 is a maximizer of both functions $\beta \longmapsto \beta^2 D^2_{yy}v(t,y,z)$ and $\beta \longmapsto \beta y D^2_{yz}v(t,y,z)$ on $[-1,1]$.

3. Let b be some given predictable process valued in $[-1,1]$, and define the sequence of stopping times

$$\tau_k \;:=\; T \wedge \inf\left\{t \geq 0 \;:\; (|Y^b_t| + |Z^b_t| \geq k\right\}\;,\quad k \in \mathbb{N}\,.$$

By Itô's lemma and (8.10), it follows that :

$$\begin{aligned} v(0, Y_0, Z_0) \;&=\; v\left(\tau_k, Y^b_{\tau_k}, Z^b_{\tau_k}\right) - \int_0^{\tau_k} \left[bD_y v + yD_z v\right]\left(t, Y^b_t, Z^b_t\right) dW_t \\ &\quad - \int_0^{\tau_k} \mathcal{L}^{b_t} v\left(t, Y^b_t, Z^b_t\right) dt \\ &\geq\; v\left(\tau_k, Y^b_{\tau_k}, Z^b_{\tau_k}\right) - \int_0^{\tau_k} \left[bD_y v + yD_z v\right]\left(t, Y^b_t, Z^b_t\right) dW_t\,. \end{aligned}$$

Taking expected values and sending k to infinity, we get by Fatou's lemma :

$$\begin{aligned} v(0, Y_0, Z_0) \;&\geq\; \liminf_{k\to\infty} E\left[v\left(\tau_k, Y^b_{\tau_k}, Z^b_{\tau_k}\right)\right] \\ &\geq\; E\left[v\left(T, Y^b_T, Z^b_T\right)\right] \;=\; E\left[e^{2\lambda Z^b_T}\right]\,, \end{aligned}$$

which proves the lemma. ♢

We are now able to prove the law of the iterated logarithm for double stochastic integrals by a direct adaptation of the case of the Brownian motion. Set

$$h(t) \;:=\; 2t \log\log\frac{1}{t} \quad \text{for} \quad t > 0\,.$$

Theorem 8.5. *Let b be a predictable process valued in a bounded interval $[\beta_0, \beta_1]$ for some real parameters $0 \leq \beta_0 < \beta_1$, and $X^b_t := \int_0^t \int_0^u b_v dW_v dW_u$. Then :*

$$\beta_0 \;\leq\; \limsup_{t \searrow 0} \frac{2X^b_t}{h(t)} \;\leq\; \beta_1 \qquad a.s.$$

Proof. We first show that the first inequality is an easy consequence of the second one. Set $\bar\beta := (\beta_0 + \beta_1)/2 \geq 0$, and set $\delta := (\beta_1 - \beta_0)/2$. By the law of the iterated logarithm for the Brownian motion, we have

$$\bar\beta \;=\; \limsup_{t \searrow 0} \frac{2X^{\bar\beta}_t}{h(t)} \;\;\leq\;\; \delta \limsup_{t \searrow 0} \frac{2X^{\tilde b}_t}{h(t)} + \limsup_{t \searrow 0} \frac{2X^b_t}{h(t)}\ ,$$

where $\tilde b := \delta^{-1}(\bar\beta - b)$ is valued in $[-1, 1]$. It then follows from the second inequality that :

$$\limsup_{t \searrow 0} \frac{2X^b_t}{h(t)} \;\;\geq\;\; \bar\beta - \delta \;=\; \beta_0\ .$$

We now prove the second inequality. Clearly, we can assume with no loss of generality that $\|b\|_\infty \leq 1$. Let $T > 0$ and $\lambda > 0$ be such that $2\lambda T < 1$. It follows from Doob's maximal inequality for submartingales that for all $\alpha \geq 0$,

$$\begin{aligned} P\left[\max_{0 \leq t \leq T} 2X^b_t \;\geq\; \alpha\right] \;&=\; P\left[\max_{0 \leq t \leq T} \exp(2\lambda X^b_t) \;\geq\; \exp(\lambda\alpha)\right] \\ &\leq\; e^{-\lambda\alpha} E\left[e^{2\lambda X^b_T}\right]\ . \end{aligned}$$

In view of Lemma 8.4, this provides :

$$\begin{aligned} P\left[\max_{0 \leq t \leq T} 2X^b_t \;\geq\; \alpha\right] \;&\leq\; e^{-\lambda\alpha} E\left[e^{2\lambda X^1_T}\right] \\ &=\; e^{-\lambda(\alpha + T)}(1 - 2\lambda T)^{-\frac{1}{2}}\ . \end{aligned} \tag{8.11}$$

We have then reduced the problem to the case of the Brownian motion, and the rest of this proof is identical to the first half of the proof of the law of the iterated logarithm for the Brownian motion. Take θ, $\eta \in (0, 1)$, and set for all $k \in \mathbb{N}$,

$$\alpha_k \;:=\; (1 + \eta)^2 h(\theta^k) \quad \text{and} \quad \lambda_k \;:=\; [2\theta^k(1 + \eta)]^{-1}\ .$$

Applying (8.11), we see that for all $k \in \mathbb{N}$,

$$P\left[\max_{0 \leq t \leq \theta^k} 2X^b_t \;\geq\; (1 + \eta)^2 h(\theta^k)\right] \;\leq\; e^{-1/2(1+\eta)}\left(1 + \eta^{-1}\right)^{\frac{1}{2}}(-k \log\theta)^{-(1+\eta)}\ .$$

Since $\sum_{k \geq 0} k^{-(1+\eta)} < \infty$, it follows from the Borel-Cantelli lemma that, for almost all $\omega \in \Omega$, there exists a natural number $K^{\theta,\eta}(\omega)$ such that for all $k \geq K^{\theta,\eta}(\omega)$,

$$\max_{0 \leq t \leq \theta^k} 2X^b_t(\omega) \;\;<\;\; (1 + \eta)^2 h(\theta^k)\,.$$

In particular, for all $t \in (\theta^{k+1}, \theta^k]$,

$$2X_t^b(\omega) \quad < \quad (1+\eta)^2 h(\theta^k) \; \leq \; (1+\eta)^2 \frac{h(t)}{\theta} \, .$$

Hence,

$$\limsup_{t \searrow 0} \frac{2X_t^b}{h(t)} \quad < \quad \frac{(1+\eta)^2}{\theta} \quad \text{a.s.}$$

and the required result follows by letting θ tend to 1 and η to 0 along the rationals. ♢

8.3 The verification argument for optimal stopping problems

8.4 Examples of optimal stopping problems with explicit solution

Chapter 9

INTRODUCTION TO VISCOSITY SOLUTIONS

Throughout this chapter, we provide the main tools from the theory of viscosity solutions for the purpose of our applications to stochastic control problems. For a deeper presentation, we refer to the excellent overview paper by Crandal, Ischii and Lions [**?**].

9.1 Intuition behind viscosity solutions

We consider a non-linear second order partial differential equation

$$\text{(E)} \quad F\left(x, u(x), Du(x), D^2u(x)\right) \;=\; 0 \;\text{ for }\; x \in \mathcal{O}$$

where $\mathcal{O}$ is an open subset of $\mathbb{R}^d$ and F is a continuous map from $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \longrightarrow \mathbb{R}$. A crucial condition on F is the so-called *ellipticity* condition :

$$F(x,r,p,A) \;\leq\; F(x,r,p,B) \quad \text{whenever} \quad A \;\geq\; B\;,$$

for all $(x,r,p) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d$. The full importance of this condition will be made clear by Proposition 9.2 below.

The first step towards the definition of a notion of weak solution to (E) is the introduction of sub and supersolutions.

Definition 9.1. *A function $u : \mathcal{O} \longrightarrow \mathbb{R}$ is a classical supersolution (resp. subsolution) of (E) if $u \in C^2(\mathcal{O})$ and*

$$F\left(x, u(x), Du(x), D^2u(x)\right) \;\geq\; \textit{(resp. } \leq\textit{)} \;\; 0 \;\; \textit{for} \;\; x \in \mathcal{O}\;.$$

The theory of viscosity solutions is motivated by the following result, whose simple proof is left to the reader.

Proposition 9.2. *Let u be a $C^2(\mathcal{O})$ function. Then the following claims are equivalent.*
(i) *u is a classical supersolution (resp. subsolution) of (E)*
(ii) *for all pairs $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that x_0 is a minimizer (resp. maximizer) of the difference $u - \varphi$) on $\mathcal{O}$, we have*

$$F\left(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \quad \geq \textit{ (resp. } \leq\textit{)} \quad 0 \, .$$

9.2 Definition of viscosity solutions

Before going any further, we need to introduce a new notation. For a locally bounded function $u : \mathcal{O} \longrightarrow \mathbb{R}$, we denote by u_* and u^* the lower and upper semicontinuous envelopes of u. We recall that u_* is the largest lower semicontinuous minorant of u, u^* is the smallest upper semicontinuous majorant of u, and

$$u_*(x) \ = \ \liminf_{x' \to x} u(x') \ , \qquad u^*(x) \ = \ \limsup_{x' \to x} u(x') \ .$$

We are now ready for the definition of viscosity solutions. Observe that Claim (ii) in the above proposition does not involve the regularity of u. It therefore suggests the following weak notion of solution to (E).

Definition 9.3. *Let F be elliptic, and $u : \mathcal{O} \longrightarrow \mathbb{R}$ be a locally bounded function.*
(i) *We say that u is a (discontinuous) viscosity supersolution of (E) if*

$$F\left(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \quad \geq \quad 0$$

for all pair $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that x_0 is a minimizer of the difference $(u_* - \varphi)$ on $\mathcal{O}$.
(ii) *We say that u is a (discontinuous) viscosity subsolution of (E) if*

$$F\left(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \quad \leq \quad 0$$

for all pair $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that x_0 is a maximizer of the difference $(u^* - \varphi)$ on $\mathcal{O}$.
(iii) *We say that u is a (discontinuous) viscosity solution of (E) if it is both a viscosity supersolution and subsolution of (E).*

Remark 9.4. An immediate consequence of Proposition 9.2 is that any classical solution of (E) is also a viscosity solution of (E).

Remark 9.5. Clearly, the above definition is not changed if the minimum or maximum are local and/or strict. Also, by a density argument, the test function can be chosen in $C^\infty(\mathcal{O})$.

Remark 9.6. Consider the equation (E$^+$): $|u'(x)| - 1 = 0$ on $\mathbb{R}$. Then

- The function $f(x) := |x|$ is not a viscosity supersolution of (E$^+$). Indeed the test function $\varphi \equiv 0$ satisfies $(f - \varphi)(0) = 0 \leq (f - \varphi)(x)$ for all $x \in \mathbb{R}$. But $|\varphi'(0)| = 0 \not\geq 1$.

- The function $g(x) := -|x|$ is a viscosity solution of (E$^+$). To see this, we concentrate on the origin which is the only critical point. The supersolution property is obviously satisfied as there is no smooth function which satisfies the minimum condition. As for the subsolution property, we observe that whenever $\varphi \in C^1(\mathbb{R})$ satisfies $(g - \varphi)(0) = \max(g - \varphi)$, then $|\varphi'(0)| \leq 1$, which is exactly the viscosity subsolution property of g.
- Similarly, the function f is a viscosity solution of the equation (E$^-$): $-|u'(x)| + 1 = 0$ on $\mathbb{R}$.

In Section 10.1, we will show that the value function V is a viscosity solution of the DPE (6.19) under the conditions of Theorem 6.6 (except the smoothness assumption on V). We also want to emphasize that proving that the value function is a viscosity solution is almost as easy as proving that it is a classical solution when V is known to be smooth.

9.3 First properties

We now turn to two important properties of viscosity solutions : the change of variable formula and the stability result.

Proposition 9.7. *Let u be a locally bounded (discontinuous) viscosity supersolution of (E). If f is a $C^1(\mathbb{R})$ function with $Df \neq 0$ on $\mathbb{R}$, then the function $v := f^{-1} \circ u$ is a (discontinuous)*

- *viscosity super-solution of $K(x, v(x), Dv(x), D^2v(x)) = 0$, $x \in \mathcal{O}$, when $Df > 0$,*
- *viscosity subsolution of $-K(x, v(x), Dv(x), D^2v(x)) = 0$, $x \in \mathcal{O}$, when $Df < 0$,*

where

$$K(x, r, p, A) \quad := \quad F\left(x, f(r), Df(r)p, D^2f(r)pp' + Df(r)A\right)$$

We leave the easy proof of this proposition to the reader. The next result shows how limit operations with viscosity solutions can be performed very easily.

Theorem 9.8. *Let u_ε be a lower semicontinuous viscosity super-solution of the equation*

$$F_\varepsilon\left(x, u_\varepsilon(x), Du_\varepsilon(x), D^2u_\varepsilon(x)\right) \;=\; 0 \quad \textit{for} \quad x \in \mathcal{O}\;,$$

where $(F_\varepsilon)_\varepsilon$ is a sequence of continuous functions satisfying the ellipticity condition. Suppose that $(\varepsilon, x) \longmapsto u_\varepsilon(x)$ and $(\varepsilon, z) \longmapsto F_\varepsilon(z)$ are locally bounded, and define

$$u_*(x) \;:=\; \liminf_{(\varepsilon, x') \to (0, x)} u_\varepsilon(x') \quad \textit{and} \quad F^*(z) \;:=\; \limsup_{(\varepsilon, z') \to (0, z)} F_\varepsilon(z')\;.$$

Then, u_* is a lower semicontinuous viscosity supersolution of the equation

$$F^*\left(x, u_*, Du_*(x), D^2u_*(x)\right) \;=\; 0 \quad \textit{for} \quad x \in \mathcal{O} \; .$$

A similar statement holds for subsolutions.

Proof. The fact that u_* is a lower semicontinuous function is left as an exercise for the reader. Let $\varphi \in C^2(\mathcal{O})$ and $\bar{x}$, be a strict minimizer of the difference $u_* - \varphi$. By definition of u_*, there is a sequence $(\varepsilon_n, x_n) \in (0,1] \times \mathcal{O}$ such that

$$(\varepsilon_n, x_n) \;\longrightarrow\; (0, \bar{x}) \quad \text{and} \quad u_{\varepsilon_n}(x_n) \;\longrightarrow\; u_*(\bar{x}) \; .$$

Consider some $r > 0$ together with the closed ball $\bar{B}$ with radius r, centered at $\bar{x}$. Of course, we may choose $|x_n - \bar{x}| < r$ for all $n \geq 0$. Let $\bar{x}_n$ be a minimizer of $u_{\varepsilon_n} - \varphi$ on $\bar{B}$. We claim that

$$\bar{x}_n \longrightarrow \bar{x} \text{ and } u_{\varepsilon_n}(\bar{x}_n) \longrightarrow u_*(\bar{x}) \quad \text{as} \quad n \to \infty. \tag{9.1}$$

Before verifying this, let us complete the proof. We first deduce that $\bar{x}_n$ is an interior point of $\bar{B}$ for large n, so that $\bar{x}_n$ is a local minimizer of the difference $u_{\varepsilon_n} - \varphi$. Then :

$$F_{\varepsilon_n}\left(\bar{x}_n, u_{\varepsilon_n}(\bar{x}_n), D\varphi(\bar{x}_n), D^2\varphi(\bar{x}_n)\right) \quad \geq \quad 0 \; ,$$

and the required result follows by taking limits and using the definition of F^*.

It remains to prove Claim (9.1). Recall that $(\bar{x}_n)_n$ is valued in the compact set $\bar{B}$. Then, there is a subsequence, still named $(\bar{x}_n)_n$, which converges to some $\tilde{x} \in \bar{B}$. We now prove that $\tilde{x} = \bar{x}$ and obtain the second claim in (9.1) as a by-product. Using the fact that $\bar{x}_n$ is a minimizer of $u_{\varepsilon_n} - \varphi$ on $\bar{B}$, together with the definition of u_*, we see that

$$\begin{aligned} 0 \;=\; (u_* - \varphi)(\bar{x}) \;&=\; \lim_{n\to\infty} \left(u_{\varepsilon_n} - \varphi\right)(x_n) \\ &\geq\; \limsup_{n\to\infty} \left(u_{\varepsilon_n} - \varphi\right)(\bar{x}_n) \\ &\geq\; \liminf_{n\to\infty} \left(u_{\varepsilon_n} - \varphi\right)(\bar{x}_n) \\ &\geq\; (u_* - \varphi)(\tilde{x}) \; . \end{aligned}$$

We now obtain (9.1) from the fact that $\bar{x}$ is a strict minimizer of the difference $(u_* - \varphi)$. ♢

Remark 9.9. The nonlinear operator F^* in the statement of Theorem 9.8 is not known to be continuous. Therefore the notion of viscosity solution is beyond the scope of our definition 9.3, and needs a further relaxation which we do not want to discuss in these notes.

Observe that the passage to the limit in partial differential equations written in the classical or the generalized sense usually requires much more technicalities, as one has to ensure convergence of all the partial derivatives involved in the equation. The above stability result provides a general method to pass to the limit when the equation is written in the viscosity sense, and its proof turns out to be remarkably simple.

A possible application of the stability result is to establish the convergence of numerical schemes. In view of the simplicity of the above statement, the notion of viscosity solutions provides a nice framework for such a numerical issue. We refer to Barles and Souganidis [**?**] who introduced the notion of monotonic schemes.

The main difficulty in the theory of viscosity solutions is the interpretation of the equation in the viscosity sense. First, by weakening the notion of solution to the second order nonlinear PDE (E), we are enlarging the set of solutions, and one has to guarantee that uniqueness still holds (in some convenient class of functions). This issue will be discussed in the subsequent Section 9.4. We conclude this section by the following result whose proof is trivial in the classical case, but needs some technicalities when stated in the viscosity sense.

Proposition 9.10. *Let $A \subset \mathbb{R}^{d_1}$ and $B \subset \mathbb{R}^{d_2}$ be two open subsets, and let $u : A \times B \longrightarrow \mathbb{R}$ be a lower semicontinuous viscosity supersolution of the equation :*

$$F\left(x, y, u(x,y), D_y u(x,y), D_y^2 u(x,y)\right) \;\geq\; 0 \quad \text{on} \quad A \times B\;,$$

where F is a continuous elliptic operator. Assume further that

$$r \;\longmapsto\; F(x,y,r,p,A) \qquad \text{is non-increasing.} \tag{9.2}$$

Then, for all fixed $x_0 \in A$, the function $v(y) := u(x_0, y)$ is a viscosity supersolution of the equation :

$$F\left(x_0, y, v(y), Dv(y), D^2 v(y)\right) \;\geq\; 0 \quad \text{on} \quad B\;.$$

If u is continuous, the above statement holds without Condition (9.2).

A similar statement holds for the subsolution property.

Proof. Fix $x_0 \in A$, set $v(y) := u(x_0, y)$, and let $y_0 \in B$ and $f \in C^2(B)$ be such that

$$(v - f)(y_0) \;<\; (v-f)(y) \quad \text{for all} \quad y \in J \setminus \{y_0\}\;, \tag{9.3}$$

where J is an arbitrary compact subset of B containing y_0 in its interior. For each integer n, define

$$\varphi_n(x,y) \;:=\; f(y) - n|x - x_0|^2 \quad \text{for} \quad (x,y) \in A \times B\;,$$

and let (x_n, y_n) be defined by

$$(u - \varphi_n)(x_n, y_n) \;=\; \min_{I \times J}(u - \varphi_n)\;,$$

where I is a compact subset of A containing x_0 in its interior. We claim that

$$(x_n, y_n) \longrightarrow (x_0, y_0) \quad \text{as} \quad n \longrightarrow \infty . \tag{9.4}$$

Before proving this, let us complete the proof. Since (x_0, y_0) is an interior point of $A \times B$, it follows from the viscosity property of u that

$$\begin{aligned} 0 &\le F\big(x_n, y_n, u(x_n, y_n), D_y\varphi_n(x_n, y_n), D_y^2\varphi_n(x_n, y_n)\big) \\ &= F\big(x_n, y_n, u(x_n, y_n), Df(y_n), D^2 f(y_n)\big) , \end{aligned}$$

and the required result follows by sending n to infinity.

We now turn to the proof of (9.4). Since the sequence $(x_n, y_n)_n$ is valued in the compact subset $A \times B$, we have $(x_n, y_n) \longrightarrow (\bar{x}, \bar{y}) \in A \times B$, after passing to a subsequence. Observe that

$$\begin{aligned} u(x_n, y_n) - f(y_n) &\le u(x_n, y_n) - f(y_n) + n|x_n - x_0|^2 \\ &= (u - \varphi_n)(x_n, y_n) \\ &\le (u - \varphi_n)(x_0, y_0) = u(x_0, y_0) - f(y_0) . \end{aligned}$$

Taking the limits, it follows from the lower semicontinuity of u that

$$u(\bar{x}, \bar{y}) - f(\bar{y}) \le u(\bar{x}, \bar{y}) - f(\bar{y}) + \liminf_{n\to\infty} n|x_n - x_0|^2 \le u(x_0, y_0) - f(y_0) .$$

Then, we must have $\bar{x} = x_0$, and

$$(v - f)(\bar{y}) = u(x_0, \bar{y}) - f(\bar{y}) \le (v - f)(y_0) ,$$

which concludes the proof of (9.4) in view of (9.3). ♢

9.4 Comparison result and uniqueness

In this section, we show that the notion of viscosity solutions is consistent with the maximum principle for a wide class of equations. We recall that the maximum principle is a stronger statement than uniqueness. Once we will have such a result, the reader must be convinced that the notion of viscosity solutions is a good weakening of the notion of classical solution.

In the viscosity solutions literature, the maximum principle is rather called *comparison principle.*

9.4.1 Comparison of classical solutions in a bounded domain

Let us first review the maxium principle in the simplest classical sense.

Proposition 9.11. *Assume that $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^d$, and the nonlinearity $F(x, r, p, A)$ is elliptic and strictly increasing in r. Let $u, v \in C^2\big(\mathrm{cl}(\mathcal{O})\big)$ be classical subsolution and supersolution of* (E)*, respectively, with $u \le v$ on $\partial\mathcal{O}$. Then $u \le v$ on* $\mathrm{cl}(\mathcal{O})$.

Proof. Our objective is to prove that

$$M \quad := \quad \sup_{\mathrm{cl}(\mathcal{O})}(u-v) \ \leq \ 0.$$

Assume to the contrary that $M > 0$. Then since $\mathrm{cl}(\mathcal{O})$ is a compact subset of $\mathbb{R}^d$, and $u - v \leq 0$ on $\partial\mathcal{O}$, it follows that

$$M = (u-v)(x_0) \text{ for some } x_0 \in \mathcal{O} \text{ with } D(u-v)(x_0) = 0,\ D^2(u-v)(x_0) \leq 0. \tag{9.5}$$

Then, it follows from the viscosity properties of u and v that:

$$\begin{aligned} F\big(x_0, u(x_0), Du(x_0), D^2u(x_0)\big) \ \leq \ 0 \quad &\leq \quad F\big(x_0, v(x_0), Dv(x_0), D^2v(x_0)\big) \\ &\leq \quad F\big(x_0, u(x_0) - M, Du(x_0), D^2u(x_0)\big), \end{aligned}$$

where the last inequality follows crucially from the ellipticity of F. This provides the desired contradiction, under our condition that F is strictly increasing in r. ♢

The objective of this section is to mimic the latter proof in the sense of viscosity solutions.

9.4.2 Semijets definition of viscosity solutions

We first need to develop a convenient alternative definition of viscosity solutions.

For $v \in \mathrm{LSC}(\mathcal{O})$, let $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ be such that x_0 is a local minimizer of the difference $(v - \varphi)$ in $\mathcal{O}$. Then, defining $p := D\varphi(x_0)$ and $A := D^2\varphi(x_0)$, it follows from a second order Taylor expansion that:

$$v(x) \geq v(x_0) + p \cdot (x - x_0) + \frac{1}{2}A(x-x_0)\cdot(x-x_0) + \circ\big(|x-x_0|^2\big). \tag{9.6}$$

Motivated by this observation, we introduce the *subjet* $J^-_{\mathcal{O}}v(x_0)$ by

$$J^-_{\mathcal{O}}v(x_0) \quad := \quad \Big\{(p,A) \in \mathbb{R}^d \times \mathcal{S}_d : (x_0, p, A) \text{ satisfies } (9.6)\Big\}. \tag{9.7}$$

Similarly, we define the *superjet* $J^+_{\mathcal{O}}u(x_0)$ of a function $u \in \mathrm{USC}(\mathcal{O})$ at the point $x_0 \in \mathcal{O}$ by

$$\begin{aligned} J^+_{\mathcal{O}}u(x_0) \quad := \quad \Big\{(p,A) \in \mathbb{R}^d \times \mathcal{S}_d : u(x) \leq u(x_0) + p\cdot(x-x_0) \\ + \frac{1}{2}A(x-x_0)\cdot(x-x_0) + \circ\big(|x-x_0|^2\big)\Big\} \end{aligned} \tag{9.8}$$

Then, one can prove that a function $v \in \mathrm{LSC}(\mathcal{O})$ is a viscosity supersolution of the equation (E) if and only if

$$F(x, v(x), p, A) \geq 0 \quad \text{for all} \quad (p,A) \in J^-_{\mathcal{O}}v(x).$$

The nontrivial implication of the latter statement requires to construct, for every $(p,A) \in J^-_{\mathcal{O}}v(x_0)$, a smooth test function φ such that the difference $(v - \varphi)$ has a local minimum at x_0. We refer to Fleming and Soner [?], Lemma V.4.1 p211.

A symmetric statement holds for viscosity subsolutions. By continuity considerations, we can even enlarge the semijets $J^{\pm}_{\mathcal{O}}w(x_0)$ to the folowing closure

$$\begin{aligned}\bar{J}^{\pm}_{\mathcal{O}}w(x) \quad := \quad & \Big\{(p,A)\in\mathbb{R}^d\times\mathcal{S}_d : (x_n,w(x_n),p_n,A_n)\longrightarrow(x,w(x),p,A) \\ & \qquad \text{for some sequence } (x_n,p_n,A_n)_n\subset \mathrm{Graph}(J^{\pm}_{\mathcal{O}}w)\Big\},\end{aligned}$$

where $(x_n,p_n,A_n)\in\mathrm{Graph}(J^{\pm}_{\mathcal{O}}w)$ means that $(p_n,A_n)\in J^{\pm}_{\mathcal{O}}w(x_n)$. The following result is obvious, and provides an equivalent definition of viscosity solutions.

Proposition 9.12. *Consider an elliptic nonlinearity F, and let $u\in\mathrm{USC}(\mathcal{O})$, $v\in\mathrm{LSC}(\mathcal{O})$.*
(i) *Assume that F is lower-semicontinuous. Then, u is a viscosity subsolution of* (E) *if and only if:*

$$F(x,u(x),p,A)\le 0 \quad \textit{for all} \quad (p,A)\in\bar{J}^{+}_{\mathcal{O}}u(x),$$

(ii) *Assume that F is upper-semicontinuous. Then, v is a viscosity supersolution of* (E) *if and only if:*

$$F(x,v(x),p,A)\ge 0 \quad \textit{for all} \quad (p,A)\in\bar{J}^{-}_{\mathcal{O}}v(x).$$

9.4.3 The Crandal-Ishii's lemma

The major difficulty in mimicking the proof of Proposition 9.11 is to derive an analogous statement to (9.5) without involving the smoothness of u and v, as these functions are only known to be upper- and lower-semicontinuous in the context of viscosity solutions.

This is provided by the following result due to M. Crandal and I. Ishii. For a symmetric matrix, we denote by $|A|:=\sup\{(A\xi)\cdot\xi : |\xi|\le 1\}$.

Lemma 9.13. *Let $\mathcal{O}$ be an open locally compact subset of $\mathbb{R}^d$. Given $u\in USC(\mathcal{O})$ and $v\in LSC(\mathcal{O})$, we assume for some $(x_0,y_0)\in\mathcal{O}^2$, $\varphi\in C^2\left(cl(\mathcal{O})^2\right)$ that:*

$$u(x_0)-v(y_0)-\varphi(x_0,y_0) \quad = \quad \max_{\mathcal{O}^2}(u-v-\varphi). \tag{9.9}$$

Then, for each $\varepsilon>0$, there exist $A,B\in\mathcal{S}_d$ such that

$$(D_x\varphi(x_0,y_0),A)\in\bar{J}^{2,+}_{\mathcal{O}}u(x_0), \qquad (-D_y\varphi(x_0,y_0),B)\in\bar{J}^{2,-}_{\mathcal{O}}v(y_0),$$

and the following inequality holds in the sense of symmetric matrices in $\mathcal{S}_{2d}$:

$$-\left(\varepsilon^{-1}+\left|D^2\varphi(x_0,y_0)\right|\right)I_{2d}\le\begin{pmatrix}A & 0\\ 0 & -B\end{pmatrix}\le D^2\varphi(x_0,y_0)+\varepsilon D^2\varphi(x_0,y_0)^2.$$

Proof. See Appendix. ♢

We will be applying Lemma 9.13 in the particular case

$$\varphi(x,y) := \frac{\alpha}{2}|x-y|^2 \quad \text{for} \quad x,y \in \mathcal{O}. \tag{9.10}$$

Intuitively, sending α to ∞, we expect that the maximization of $(u(x) - v(y) - \varphi(x,y)$ on $\mathcal{O}^2$ reduces to the maximization of $(u-v)$ on $\mathcal{O}$ as in (9.5). Then, taking $\varepsilon^{-1} = \alpha$, we directly compute that the conclusions of Lemma 9.13 reduce to

$$(\alpha(x_0 - y_0), A) \in \bar{J}^{2,+}_{\mathcal{O}} u(x_0), \qquad (\alpha(x_0 - y_0), B) \in \bar{J}^{2,-}_{\mathcal{O}} v(y_0), \tag{9.11}$$

and

$$-3\alpha \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} \le \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \le 3\alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}. \tag{9.12}$$

Remark 9.14. If u and v were C^2 functions in Lemma 9.13, the first and second order condition for the maximization problem (9.9) with the test function (9.10) is $Du(x_0) = \alpha(x_0 - y_0)$, $-Dv(x_0) = \alpha(x_0 - y_0)$, and

$$\begin{pmatrix} D^2u(x_0) & 0 \\ 0 & -D^2v(y_0) \end{pmatrix} \le \alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}.$$

Hence, the right-hand side inequality in (9.12) is worsening the latter second order condition by replacing the coefficient α by 3α. ♢

Remark 9.15. The right-hand side inequality of (9.12) implies that

$$A \le B. \tag{9.13}$$

To see this, take an arbitrary $\xi \in \mathbb{R}^d$, and denote by ξ^{T} its transpose. From right-hand side inequality of (9.12), it follows that

$$0 \ge (\xi^{\mathrm{T}}, \xi^{\mathrm{T}}) \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = (A\xi)\cdot\xi - (B\xi)\cdot\xi.$$

♢

9.4.4 Comparison of viscosity solutions in a bounded domain

We now prove a comparison result for viscosity sub- and supersolutions by using Lemma 9.13 to mimic the proof of Proposition 9.11. The statement will be proved under the following conditions on the nonlinearity F which will be used at the final Step 3 of the subsequent proof.

Assumption 9.16. (i) *There exists* $\gamma > 0$ *such that*

$$F(x,r,p,A) - F(x,r',p,A) \ge \gamma(r-r') \text{ for all } r \ge r',\ (x,p,A) \in \mathcal{O} \times \mathbb{R}^d \times \mathcal{S}_d.$$

(ii) *There is a function* $\varpi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *with* $\varpi(0+) = 0$*, such that*

$$F(y,r,\alpha(x-y),B) - F(x,r,\alpha(x-y),A) \le \varpi\left(\alpha|x-y|^2 + |x-y|\right)$$
$$\text{for all } x,y \in \mathcal{O},\ r \in \mathbb{R},\ \alpha \in \mathbb{R}^+ \text{ and } A,B \text{ satisfying (9.12)}.$$

Remark 9.17. Assumption 9.16 (ii) implies that the nonlinearity F is elliptic. To see this, notice that for $A \le B$, $\xi, \eta \in \mathbb{R}^d$, and $\varepsilon > 0$, we have

$$\begin{aligned} A\xi\cdot\xi - (B+\varepsilon I_d)\eta\cdot\eta &\le B\xi\cdot\xi - (B+\varepsilon I_d)\eta\cdot\eta \\ &= 2\eta\cdot B(\xi-\eta) + B(\xi-\eta)\cdot(\xi-\eta) - \varepsilon|\eta|^2 \\ &\le \varepsilon^{-1}|B(\xi-\eta)|^2 + B(\xi-\eta)\cdot(\xi-\eta) \\ &\le |B|\left(1+\varepsilon^{-1}|B|\right)|\xi-\eta|^2. \end{aligned}$$

For $3\alpha \ge (1+\varepsilon^{-1}|B|)|B|$, the latter inequality implies the right-hand side of (9.12) holds true with $(A, B+\varepsilon I_d)$. For ε sufficiently small, the left-hand side of (9.12) is also true with $(A, B+\varepsilon I_d)$ if in addition $\alpha > |A| \vee |B|$. Then

$$F(x-\alpha^{-1}p, r, p, B+\varepsilon I) - F(x,r,p,A) \le \varpi\left(\alpha^{-1}(|p|^2+|p|)\right),$$

which provides the ellipticity of F by sending $\alpha \to \infty$ and $\varepsilon \to 0$. ♢

Theorem 9.18. *Let* $\mathcal{O}$ *be an open bounded subset of* $\mathbb{R}^d$ *and let* F *be an elliptic operator satisfying Assumption 9.16. Let* $u \in USC(\mathcal{O})$ *and* $v \in LSC(\mathcal{O})$ *be viscosity subsolution and supersolution of the equation (E), respectively. Then*

$$u \le v \text{ on } \partial\mathcal{O} \implies u \le v \text{ on } \bar{\mathcal{O}} := cl(\mathcal{O}).$$

Proof. As in the proof of Proposition 9.11, we assume to the contrary that

$$\delta := (u-v)(z) > 0 \quad \text{for some} \quad z \in \mathcal{O}. \tag{9.14}$$

Step 1: For every $\alpha > 0$, it follows from the upper-semicontinuity of the difference $(u-v)$ and the compactness of $\bar{\mathcal{O}}$ that

$$\begin{aligned} M_\alpha &:= \sup_{\mathcal{O}\times\mathcal{O}} \left\{u(x) - v(y) - \frac{\alpha}{2}|x-y|^2\right\} \\ &= u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \end{aligned} \tag{9.15}$$

for some $(x_\alpha, y_\alpha) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}}$. Since $\bar{\mathcal{O}}$ is compact, there is a subsequence $(x_n, y_n) := (x_{\alpha_n}, y_{\alpha_n})$, $n \ge 1$, which converges to some $(\hat{x}, \hat{y}) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}}$. We shall prove in Step 4 below that

$$\hat{x} = \hat{y},\ \alpha_n|x_n - y_n|^2 \longrightarrow 0, \text{ and } M_{\alpha_n} \longrightarrow (u-v)(\hat{x}) = \sup_{\mathcal{O}}(u-v). \tag{9.16}$$

Then, since $u \leq v$ on $\partial\mathcal{O}$ and

$$\delta \leq M_{\alpha_n} = u(x_n) - v(y_n) - \frac{\alpha_n}{2}|x_n - y_n|^2 \tag{9.17}$$

by (9.14), it follows from the first claim in (9.16) that $(x_n, y_n) \in \mathcal{O} \times \mathcal{O}$.
Step 2: Since the maximizer (x_n, y_n) of M_{α_n} defined in (9.15) is an interior point to $\mathcal{O} \times \mathcal{O}$, it follows from Lemma 9.13 that there exist two symmetric matrices $A_n, B_n \in \mathcal{S}_n$ satisfying (9.12) such that $(x_n, \alpha_n(x_n - y_n), A_n) \in \bar{J}_{\mathcal{O}}^{2,+}u(x_n)$ and $(y_n, \alpha_n(x_n - y_n), B_n) \in \bar{J}_{\mathcal{O}}^{2,-}v(y_n)$. Then, since u and v are viscosity subsolution and supersolution, respectively, it follows from the alternative definition of viscosity solutions in Proposition 9.12 that:

$$F(x_n, u(x_n), \alpha_n(x_n - y_n), A_n) \leq 0 \leq F(y_n, v(x_n), \alpha_n(x_n - y_n), B_n). \tag{9.18}$$

Step 3: We first use the strict monotonicity Assumption 9.16 (i) to obtain:

$$\begin{aligned}\gamma\delta \;\leq\; \gamma\big(u(x_n) - v(x_n)\big) \;&\leq\; F(x_n, u(x_n), \alpha_n(x_n - y_n), A_n) \\ &\quad - F(x_n, v(x_n), \alpha_n(x_n - y_n), A_n).\end{aligned}$$

By (9.18), this provides:

$$\gamma\delta \;\leq\; F(y_n, v(x_n), \alpha_n(x_n - y_n), B_n) - F(x_n, v(x_n), \alpha_n(x_n - y_n), A_n).$$

Finally, in view of Assumption 9.16 (ii) this implies that:

$$\gamma\delta \;\leq\; \varpi\left(\alpha_n|x_n - y_n|^2 + |x_n - y_n|\right).$$

Sending n to infinity, this leads to the desired contradiction of (9.14) and (9.16).
Step 4: It remains to prove the claims (9.16). By the upper-semicontinuity of the difference $(u - v)$ and the compactness of $\bar{\mathcal{O}}$, there exists a maximizer x^* of the difference $(u - v)$. Then

$$(u - v)(x^*) \leq M_{\alpha_n} = u(x_n) - v(y_n) - \frac{\alpha_n}{2}|x_n - y_n|^2.$$

Sending $n \to \infty$, this provides

$$\begin{aligned}\bar{\ell} := \frac{1}{2}\limsup_{n\to\infty} \alpha_n|x_n - y_n|^2 \;&\leq\; \limsup_{n\to\infty} u(x_{\alpha_n}) - v(y_{\alpha_n}) - (u - v)(x^*) \\ &\leq\; u(\hat{x}) - v(\hat{y}) - (u - v)(x^*);\end{aligned}$$

in particular, $\bar{\ell} < \infty$ and $\hat{x} = \hat{y}$. Moreover, denoting $2\underline{\ell} := \liminf_n \alpha_n|x_n - y_n|^2$, and using the definition of x^* as a maximizer of $(u - v)$, we see that:

$$0 \;\leq\; \underline{\ell} \;\leq\; \bar{\ell} \;\leq\; (u - v)(\hat{x}) - (u - v)(x^*) \;\leq\; 0.$$

Then $\hat{x}$ is a maximizer of the difference $(u - v)$ and $M_{\alpha_n} \longrightarrow \sup_{\mathcal{O}}(u - v)$. ♢

We list below two interesting examples of operators F which satisfy the conditions of the above theorem:

Example 9.19. *Assumption 9.16 is satisfied by the nonlinearity*

$$F(x,r,p,A) \quad = \quad \gamma r + H(p)$$

for any continuous function $H\ :\mathbb{R}^d \longrightarrow \mathbb{R}$, *and* $\gamma > 0$.

In this example, the condition $\gamma > 0$ *is not needed when* H *is a convex and* $H(D\varphi(x)) \le \alpha < 0$ *for some* $\varphi \in C^1(\mathcal{O})$. *This result can be found in [**?**].*

Example 9.20. *Assumption 9.16 is satisfied by*

$$F(x,r,p,A) \quad = \quad -Tr(\sigma\sigma'(x)A) + \gamma r,$$

where $\sigma : \mathbb{R}^d \longrightarrow \mathcal{S}_d$ *is a Lipschitz function, and* $\gamma > 0$. *Condition (i) of Assumption 9.16 is obvious. To see that Condition (ii) is satisfied, we consider* $(A,B,\alpha) \in \mathcal{S}_d \times \mathcal{S}_d \times \mathbb{R}_+^*$ *satisfying* (9.12). *We claim that*

$$\mathrm{Tr}[MM^TA - NN^TB] \quad \le \quad 3\alpha|M-N|^2 \;=\; 3\alpha\sum_{i,j=1}^{d}(M-N)_{ij}^2.$$

To see this, observe that the matrix

$$C \quad := \quad \begin{pmatrix} NN^T & NM^T \\ MN^T & MM^T \end{pmatrix}$$

is a non-negative matrix in $\mathcal{S}_d$. *From the right hand-side inequality of* (9.12), *this implies that*

$$\begin{aligned}
\mathrm{Tr}[MM^TA - NN^TB] \quad &= \quad \mathrm{Tr}\left[C\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}\right] \\
&\le \quad 3\alpha\mathrm{Tr}\left[C\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}\right] \\
&= \quad 3\alpha\mathrm{Tr}\left[(M-N)(M-N)^T\right] \;=\; 3\alpha|M-N|^2.
\end{aligned}$$

9.5 Comparison in unbounded domains

When the domain $\mathcal{O}$ is unbounded, a growth condition on the functions u and v is needed. Then, by using the growth at infinity, we can build on the proof of Theorem 9.18 to obtain a comparison principle. The following result shows how to handle this question in the case of a sub-quadratic growth. We emphasize that the present argument can be adapted to alternative growth conditions.

The following condition differs from Assumption 9.16 only in its part (ii) where the constant 3 in (9.12) is replaced by 4 in (9.19). Thus the following Assumption 9.21 (ii) is slightly stronger than Assumption 9.16 (ii).

Assumption 9.21. (i) *There exists* $\gamma > 0$ *such that*

$$F(x,r,p,A) - F(x,r',p,A) \ge \gamma(r-r') \text{ for all } r \ge r',\ (x,p,A) \in \mathcal{O}\times\mathbb{R}^d\times\mathcal{S}_d.$$

(ii) *There is a function* $\varpi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *with* $\varpi(0+) = 0$*, such that*

$$F(y,r,\alpha(x-y),B) - F(x,r,\alpha(x-y),A) \le \varpi\left(\alpha|x-y|^2 + |x-y|\right)$$

$$\text{for all } x,y \in \mathcal{O},\ r \in \mathbb{R} \text{ and } A,B \text{ satisfying}$$

$$-4\alpha \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} \le \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \le 4\alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}. \qquad (9.19)$$

Theorem 9.22. *Let* F *be a uniformly continuous elliptic operator satisfying Assumption 9.21. Let* $u \in USC(\mathcal{O})$ *and* $v \in LSC(\mathcal{O})$ *be viscosity subsolution and supersolution of the equation (E), respectively, with* $|u(x)| + |v(x)| = \circ(|x|^2)$ *as* $|x| \to \infty$*. Then*

$$u \le v \text{ on } \partial\mathcal{O} \implies u \le v \text{ on } cl(\mathcal{O}).$$

Proof. We assume to the contrary that

$$\delta := (u-v)(z) > 0 \quad \text{for some} \quad z \in \mathbb{R}^d, \qquad (9.20)$$

and we work towards a contradiction. Let

$$M_\alpha \quad := \quad \sup_{x,y \in \mathbb{R}^d} u(x) - v(y) - \phi(x,y),$$

where

$$\phi(x,y) \quad := \quad \frac{1}{2}\left(\alpha|x-y|^2 + \varepsilon|x|^2 + \varepsilon|y|^2\right).$$

1. Since $u(x) = \circ(|x|^2)$ and $v(y) = \circ(|y|^2)$ at infinity, there is a maximizer (x_α, y_α) for the previous problem:

$$M_\alpha \quad = \quad u(x_\alpha) - v(y_\alpha) - \phi(x_\alpha, y_\alpha).$$

Moreover, there is a sequence $\alpha_n \to \infty$ such that

$$(x_n, y_n) := (x_{\alpha_n}, y_{\alpha_n}) \quad \longrightarrow \quad (\hat{x}, \hat{y}),$$

and, similar to Step 4 of the proof of Theorem 9.18, we can prove that $\hat{x} = \hat{y}$,

$$\alpha_n|x_n - y_n|^2 \longrightarrow 0, \text{ and } M_{\alpha_n} \longrightarrow M_\infty := \sup_{x \in \mathbb{R}^n} (u-v)(x) - \varepsilon|x|^2. \qquad (9.21)$$

Notice that

$$\begin{aligned}
\limsup_{n\to\infty} M_{\alpha_n} \quad &= \quad \limsup_{n\to\infty} \{u(x_n) - v(y_n) - \phi(x_n, y_n)\} \\
&\le \quad \limsup_{n\to\infty} \{u(x_n) - v(y_n))\} \\
&\le \quad \limsup_{n\to\infty} u(x_n) - \liminf_{n\to\infty} v(y_n) \\
&\le \quad (u-v)(\hat{x}).
\end{aligned}$$

Since $u \leq v$ on $\partial\mathcal{O}$, and

$$M_{\alpha_n} \quad \geq \quad \delta - \varepsilon|z|^2 \; > \; 0,$$

by (9.20), we deduce that $\hat{x} \notin \partial\mathcal{O}$ and therefore (x_n, y_n) is a local maximizer of $u - v - \phi$.

2. By the Crandal-Ishii Lemma 9.13, there exist $A_n, B_n \in \mathcal{S}_n$, such that

$$\begin{array}{rcl} (D_x\phi(x_n,y_n), A_n) & \in & \bar{\mathcal{J}}^{2,+}_{\mathcal{O}} u(t_n, x_n), \\ (-D_y\phi(x_n,y_n), B_n) & \in & \bar{\mathcal{J}}^{2,-}_{\mathcal{O}} v(t_n, y_n), \end{array} \tag{9.22}$$

and

$$-(\alpha + |D^2\phi(x_0,y_0)|)I_{2d} \leq \begin{pmatrix} A_n & 0 \\ 0 & -B_n \end{pmatrix} \leq D^2\phi(x_n,y_n) + \frac{1}{\alpha}D^2\phi(x_n,y_n)^2. \tag{9.23}$$

In the present situation, we immediately calculate that

$$D_x\phi(x_n,y_n) = \alpha(x_n - y_n) + \varepsilon x_n, \;\; -D_y\phi(x_n,y_n) = \alpha(x_n - y_n) - \varepsilon y_n$$

and

$$D^2\phi(x_n,y_n) \quad = \quad \alpha\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + \varepsilon\, I_{2d},$$

which reduces the right hand-side of (9.23) to

$$\begin{pmatrix} A_n & 0 \\ 0 & -B_n \end{pmatrix} \quad \leq \quad (3\alpha + 2\varepsilon)\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + \left(\varepsilon + \frac{\varepsilon^2}{\alpha}\right)I_{2d}, \tag{9.24}$$

while the left land-side of (9.23) implies:

$$-3\alpha I_{2d} \quad \leq \quad \begin{pmatrix} A_n & 0 \\ 0 & -B_n \end{pmatrix} \tag{9.25}$$

3. By (9.22) and the viscosity properties of u and v, we have

$$\begin{array}{rcl} F(x_n, u(x_n), \alpha_n(x_n - y_n) + \varepsilon x_n, A_n) & \leq & 0, \\ F(y_n, v(y_n), \alpha_n(x_n - y_n) - \varepsilon y_n, B_n) & \geq & 0. \end{array}$$

Using Assumption 9.21 (i) together with the uniform continuity of H, this implies that:

$$\begin{array}{rcl} \gamma\big(u(x_n) - v(x_n)\big) & \leq & F\big(y_n, u(x_n), \alpha_n(x_n - y_n), \tilde{B}_n\big) \\ & & -F\big(x_n, u(x_n), \alpha_n(x_n - y_n), \tilde{A}_n\big) + c(\varepsilon) \end{array}$$

where $c(.)$ is a modulus of continuity of F, and $\tilde{A}_n := A_n - 2\varepsilon I_n$, $\tilde{B}_n := B_n + 2\varepsilon I_n$. By (9.24), we have

$$-4\alpha I_{2d} \;\leq\; \begin{pmatrix} \tilde{A}_n & 0 \\ 0 & -\tilde{B}_n \end{pmatrix} \;\leq\; 4\alpha\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix},$$

for small ε. Then, it follows from Assumption 9.21 (ii) that

$$\gamma\big(u(x_n) - v(x_n)\big) \quad \leq \quad \varpi\big(\alpha_n|x_n - y_n|^2 + |x_n - y_n|\big) + c(\varepsilon).$$

By sending n to infinity, it follows from (9.21) that:

$$c(\varepsilon) \geq \gamma\big(M_\infty + |\hat{x}|^2\big) \geq \gamma M_\infty \geq \gamma\big(u(z) - v(z) - \varepsilon|z|^2\big),$$

and we get a contradiction of (9.20) by sending ε to zero. ♢

9.6 Useful applications

We conclude this section by two consequences of the above comparison results, which are trivial properties in the context of classical solutions.

Lemma 9.23. *Let $\mathcal{O}$ be an open interval of $\mathbb{R}$, and $U : \mathcal{O} \longrightarrow \mathbb{R}$ be a lower semicontinuous viscosity supersolution of the equation $DU \geq 0$ on $\mathcal{O}$. Then U is nondecreasing on $\mathcal{O}$.*

Proof. For each $\varepsilon > 0$, define $W(x) := U(x) + \varepsilon x$, $x \in \mathcal{O}$. Then W satisfies in the viscosity sense $DW \geq \varepsilon$ in $\mathcal{O}$, i.e. for all $(x_0, \varphi) \in \mathcal{O} \times C^1(\mathcal{O})$ such that

$$(W - \varphi)(x_0) \quad = \quad \min_{x \in \mathcal{O}} (W - \varphi)(x), \tag{9.26}$$

we have $D\varphi(x_0) \geq \varepsilon$. This proves that φ is strictly increasing in a neighborhood $\mathcal{V}$ of x_0. Let $(x_1, x_2) \subset \mathcal{V}$ be an open interval containing x_0. We intend to prove that

$$W(x_1) \quad < \quad W(x_2), \tag{9.27}$$

which provides the required result from the arbitrariness of $x_0 \in \mathcal{O}$.

To prove (9.27), suppose to the contrary that $W(x_1) \geq W(x_2)$, and the consider the function $v(x) = W(x_2)$ which solves the equation

$$Dv \;=\; 0 \quad \text{on the open interval} \quad (x_1, x_2).$$

together with the boundary conditions $v(x_1) = v(x_2) = W(x_2)$. Observe that W is a lower semicontinuous viscosity supersolution of the above equation. From the comparison theorem of Remark **??**, this implies that

$$\sup_{[x_1, x_2]} (v - W) \quad = \quad \max\left\{(v - W)(x_1), (v - W)(x_2)\right\} \;\leq\; 0.$$

Hence $W(x) \geq v(x) = W(x_2)$ for all $x \in [x_1, x_2]$. Applying this inequality at $x_0 \in (x_1, x_2)$, and recalling that the test function φ is strictly increasing on $[x_1, x_2]$, we get :

$$(W - \varphi)(x_0) \quad > \quad (W - \varphi)(x_2),$$

contradicting (9.26). ♢

Lemma 9.24. *Let $\mathcal{O}$ be an open interval of $\mathbb{R}$, and $U : \mathcal{O} \longrightarrow \mathbb{R}$ be a lower semicontinuous viscosity supersolution of the equation $-D^2U \geq 0$ on $\mathcal{O}$. Then U is concave on $\mathcal{O}$.*

Proof. Let $a < b$ be two arbitrary elements in $\mathcal{O}$, and consider some $\varepsilon > 0$ together with the function

$$v(s) \;:=\; \frac{U(a)\left(e^{\sqrt{\varepsilon}(b-s)}-1\right)+U(b)\left(e^{\sqrt{\varepsilon}(s-a)}-1\right)}{e^{\sqrt{\varepsilon}(b-a)}-1} \quad \text{for} \quad a \leq s \leq b.$$

Clearly, v solves the equation

$$(\varepsilon v - D^2 v)(t,s) = 0 \quad \text{on} \quad (a,b).$$

Since U is lower semicontinuous it is bounded from below on the interval $[a,b]$. Therefore, by possibly adding a constant to U, we can assume that $U \geq 0$, so that U is a lower semicontinuous viscosity supersolution of the above equation. It then follows from the comparison theorem 10.6 that :

$$\sup_{[a,b]}(v-U) \;=\; \max\left\{(v-U)(a),(v-U)(b)\right\} \;\leq\; 0.$$

Hence,

$$U(s) \;\geq\; v(s) \;=\; \frac{U(a)\left(e^{\sqrt{\varepsilon}(b-s)}-1\right)+U(b)\left(e^{\sqrt{\varepsilon}(s-a)}-1\right)}{e^{\sqrt{\varepsilon}(b-a)}-1}$$

and by sending ε to zero, we see that

$$U(s) \;\geq\; (U(b)-U(a))\frac{s-a}{b-a}+U(a)$$

for all $s \in [a,b]$. Let λ be an arbitrary element of the interval [0,1], and set $s := \lambda a + (1-\lambda)b$. The last inequality takes the form :

$$U\big(\lambda a + (1-\lambda)b\big) \;\geq\; \lambda U(a) + (1-\lambda)U(b)\ ,$$

proving the concavity of U. ♢

9.7 Appendix: proof of the Crandal-Ishii's lemma

Chapter 10

Dynamic Programming Equation in Viscosity Sense

10.1 DPE for stochastic control problems

We now turn to the stochastic control problem introduced in Section 6.1. The chief goal of this paragraph is to use the notion of viscosity solutions in order to relax the smoothness condition on the value function V in the statement of Propositions 6.5 and 6.4. Notice that the following proofs are obtained by slight modification of the corresponding proofs in the smooth case.

Remark 10.1. Recall that the general theory of viscosity solutions applies for nonlinear partial differential equations on an open domain $\mathcal{O}$. This indeed ensures that the optimizer in the definition of viscosity solutions is an interior point. In the setting of control problems with finite horizon, the time variable moves forward so that the zero boundary is not relevant. We shall then write the DPE on the domain $[0,T) \times \mathbb{R}^n$. Although this is not an open domain, the general theory of viscosity solutions is still valid.

Proposition 10.2. *Assume that V is locally bounded on $[0,T) \times \mathbb{R}^n$, and let the coefficients $k(\cdot,\cdot,u)$ and $f(\cdot,\cdot,u)$ be continuous in (t,x) for all fixed $u \in U$. Then, the value function V is a (discontinuous) viscosity supersolution of the equation*

$$-\partial_t V(t,x) - H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \quad \geq \quad 0 \qquad (10.1)$$

on $[0,T) \times \mathbb{R}^n$.

Proof. Let $(t,x) \in Q := [0,T) \times \mathbb{R}^n$ and $\varphi \in C^2(Q)$ be such that

$$0 \;=\; (V_* - \varphi)(t,x) \quad = \quad \max_{Q}\,(V_* - \varphi). \qquad (10.2)$$

Let $(t_n, x_n)_n$ be a sequence in Q such that

$$(t_n, x_n) \longrightarrow (t, x) \quad \text{and} \quad V(t_n, x_n) \longrightarrow V_*(t, x).$$

Since φ is smooth, notice that

$$\eta_n := V(t_n, x_n) - \varphi(t_n, x_n) \longrightarrow 0.$$

Next, let $u \in U$ be fixed, and consider the constant control process $\nu = u$. We shall denote by $X^n := X^{t_n, x_n, u}$ the associated state process with initial data $X^n_{t_n} = x_n$. Finally, for all $n > 0$, we define the stopping time :

$$\theta_n := \inf \{ s > t_n \; : \; (s - t_n, X^n_s - x_n) \notin [0, h_n) \times \alpha B \},$$

where $\alpha > 0$ is some given constant, B denotes the unit ball of $\mathbb{R}^n$, and

$$h_n := \sqrt{\eta_n} \mathbf{1}_{\{\eta_n \neq 0\}} + n^{-1} \mathbf{1}_{\{\eta_n = 0\}}.$$

Notice that $\theta_n \longrightarrow t$ as $n \longrightarrow \infty$.

1. From the dynamic programming principle, it follows that:

$$0 \le \mathbb{E} \left[V(t_n, x_n) - \beta(t_n, \theta_n) V_*(\theta_n, X^n_{\theta_n}) - \int_{t_n}^{\theta_n} \beta(t_n, r) f(r, X^n_r, \nu_r) dr \right].$$

Now, in contrast with the proof of Proposition 6.4, the value function is not known to be smooth, and therefore we can not apply Itô's formula to V. The main trick of this proof is to use the inequality $V^* \le \varphi$ on Q, implied by (10.2), so that we can apply Itô's formula to the smooth test function φ:

$$\begin{aligned} 0 &\le \eta_n + \mathbb{E} \left[\varphi(t_n, x_n) - \beta(t_n, \theta_n) \varphi(\theta^n_h, X^n_{\theta_n}) - \int_{t_n}^{\theta_n} \beta(t_n, r) f(r, X^n_r, \nu_r) dr \right] \\ &= \eta_n - \mathbb{E} \left[\int_{t_n}^{\theta_n} \beta(t_n, r)(\partial_t \varphi + \mathcal{L}^{\cdot} \varphi - f)(r, X^n_r, u) dr \right] \\ &\quad - \mathbb{E} \left[\int_{t_n}^{\theta_n} \beta(t_n, r) D\varphi(r, X^n_r) \sigma(r, X^n_r, u) dW_r \right], \end{aligned}$$

where $\partial_t \varphi$ denotes the partial derivative with respect to t.

2. We now continue exactly along the lines of the proof of Proposition 6.5. Observe that $\beta(t_n, r) D\varphi(r, X^n_r) \sigma(r, X^n_r, u)$ is bounded on the stochastic interval $[t_n, \theta_n]$. Therefore, the second expectation on the right hand-side of the last inequality vanishes, and :

$$\frac{\eta_n}{h_n} - \mathbb{E} \left[\frac{1}{h_n} \int_{t_n}^{\theta_n} \beta(t_n, r)(\partial_t \varphi + \mathcal{L}^{\cdot} \varphi - f)(r, X_r, u) dr \right] \ge 0.$$

We now send n to infinity. The a.s. convergence of the random value inside the expectation is easily obtained by the mean value Theorem; recall that for $n \geq N(\omega)$ sufficiently large, $\theta_n(\omega) = h_n$. Since the random variable $h_n^{-1}\int_t^{\theta_n}\beta(t_n,r)(\mathcal{L}^{\cdot}\varphi - f)(r,X_r^n,u)dr$ is essentially bounded, uniformly in n, on the stochastic interval $[t_n,\theta_n]$, it follows from the dominated convergence theorem that :

$$-\partial_t\varphi(t,x) - \mathcal{L}^u\varphi(t,x) - f(t,x,u) \quad \geq \quad 0,$$

which is the required result, since $u \in U$ is arbitrary. ♢

We next wish to show that V satisfies the nonlinear partial differential equation (10.1) with equality, in the viscosity sense. This is also obtained by a slight modification of the proof of Proposition 6.5.

Proposition 10.3. *Assume that the value function V is locally bounded on $[0,T)\times\mathbb{R}^n$. Let the function H be continuous, and $\|k^+\|_\infty < \infty$. Then, V is a (discontinuous) viscosity subsolution of the equation*

$$-\partial_t V(t,x) - H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \quad \leq \quad 0 \qquad (10.3)$$

on $[0,T)\times\mathbb{R}^n$.

Proof. Let $(t_0,x_0) \in Q := [0,T)\times\mathbb{R}^n$ and $\varphi \in C^2(Q)$ be such that

$$0 \;=\; (V^*-\varphi)(t_0,x_0) \;>\; (V^*-\varphi)(t,x) \quad \text{for} \quad (t,x) \in Q\setminus\{(t_0,x_0)\} \qquad (10.4)$$

In order to prove the required result, we assume to the contrary that

$$h(t_0,x_0) \;:=\; \partial_t\varphi(t_0,x_0) + H\left(t_0,x_0,\varphi(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0)\right) \quad > \quad 0,$$

and work towards a contradiction.

1. Since H is continuous, there exists an open neighborhood of (t_0,x_0) :

$$\mathcal{N}_\eta \quad := \quad \{(t,x)\;:\;(t-t_0,x-x_0) \in (-\eta,\eta)\times\eta B \text{ and } h(t,x) > 0\}\ ,$$

for some $\eta > 0$. From (10.4), it follows that

$$-3\gamma e^{\eta\|k^+\|_\infty} \quad := \quad \max_{\partial\mathcal{N}_\eta}(V^*-\varphi) \;<\; 0\ . \qquad (10.5)$$

Next, let $(t_n,x_n)_n$ be a sequence in $\mathcal{N}_h$ such that

$$(t_n,x_n) \;\longrightarrow\; (t_0,x_0) \quad \text{and} \quad V(t_n,x_n) \;\longrightarrow\; V_*(t_0,x_0)\ .$$

Since $(V-\varphi)(t_n,x_n) \longrightarrow 0$, we can assume that the sequence (t_n,x_n) also satisfies :

$$|(V-\varphi)(t_n,x_n)| \;\leq\; \gamma \quad \text{for all} \quad n \geq 1\ . \qquad (10.6)$$

Finally, we introduce a $\gamma-$optimal control $\tilde{\nu}^n$ for the problem $V(t_n,x_n)$, i.e.

$$J(t_n,x_n,\tilde{\nu}^n) \quad \geq \quad V(t_n,x_n) - \gamma. \qquad (10.7)$$

We shall denote by $\tilde{X}^n$ and $\tilde{\beta}^n$ the controlled process and the discount factor defined by the control $\tilde{\nu}^n$ and the initial data $\tilde{X}^n_{t_n} = x_n$.

3. Consider the stopping time

$$\theta_n \quad := \quad \inf\left\{ s > t_n \ : \ (s, \tilde{X}^n_s) \not\in \mathcal{N}_\eta \right\} \ ,$$

and observe that, by continuity of the state process, $(\theta_n, \tilde{X}^n_{\theta_n}) \in \partial\mathcal{N}_\eta$, so that :

$$(V^* - \varphi)(\theta_n, \tilde{X}^n_{\theta_n}) \quad \geq \quad 3\gamma e^{-\eta\|k^+\|_\infty} \tag{10.8}$$

by (10.5). Then, it follows from (10.8) and (10.6) that :

$$\begin{aligned} & \tilde{\beta}^n(t_n,\theta_n)V^*(\theta_n,\tilde{X}^n_{\theta_n}) - V(t_n,x_n) \\ \leq \ & \tilde{\beta}^n(t_n,\theta_n)\varphi^*(\theta_n,\tilde{X}^n_{\theta_n}) - \varphi(t_n,x_n) - 3\gamma e^{\eta\|k^+\|_\infty}\tilde{\beta}^n(t_n,\theta_n) + \gamma \\ \leq \ & -2\gamma + \int_{t_n}^{\theta_n} d[\tilde{\beta}^n(t_n,r)\varphi(r,\tilde{X}^n_r)]. \end{aligned}$$

By Itô's formula, this provides :

$$V(t_n,x_n) \quad \geq \quad 2\gamma + \mathbb{E}_{t_n,x_n}\left[\tilde{\beta}^n(t_n,\theta_n)V^*(\theta_n,\tilde{X}^n_{\theta_n}) - \int_{t_n}^{\theta_n} (\partial_t\varphi + \mathcal{L}^{\tilde{\nu}^n_r}\varphi)(r,\tilde{X}^n_r)dr \right],$$

where the stochastic term has zero mean, as its integrand is bounded on the stochastic interval $[t_n,\theta_n]$. Observe also that $(\partial_t\varphi + \mathcal{L}^{\tilde{\nu}_r}\varphi)(r,\tilde{X}^n_r) + f(r,\tilde{X}^n_r,\tilde{\nu}^n_r)$ $\geq h(r,\tilde{X}^n_r) \geq 0$ on the stochastic interval $[t_n,\theta_n]$. We therefore deduce that :

$$V(t_n,x_n) \quad \leq \quad 2\gamma + \mathbb{E}_{t_n,x_n}\left[\int_{t_n}^{\theta_n} \tilde{\beta}^n(t_n,r)f(r,\tilde{X}_r,\tilde{\nu}_r) + \tilde{\beta}^n(t_n,\theta_n)V^*(\theta_n,\tilde{X}^n_{\theta_n}) \right].$$

Since $V^*(\theta_n,\tilde{X}^n_{\theta_n}) \geq V(\theta_n,\tilde{X}^n_{\theta_n}) \geq J(\theta_n,\tilde{X}^n_{\theta_n},\tilde{\nu}^n)$, this provides:

$$V(t_n,x_n) \ \leq \ 2\gamma + J(t_n,x_n,\tilde{\nu}) \ \leq \ \gamma + V(t_n,x_n),$$

where the last inequality follows by (10.7). This completes the proof. ♢

As a consequence of Propositions 10.3 and 10.2, we have the main result of this section :

Theorem 10.4. *Let the conditions of Propositions 10.3 and 10.2 hold. Then, the value function V is a (discontinuous) viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$-\partial_t V(t,x) - H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \quad = \quad 0 \tag{10.9}$$

on $[0,T) \times \mathbb{R}^n$.

The partial differential equation (10.9) has a very simple and specific dependence in the time-derivative term. Because of this, it is usually referred to as a *parabolic* equation.

In order to a obtain a characterization of the value function by means of the dynamic programming equation, the latter viscosity property needs to be complemented by a uniqueness result. This is usually obtained as a consequence of a comparison result.

In the present situation, one may verify the conditions of Theorem 9.22. For completeness, we report a comparison result which is adapted for the class of equations corresponding to stochastic control problems.

Consider the parabolic equation:

$$\partial_t u + G\left(t, x, Du(t,x), D^2u(t,x)\right) \;=\; 0 \quad \text{on} \quad Q \;:=\; [0,T) \times \mathbb{R}^n \;, \tag{10.10}$$

where G is elliptic and continuous. For $\gamma > 0$, set

$$\begin{aligned} G^{+\gamma}(t,x,p,A) &:= \sup\left\{G(s,y,p,A) \;:\; (s,y) \in B_Q(t,x;\gamma)\right\} \;, \\ G^{-\gamma}(t,x,p,A) &:= \inf\left\{G(s,y,p,A) \;:\; (s,y) \in B_Q(t,x;\gamma)\right\} \;, \end{aligned}$$

where $B_Q(t,x;\gamma)$ is the collection of elements (s,y) in Q such that $|t-s|^2+|x-y|^2 \le \gamma^2$. We report, without proof, the following result from [**?**] (Theorem V.8.1 and Remark V.8.1).

Assumption 10.5. *The above operators satisfy:*

$$\begin{array}{l} \limsup_{\varepsilon\searrow 0}\left\{G^{+\gamma_\varepsilon}(t_\varepsilon,x_\varepsilon,p_\varepsilon,A_\varepsilon) - G^{-\gamma_\varepsilon}(s_\varepsilon,y_\varepsilon,p_\varepsilon,B_\varepsilon)\right\} \\ \le\; Const\left(|t_0-s_0|+|x_0-y_0|\right)\left[1+|p_0|+\alpha\left(|t_0-s_0|+|x_0-y_0|\right)\right] \end{array} \tag{10.11}$$

for all sequences $(t_\varepsilon,x_\varepsilon), (s_\varepsilon,y_\varepsilon) \in [0,T)\times\mathbb{R}^n$, $p_\varepsilon \in \mathbb{R}^n$, *and* $\gamma_\varepsilon \ge 0$ *with :*

$$((t_\varepsilon,x_\varepsilon),(s_\varepsilon,y_\varepsilon),p_\varepsilon,\gamma_\varepsilon) \;\longrightarrow\; ((t_0,x_0),(s_0,y_0),p_0,0) \quad as \quad \varepsilon \searrow 0,$$

and symmetric matrices $(A_\varepsilon, B_\varepsilon)$ *with*

$$-KI_{2n} \;\le\; \begin{pmatrix} A_\varepsilon & 0 \\ 0 & -B_\varepsilon \end{pmatrix} \;\le\; 2\alpha\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}$$

for some α *independent of* ε.

Theorem 10.6. *Let Assumption 10.5 hold true, and let* $u \in USC([0,T]\times\mathbb{R}^n)$, $v \in LSC([0,T]\times\mathbb{R}^n)$ *be viscosity subsolution and supersolution of* (10.10), *respectively. Then*

$$\sup_{\bar{Q}}(u-v) \;=\; \sup_{\mathbb{R}^n}(u-v)(T,\cdot).$$

A sufficient condition for (10.11) to hold is that $f(\cdot,\cdot,u)$, $k(\cdot,\cdot,u)$, $b(\cdot,\cdot,u)$, and $\sigma(\cdot,\cdot,u) \in C^1(\bar{Q})$ with

$$\begin{array}{r} \|b_t\|_\infty + \|b_x\|_\infty + \|\sigma_t\|_\infty + \|\sigma_x\|_\infty \;<\; \infty \\ |b(t,x,u)| + |\sigma(t,x,u)| \;\le\; Const(1+|x|+|u|) \;; \end{array}$$

see [**?**], Lemma V.8.1.

10.2 DPE for optimal stopping problems

We first recall the optimal stopping problem considered in Section 7.1. For $0 \le t \le T \le \infty$, the set $\mathcal{T}_{[t,T]}$ denotes the collection of all $\mathbb{F}-$stopping times with values in $[t,T]$. The state process X is defined by the SDE:

$$dX_t \;=\; \mu(t,X_t)dt + \sigma(t,X_t)dW_t, \tag{10.12}$$

where μ and σ are defined on $\bar{\mathbf{S}} := [0,T) \times \mathbb{R}^n$, take values in $\mathbb{R}^n$ and $\mathcal{S}_n$, respectively, and satisfy the usual Lipschitz and linear growth conditions so that the above SDE has a unique strong solution satisfying the integrability of Theorem 5.2.

For a measurable function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, satisfying $\mathbb{E}\left[\sup_{0\le t<T} |g(X_t)|\right] < \infty$, the gain criterion is given by:

$$J(t,x,\tau) := \mathbb{E}\left[g\left(X_\tau^{t,x}\right)\mathbf{1}_{\tau<\infty}\right] \quad \text{for all} \quad (t,x) \in \mathbf{S},\ \tau \in \mathcal{T}_{[t,T]}. \tag{10.13}$$

Here, $X^{t,x}$ denotes the unique strong solution of (7.1) with initial condition $X_t^{t,x} = x$. Then, the optimal stopping problem is defined by:

$$V(t,x) := \sup_{\tau\in\mathcal{T}_{[t,T]}} J(t,x,\tau) \quad \text{for all} \quad (t,x) \in \mathbf{S}. \tag{10.14}$$

The next result derives the dynamic programming equation for the latter optimal stopping problem in the sense of viscosity solution, thus relaxing the $C^{1,2}$ regularity condition in the statement of Theorem 7.4.

Theorem 10.7. *Assume that V is locally bounded, and let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous. Then V is a viscosity solution of the obstacle problem:*

$$\min\left\{-(\partial_t + \mathcal{A})V\ ,\ V - g\right\} \;=\; 0 \quad on \quad \mathbf{S}. \tag{10.15}$$

Proof. (i) We first show that V_* is a viscosity supersolution. As in the proof of Theorem 7.13, the inequality $V - g \ge 0$ is obvious, and implies that $V_* \ge g$. Let $(t_0,x_0) \in \mathbf{S}$ and $\varphi \in C^2(\mathbf{S})$ be such that

$$0 = (V_* - \varphi)(t_0,x_0) \;=\; \min_{\mathbf{S}}(V_* - \varphi).$$

To prove that $-(\partial_t + \mathcal{A})\varphi(t_0,x_0) \ge 0$, we consider a sequence $(t_n,x_n)_{n\ge1} \subset [t_0-h,t_0+h] \times B$, for some small $h>0$, such that

$$(t_n,x_n) \longrightarrow (t_0,x_0) \quad \text{and} \quad V(t_n,x_n) \longrightarrow V_*(t_0,x_0).$$

Let $(h_n)_n$ be a sequence of positive scalars converging to zero, to be fixed later, and introduce the stopping times

$$\theta_{h_n}^n := \inf\left\{t > t_n :\ (t, X_t^{t_n,x_n}) \notin [t_0-h_n, t_0+h_n] \times B\right\}.$$

Then $\theta_{h_n} \in \mathcal{T}_{[t,T]}^t$ for sufficiently small h, and it follows from (7.10)that:

$$V(t_n,x_n) \;\ge\; \mathbb{E}\left[V_*\left(\theta_h^n, X_{\theta_h^n}\right)\right].$$

Since $V_* \geq \varphi$, and denoting $\eta_n := (V-\varphi)(t_n,x_n)$, this provides

$$\eta_n + \varphi(t_n,x_n) \geq \mathbb{E}\left[\varphi\left(\theta_h^n, X_{\theta_h^n}\right)\right] \quad \text{where} \quad \eta_n \longrightarrow 0.$$

We continue by fixing

$$h_n \quad := \quad \sqrt{\eta_n}\mathbf{1}_{\{\eta_n \neq 0\}} + n^{-1}\mathbf{1}_{\{\eta_n = 0\}},$$

as in the proof of Proposition 10.2. Then, the rest of the proof follows exactly the line of argument of the proof of Theorem 7.4 combined with that of Proposition 10.2.

(ii) We next prove that V^* is a viscosity subsolution of of the equation (10.15). Let $(t_0,x_0) \in \mathbf{S}$ and $\varphi \in C^2(\mathbf{S})$ be such that

$$0 \quad = \quad (V^*-\varphi)(t_0,x_0) \; = \; \text{strict} \; \max_{\mathbf{S}} \, (V^*-\varphi),$$

assume to the contrary that

$$(V^*-g)(t_0,x_0) > 0 \quad \text{and} \quad -(\partial_t + \mathcal{A})\varphi(t_0,x_0) > 0,$$

and let us work towards a contradiction of the weak dynamic programming principle.

Since g is continuous, and $V^*(t_0,x_0) = \varpi(t_0,x_0)$, we may finds constants $h>0$ and $\delta>0$ so that

$$\varphi \geq g + \delta \text{ and } -(\partial_t + \mathcal{A})\varphi \geq 0 \quad \text{on} \quad \mathcal{N}_h := [t_0, t_0+h] \times hB, \tag{10.16}$$

where B is the unit ball centered at x_0. Moreover, since (t_0,x_0) is a strict maximizer of the difference $V^*-\varphi$:

$$-\gamma \quad := \quad \max_{\partial \mathcal{N}_h}(V^*-\varphi) \; < \; 0. \tag{10.17}$$

let (t_n,x_n) be a sequence in $\mathbf{S}$ such that

$$(t_n,x_n) \longrightarrow (t_0,x_0) \quad \text{and} \quad V(t_n,x_n) \longrightarrow V^*(t_0,x_0).$$

We next define the stopping times:

$$\theta_n \quad := \quad \inf\left\{t > t_n : \; \left(t, X_t^{t_n,x_n}\right) \notin \mathcal{N}_h\right\},$$

and we continue as in Step 2 of the proof of Theorem 7.4. We denote $\eta_n := V(t_n,x_n) - \varphi(t_n,x_n)$, and we compute by Itô's formula that for an arbitrary stopping rule $\tau \in \mathcal{T}_{[t,T]}^t$:

$$\begin{aligned} V(t_n,x_n) \quad &= \quad \eta_n + \varphi(t_n,x_n) \\ &= \quad \eta_n + \mathbb{E}\left[\varphi\left(\tau\wedge\theta_n, X_{\tau\wedge\theta_n}\right) - \int_{t_n}^{\tau\wedge\theta_n} (\partial_t + \mathcal{A})\varphi(t,X_t)dt\right], \end{aligned}$$

where diffusion term has zero expectation because the process $(t, X_t^{t_n,x_n})$ is confined to the compact subset $\mathcal{N}_h$ on the stochastic interval $[t_n, \tau \wedge \theta_n]$. Since $-(\partial_t + \mathcal{A})\varphi \geq 0$ on $\mathcal{N}_h$ by (10.16), this provides:

$$\begin{array}{rcl} V(t_n, x_n) & \geq & \eta_n + \mathbb{E}\left[\varphi\left(\tau, X_\tau\right)\mathbf{1}_{\{\tau<\theta_n\}} + \varphi\left(\theta_n, X_{\theta_n}\right)\mathbf{1}_{\tau \geq \{\theta_n\}} + \right] \\ & \geq & \mathbb{E}\left[\left(g\left(X_\tau\right) + \delta\right)\mathbf{1}_{\{\tau<\theta_n\}} + \left(V^*\left(\theta_n, X_{\theta_n}\right) + \gamma\right)\mathbf{1}_{\{\theta_n \geq \tau\}}\right] \\ & \geq & \gamma \wedge \delta + \mathbb{E}\left[g\left(X_\tau\right)\mathbf{1}_{\{\tau<\theta_n\}} + V^*\left(\theta_n, X_{\theta_n}\right)\mathbf{1}_{\{\theta_n \geq \tau\}}\right], \end{array}$$

where we used the fact that $\varphi \geq g + \delta$ on $\mathcal{N}_h$ by (10.16), and $\varphi \geq V^* + \gamma$ on $\partial\mathcal{N}_h$ by (10.17). Since $\eta_n := (V - \varphi)(t_n, x_n) \longrightarrow 0$ as $n \to \infty$, and $\tau \in \mathcal{T}_{[t,T]}^t$ is arbitrary, this provides the desired contradiction of (7.9). ♢

10.3 A comparison result for obstacle problems